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Typical growth behavior of the Out-of-Time-Ordered Commutator in Many-Body Localized systems

Dong-Hee Kim

Department of Physics and Photon Science
Gwangju Institute of Science and Technology

Outline

- **Many-Body Localization (MBL)**
  - Thermal. vs. Anderson Localization vs. MBL in disordered systems
  - How can we distinguish them?
  - One of the candidates: growth of **OTOC**

- **Out-of-Time-Ordered Commutator / Correlator**
  - A typical characteristic behavior of OTOC in MBL systems

  Q. effective model vs. realistic quantum spin chain

  - It does exists but cannot survive in disorder averages.

REF: J. Lee, D. Kim, and D.-H. Kim, PRB 99, 184202 (2019).

"Usual" ingredients for MBL

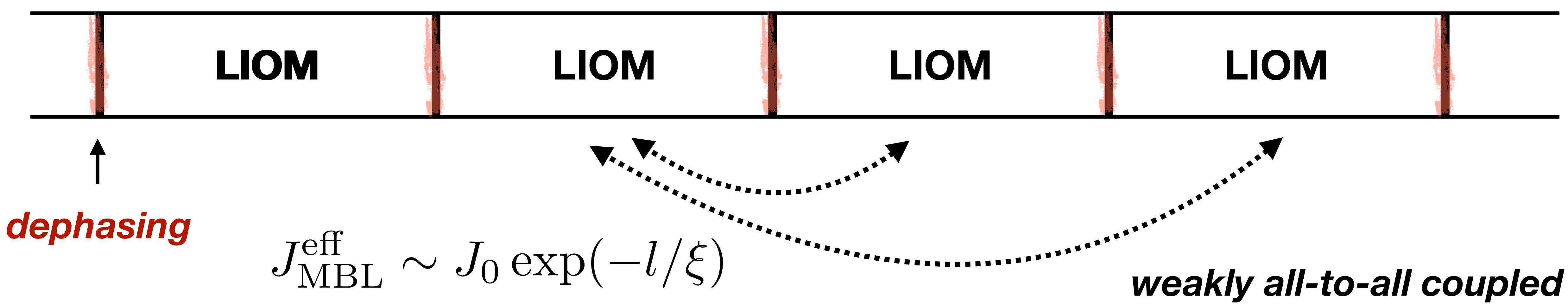
*Phase diagram of the disordered **XXZ** chain*

- **Disorders**
  - onsite energy, hopping strength, quasi-periodicity
  - cf. MBL without disorder may need an effective disorder.

- **Interactions**
  - A *naive* version of MBL = Anderson localization + Interactions
  - What's essential: "*dephasing*" -> *Information spreading/scrambling*

Many-Body Localization (with interactions)

Anderson Localization (non-interacting)



$$J_{\rm MBL}^{\rm eff} \sim J_0 \exp(-l/\xi)$$

Phenomenological Comparisons

R. Nandkishore and D. A. Huse, Annu. Rev. Condens. Matter Phys. 6, 15 (**2015**).

Thermal phase	Anderson Localization	Many-Body Localization
Thermal phase	Single-particle localized	Many-body localized
Memory of initial conditions hidden in global operators at long times	Some memory of local initial conditions preserved in local observables at long times	Some memory of local initial conditions preserved in local observables at long times
Eigenstate thermalization hypothesis (ETH) true	ETH false	ETH false
May have nonzero DC conductivity	Zero DC conductivity	Zero DC conductivity
Continuous local spectrum	Discrete local spectrum	Discrete local spectrum
Eigenstates with volume-law entanglement	Eigenstates with area-law entanglement	Eigenstates with area-law entanglement
Power-law spreading of entanglement from nonentangled initial condition	No spreading of entanglement	Logarithmic spreading of entanglement from nonentangled initial condition
Dephasing and dissipation	No dephasing, no dissipation	Dephasing but no dissipation

Only EE growth distinguishes MBL from AL in this table.

Q. Can OTOC do the same thing?

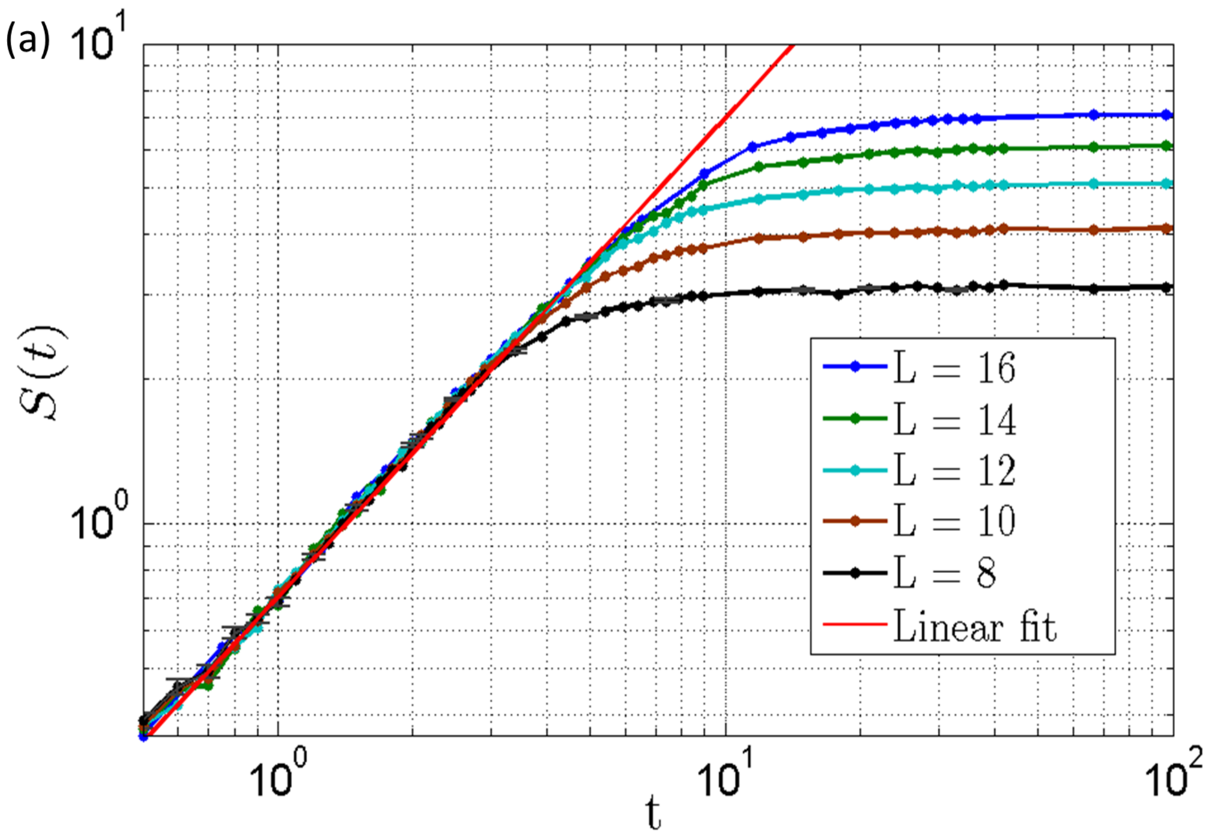
Entanglement Entropy Growth: Thermal vs. MBL

Thermal : $S(t) \sim t$

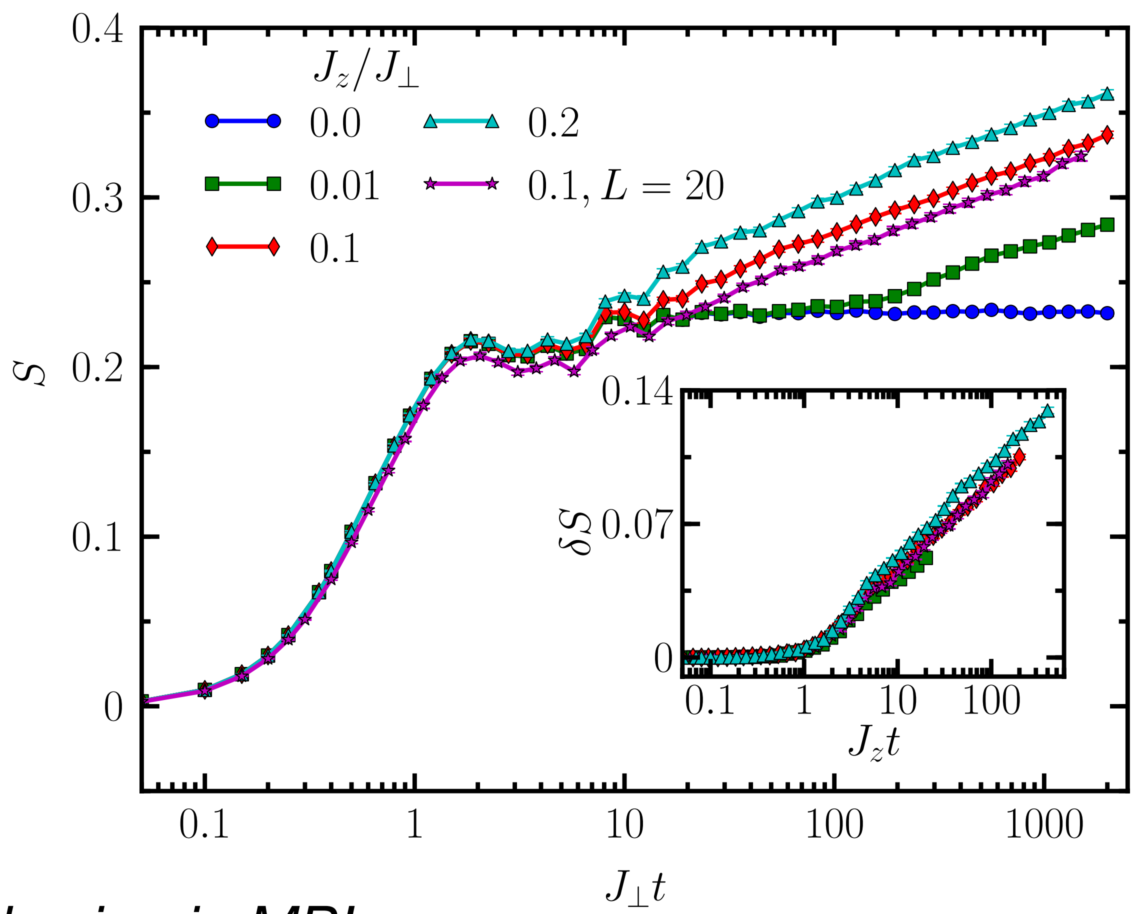
MBL : $S(t) \sim \ln t$

c.f. ***AL:*** *S(t) ~ constant*

H. Kim and D. A. Huse, PRL 111, 127205 (2013)



Bardarson, Pullman, Moore, PRL 109, 017202 (2012)



Time scale: $Jt \sim 1$

Dephasing in MBL:

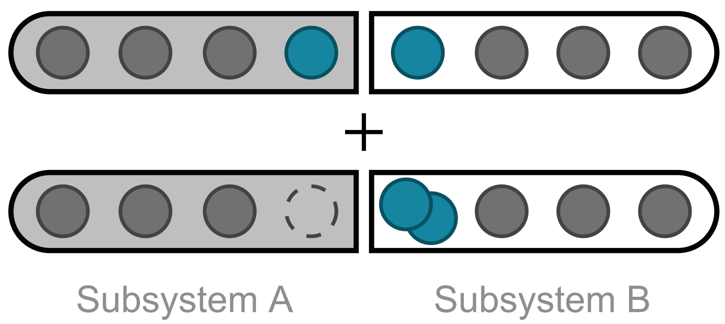
$$J^{\text{eff}}_{\text{MBL}} \sim J_0 \exp(-l/\xi)$$

Experiment to measure EE (a sort of)

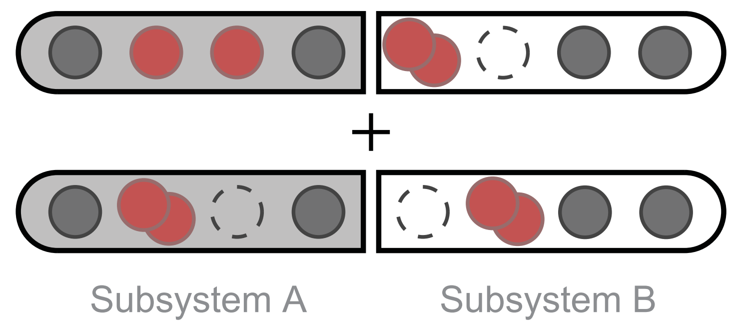
[Aubry-Andre, boson (^{87}Rb)]

Lukin et. al., Science 364, 256 (2019).

Number entanglement



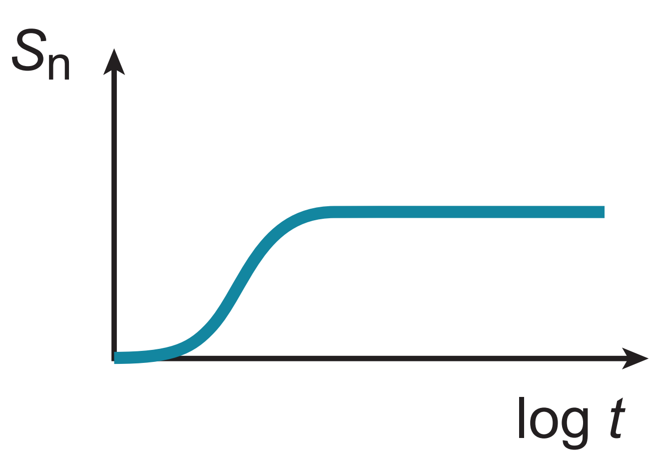
Configurational entanglement

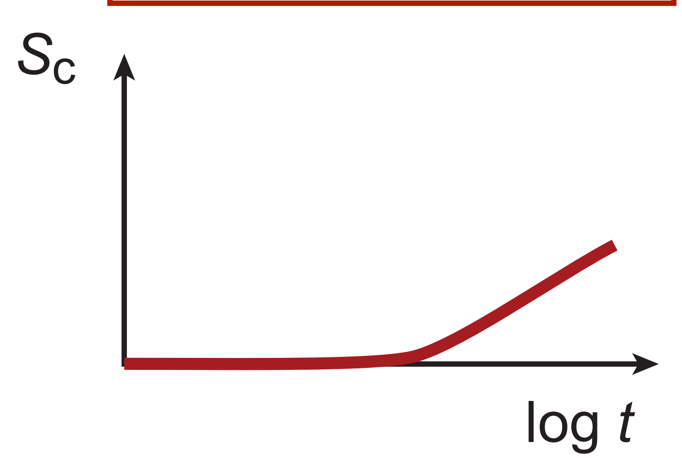


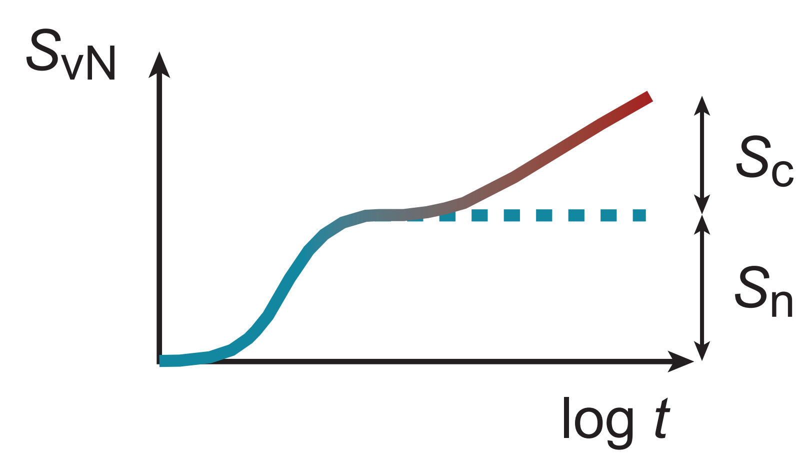
~~Entanglement~~ Correlation

$$C = \sum_{n=0}^{N} p_n \sum_{\{A_n\},\{B_n\}} |p(A_n \otimes B_n) - p(A_n)p(B_n)|$$

inhibition of transport + buildup of non-local entanglement = logarithmically growing entanglement







Out-of-Time-Ordered Commutator / Correlator

A measure of quantum chaos

Quantum Chaos: $C(t) \propto \exp[\lambda_L t]$

A. I. Larkin and Y. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. 55, 2262 (**1968**) .

A. Kitaev, a talk in Fundamental Physics Prize Symposium (2014).

An *alternative* measure of MBL

MBL: slow growth/spreading

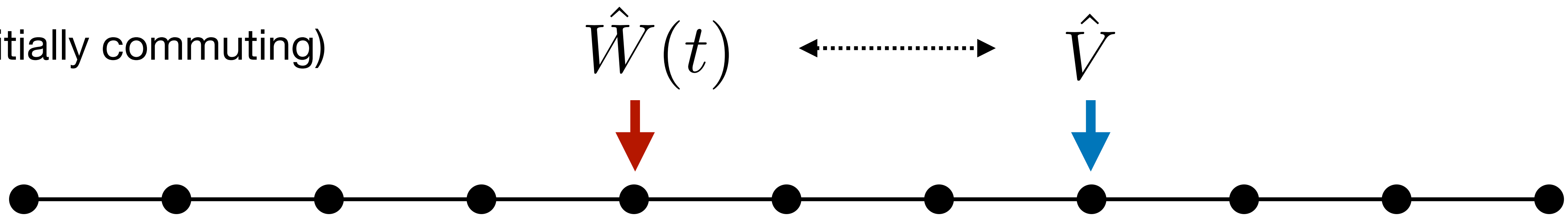
B. Swingle and D. Chowdhury, Phys. Rev. B 95, 060201(R) (2017).
R. Fan, P. Zhang, H. Shen, and H. Zhai, Sci. Bull. 62, 707 (2017).
X. Chen, T. Zhou, D. A. Huse, and E. Fradkin, Ann. Phys. (Berlin) 529, 1600332 (2017).
R.-Q. He and Z.-Y. Lu, Phys. Rev. B 95, 054201 (2017).
Y. Chen, arXiv:1608.02765.
Y. Huang, Y.-L. Zhang, and X. Chen, Ann. Phys. (Berlin) 529, 1600318 (2017).
K. Slagle, Z. Bi, Y.-Z. You, and C. Xu, Phys. Rev. B 95, 165136 (2017).
P. Bordia, F. Alet, and P. Hosur, Phys. Rev. A 97, 030103(R) (2018).

and many other OTOC studies for quantum chaos and non-MBL systems.

Out-of-Time-Ordered Commutator: the definition

unitary "local" operators

(initially commuting)



$$C(t) = \frac{1}{2}\langle [\hat{W}(t), \hat{V}]^\dagger [\hat{W}(t), \hat{V}] \rangle = 1 - \mathrm{Re}[F(t)]$$

OTO "commutator"

OTO "correlator"

$$F(t) = \langle \hat{W}^\dagger(t) \hat{V}^\dagger \hat{W}(t) \hat{V} \rangle$$

Experiments : NMR, trapped ions, ultracold gases

Li et al., PRX 7, 031011 (2017).

Gärttner et al., Nat. Phys. 13, 781 (2017).

Meier et al., PRA 100, 013623 (2019).

*Proposals for measurements

Swingle, et. al., PRA (2016); Yao et. al., arXiv:1607.01801, Zhu et. al. PRA (2016), Yunger Halpern, PRA (2017); …

Our system: Heisenberg XXZ chain

$$\mathcal{H} = -\sum_{i=1}^{L-1}\Big[J\big(\hat{S}_i^x\hat{S}_{i+1}^x + \hat{S}_i^y\hat{S}_{i+1}^y\big) + J_z\hat{S}_i^z\hat{S}_{i+1}^z\Big] + \sum_{i=1}^{L} h_i\hat{S}_i^z$$

interaction (J_z)

disorder (h_i)

$$h_i \in [-\eta, \eta]$$

Operator choice : $\hat{W} = \hat{\sigma}_3^x,\ \hat{V} = \hat{\sigma}_0^x$

$$C(t) = \frac{1}{2}\big\langle[\hat{\sigma}_3^x(t), \hat{\sigma}_0^x]^\dagger[\hat{\sigma}_3^x(t), \hat{\sigma}_0^x]\big\rangle$$

State choice: **1. Maximally mixed state (β=0)** $\hat{\rho} = 1/d$

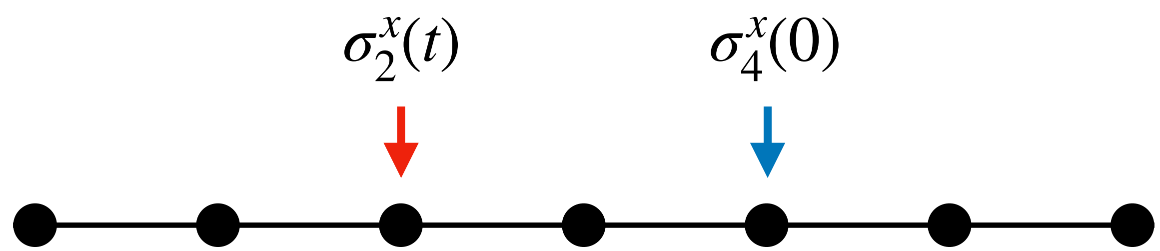
$\langle\cdots\rangle \equiv \mathrm{Tr}[\hat{\rho}\cdots]$

2. Random product state $|\Psi_v\rangle = \bigotimes_{i=1}^{L}\left(\cos\frac{\theta_i}{2}|\uparrow\rangle + e^{i\phi_i}\sin\frac{\theta_i}{2}|\downarrow\rangle\right)$

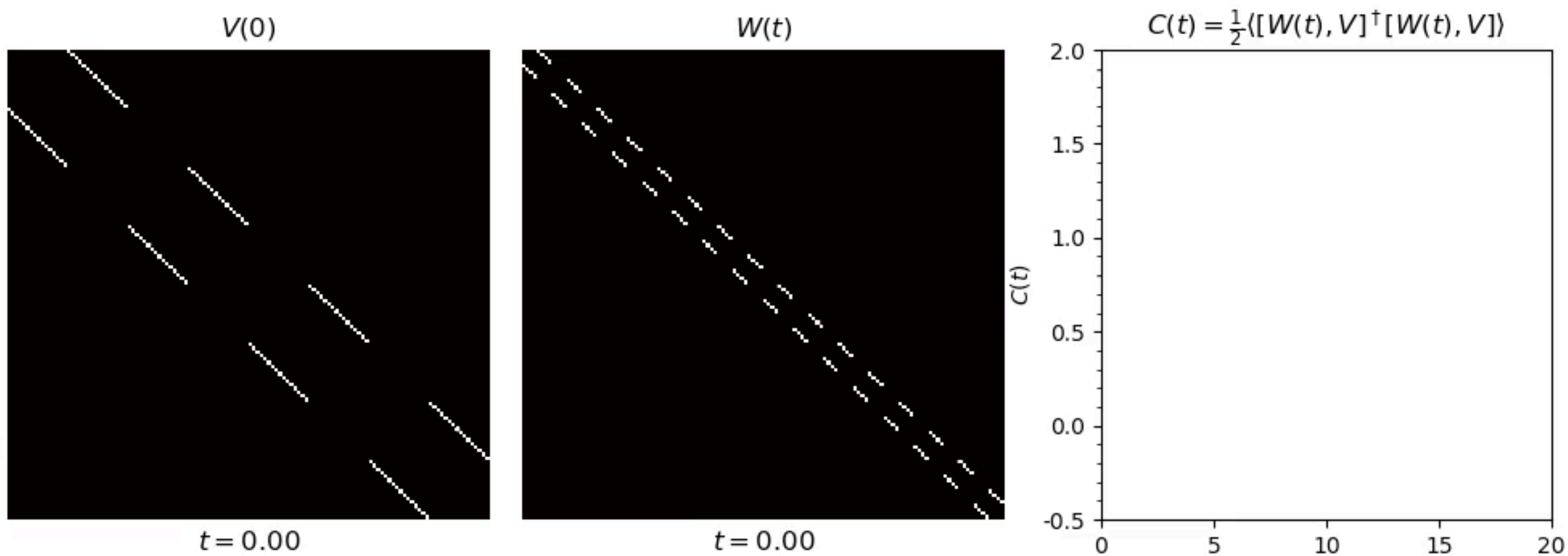
Heisenberg XXZ chain in a thermal phase

Chaotic

$$C(t) = \frac{1}{2}\langle[\hat{W}(t),\hat{V}]^\dagger[\hat{W}(t),\hat{V}]\rangle$$

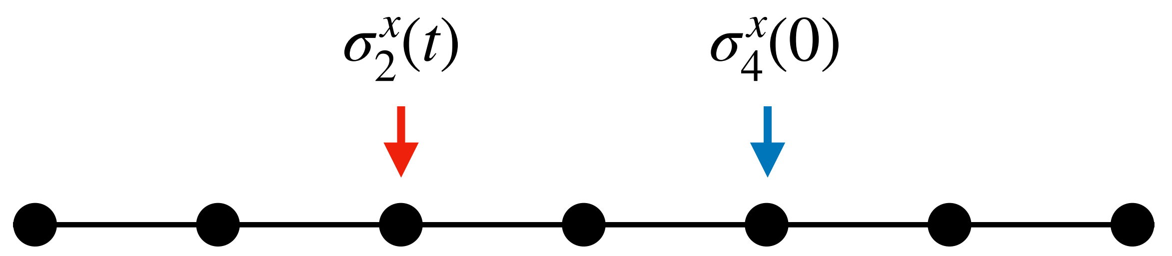


$$[\sigma_2^x(t), \sigma_4^x(0)]^2$$

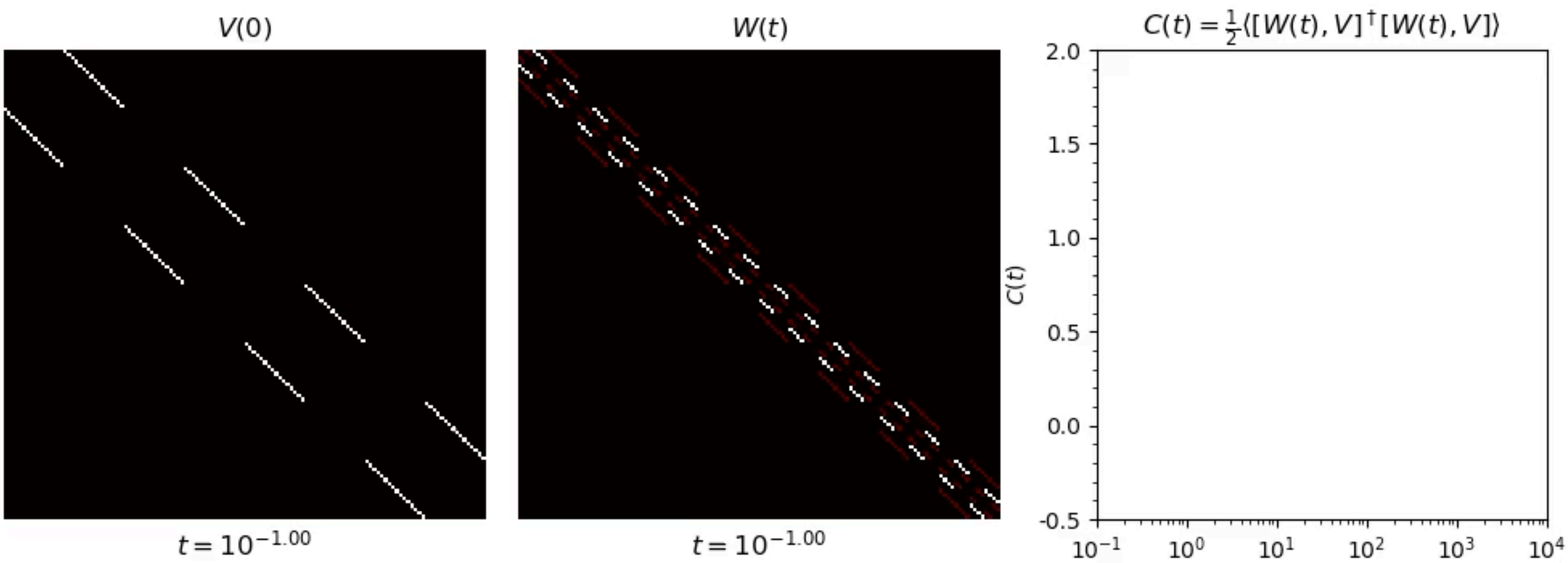


$$L = 7,\ \hat{W} = \sigma_2^x,\ \hat{V} = \sigma_4^x,\ J_z = 1,\ \eta = 1$$

Anderson Localized

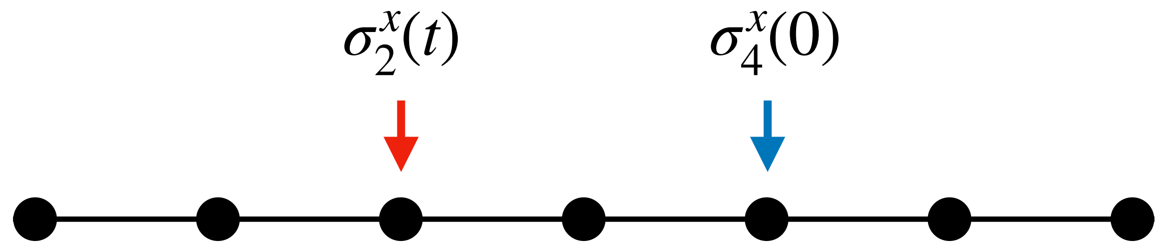


$[\sigma_2^x(t), \sigma_4^x(0)]^2$

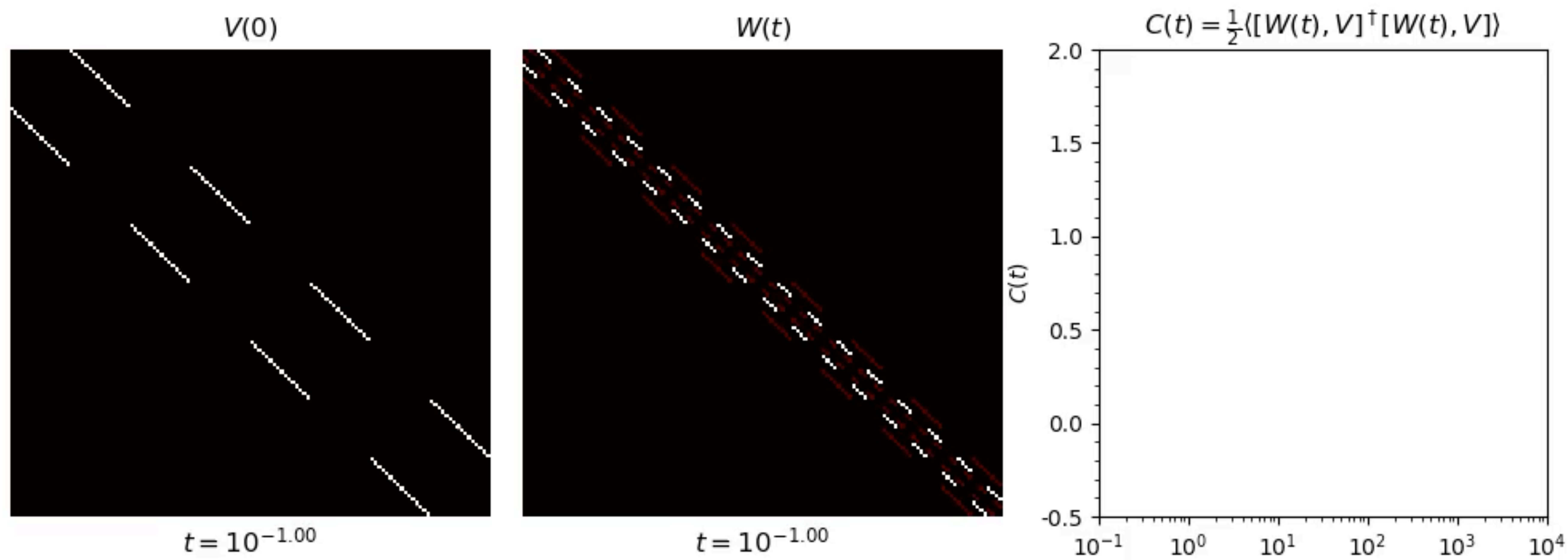


$$L = 7,\ \hat{W} = \sigma_2^x,\ \hat{V} = \sigma_4^x,\ J_z = 0,\ \eta = 10$$

Many-Body Localized



$$[\sigma_2^x(t), \sigma_4^x(0)]^2$$

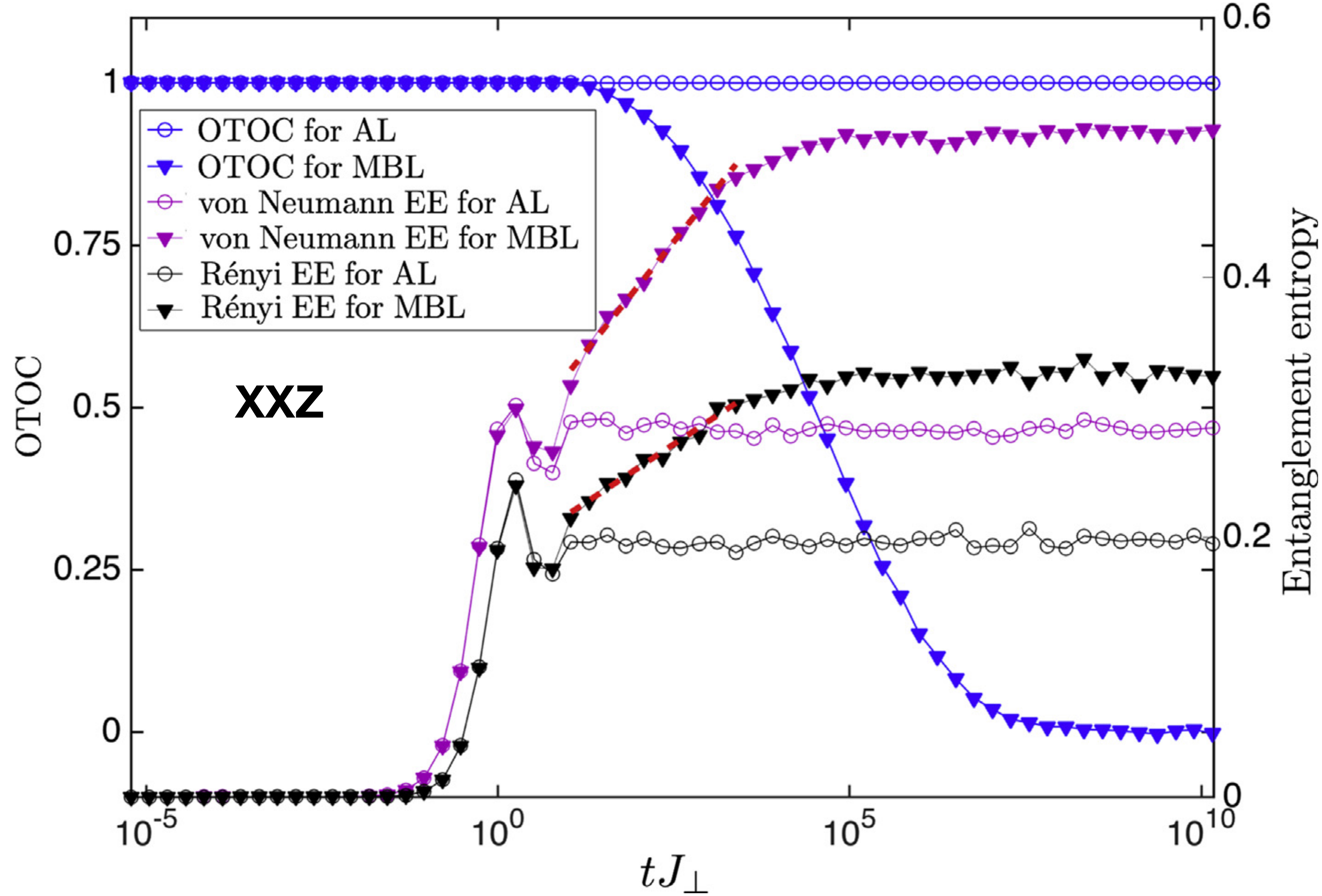


$$L = 7,\ \hat{W} = \sigma_2^x,\ \hat{V} = \sigma_4^x,\ J_z = 1,\ \eta = 10$$

OTO Correlator ⟷ Rényi Entropy

AL vs. MBL: The OTO correlator would work like EE.

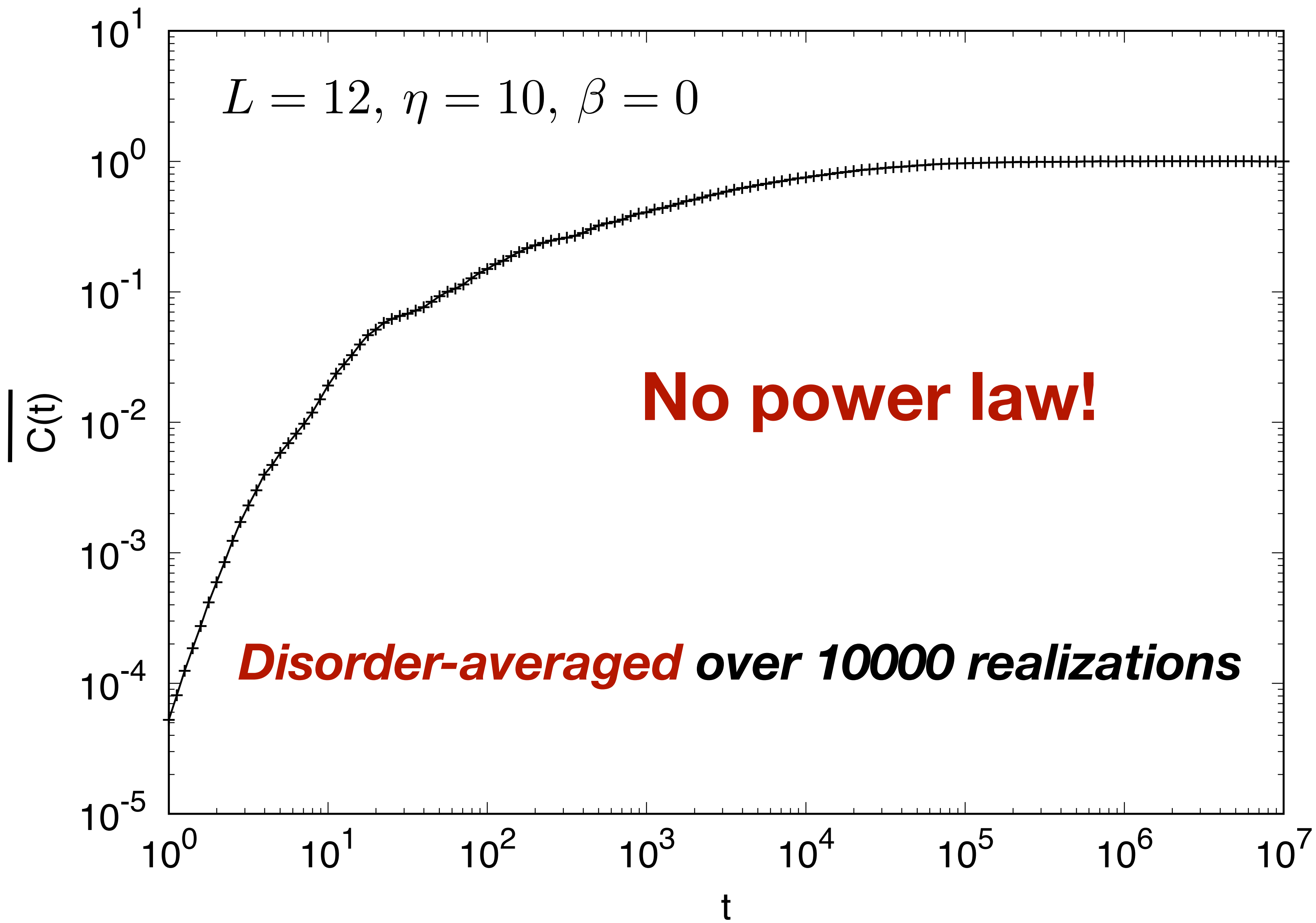
R. Fan, P. Zhang, H. Shen, H. Zhai, Sci. Bull. 62, 707 (2017).



Chaotic vs. MBL vs. AL in OTOC

	Growth	Particle transport	Models	Note
Chaotic (Thermal)	$C(t) \propto \exp[\lambda_L t]$ (early time)	Yes	Semiclassical, Large-N limit, SYK, black hole.	$\lambda_L \leq 2\pi T$
Many-Body Localization (MBL)	$C(t) \propto t^2$ (early time)	No	**Effective l-bit model**	**Can we see it in realistic systems?**
Anderson Localization (AL)	$C(t) \approx 0$	No	All	All frozen

But, No t^2 growth in the XXZ?



The t^2 behavior is derived in the l-bit model:

Swingle and Chowdhury, PRB (2017)

Fan et al., Sci. Bull. (2017)

t^2 growth has not been shown with disorder averaging in any quantum spin models.

MBL studies with OTOC:

Chen et al., Ann. Phys. (2017)

He and Lu, PRB (2017)

Huang et al., Ann. Phys. (2017)

and more.

OTOC growth: the effective l-bit model of MBL

Swingle and Chowdhury, PRB 95, 060201(R) (2017)
R. Fan, P. Zhang, H. She, H. Zhai, Sci. Bull. 62, 707 (2017).

Fully MBL

$$\mathcal{H} = \sum_i h_i \hat{\tau}_i^z + \sum_{\{i,j\}} J_{ij} \hat{\tau}_i^z \hat{\tau}_j^z + \sum_{\{i,j,k\}} K_{ijk} \hat{\tau}_i^z \hat{\tau}_j^z \hat{\tau}_k^z + \cdots$$

OTO correlator $F(t) = \langle \hat{W}^\dagger(t) \hat{V}^\dagger \hat{W}(t) \hat{V} \rangle$ **for** $\hat{W} = \hat{\tau}_a^x \quad \hat{V} = \hat{\tau}_b^x$

$$\longrightarrow \quad F(t) = \langle e^{i\mathcal{H}t} \hat{\tau}_a^x e^{-i\mathcal{H}t} \hat{\tau}_b^x e^{i\mathcal{H}t} \hat{\tau}_a^x e^{-i\mathcal{H}t} \hat{\tau}_b^x \rangle$$

just a Ising spin flip

Energy difference: $2 \times 2 \hat{J}_{ab}^{\text{eff}} \hat{\tau}_a^z \hat{\tau}_b^z$

$$\longrightarrow \quad \boxed{F(t) = \left\langle \exp\left(it \cdot 4 \hat{J}_{ab}^{\text{eff}} \hat{\tau}_a^z \hat{\tau}_b^z \right) \right\rangle}$$

Effective interaction: $\hat{J}_{ab}^{\text{eff}} = J_{ab} + \sum_k{}' K_{abk} \hat{\tau}_k^z + \sum_{\{k,l\}}{}' Q_{abkl} \hat{\tau}_k^z \hat{\tau}_l^z + \cdots$

Quadratic Growth form of OTOC in the l-bit model

The t² behavior occurs at any disorder and any state preparation.

OTO correlator $$F(t) = \langle \exp\left(it \cdot 4\hat{J}^{\text{eff}}_{ab}\hat{\tau}^z_a\hat{\tau}^z_b\right)\rangle$$

For a given disorder realization,

Swingle and Chowdhury, PRB (2017)

Fan et al., Sci. Bull. (2017)

OTO "commutator"

$$\begin{aligned} C(t) &= 1 - \text{Re}\left[\langle \exp\left(it \cdot 4\hat{J}^{\text{eff}}_{ab}\hat{\tau}^z_a\hat{\tau}^z_b\right)\rangle\right] \\ &\simeq 1 - \cos\left(4t\langle \hat{J}^{\text{eff}}_{ab}\rangle\right)\exp\left[-8t^2\left(\langle[\hat{J}^{\text{eff}}_{ab}]^2\rangle - \langle\hat{J}^{\text{eff}}_{ab}\rangle^2\right)\right] \end{aligned}$$

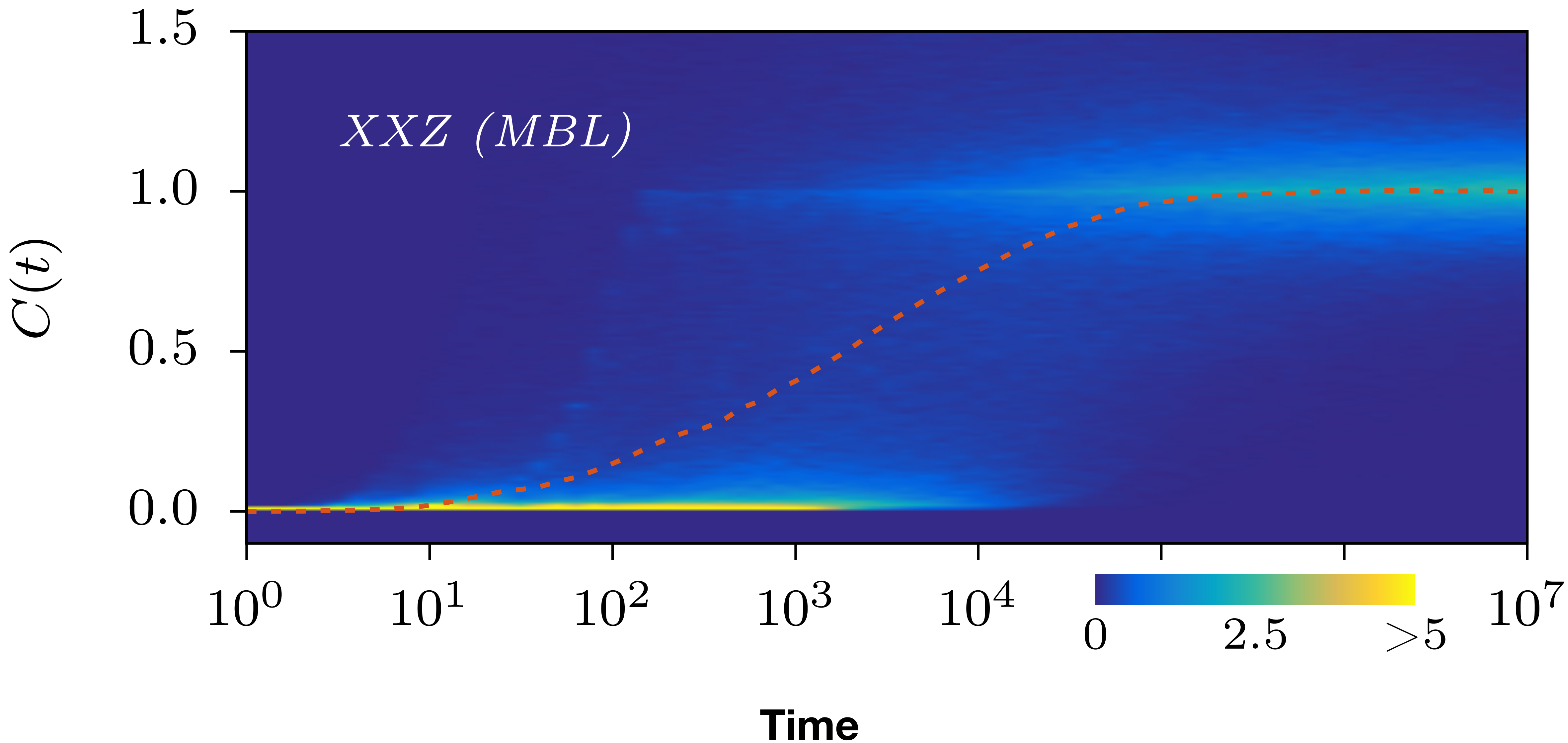
Measured with an eigenstate, $C(t) \simeq 1 - \cos\left(4t\langle\hat{J}^{\text{eff}}_{ab}\rangle\right)$

At very early times,

$$C(t) = 8\langle[\hat{J}^{\text{eff}}_{ab}]^2\rangle t^2 + O(t^4)$$

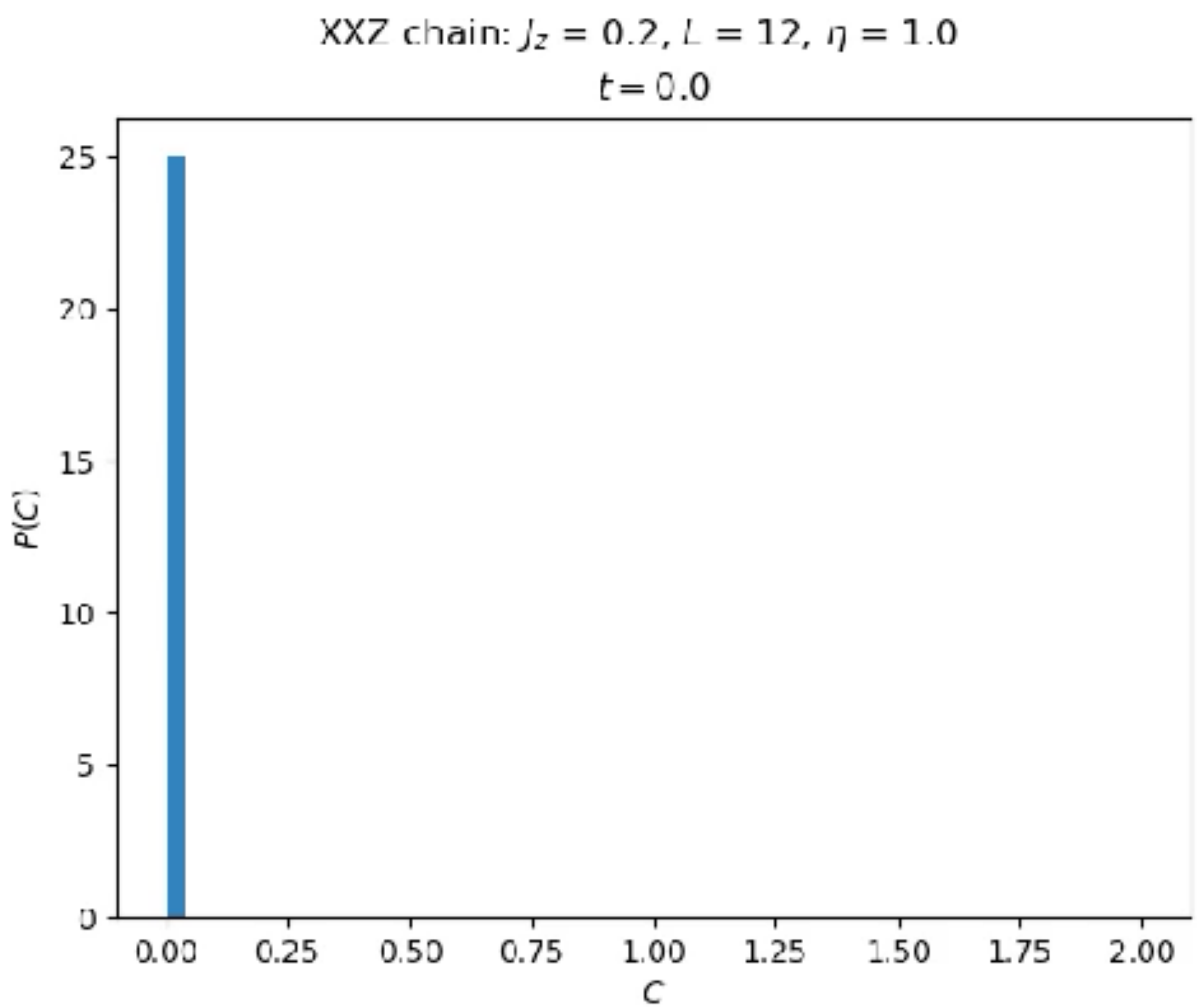
Disorder average?

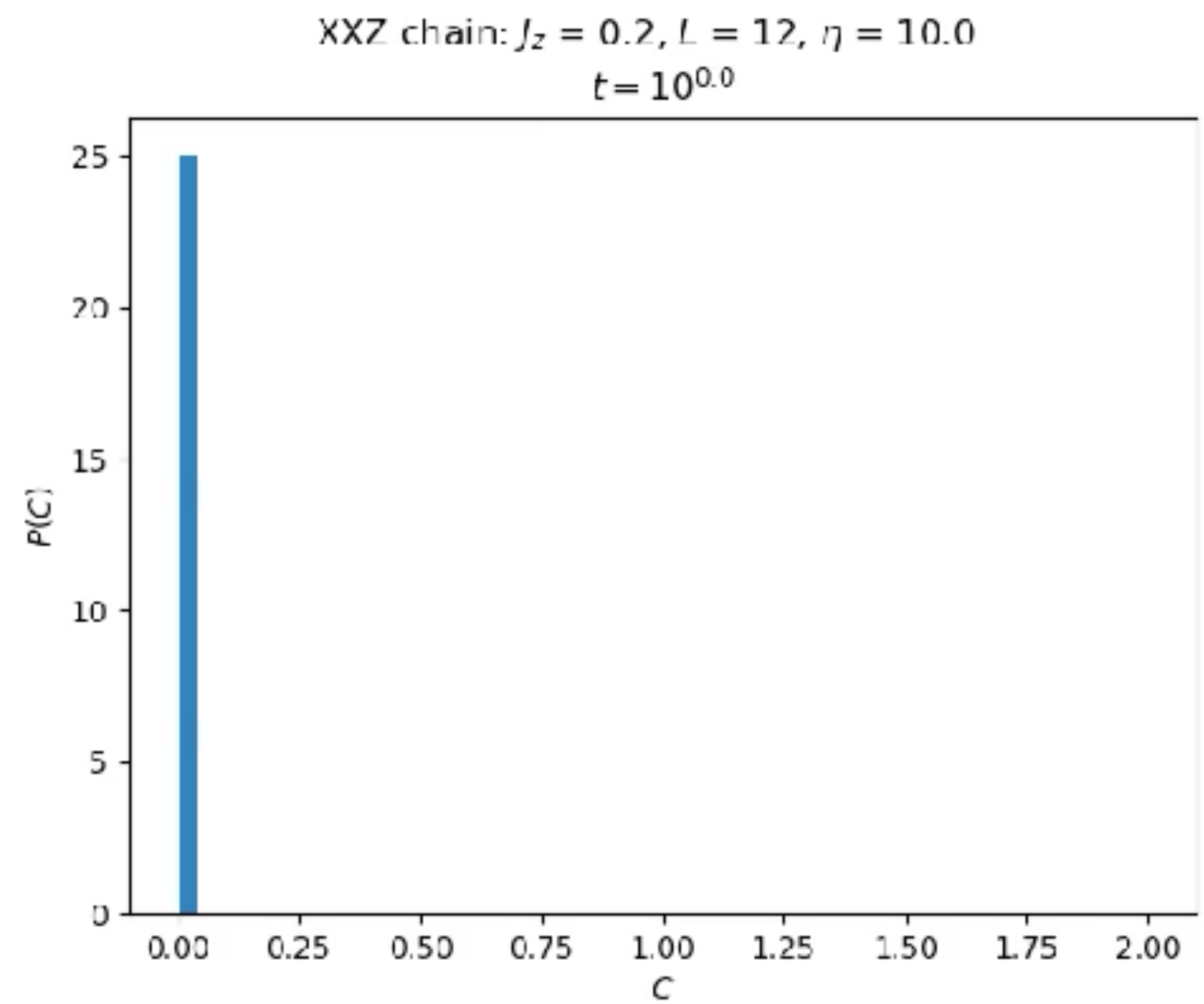
The distribution of C(t) looks like this:



GIST

Time-evolving distribution of OTOC: Thermal vs. MBL





$\eta = 1$ **Thermal**

$\eta = 10$ **MBL**

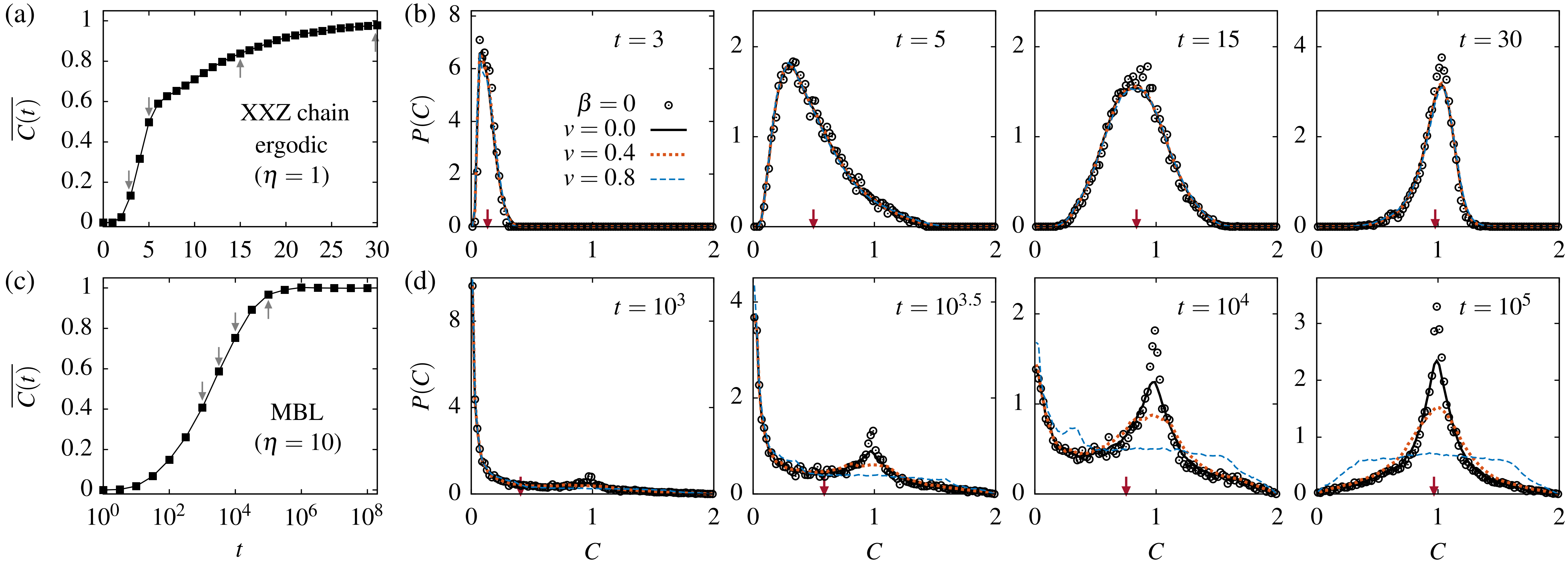
Unimodal distribution

Bimodal distribution

c.f. l-bit model: It's Gaussian (CLT).

Time-evolving distribution of OTOC: Thermal vs. MBL

The average is meaningless in the MBL phase.



A double-peak distribution appears in the MBL phase.

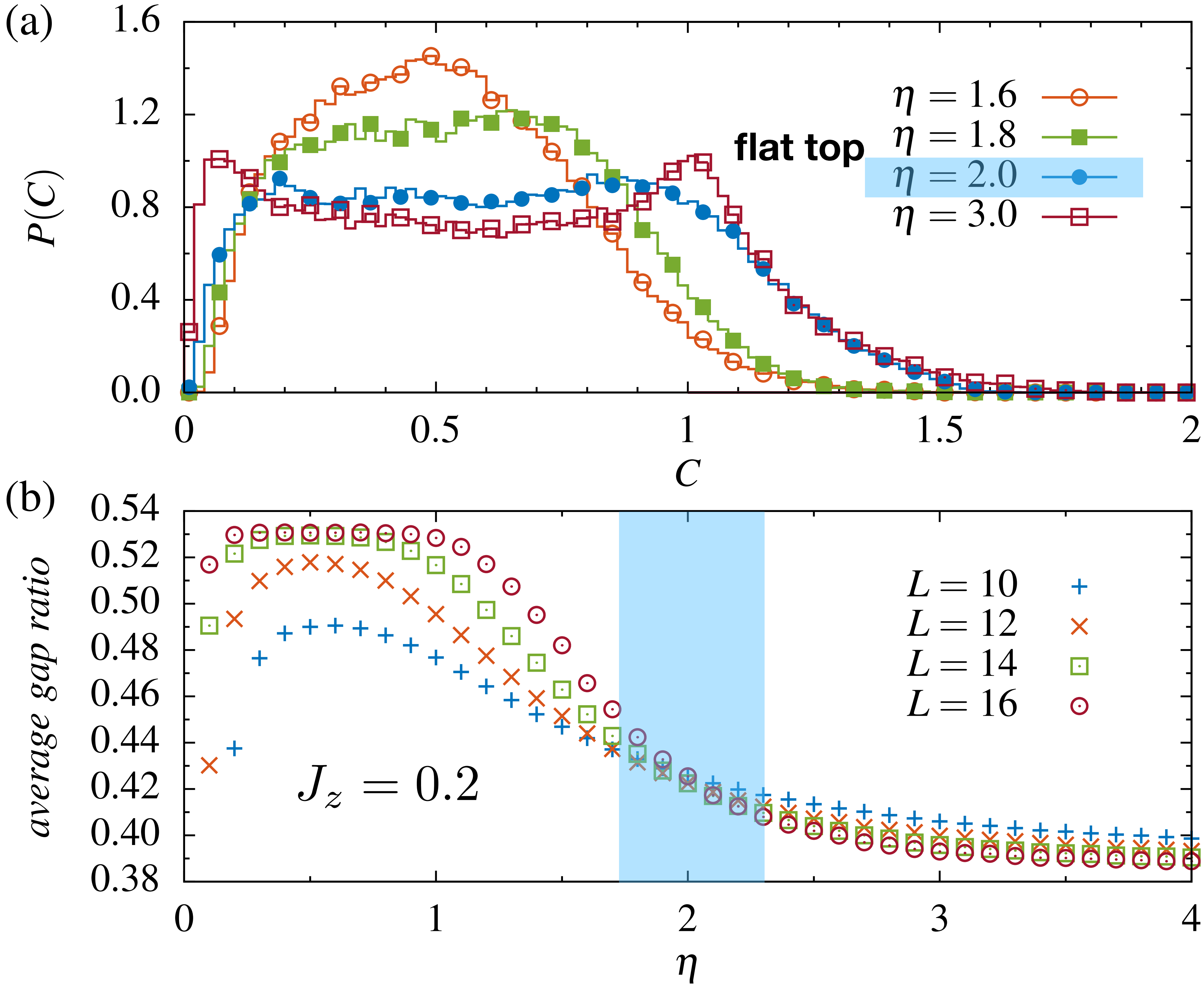
Q. How can we understand the discrepancy?

Q. How can we use this?

Unimodal vs. Bimodal: thermal-to-MBL transition

L = 14

A possible indicator of the MBL transition?

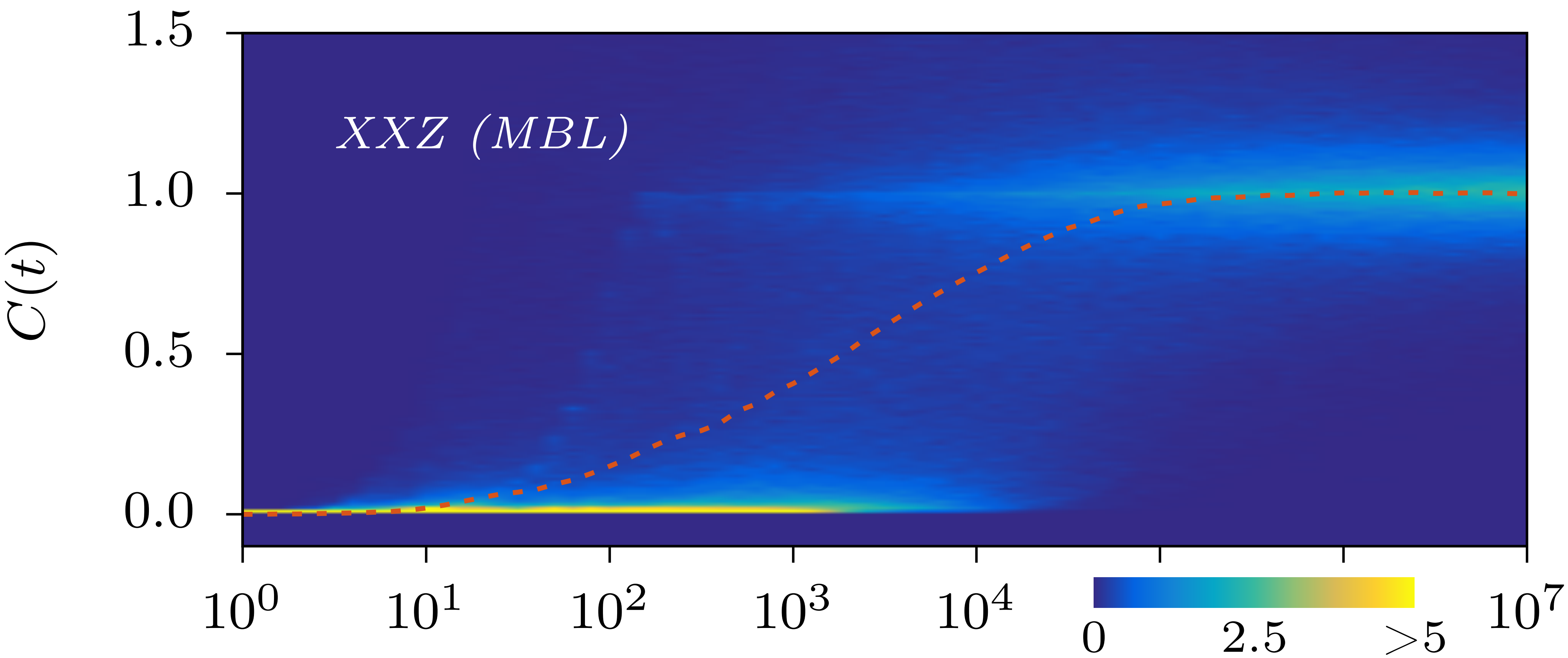


$P(C)$

at

$\overline{C} \approx 0.6$

Level spacing statistics



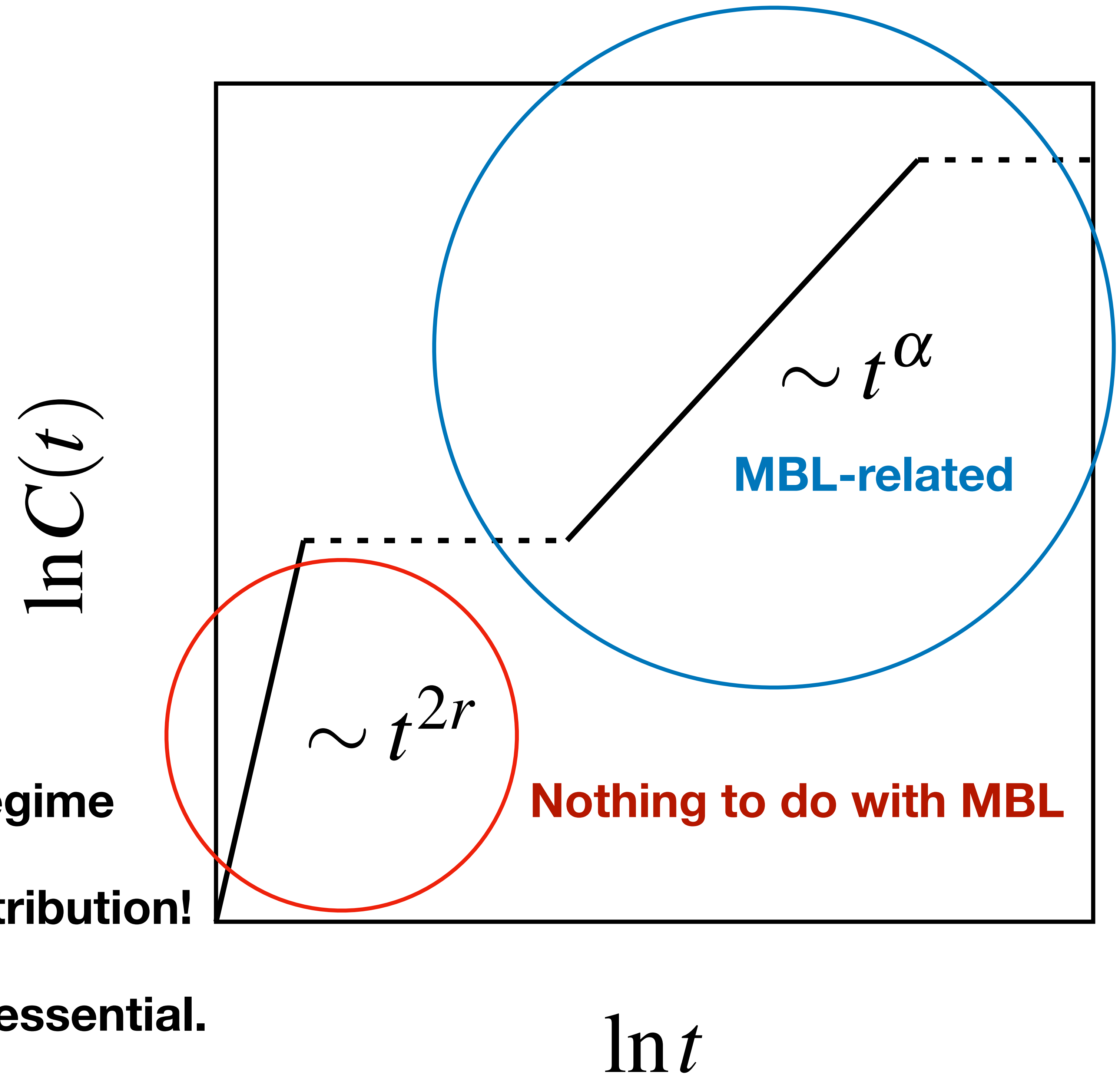
Is the t^2 growth gone, really?

Let's look an individual disorder realization.

Observation at an individual disorder realization

Large deviations in time scales between different disorder realizations

At a given time, one is at stage 1 while the other is stage 2.



Disorder dependent

Exponent is about 2 (a power-law fit)

$$\alpha \approx 2$$

Any relevance to the l-bit model?

Perturbative regime

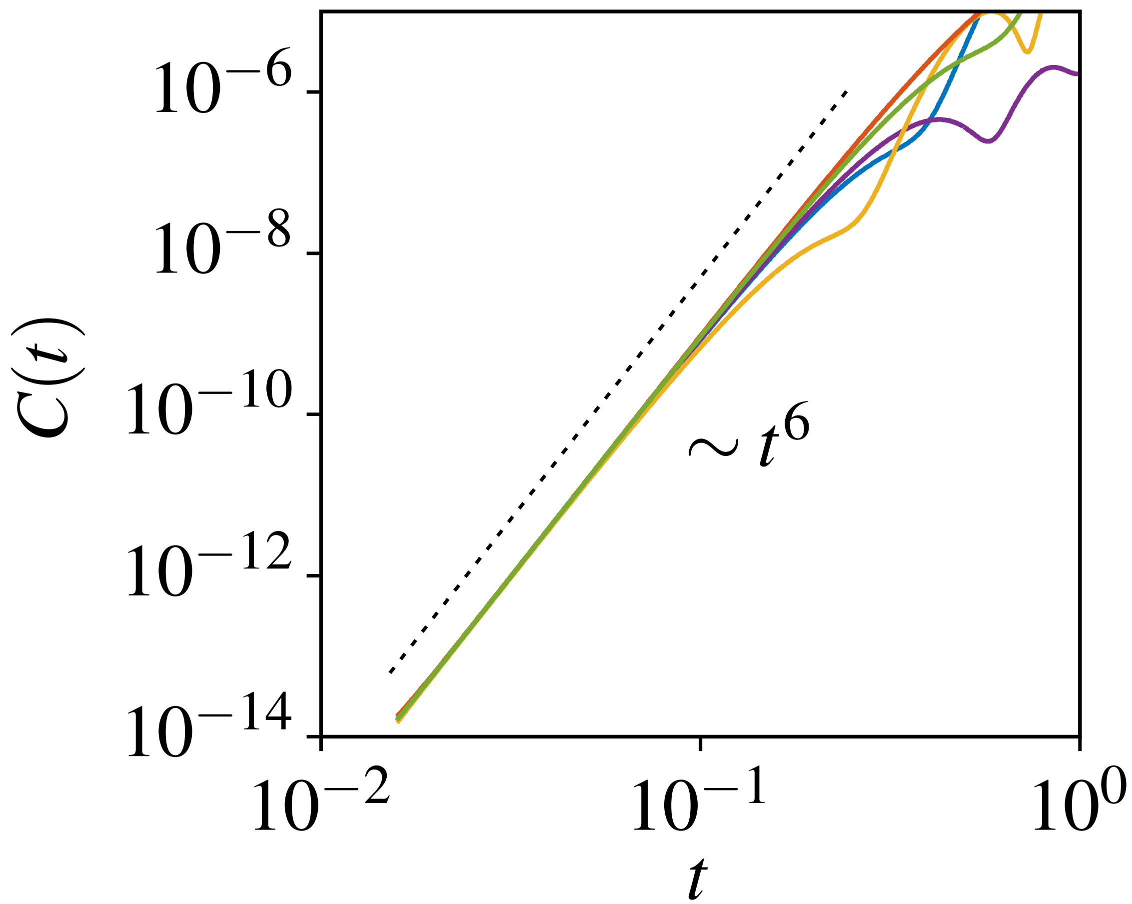
No disorder contribution!

Interacton is not essential.

Early-time growth

$$\hat{\sigma}^x_{r+1}(t) = \hat{\sigma}^x_{r+1} + it[\mathcal{H}, \hat{\sigma}^x_{r+1}] + \frac{(it)^2}{2!}[\mathcal{H}, [\mathcal{H}, \hat{\sigma}^x_{r+1}]] + \cdots$$

The lowest-order term with $\hat{\sigma}^x_1$ **appears at** t^r



Squared-commutator (OTOC)

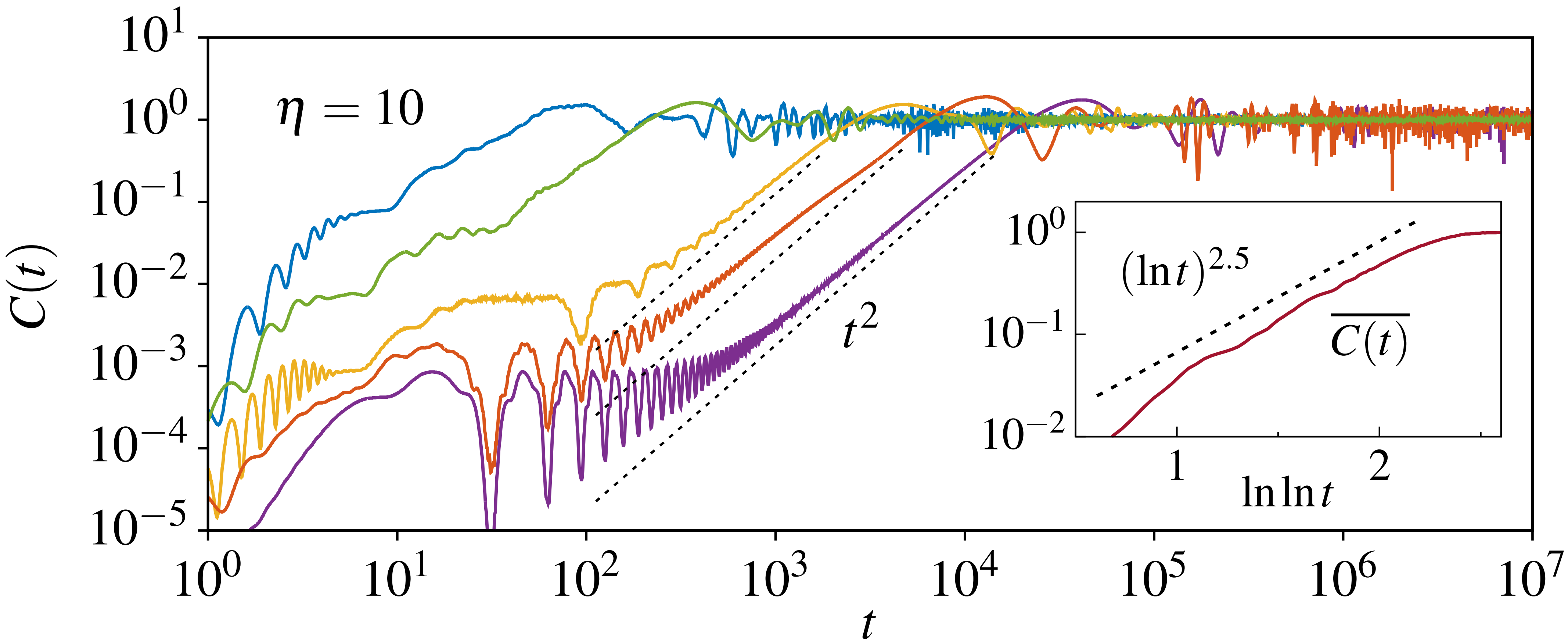
$$[\hat{\sigma}^x_4, \hat{\sigma}^x_1] \sim t^3 \rightarrow C(t) \sim t^6$$

This is an intrinsic property of the XXZ model!

No influence of disorder

Interaction is not essential.

Intermediate-time behavior



Some disorder realizations give a power-law behavior very close to t².

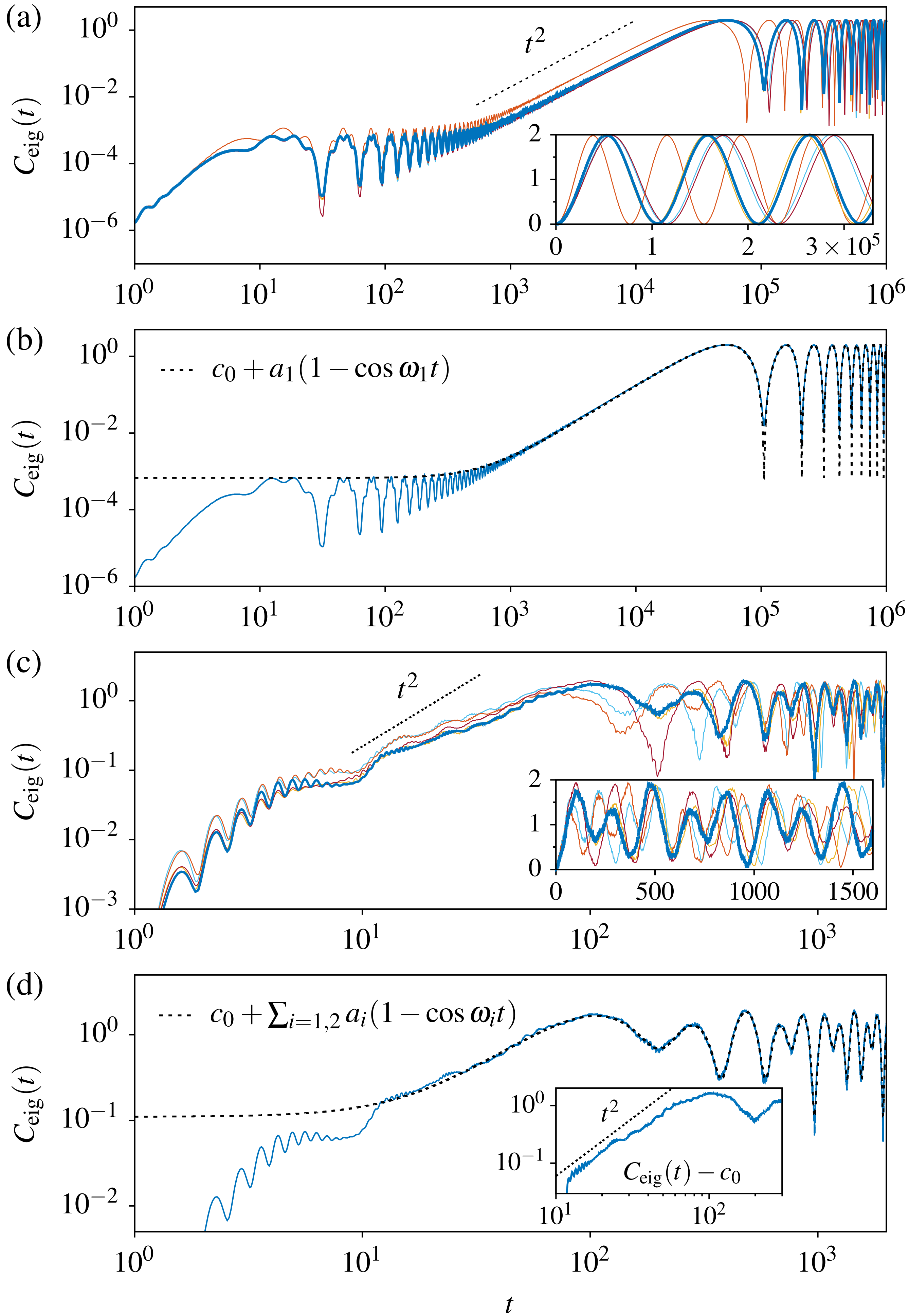
$$C(t) = c_0 + \varepsilon t^2$$

Early-time non-MBL part

1-cos(ωt) activated with a delay time

DISORDER-DEPENDENT OFFSET!

(from the fixed-point H)



Eigenstate-OTOC

Measured at an eigenstate

$$C_{\mathrm{eig}}(t) = 1 - \mathrm{Re}\Big[\sum_{\beta,\gamma,\delta} e^{it(E_\alpha - E_\beta + E_\gamma - E_\delta)} s_{\alpha\beta\gamma\delta}\Big]$$

$$s_{\alpha\beta\gamma\delta} = \langle\alpha|\hat{\sigma}_3^x|\beta\rangle\langle\beta|\hat{\sigma}_0^x|\gamma\rangle\langle\gamma|\hat{\sigma}_3^x|\delta\rangle\langle\delta|\hat{\sigma}_0^x|\alpha\rangle$$

Dominant modes with a few smallest ω_i

$$C_{\mathrm{eig}}(t) \sim c_0 + \sum_i a_i(1 - \cos\omega_i t)$$

fast components

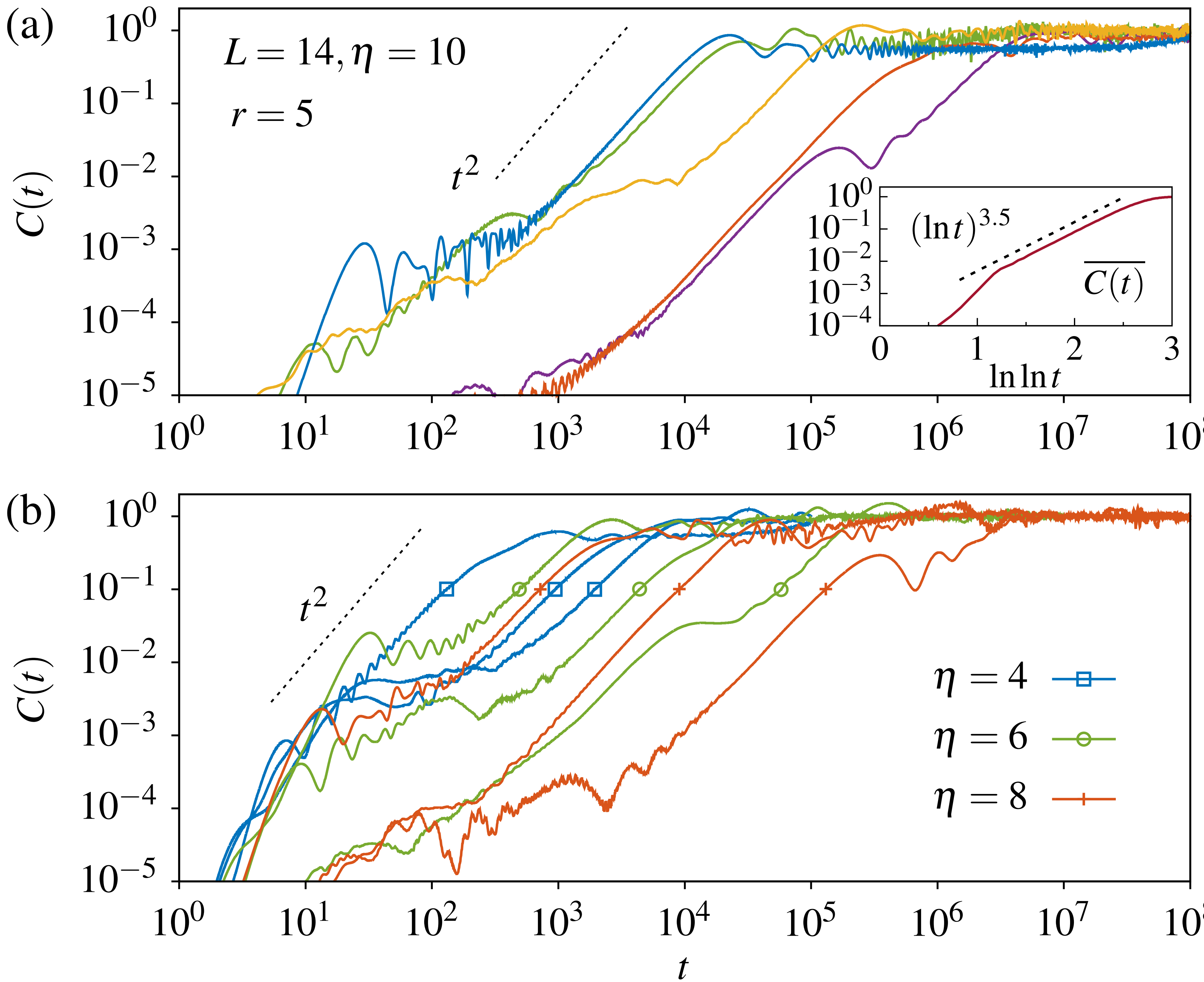
slow components

l-bit: $C(t) \simeq 1 - \cos\left(4t\langle\hat{J}_{ab}^{\mathrm{eff}}\rangle\right)$

Finite-size effects?

Disordered XXZ model

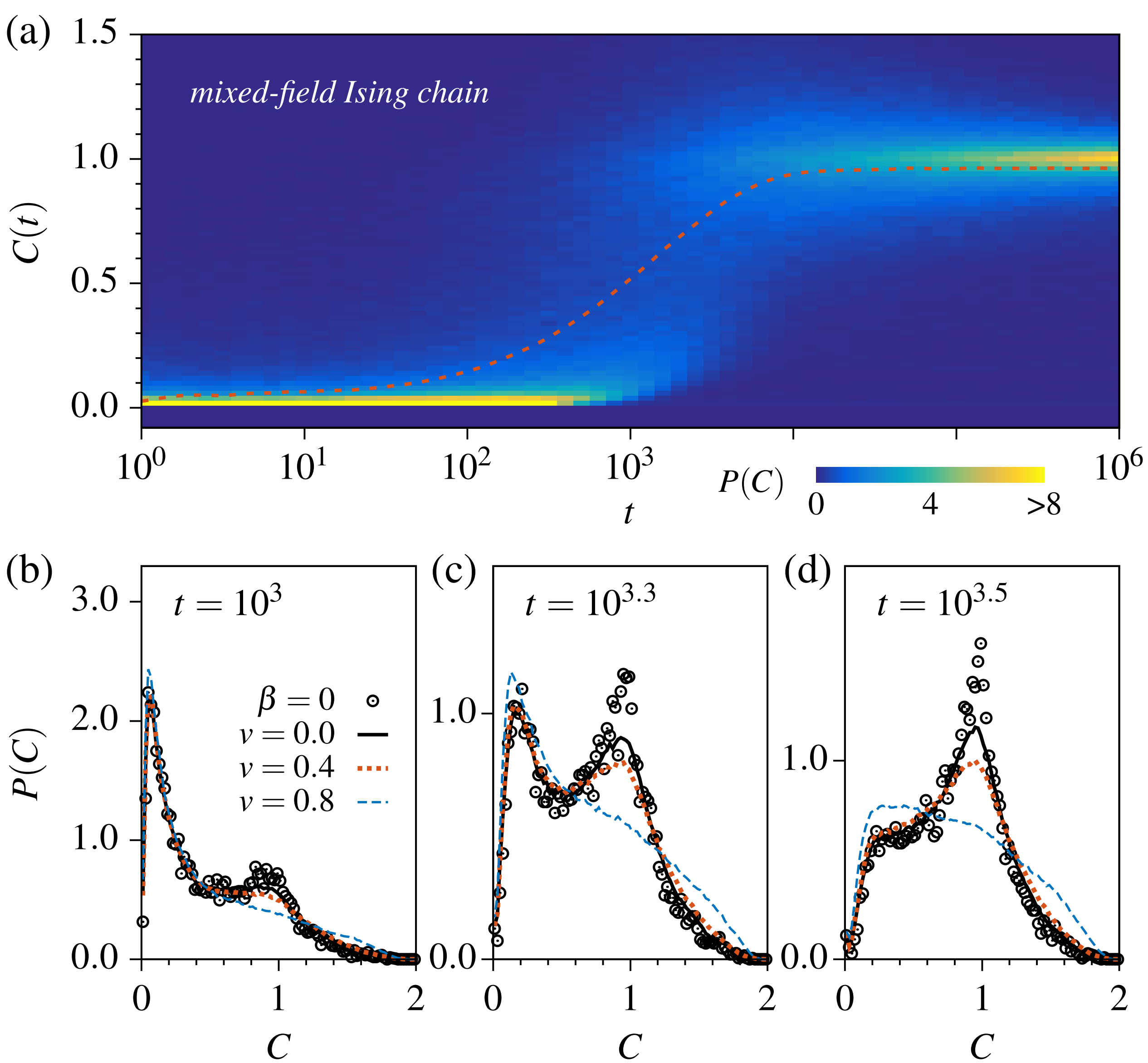
No systematic tests available, but it is the same in a larger system.



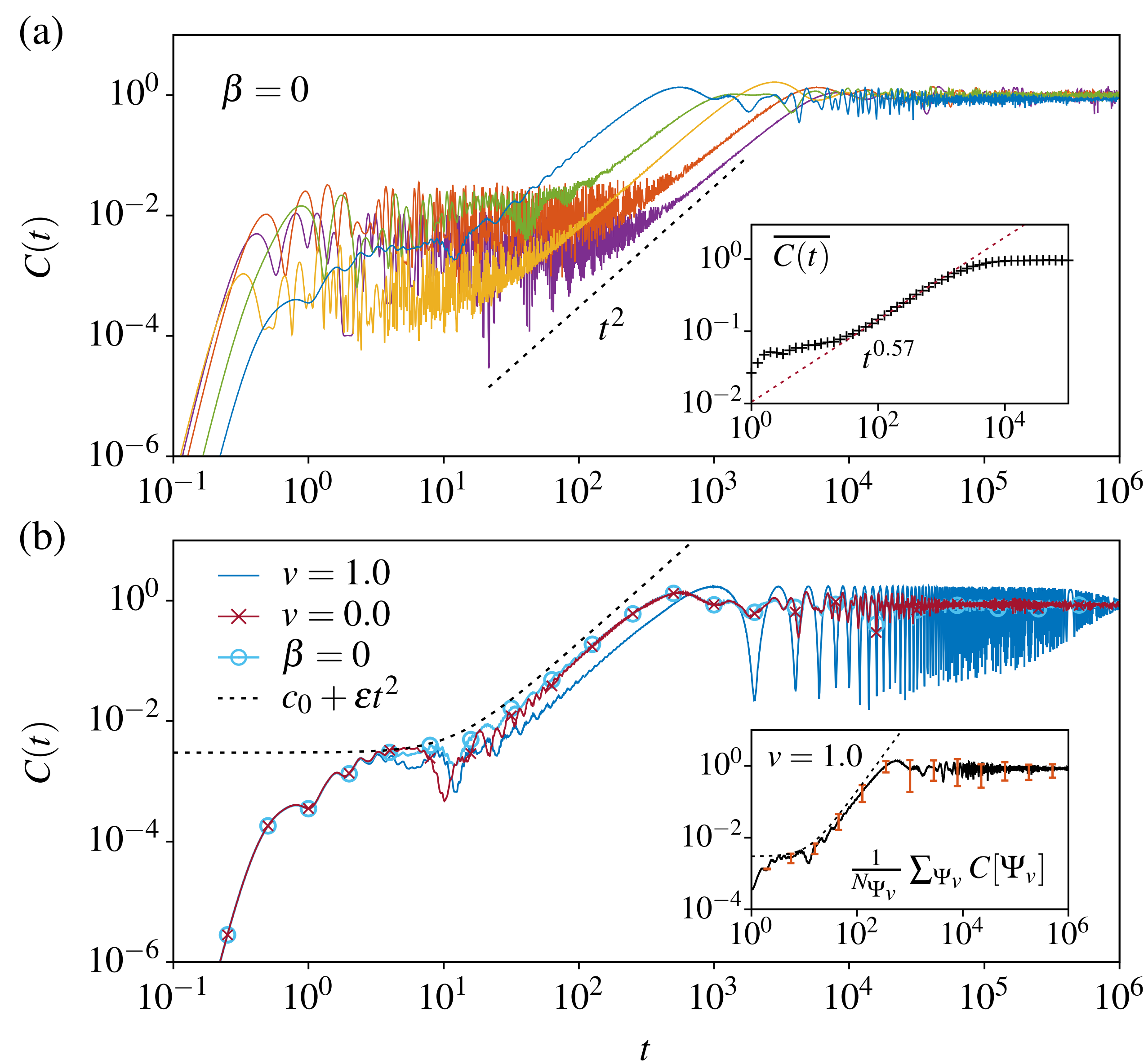
Another system: Mixed-Field Ising chain

$$\mathcal{H} = -\sum_{i=1}^{L-1} \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z - \sum_{i=1}^{L} h_i \hat{\sigma}_i^x - h_z \sum_{i=1}^{L} \hat{\sigma}_i^z \qquad \hat{W} = \hat{\sigma}_3^z \qquad \hat{V} = \hat{\sigma}_0^z$$

Doubly peaked distribution



t² growth!



Summary

J. Lee, D. Kim, and D.-H. Kim, PRB 99, 184202 (2019)

- Out-of-time-ordered commutator is examined as a measure of MBL.

- Beyond the effective l-bit model
  - **characteristic quadratic growth** in the MBL phase
    : Go for an *individual* disorder realization!
    Do not try disorder-averaging.
  - **unimodal-to-flat-to-bimodal** distribution
    : a possible indicator of the ergodic-MBL transition.

- Spectral characteristics of OTOC (on-going)
  - **A single dominant frequency mode exists.**
    Perturbation calculations may explain larger deviations.

- Future works: finite-size effects, more systematic analysis of the OTOC spectrum, different settings of disorders, a more experiment-friendly OTOC, …