WHAT CAN BE LEARNED FROM
\[ \tau \rightarrow \pi \pi \pi \nu_\tau \]
FOR THE AXIAL FORM
FACTOR OF THE NUCLEON

SERGI GONZÀLEZ-SOLÍS

DEPARTMENT OF PHYSICS, INDIANA UNIVERSITY
BLOOMINGTON, IN 47405, USA

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Outline

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INTRODUCTION AND MOTIVATION
Neutrino-Nucleus Scattering

- Accurate neutrino measurements: oscillations, mass hierarchy, CP violation etc.
  \[ \Rightarrow \text{need to know the cross section very precisely} \]

- \[ \sigma \sim \text{flux (\# of } \nu \text{'s}) \times |\text{nucleon amplitude}|^2 \times \text{nuclear effects} \]

- One can distinguish between different energy regions
  - Quasi elastic scattering
    \[ \nu_\mu n \rightarrow \mu^- p \]
    (used to study the nucleon form factors)
  - Resonance production
  - Deep inelastic scattering
Form Factors of the Nucleon

- Nucleon matrix element

\[
\langle N(p')| J_{em}^{\mu}| N(p) \rangle = \bar{u}(p') \left[ \gamma^{\mu} F_1(t) + \frac{i}{2m_N} \sigma^{\mu\nu} (p' - p)_\nu F_2(t) \right] u(p) ,
\]

\[
\langle N(p')| J_A^{\mu a}| N(p) \rangle = \bar{u}(p') \frac{\tau^a}{2} \gamma_5 \left[ \gamma^{\mu} F_A(t) + \frac{(p' - p)_{\mu}}{2m_N} F_P(t) \right] u(p) ,
\]

- Four Form Factor to determine \((t = (p' - p)^2)\)

  - \(F_1(t)\) and \(F_2(t)\): Dirac and Pauli Form Factors
  - \(G_E(t)\) and \(G_M(t)\): electric and magnetic (Sachs) FFs, well-known
    \[
    G_E(t) = F_1(t) + \frac{t}{4m_N^2} F_2(t) , \quad G_M(t) = F_1(t) + F_2(t)
    \]
  - \(F_A(t)\): Axial Form Factor: main unknown
  - \(F_P(t)\): Pseudo-scalar Form Factor. It can be related to the Axial FF using PCAC or pion-pole approximation
    \[
    F_P(t) = \frac{2m_N^2}{m^2_\pi - t} F_A(t)
    \]
**Electromagnetic Form Factors: Time-like**

- **Dispersive parametrizations**
  (Belushkin’06, Lorenz’12, Hoferichter’16, Leupold’17, Alarcon’18)

\[
G_{E,M}(t) = \frac{1}{\pi} \int_{s_{th}}^{\infty} ds \frac{\Im G_{E,M}^{p,n}(s)}{s - t - i\epsilon},
\]

\[
G_{E,M} = \frac{1}{2} \left( G_E^p - G_E^n \right),
\]

\[
\Im G_E^V(t) = \frac{k_{cm}^3}{m_N \sqrt{t}} f_+(t) F_{\pi}^V(t)
\]

\[
\Im G_M^V(t) = \frac{k_{cm}^3}{\sqrt{2t}} f_+(t) F_{\pi}^{V*}(t)
\]

where \(k_{cm} = \sqrt{t/4 - m_{\pi}^2}\)
Electromagnetic Form Factors: Space-like

\[ Q^2 \equiv -t \geq 0 \]
With very precise data dipole parametrizations fail for e.m. FFs
**Axial Vector Form Factor**

- $F_A(t)$: Main Unknown

- How to determine $F_A(t)$ experimentally?
  - $\nu_\mu n \rightarrow \mu p$: Best way but old data from the 80s
  - $e^- p \rightarrow e^- n \pi^+$ pion electroproduction using ChPT at threshold
    - This low-energy theorem is strictly valid in the chiral limit $m_\pi = 0$ (extrapolation using ChPT)
    - Problem: most of experimental data is outside of ChPT range
  - $\mu$ capture: allow to determine $F_P(t)$

- What do we know theoretically on the form factors?
  - Their low-energy behaviour: given by ChPT
  - Their high-energy behaviour: given by pQCD ($\sim 1/Q^4$)
  - For the intermediate energy region: models i.e. VMD
**Representations of the Axial Vector Form Factor**

- **Dispersive representation**

\[
F_A(t) = \frac{1}{\pi} \int_{9m^2_\pi}^{\infty} ds' \frac{\text{Im} F_A(t')}{t' - t - i\varepsilon},
\]

\[
\text{Im} F_A(t) = \pi \rho_A(t),
\]

- **Warm-up: Axial Meson Dominance of a** \( J^{PC} = 1^{++} \) **state,** \( a_1(1260) \)

\[
\rho_A(t) = g_A m_A^2 \delta(t - m_A^2)
\]

\[
F_A(Q^2) = g_A \frac{m_A^2}{m_A^2 + Q^2}, \quad (Q^2 \equiv -t)
\]

**Drawbacks:**
- Neglects \( a_1(1640) \) etc.
- It cannot be used above \( 9m^2_\pi \)
- It does not falls as \( 1/Q^4 \) (pQCD)
Representations of the Axial Vector Form Factor

Dipole parametrization

\[ F_A(Q^2) = \frac{1}{(1 + Q^2/m_A^2)^2}, \]

Simple ansatz that fulfills pQCD

two degenerate resonances

\[ \rho_A(t) \propto \delta(t - m_A) + \delta(t - m'_A) \]

with residues of equal magnitude and opposite signs
Dipole parametrization

\[ F_A(Q^2) = \frac{1}{(1 + Q^2/m_A^2)^2}, \]

Shortcomings:

- Introduces a tight and unnatural artificial bias connecting high-and-low regions of \( Q^2 \)
- Respects pQCD \((1/Q^4)\)...  
- ...but scaling effects take place well outside the energy window accessible at \( \nu N \) experiments
- No solid reason to believe that it will be a precise description of the physics at high-energies
Dipole parametrization

\[ F_A(Q^2) = \frac{1}{(1 + Q^2/m_A^2)^2}, \]

► Recent Lattice results: reason to consider the Axial Form Factor seriously
Expanding the normalized axial form factor around $Q^2 = 0$

$$\tilde{F}_A(Q^2) = 1 - \frac{1}{6} \langle r_A^2 \rangle Q^2 + \frac{1}{120} \langle r_A^4 \rangle (Q^2)^2 + \ldots$$

For the dipole form factor we have:

$$\langle r_A^2 \rangle = \frac{12}{m_A^2}, \quad \langle r_A^4 \rangle = \frac{360}{m_A^4},$$

The slope and curvature are the only relevant shape parameters at very low-$Q^2$

Their definition is model-independent: allows to compare different theoretical approaches
Parametrizations abandoning the dipole for the low-$Q^2$

- Two monopoles including $a_1(1260)$ and $a_1(1640)$ (Masjuan’12):
  \[
  \tilde{F}_A(Q^2) = \frac{1}{(1 + Q^2/m_{a_1}^2) \left(1 + Q^2/m_{a_1'}^2\right)}.
  \]

- Chiral Effective Lagrangian model (Scherer’18)
  \[
  \tilde{F}_A(Q^2) = 1 + \tilde{c}_1 Q^2 - \tilde{c}_2 \frac{(Q^2)^2}{M_A^2(M_A^2 + Q^2)},
  \]

- Taylor expansion in terms of a conformal complex variable $z$
  \[
  F_A(t) = \sum_{k=0}^{\infty} a_k z(t)^k, \quad z(t, 9m_{\pi}^2, t_0) = \frac{\sqrt{9m_{\pi}^2 - t} - \sqrt{9m_{\pi}^2 - t_0}}{\sqrt{9m_{\pi}^2 - t} + \sqrt{9m_{\pi}^2 - t_0}},
  \]

Bounds on $a_k$ have been set by imposing a Breit-Wigner

\[
F_A(t) = \frac{m_{a_1}^2}{m_{a_1}^2 - t - im_{a_1} \Gamma_{a_1}},
\]
Our proposal:

- To investigate the axial-vector weak hadronic current through the \( \tau \rightarrow 3\pi \nu_\tau \) axial-vector spectral function

- \( 3\pi \) system (predominantly) in \( J^{PC} = 1^{++} \) produced through the \( a_1(1260) \)

- To see what can be learned for the low-\( Q^2 \) behavior of the axial form factor of the nucleon
\[ \tau \rightarrow 3\pi \nu_{\tau} \]
\( \tau^- \to (PPP)^- \nu_{\tau} \): BASICS

- **Generic Amplitude for a 3-meson decay of the \( \tau \)**

\[
M(\tau^- \to (PPP)^- \nu_{\tau}) = \frac{G_F}{\sqrt{2}} |V_{ij}| \bar{u}_\nu \gamma_\mu (1 - \gamma_5) u_{\tau} \langle (PPP)^- |(V - A)^\mu |0 \rangle ,
\]

- **Hadronic matrix element in terms of four form factors**

\[
\langle (P(p_1)P(p_2)P(p_3))^-(V - A)^\mu |0 \rangle = V_1^\mu F_1^A (Q^2, s_1, s_2) + V_2^\mu F_2^A (Q^2, s_1, s_2) ,
\]

\[
+ Q^\mu F_3^A (Q^2, s_1, s_2) + iV_4^\mu F_4^V (Q^2, s_1, s_2) ,
\]

where

\[
V_1^\mu = (g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}) (p_1 - p_3)_{\nu} , \quad V_2^\mu = (g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}) (p_2 - p_3)_{\nu} ,
\]

\[
V_4^\mu = \varepsilon^{\mu\alpha\beta\gamma} p_{1\alpha} p_{2\beta} p_{3\gamma} , \quad Q^\mu = (p_1 + p_2 + p_3)^\mu , \quad s_i = (Q - p_i)^2 ,
\]
\( \tau^- \rightarrow (PPP)^- \nu_\tau: \text{BASICS} \)

- **Generic Amplitude for a 3-meson decay of the \( \tau \)**

\[
M(\tau^- \rightarrow (PPP)^- \nu_\tau) = \frac{G_F}{\sqrt{2}} |V_{ij}| \bar{u}_\nu \gamma_\mu (1 - \gamma_5) u_\tau \langle (PPP)^- | (V - A)^\mu | O \rangle ,
\]

- **hadronic matrix element in terms of four form factors**

\[
\langle (P(p_1)P(p_2)P(p_3))^-(V - A)^\mu | O \rangle = V_1^\mu F_1^A(Q^2, s_1, s_2) + V_2^\mu F_2^A(Q^2, s_1, s_2),
\]

\[+ Q^\mu F_3^A(Q^2, s_1, s_2) + iV_4^\mu F_4^V(Q^2, s_1, s_2),\]

- **\( F_1^A(Q^2, s_1, s_2) \):** \( J^P = 1^+ \) transition (axial-vector form factors)
- **\( F_3^A(Q^2, s_1, s_2) \):** \( J^P = 0^- \) transition (pseudoscalar form factor)
- **\( F_4^V(Q^2, s_1, s_2) \):** \( J^P = 1^- \) transition (vector form factor)
\( \tau \rightarrow \pi^- \pi^+ \pi^- \nu_\tau \)

- Bose symmetry: \( F_A^1(Q^2, s_1, s_2) = F_A^2(Q^2, s_2, s_1) \equiv F_A(Q^2, s_1, s_2) \)
- Conservation of \( J_A^\mu \) in the chiral limit: \( F_A^3(Q^2, s_1, s_2) \) must vanish with \( m_\pi^2 \)
- \( G \)-parity conservation: \( F_A^V(Q^2, s_1, s_2) = 0 \)
- The axial-vector hadronic current takes the form

\[
J_A^\mu = F_A(Q^2, s_1, s_2) V_1^{\mu} + F_A(Q^2, s_2, s_1) V_2^{\mu},
\]

- Decay rate

\[
\frac{d\Gamma(\tau^- \rightarrow \pi^- \pi^- \pi^+ \nu_\tau)}{dQ^2} = \frac{G_F^2 |V_{ud}|^2}{32\pi^2 M_\tau} \left(M_\tau^2 - Q^2\right)^2 \left(1 + \frac{2Q^2}{M_\tau^2}\right) a_1(Q^2),
\]
\[ \tau \rightarrow \pi^- \pi^+ \pi^- \nu_\tau \]

**Spectral function**

\[ a_1(Q^2) = \frac{1}{768\pi^3} \frac{1}{Q^4} \int_{s_1,\text{min}}^{s_1,\text{max}} ds_1 \int_{s_2,\text{min}}^{s_2,\text{max}} ds_2 \ W_A. \]

where

\[ W_A = - \left[ V_1^\mu F_A(Q^2, s_1, s_2) + V_2^\mu F_A(Q^2, s_2, s_1) \right] \times \]

\[ \left[ V_1^\mu F_A(Q^2, s_1, s_2) + V_2^\mu F_A(Q^2, s_2, s_1) \right], \]

**For comparison of theory and experiment**

\[ \frac{\Gamma(\tau^- \rightarrow (3\pi)^- \nu_\tau)}{\Gamma(\tau^- \rightarrow e^- \bar{\nu}_e \nu_\tau)} \frac{1}{N_{\text{events}}} \frac{dN_{\text{events}}}{dQ^2} = \frac{6\pi |V_{ub}|^2 S_{\text{EW}}}{M_\tau^2} \left( 1 - \frac{Q^2}{M_\tau^2} \right)^2 \left( 1 + 2 \frac{Q^2}{M_\tau^2} \right) a_1(Q^2) \]
Evaluation of the Axial-Vector Form Factor

- Factorization ansatz: $\tau \rightarrow \nu_\tau a_1 \rightarrow \nu_\tau \rho \pi \rightarrow 3\pi$

$$F_A(Q^2, s_1, s_2) = F_{a_1}(Q^2)F_\pi^\pi(s_2),$$

- $F_{a_1}(Q^2)$ accounts for the $a_1(1260)$-resonance production
- $F_\pi^\pi(s_i)$: $\rho \rightarrow \pi\pi$ decay (ansatz pion vector FF)

$$F_\pi^\pi(s_i) \rightarrow 1, \quad s_i \rightarrow 0.$$

- Axial-vector current

$$J_A^\mu = F_{a_1}(Q^2) [F_\pi^\pi(s_2)V_{1\mu} + F_\pi^\pi(s_1)V_{2\mu}],$$

- Isospin relation: same predictions for both modes (in the isospin limit)

$$F_A(Q^2, s_1, s_2) = -F_{1\pi^0\pi^0\pi}(Q^2, s_1, s_2),$$
\textbf{EVALUATION OF THE AXIAL-VECTOR FORM FACTOR}

- ChPT prediction at $O(p^2)$

\[ J_A^{\mu} \bigg|_{\text{ChPT}}^{O(p^2)} = -\frac{2\sqrt{2}}{3F_\pi} \left( V_{1\mu} + V_{2\mu} \right), \]

- Normalization of $F_{a_1}(Q^2)$

\[ F_{a_1}(Q^2) = -\frac{2\sqrt{2}}{3F_\pi} f_{a_1}(Q^2), \quad f_{a_1}(Q^2 \to 0) \to 1, \]

- In this framework, the axial spectral function reads:

\[ a_1(Q^2) = \frac{1}{768\pi^3} \left( -\frac{2\sqrt{2}}{3F_\pi} \right)^2 |f_{a_1}(Q^2)|^2 \frac{g(Q^2)}{Q^2}, \]

where

\[ g(Q^2) = \frac{1}{Q^2} \int_{s_{1,\text{min}}}^{s_{1,\text{max}}} ds_1 \int_{s_{2,\text{min}}}^{s_{2,\text{max}}} ds_2 \left\{ - V_1^2 |F_\pi^V(s_2)|^2 - V_2^2 |F_\pi^V(s_1)|^2 \right. \]

\[ \left. - 2V_1V_2 \text{Re} \left[ F_\pi^V(s_1)(F_\pi^V(s_2))^* \right] \right\}, \]
EVALUATION OF THE AXIAL-VECTOR FORM FACTOR

\[ g(Q^2) = \frac{\rho}{\rho}, \frac{\rho'}{\rho'}, \frac{\rho''}{\rho''} \]
Evaluation of the axial-vector form factor

Breit-Wigner 1: $a_1(1260)$

$$f_{a_1}(Q^2) = \frac{m_{a_1}^2}{m_{a_1}^2 - Q^2 - i m_{a_1} \Gamma_{a_1}(Q^2)},$$

where the energy dependent width is given by

$$\Gamma_{a_1}(Q^2) = \gamma_{a_1} \frac{g(Q^2)}{g(m_{a_1}^2)}.$$

Breit-Wigner 2: $a_1(1260) + a_1(1640)$

$$f_{a_1}(Q^2) = \frac{m_{a_1}^2}{m_{a_1}^2 - Q^2 - i m_{a_1} \Gamma_{a_1}(Q^2)} + \kappa e^{i \phi} \frac{m_{a_1'}^2}{m_{a_1'}^2 - Q^2 - i m_{a_1'} \Gamma_{a_1'}(Q^2)},$$
Breit-Wigner 1: $m_{a_1(1260)}^{\text{fit}} = 1297(5) \text{ MeV}, \gamma_{a_1(1260)}^{\text{fit}} = 665(14) \text{ MeV}, \chi_{\text{dof}}^2 = 1.14$

Breit-Wigner 2: $m_{a_1(1260)}^{\text{fit}} = 1254(3) \text{ MeV}, \gamma_{a_1(1260)}^{\text{fit}} = 589(10) \text{ MeV}, \chi_{\text{dof}}^2 = 1.91$
Fits to $\tau \to 3\pi \nu_\tau$: Low-$Q^2$ Region

Dominant contribution arise from resonance exchange

<table>
<thead>
<tr>
<th>$a_1(s)$</th>
<th>0.002</th>
<th>0.004</th>
<th>0.006</th>
<th>0.008</th>
<th>0.010</th>
</tr>
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<tr>
<td>$s$ [GeV$^2$]</td>
<td>0.0</td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
</tr>
</tbody>
</table>

- ALEPH (2013) $\tau^- \to \pi^0 \pi^0 \pi^- \nu_\tau$
- ALEPH (2013) $\tau^- \to \pi^- \pi^+ \pi^- \nu_\tau$
- Tree-level ChPT at NLO
- Breit-Wigner $a_1$ & $\rho, \rho', \rho''$ ($\tau^- \to \pi^0 \pi^0 \pi^- \nu_\tau$)
- Breit-Wigner $a_1$ & $\rho, \rho', \rho''$ ($\tau^- \to \pi^- \pi^+ \pi^- \nu_\tau$)
Fits to the $\tau \rightarrow 3\pi \nu_\tau$ Axial Spectral Function

Breit-Wigner 2: $m_{a_1(1260)}^{\text{fit}} = 1352(13) \text{ MeV}, \gamma_{a_1(1260)}^{\text{fit}} = 622(15) \text{ MeV}, \chi^2_{\text{dof}} = 0.70$

$m_{a_1(1640)}^{\text{PDG}} = 1647 \text{ MeV}, \gamma_{a_1(1640)}^{\text{PDG}} = 254 \text{ MeV}, \kappa = -0.37(1), \phi = 0.73(2)$
Breit-Wigner 2: $m_{a_1(1260)}^{\text{fit}} = 1295(7)\text{ MeV}, \gamma_{a_1(1260)}^{\text{fit}} = 583(12)\text{ MeV}, \chi^2_{\text{dof}} = 1.58$

$m_{a_1(1640)}^{\text{PDG}} = 1647\text{ MeV}, \gamma_{a_1(1640)}^{\text{PDG}} = 254\text{ MeV}, \kappa = -0.32(1), \phi = 0.78(2)$
Drawbacks associated with the use of Breit-Wigners

- Constraints from analyticity and unitarity not fully respected
- Does not incorporate the constraints from chiral symmetry

Dispersive approach (in progress)

\[
f_{a_1}(Q^2) = \mathcal{P}_n(Q^2) \exp \left[ \frac{(Q^2 - s_o)^n}{\pi} \int_{9m_{\pi}^2}^{\infty} ds' \frac{\delta(s')}{(s' - s_o)^n(s' - Q^2 - i\alpha)} \right],
\]

\[
\tan \delta(Q^2) = \frac{\text{Im} f_{a_1}^{\text{BW}}(Q^2)}{\text{Re} f_{a_1}^{\text{BW}}(Q^2)},
\]

\[
\mathcal{P}_n(Q^2): \text{related to chiral low-energy observables}
\]

\[
f_{a_1}(Q^2) = 1 - \frac{1}{6} \langle r_A^2 \rangle Q^2 + \cdots
\]
AXIAL FORM FACTOR OF THE NUCLEON
Analytical continuation to the space-like Axial form factor of the nucleon

\[ F_A(Q^2) = f_{a_1}(Q^2) P_{a_1NN}(Q^2), \]

- \( f_{a_1}(Q^2) \): from \( \tau \rightarrow 3\pi \)
- \( P_{a_1NN}(Q^2) \): \( a_1 \)-NN vertex function
- if \( P_{a_1NN}(Q^2) = 1 \): direct extrapolation of \( f_{a_1}(Q^2) \)
Analytical continuation to the space-like

Axial form factor of the nucleon

\[ F_A(Q^2) = f_{a_1}(Q^2)P_{a_1 NN}(Q^2) = f_{a_1}(Q^2)(1 + \langle r_A^2 \rangle_{a_1 NN} \times Q^2), \]

\[ \langle r_A^2 \rangle = \langle r_A^2 \rangle_{a_1} + \langle r_A^2 \rangle_{a_1 NN} = 0.14(1) \text{ [fm}^2\text{]} + 0.09(1) \text{ [fm}^2\text{]} = 0.23(1) \text{ [fm}^2\text{]} \]
Expanding the normalized axial form factor around $Q^2 = 0$

$$\tilde{F}_A(Q^2) = 1 - \frac{1}{6} \langle r_A^2 \rangle Q^2 + \frac{1}{120} \langle r_A^4 \rangle (Q^2)^2 + \cdots$$

<table>
<thead>
<tr>
<th>Reference</th>
<th>$m_A$ [GeV]</th>
<th>$\langle r_A^2 \rangle$ [fm$^2$]</th>
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<tbody>
<tr>
<td>K2K</td>
<td>1.20 ± 0.12</td>
<td>0.32 ± 0.06</td>
</tr>
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<td>1.05 ± 0.06</td>
<td>0.42 ± 0.05</td>
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<td>0.26 ± 0.06</td>
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<td>MINERvA</td>
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<td>0.48</td>
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<tr>
<td>MINOS</td>
<td>1.23$^{+0.13}_{-0.09}$</td>
<td>0.31$^{+0.07}_{-0.05}$</td>
</tr>
<tr>
<td>This work (preliminary)</td>
<td>—</td>
<td>0.23(1)</td>
</tr>
</tbody>
</table>

**Table:** Axial mass and squared axial radius determinations from neutrino scattering experiments.
CONCLUSIONS AND OUTLOOK
Conclusions and Outlook

- We need to work on improving the prediction for neutrino-nucleon cross section.

- This requires to improve our knowledge on the axial form factor of the nucleon:
  - Lattice QCD
  - Measurements of $\nu N$ scattering with the smallest possible nuclear effects
  - Analytically: $3\pi \to NN$ amplitude

- $\tau \to 3\pi \nu_\tau$ as an advantageous laboratory to get valuable information for the axial form factor of the nucleon.
THANKS FOR YOUR ATTENTION