

FEW-BODY METHODS FOR LIGHT NUCLEI & HYPERNUCLEI

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LECTURES AT ITP, CAS, BEIJING, CHINA

INTRODUCTION & OUTLINE

- Introduction to chiral nuclear interactions
- Numerical solution of the two-nucleon problem & quality of NN interactions
- Three-nucleon bound state problem in momentum space & size of three-nucleon forces
- Three-nucleon scattering problem
- Four-nucleons & extension to hypernuclei
- Jacobi no-core shell model
- Similarity renormalization group: unitary transformations for soft interactions
- Extension to $S = -1$ hypernuclei
- Extension to $S = -2$ hypernuclei
- Outlook

BRIEF INTRO: NUCLEAR INTERACTIONS

Symmetries of QCD

QCD Lagrangian, light quark sector

$$\mathcal{L}_{\text{QCD}} = \bar{q}(x) iD q(x) - \bar{q}(x) M q(x) \quad \text{with} \quad q(x) = \begin{pmatrix} u(x) \\ d(x) \\ s(x) \end{pmatrix} \quad M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \quad D^\mu = \partial^\mu - ig G_\alpha^\mu(x) \frac{\lambda_\alpha}{2}$$

only flavor space shown
4-spinor and 3-vector in color space!

Gell-Mann matrices in color space
color indices $\alpha, \dots$

mass matrix defined in flavor space

define left and right handed quark states

$$q_{\text{R/L}}(x) = \frac{1}{2} (\mathbb{I} \pm \gamma_5) q(x)$$

$$\mathcal{L}_{\text{QCD}} = \bar{q}_{\text{R}}(x) i \not{D} q_{\text{R}}(x) + \bar{q}_{\text{L}}(x) iD q_{\text{L}}(x) - \bar{q}_{\text{L}}(x) M q_{\text{R}}(x) - \bar{q}_{\text{R}}(x) M q_{\text{L}}(x)$$

invariant under independent flavor rotations L and R of left and right handed quarks

breaks symmetry under independent flavor rotations L and R
symmetry under flavor rotation if masses are equal
approximate symmetry under both if mass term is **small**

Symmetry implies approximately conserved Noether currents

$$L_a^\mu(x) = \bar{q}_{\text{L}}(x) \gamma^\mu \frac{\lambda_a}{2} q_{\text{L}}(x) \quad \text{and} \quad R_a^\mu(x) = \bar{q}_{\text{R}}(x) \gamma^\mu \frac{\lambda_a}{2} q_{\text{R}}(x)$$

or in terms of vector and axial vector currents

$$V_a^\mu(x) = R_a^\mu(x) + L_a^\mu(x) = \bar{q}(x) \gamma^\mu \frac{\lambda_a}{2} q(x)$$

$$A_a^\mu(x) = R_a^\mu(x) - L_a^\mu(x) = \bar{q}(x) \gamma^\mu \gamma_5 \frac{\lambda_a}{2} q(x)$$

Approximate current conservation

$$\partial_\mu V_a^\mu(x) = i\bar{q}(x) \left[M, \frac{\lambda_a}{2} \right] q(x)$$

$$\partial_\mu A_a^\mu(x) = i\bar{q}(x) \gamma_5 \left\{ M, \frac{\lambda_a}{2} \right\} q(x)$$

conserved for equal quark masses
(isospin/SU(3)-flavor symmetry)

$$\partial_\mu V^\mu(x) = 0$$

always conserved (baryon number)

$$\partial_\mu A^\mu(x) = 2i\bar{q}(x) \gamma_5 M q(x) + \frac{N_f}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} G_\alpha^{\mu\nu} G_\alpha^{\rho\sigma}$$

conserved for zero quark masses

not conserved, axial anomaly

Approximately conserved charges: $Q_{Va}(t) = \int d^3x V_a^0(\vec{x}, t)$ $Q_{Aa}(t) = \int d^3x A_a^0(\vec{x}, t)$ $[H, Q_{Va}(t)] \approx 0$ and $[H, Q_{Aa}(t)] \approx 0$

Vector symmetry experimentally observed (flavor multiplets)

Axial symmetry experimentally **not** observed (no parity multiplets!)

→ spontaneously broken down to vector symmetry $SU(3)_L \otimes SU(3)_R \longrightarrow SU(3)_V$

Spontaneously-broken **chiral symmetry** imply

- **8 (nearly-)massless (pseudo-)Goldstone bosons: π, K, η**
- in chiral limit vanishing coupling of Goldstone bosons to other nucleons, hyperons

Chiral Effective Field Theory

see Scherer arXiv:hep-ph/0210398 [hep-ph]

Epelbaum arXiv:1001.3229 [nucl-th]

for a general introduction to EFTs: see also Meißner, Rusetsky, Effective Field Theories, Cambridge University Press, 2022

QCD non-perturbative $\longrightarrow$ Solutions only for $A \lesssim 1$ (Lattice QCD)
Confinement: relevant degrees of freedom are hadrons
Goldstone bosons: "nearly" massless, relevant for almost all reactions

$\longrightarrow$ formulate perturbative effective field theory of hadrons and π, K, η :
chiral perturbation theory (ChPT)

EFT idea:

- choose fields that represent relevant degrees of freedom (here in terms of π, K, η and nucleon & hyperon)
- formulate **most general** Lagrangian in terms of these fields with **same symmetries** as QCD
- sort **importance of terms** identifying a small parameter and its powers (power counting)
- evaluate all Feynman diagrams **up to a given order** in this power counting

Difficulties

- need to realize chiral symmetry for effective fields
- hadron mass introduces high energy scale
- for $A \geq 2$ problem again non-perturbative

Non-linear realization of chiral symmetry for effective fields

for $L, R \in SU(3)$ and defining for $\Phi(x) = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}$ the unitary matrix $U(x) = \exp\left(i\frac{\Phi(x)}{F}\right)$

specific, non-unique realization

pion-decay constant

and defining fields for the baryon octet $B(x) = \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix} = \sum_a B_a(x) \frac{\lambda_a}{\sqrt{2}}$ or $\psi = \begin{pmatrix} p \\ n \end{pmatrix}$

$SU(3)$

$SU(2)$

the mesons transform as $U \longrightarrow U' = RUL^\dagger$

with $K = \sqrt{RUL^\dagger}^{-1} R \sqrt{U}$ the baryons transform as

$$B \longrightarrow B' = KBK^\dagger$$

$$\psi \longrightarrow \psi' = K\psi$$

see lectures on ChPT for more explanations!

External sources in QCD & EFT

Chirally symmetrical EFT built from blocks with definite, simple transformation properties e.g. from U and B .

Question: how to relate fields beyond quark in QCD and EFT $\longrightarrow$ method of external sources

let action depend on external sources (matrix valued)

$$v_\mu(x) = v_\mu^a(x) \frac{\lambda^a}{2} \quad a_\mu(x) = a_\mu^a(x) \frac{\lambda^a}{2} \quad s(x) = s^a(x) \frac{\lambda^a}{2} \quad p(x) = p^a(x) \frac{\lambda^a}{2} \quad \text{or} \quad r_\mu(x) = v_\mu(x) + a_\mu(x) \quad \text{and} \quad l_\mu(x) = v_\mu(x) - a_\mu(x)$$

$$\longrightarrow \mathcal{Z}(s, p, v, a) = \int dq d\bar{q} dG_\mu \exp \left(i \int d^4x \left[\mathcal{L}_G + \mathcal{L}_q + \text{gauge fixing} + \text{ghosts} + \dots \right] \right)$$

sources enter in quark Lagrangian

$$\mathcal{L}_q = \bar{q}(x) \left[\gamma_\mu \left(iD_\mu + v_\mu(x) + \gamma_5 a_\mu(x) \right) - s(x) + i\gamma_5 p(x) \right] q(x)$$

by construction (not nature): quark Lagrangian is invariant under local chiral transformations!

$\longrightarrow$ read off necessary trafos of sources

$$\begin{aligned} l_\mu(x) &\xrightarrow{(L(x), R(x))} L(x) l_\mu(x) L^\dagger(x) + iL(x) \partial_\mu L^\dagger(x) \\ r_\mu(x) &\xrightarrow{(L(x), R(x))} R(x) r_\mu(x) R^\dagger(x) + iR(x) \partial_\mu R^\dagger(x) \\ s(x) + ip(x) &\xrightarrow{(L(x), R(x))} R(x) [s(x) + ip(x)] L^\dagger(x) \\ s^\dagger(x) - ip^\dagger(x) &\xrightarrow{(L(x), R(x))} L(x) [s^\dagger(x) - ip^\dagger(x)] R^\dagger(x) \end{aligned}$$

chiral symmetry with external sources in EFT

e.g. use $\chi = 2B_0 (s + ip)$

a priori unknown (low energy) constant (LEC)

building block with external sources

then $\mathcal{L}_{eff} = \frac{F^2}{4} \text{Tr} \left(\partial_\mu U \partial^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U \right)$ is invariant under chiral transformations (for $r_\mu, l_\mu = 0$)

now compare to QCD again

$$\mathcal{L}_{\text{QCD}} = \bar{q}(x) i \not{D} q(x) - \bar{q}(x) M q(x)$$

$$\mathcal{L}_q = \bar{q}(x) \left[\gamma_\mu \left(iD_\mu + v_\mu(x) + \gamma_5 a_\mu(x) \right) - s(x) + i\gamma^5 p(x) \right] q(x)$$

mass is scalar source

→ $\mathcal{L}_{eff} = \frac{F^2}{4} \text{Tr} \left(\partial_\mu U \partial^\mu U^\dagger + 2B_0 M U^\dagger + 2B_0 M U \right)$ and don't care about chiral symmetry ...

not the same!!!

Systematic way to introduce chiral symmetry breaking terms in EFT that break symmetry the same way in QCD

also works for electroweak gauge boson fields attached to quarks using the covariant derivative

$$D_\mu A(x) = \partial_\mu A(x) + i A(x) l_\mu(x) - i r_\mu(x) A(x)$$

with $D_\mu A(x) \longrightarrow D'_\mu A'(x) = R(x) D_\mu A(x) L^\dagger(x)$

and $A(x) \longrightarrow A'(x) = R(x) A(x) L^\dagger(x)$

Leading Lagrangian & building blocks

Chiral connection defines covariant derivative for baryon fields

$$D_\mu = \partial_\mu + \Gamma_\mu \quad \text{with} \quad \Gamma_\mu = \frac{1}{2} \left[u \left[\partial_\mu - i l_\mu \right] u^\dagger + u^\dagger \left[\partial_\mu - i r_\mu \right] u \right]$$
$$\text{and} \quad u = \sqrt{U} = \exp \left(i \frac{\Phi}{2F_0} \right)$$

Chiral vielbein has simple transformation properties

building block	chiral trafo	P	C
U	$RUL^\dagger$	$U^\dagger$	U^T
$D_{\mu_1} \dots D_{\mu_n} U$	$RD_{\mu_1} \dots D_{\mu_n} UL^\dagger$	$[D^{\mu_1} \dots D^{\mu_n} U]^\dagger$	$[D_{\mu_1} \dots D_{\mu_n} U]^T$
χ	$R\chi L^\dagger$	$\chi^\dagger$	χ^T
$D_{\mu_1} \dots D_{\mu_n} \chi$	$RD_{\mu_1} \dots D_{\mu_n} \chi L^\dagger$	$[D^{\mu_1} \dots D^{\mu_n} \chi]^\dagger$	$[D_{\mu_1} \dots D_{\mu_n} \chi]^T$
r_μ	$Rr_\mu R^\dagger + iR\partial_\mu R^\dagger$	l^μ	$-l_\mu^T$
l_μ	$Ll_\mu L^\dagger - iL\partial_\mu L^\dagger$	r^μ	$-r_\mu^T$
$f_{R\mu\nu}$	$Rf_{R\mu\nu}R^\dagger$	$f_L^{\mu\nu}$	$-f_{L\mu\nu}^T$
$f_{L\mu\nu}$	$Lf_{L\mu\nu}L^\dagger$	$f_R^{\mu\nu}$	$-f_{R\mu\nu}^T$

$$u_\mu = i \left(u^\dagger \left[\partial_\mu - i r_\mu \right] u - u \left[\partial_\mu - i l_\mu \right] u^\dagger \right)$$

leading Goldstone boson part

$$\mathcal{L}^{(2)} = \frac{F_0^2}{4} \text{Tr} \left(D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U \right)$$

leading baryonic part

$SU(2)$ case:

$$\mathcal{L}_{\pi N}^{(1)} = \bar{\psi} \left[iD_\mu \gamma^\mu - m_N^0 + \frac{g_a}{2} \gamma^\mu \gamma_5 u_\mu \right] \psi$$

invariant under P, C and T !

$SU(3)$ case:

$$\mathcal{L}_{\Phi B}^{(1)} = \text{Tr} \left(\bar{B} \left[iD_\mu \gamma^\mu - m_0 \right] B \right) - \frac{D}{2} \text{Tr} \left(\bar{B} \gamma^\mu \gamma_5 \left\{ u_\mu, B \right\} \right) - \frac{F}{2} \text{Tr} \left(\bar{B} \gamma^\mu \gamma_5 \left[u_\mu, B \right] \right)$$

building block	chiral trafo
B	$KBK^\dagger$
$D_{\mu_1} \dots D_{\mu_n} B$	$KD_{\mu_1} \dots D_{\mu_n} BK^\dagger$
u	$RuK^\dagger$
u_μ	$Ku_\mu K^\dagger$

$SU(2)$ case: two parameters / LECs

axial-vector coupling: parity invariance, derivative coupling

$SU(3)$ case: three parameters / LECs

$1/m$ expansion of hadron (baryon) mass

problematic: time-derivative of baryon field and mass not suppressed

mostly used solution: Heavy Baryon ChiPT (HBChPT)

$P_{v\pm} = \frac{1}{2}(1 \pm \not{v})$ are projection operators

simple choice: $v^\mu = (1, \vec{0})$ or $v = p/m_N$

heavy & light component (lower & upper components)

$\mathcal{N}_v(x) = e^{im_N vx} P_{v+} \psi(x)$ $\mathcal{H}_v(x) = e^{im_N vx} P_{v-} \psi(x)$

dominant part of time-variation

$\longrightarrow \mathcal{L}_{\pi N}^{\text{eff}} = \overline{\mathcal{N}}_v \left(i v \cdot D + \frac{g_A}{2} u_{\perp, \mu} \gamma^\mu \gamma_5 \right) \mathcal{N}_v$ with $u_{\perp, \mu} = u_\mu - (v \cdot u) v_\mu$

leading HBChPT Lagrangian

Chiral EFT for $A \geq 2$

relevant Feynman rules (here nucleons and pions, $SU(2)$ case)

naive power counting defines scaling
of arbitrary diagram with respect to external momenta
and Goldstone boson masses (chiral dimension)

$$D = 1 + 2N_L + \sum_{k=1}^{\infty} N_{2k}^M (2k - 2) + \sum_{k=1}^{\infty} N_k^B (k - 1) + \text{two- and more-baryon}$$

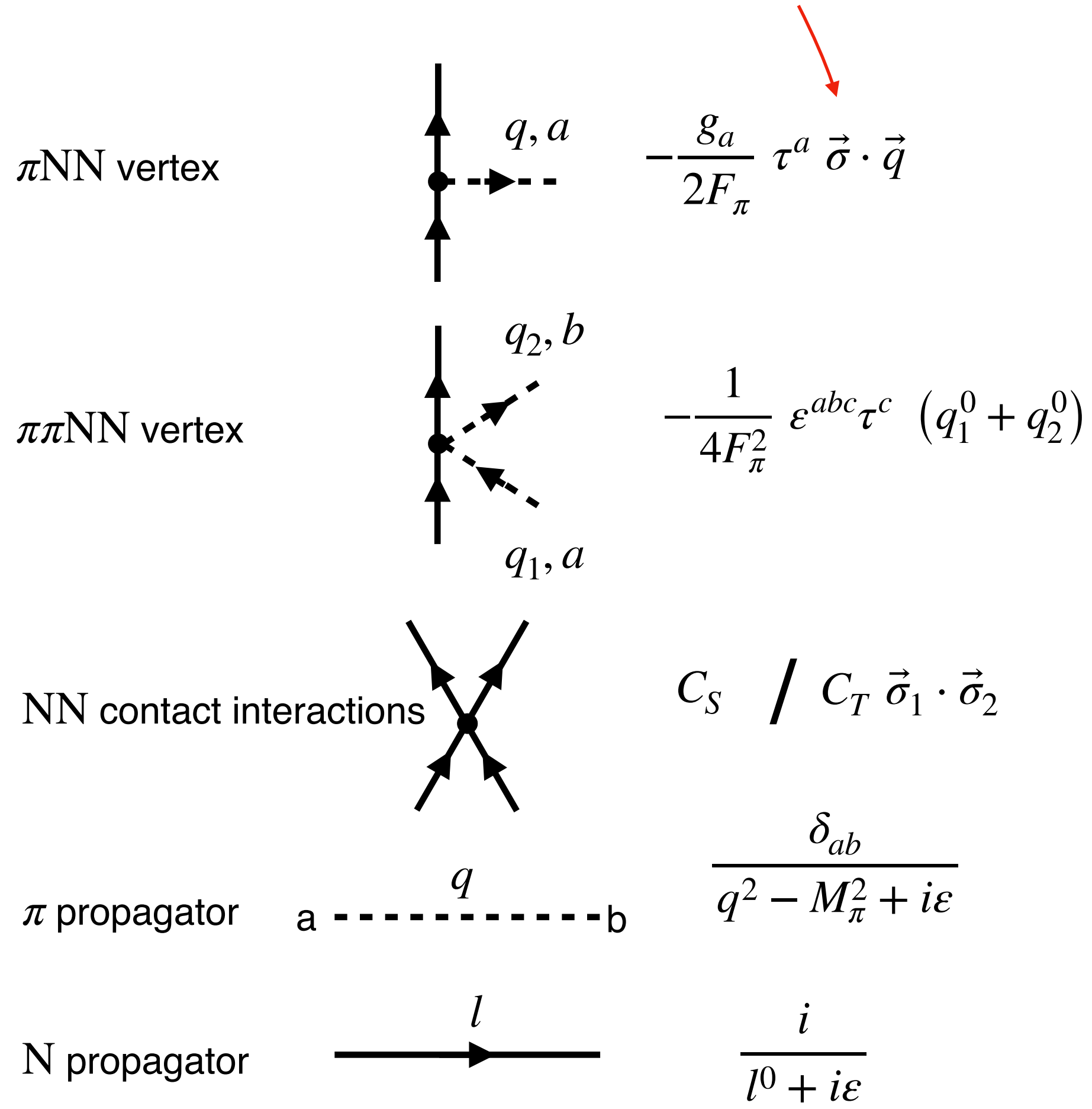
of loops

order $2k$ meson vertices

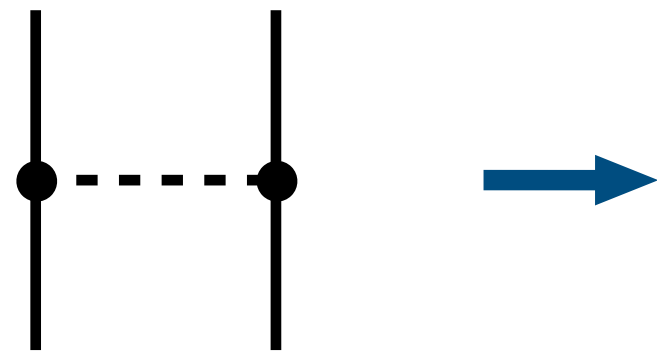
order k meson-baryon vertices

→ diagram scales like Q^D

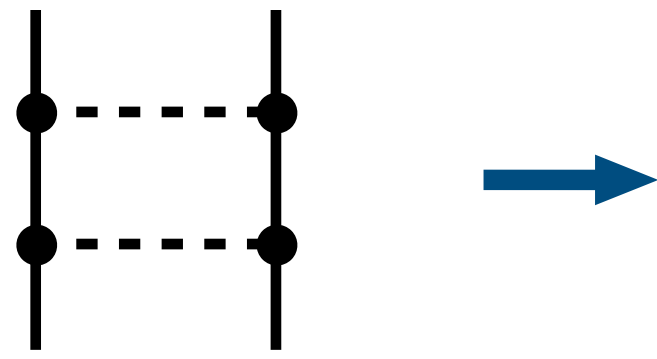
N is a 2-component spinor and 2-component isospin vector



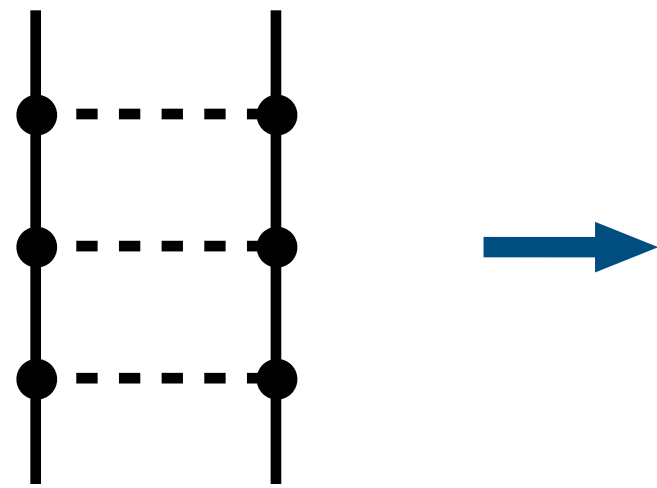
Naive power counting in the NN system (example: iterations of 1π exchange):



$$\underbrace{\frac{1}{Q^2}}_{\text{propagator}} \cdot \underbrace{Q \cdot Q}_{\text{vertices}} \propto \mathcal{O}(Q^0)$$



$$\underbrace{\left(\frac{1}{Q}\right)^2 \left(\frac{1}{Q^2}\right)^2}_{\text{propagator}} \cdot \underbrace{Q \cdot Q \cdot Q \cdot Q}_{\text{vertices}} \cdot \underbrace{Q^4}_{\text{loop}} \propto \mathcal{O}(Q^2)$$



$$\underbrace{\left(\frac{1}{Q}\right)^4 \left(\frac{1}{Q^2}\right)^3}_{\text{propagator}} \cdot \underbrace{Q^6}_{\text{vertices}} \cdot \underbrace{Q^8}_{\text{loop}} \propto \mathcal{O}(Q^4)$$

Power counting indicates that nuclear interactions are perturbative
In contradiction with the observation of bound states

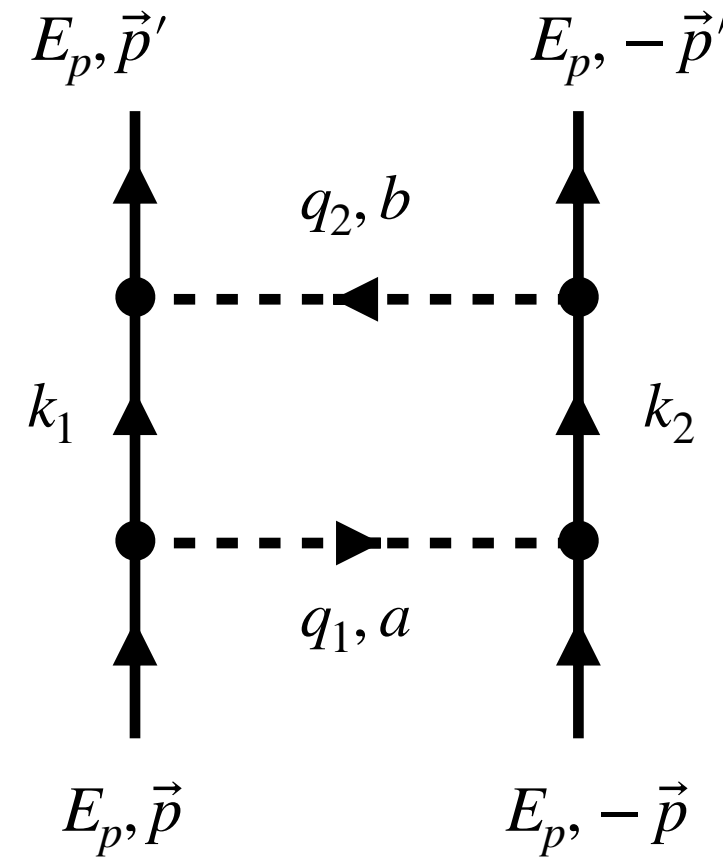
Let us look in more detail into the 2π exchange ("box") diagram

The amplitude is then

$$i\mathcal{M} = i \int \frac{d^4l}{(2\pi)^4} \left(-\frac{g_a}{2F_\pi} \right)^4 \frac{1}{q_1^2 - M_\pi^2 + i\varepsilon} \frac{1}{q_2^2 - M_\pi^2 + i\varepsilon} \tau_1^b \vec{\sigma}_1 \cdot (-\vec{q}_2) \frac{i}{E_p - l^0 - \frac{\vec{l}^2}{2m_N} + i\varepsilon} \tau_1^a \vec{\sigma}_1 \cdot \vec{q}_1 \tau_2^b \vec{\sigma}_2 \cdot \vec{q}_2 \frac{i}{E_p + l^0 - \frac{\vec{l}^2}{2m_N} + i\varepsilon} \tau_2^a \vec{\sigma}_2 \cdot (-\vec{q}_1)$$

spin/isospin space of nucleon 1 spin/isospin space of nucleon 2

additional term is higher order in $1/m_N$ expansion



$$q_1 = (l^0, \vec{p} - \vec{l})$$

$$q_2 = (l^0, \vec{p}' - \vec{l})$$

$$k_1 = (E_p - l_0, \vec{l})$$

$$k_2 = (E_p + l_0, -\vec{l})$$

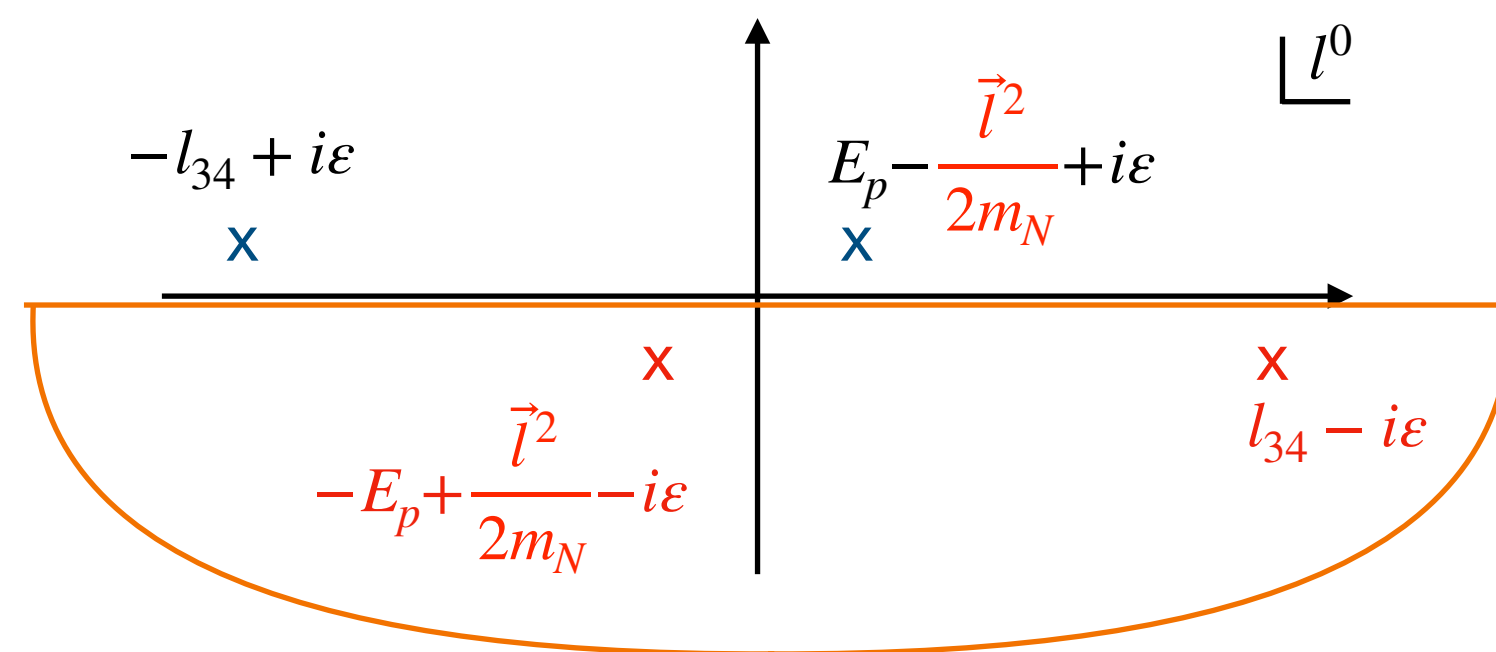
$$E_p = \frac{\vec{p}^2}{2m_N} = \frac{\vec{p}'^2}{2m_N}$$

Now let us look specifically at l^0 integral combine the mesonic propagators using the Feynman trick

$$\int \frac{dl^0}{2\pi} \frac{1}{E_p - l^0 - \frac{\vec{l}^2}{2m_N} + i\varepsilon} \frac{1}{E_p + l^0 - \frac{\vec{l}^2}{2m_N} + i\varepsilon} \frac{1}{(l^0)^2 - \underbrace{\left((\vec{p} - \vec{l})^2 + M_\pi^2 - i\varepsilon \right)}_{\equiv \omega_1^2}} \frac{1}{(l^0)^2 - \underbrace{\left((\vec{p}' - \vec{l})^2 + M_\pi^2 - i\varepsilon \right)}_{\equiv \omega_2^2}}$$

$$= \int_0^1 dx \int \frac{dl^0}{2\pi} \frac{1}{E_p - l^0 - \frac{\vec{l}^2}{2m_N} + i\varepsilon} \frac{1}{E_p + l^0 - \frac{\vec{l}^2}{2m_N} + i\varepsilon} \frac{1}{\left[(l^0)^2 - \underbrace{\left[x\omega_1^2 + (1-x)\omega_2^2 - i\varepsilon \right]}_{\equiv l_{34}^2} \right]^2} = \int_0^1 dx \int \frac{dl^0}{2\pi} \frac{1}{E_p - l^0 - \frac{\vec{l}^2}{2m_N} + i\varepsilon} \frac{1}{E_p + l^0 - \frac{\vec{l}^2}{2m_N} + i\varepsilon} \frac{1}{(l^0 - l_{34})^2} \frac{1}{(l^0 + l_{34})^2}$$

With partial fraction decomposition and residue theorem



$$= -i \int_0^1 dx \left[\frac{1}{2E_p - \frac{\vec{l}^2}{m_N}} \frac{1}{\left(E_p - \frac{\vec{l}^2}{2m_N} + l_{34} \right)^2} \frac{1}{\left(E_p - \frac{\vec{l}^2}{2m_N} - l_{34} \right)^2} + \frac{-(E_p - \frac{\vec{l}^2}{2m_N})^2 + 3l_{34}^2}{4 \left(E_p - \frac{\vec{l}^2}{2m_N} + l_{34} \right)^2} \frac{1}{\left(E_p - \frac{\vec{l}^2}{2m_N} - l_{34} \right)^2} \frac{1}{l_{34}^3} \right]$$

Let us do power counting again!

$$\int_0^1 dx \left[\frac{1}{2E_p} \frac{1}{(E_p + l_{34})^2} \frac{1}{(E_p - l_{34})^2} + \frac{-E_p^2 + 3l_{34}^2}{4(E_p + l_{34})^2 (E_p - l_{34})^2} \frac{1}{l_{34}^3} \right]$$

$\frac{m_N}{Q^2} \frac{1}{Q^2} \frac{1}{Q^2} Q^3 Q^4 \quad Q^2 \frac{1}{Q^2} \frac{1}{Q^2} \frac{1}{Q^3} Q^3 Q^4$

$\propto \mathcal{O}\left(\frac{m_N}{Q} Q^2\right) \quad \propto \mathcal{O}(Q^2)$

numerator
 loop

We observe that the part related to the baryon poles is enhanced by a factor m_N/Q .

This enhancement factor is sufficient to turn the **problem non-perturbative!**

(rewriting the amplitude of the enhanced part of the example results purely in iterated 1π exchange)

➔ general solution:

- define potential using 2-Baryon irreducible diagrams (i.e. pole not related to baryons, but Goldstone bosons)
- sum iterated potential using a **Schrödinger** or **Lippmann-Schwinger equation**

To see the last statement, let us look at the enhanced part for the 2π exchange

$$i\mathcal{M} = i \left(-\frac{g_a}{2F_\pi} \right)^4 \int \frac{d^3l}{(2\pi)^3} \int_0^1 dx \left[\frac{1}{\frac{\vec{p}^2}{m_N} - \frac{\vec{l}^2}{m_N} + i\varepsilon} \frac{1}{(l_{34})^4} \right] \vec{\tau}_1 \cdot \vec{\tau}_2 \vec{\tau}_1 \cdot \vec{\tau}_2 \vec{q}_1 \cdot \sigma_1 \vec{q}_1 \cdot \sigma_2 \vec{q}_2 \cdot \sigma_1 \vec{q}_2 \cdot \sigma_2$$

add the numerator again to recognize the 1π exchange

$m_N \rightarrow \infty$ whenever
leading term is m_N
independent.

Reverse Feynman trick

$$= i \left(-\frac{g_a}{2F_\pi} \right)^4 \int \frac{d^3l}{(2\pi)^3} \left[\frac{1}{\frac{\vec{p}^2}{m_N} - \frac{\vec{l}^2}{m_N} + i\varepsilon} \frac{\vec{\tau}_1 \cdot \vec{\tau}_2 (\vec{q}_1 \cdot \vec{\sigma}_1)(\vec{q}_1 \cdot \vec{\sigma}_2)}{(\omega_1^2 - i\varepsilon)} \frac{\vec{\tau}_1 \cdot \vec{\tau}_2 (\vec{q}_2 \cdot \vec{\sigma}_1)(\vec{q}_2 \cdot \vec{\sigma}_2)}{(\omega_2^2 - i\varepsilon)} \right]$$

Insert $\omega_i^2 = \vec{q}_i^2 + M_\pi^2$

$$= i \left(-\frac{g_a}{2F_\pi} \right)^4 \int \frac{d^3l}{(2\pi)^3} \left[\frac{\vec{\tau}_1 \cdot \vec{\tau}_2 (\vec{q}_1 \cdot \vec{\sigma}_1)(\vec{q}_1 \cdot \vec{\sigma}_2)}{\vec{q}_1^2 + M_\pi^2} \frac{1}{\frac{\vec{p}^2}{m_N} - \frac{\vec{l}^2}{m_N} + i\varepsilon} \frac{\vec{\tau}_1 \cdot \vec{\tau}_2 (\vec{q}_2 \cdot \vec{\sigma}_1)(\vec{q}_2 \cdot \vec{\sigma}_2)}{\vec{q}_2^2 + M_\pi^2} \right]$$

In this form one recognizes the iterated 1π exchange using a non-relativistic Lippmann-Schwinger equation (equivalent to Schrödinger equation).

SCHRÖDINGER EQUATION

[\(back\)](#)

Configuration and momentum space basis

Quantum mechanical two-body state problem: stationary **Schrödinger equation** (non-rel., for short range interactions, no Coulomb)

$$\hat{H} |\Psi\rangle = E |\Psi\rangle$$

E < 0 for the bound state (Why is this necessarily the case?)

Hamiltonian separated in kinetic energy $\hat{T}$ and potential $\hat{V}$: $\hat{H} = \hat{T} + \hat{V}$

Single particle coordinates, e.g. position or momentum:

$$\vec{r}_1, \vec{r}_2 \quad \text{or} \quad \vec{k}_1, \vec{k}_2$$

Exploit translation invariance  **Jacobi coordinates**

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\vec{R} = \frac{1}{m_1 + m_2} (m_1 \vec{r}_1 + m_2 \vec{r}_2)$$

$$\vec{p} = \frac{1}{m_1 + m_2} (m_2 \vec{k}_1 - m_1 \vec{k}_2)$$

$$\vec{P} = \vec{k}_1 + \vec{k}_2$$

(convince yourself that $\vec{k}_1 \vec{r}_1 + \vec{k}_2 \vec{r}_2 = \vec{p} \vec{r} + \vec{P} \vec{R}$,

also note that the **Jacobian** for changing variables

$\vec{r}_1, \vec{r}_2 \leftrightarrow \vec{r}, \vec{R}$ and $\vec{k}_1, \vec{k}_2 \leftrightarrow \vec{p}, \vec{P}$ is **one**)

Express the Schrödinger equation using **Jacobi momentum eigenstates** $|\vec{p} \vec{P}\rangle$ (unity operator) $\int d^3p' d^3P' |\vec{p}' \vec{P}'\rangle \langle \vec{p}' \vec{P}'| = \mathbb{1}$

$$\left(\frac{\vec{p}^2}{2\mu_{12}} + \frac{\vec{P}^2}{2M_{12}} \right) \psi(\vec{p}, \vec{P}) + \int d^3p' \langle \vec{p} | \hat{V} | \vec{p}' \rangle \psi(\vec{p}', \vec{P}) = E \underbrace{\psi(\vec{p}, \vec{P})}_{\psi(\vec{p})\phi(\vec{P})}$$

Separate off the **center of mass (CM)** motion: $\left(\bar{E} - \frac{\vec{p}^2}{2\mu_{12}} \right) \psi(\vec{p}) = \int d^3p' \langle \vec{p} | \hat{V} | \vec{p}' \rangle \psi(\vec{p}') \quad (*)$

Intrinsic energy: $\bar{E} = E - \frac{P^2}{2M_{12}}$ reduced mass: $\mu_{12} = \frac{m_1 m_2}{m_1 + m_2}$ and $M_{12} = m_1 + m_2$ (from now on: use E for the intrinsic energy)

For numerical solution: equation (*) is still **not** suitable.

- spectrum of the kinetic operator (and also Hamiltonian) is unbound.
- not interested in states with larger eigenvalues (high excitations, we need the ground state and low excitations).

Equation can be easily transformed so that the **eigenvalue spectrum is bounded**:

$$\psi(\vec{p}) = \frac{1}{\bar{E} - \frac{\vec{p}^2}{2\mu_{12}}} \int d^3p' \langle \vec{p} | \hat{V} | \vec{p}' \rangle \psi(\vec{p}')$$

- Zero is only one accumulation point of the eigenvalues of this kernel.
- Equation is compact and suitable for **solution by discretization**.

Two nucleon/baryon system

basis states include **spin**, particle kind/**isospin** (particle and isospin basis can be separately considered)

Partial wave decomposition to reduce the dimensionality of the problem:

Orbital part in momentum space.

$$\langle \vec{p} | \tilde{p} l m \rangle = \frac{\delta(p - \tilde{p})}{p \tilde{p}} Y_{lm}(\hat{p}) \quad \text{or} \quad \sum_{ml} \int d\tilde{p} \tilde{p}^2 | \tilde{p} l m \rangle \langle \tilde{p} l m | = \mathbb{1}$$

Now add spin $|m_1^s m_2^s\rangle$ and define total angular momentum using Clebsch-Gordan coefficients

$$|p (l(s_1 s_2) s) j m\rangle = \sum (s_1 s_2 s, m_1^s m_2^s m_1^s + m_2^s) (l s j, m - m_1^s - m_2^s m_1^s + m_2^s m) |p l m - m_1^s - m_2^s\rangle |m_1^s m_2^s\rangle$$

Rotational invariance and parity

$$\langle (l' s') j' m' | V | (l s) j m \rangle = V_{l'l}^{js}(p, p') \delta_{jj'} \delta_{ss'} \delta_{mm'}$$

rotational invariance: j conserved
parity: $l - l'$ even

Schrödinger equation reduces to **one dimensional coupled** integral equation

$$\psi_{l'}^{js}(p') = \frac{1}{\bar{E} - \frac{p^2}{2\mu_{12}}} \sum_l \int dp p^2 V_{l'l}^{js}(p', p) \psi_l^{js}(p)$$

summation of channels
can in general include particle channels

Numerical solution

1. partial wave representation of the interaction

use predefined library (usually FORTRAN routines, can be wrapped for, e.g., Python) $\longrightarrow V_{l'l}^{js}(p, p')$

2. **Standard numerical libraries** (Numerical recipes, gsl, numpy, ...): routines to define grid points and integration weights for **Gauss-Legendre integration**.

By definition, these grid points are suitable for an integration interval $[-1, 1]$.

For integration over an interval $[0, p_b]$, we map these grid points for example using $x \longrightarrow p(x) = \frac{1+x}{\frac{1}{p_a} - \left(\frac{1}{p_a} - \frac{2}{p_b}\right)x}$

$$\longrightarrow \int_0^{p_b} dp f(p) = \int dx p'(x) f(p(x)) \approx \sum_{i=1}^N \underbrace{\omega_i}_{\omega_i} \underbrace{p'(x_i)}_{p_i} f(\underbrace{p(x_i)}_{p_i}) \quad (\text{note that } [-1, 0] \longrightarrow [0, p_a] \text{ and } [0, 1] \longrightarrow [p_a, p_b])$$

Also linear transformation: $x \longrightarrow p(x) = \frac{p_b + p_c}{2} + \frac{p_c - p_b}{2}x$ (maps $[-1, 1] \longrightarrow [p_b, p_c]$)

3. **Use standard numerical libraries** to get eigen values/vectors of the discretized problem

$$\lambda(\bar{E}) v_{j\alpha'} \equiv \lambda(\bar{E}) \psi_{l'}^{js}(p_j) = \sum_{i\alpha} \underbrace{\frac{1}{\bar{E} - \frac{p_j^2}{2\mu_{12}}} \omega_i p_i^2 V_{l'l}^{js}(p_j, p_i)}_{\equiv A_{j\alpha', i\alpha}(\bar{E})} v_{i\alpha}$$

4. Eigenvalue spectrum of the matrix A is bounded even when the dimensionality increases.
One accumulation point at zero.

Considering a scaled potential: **ground state: largest (positive) eigenvalue**, first excited state: second one, etc.

➡ required are **few, largest eigenvalues and their dependence on $\bar{E}$** for negative energies
n-th bound state $\longleftrightarrow$ n-th eigenvalue $\lambda(\bar{E}) = 1$.

Simple numerical method to find $\bar{E}$: **secant method**

Linear approximation, starting from two estimated solutions E_1 and E_2

➡ improved approximation to the solution by

$$E = E_2 + \frac{E_1 - E_2}{\lambda(E_1) - \lambda(E_2)}(1 - \lambda(E_2))$$

The approximation can then be further improved by repeating the improvement for $E_1 = E_2$ and $E_2 = E$.

LIPPMANN-SCHWINGER EQUATION

2-body scattering problem

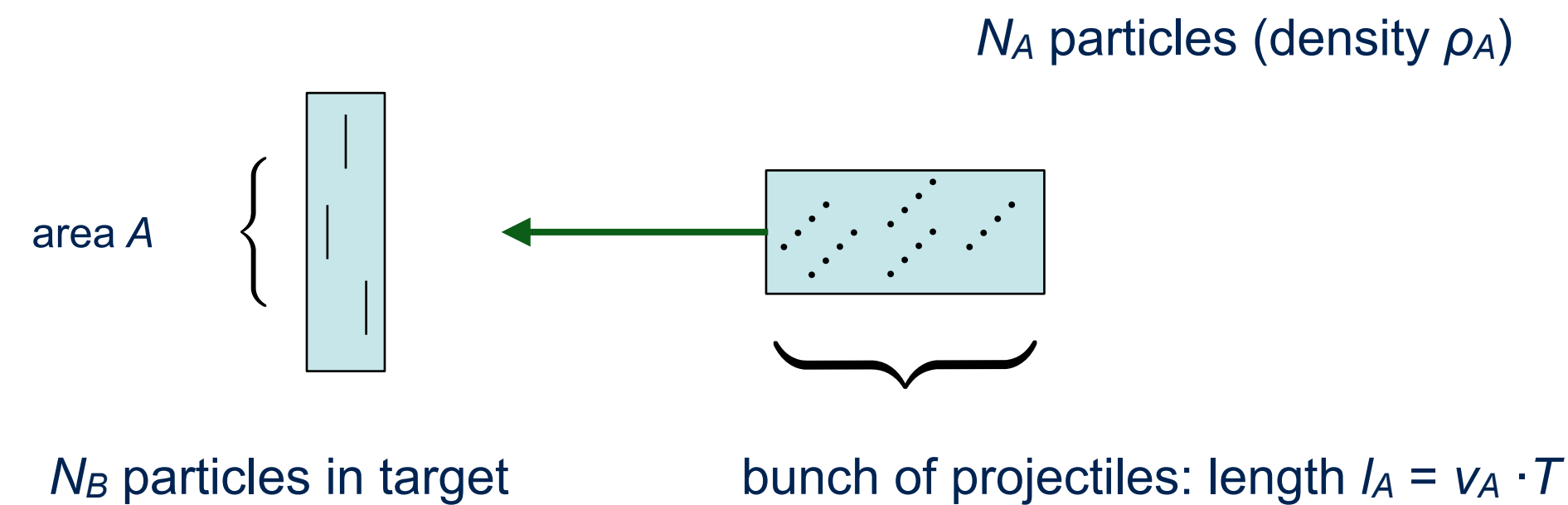
We now turn to the scattering problem. In this lecture, we will briefly introduce the underlying equations.

We will again use momentum space, which allows one directly include the boundary conditions into the resulting integral equation.

Technically, the main complication is the treatment of singularities of the integral equations.

The final observable that needs to be calculate will be the **cross section**.

A sketch of a typical scattering experiment

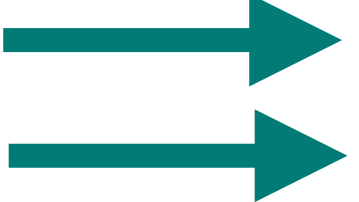


$$\sigma = \frac{\# \text{ of events}}{T} \frac{1}{N_B} \underbrace{\frac{1}{\rho_A v_A}}_{\text{particle current } j_A} = \underbrace{\frac{\# \text{ of events}}{T}}_{\text{transition probability per unit time}} \frac{1}{j_A}$$

"in" and "out" states

Need to calculate the **transition probability per unit time**.

The key for this is the definition of so called **"in" and "out" states**.

Start at large negative times $T \rightarrow -\infty$  particles well separated from each other
time dependence given by free Hamiltonian
"initial" state 

drop spin & particle indices for the time-being to simplify notation

$$|i\rangle = \int d^3k_1 d^3k_2 f(\vec{k}_1) \tilde{f}(\vec{k}_2) |\vec{k}_1 \vec{k}_2\rangle$$

Full time dependence given by full Hamiltonian: $i \frac{\partial}{\partial t} |\Psi\rangle = \hat{H} |\Psi\rangle$  $|\Psi(t)\rangle = \exp(-i \hat{H} t) |\Psi(0)\rangle \equiv \hat{U}(t) |\Psi(0)\rangle$

The aim is to define a scattering state $|\Psi(t)\rangle$ that corresponds to $|i\rangle$ at large negative times
(use $\hat{U}_0(t)$ for time evolution operator only involving the kinetic energy).

$$\hat{U}(T) |\Psi(0)\rangle \xrightarrow{T \rightarrow -\infty} \hat{U}_0(T) |i\rangle$$

This idea defines "in" and "out" states (for $T \rightarrow +\infty$):

$$|i, in\rangle \equiv \underbrace{\lim_{T \rightarrow -\infty} \hat{U}^\dagger(T) \hat{U}_0(T)}_{\Omega_+ - \text{Møller operator}} |i\rangle$$

$$|f, out\rangle \equiv \underbrace{\lim_{T \rightarrow \infty} \hat{U}^\dagger(T) \hat{U}_0(T)}_{\Omega_- - \text{Møller operator}} |f\rangle$$

"final" state

Given the "in" and "out" states, the transition probability from an "initial" to "final" state is given by the S operator (or in momentum space) **S-matrix**

$$|\underbrace{\langle f, out | i, in \rangle}_{S_{fi}}|^2 = |\langle f | \underbrace{\Omega_-^\dagger \Omega_+}_{\text{S operator}} | i \rangle|^2$$

Trick to perform limits in time $T \rightarrow \pm \infty$:

$$\lim_{T \rightarrow -\infty} f(T) = \lim_{\varepsilon \rightarrow 0^+} \varepsilon \int_{-\infty}^0 dt \exp(\varepsilon t) f(t)$$

$$\lim_{T \rightarrow \infty} f(T) = \lim_{\varepsilon \rightarrow 0^+} \varepsilon \int_0^{\infty} dt \exp(-\varepsilon t) f(t)$$

This trick allows to rewrite the Möller operator in a **time-independent form**

$$| i, in \rangle = \lim_{T \rightarrow -\infty} \hat{U}^\dagger(T) \hat{U}_0(T) | i \rangle = \lim_{\varepsilon \rightarrow 0^+} \varepsilon \int_{-\infty}^0 dt \exp(\varepsilon t) \hat{U}^\dagger(t) \hat{U}_0(t) | i \rangle$$

$$= \lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^0 dt \exp(\varepsilon t) \exp(+i\hat{H}t) \exp(-i\hat{H}_0t) \int d^3k_1 d^3k_2 f(\vec{k}_1) \tilde{f}(\vec{k}_2) | \vec{k}_1 \vec{k}_2 \rangle = \int d^3k_1 d^3k_2 f(\vec{k}_1) \tilde{f}(\vec{k}_2) \underbrace{\lim_{\varepsilon \rightarrow 0^+} \frac{i\varepsilon}{E_{k_1 k_2} + i\varepsilon - \hat{H}} | \vec{k}_1 \vec{k}_2 \rangle}_{\equiv | \vec{k}_1 \vec{k}_2 \rangle^{(+)}}$$

The calculation of these "in" states is a **time-independent problem!**

"in" state of $| \vec{k}_1 \vec{k}_2 \rangle$



Analogously, one finds

$$|f, out\rangle = \int d^3k_1 d^3k_2 f(\vec{k}_1) \tilde{f}(\vec{k}_2) \underbrace{\lim_{\varepsilon \rightarrow 0^+} \frac{-i\varepsilon}{E_{k_1 k_2} - i\varepsilon - \hat{H}}}_{\equiv |\vec{k}_1 \vec{k}_2\rangle^{(-)}} |\vec{k}_1 \vec{k}_2\rangle$$

“out” state of $|\vec{k}_1 \vec{k}_2\rangle$



Problem of finding "in" and "out" states is driven by **resolvent operators**

$$\hat{G}^{(\pm)} \equiv \frac{1}{E_{k_1 k_2} \pm i\varepsilon - \hat{H}}$$

Now interested in expressing S-matrix ${}^{(-)}\langle \vec{k}'_1 \vec{k}'_2 | \vec{k}_1 \vec{k}_2 \rangle^{(+)}$ in terms of $\hat{G}^{(\pm)}$

Translation invariance the operator conserves total momentum use **relative coordinates** $S_{fi} = {}^{(-)}\langle \vec{p}' | \vec{p} \rangle^{(+)}$

$$\hat{G}^{(\pm)} \equiv \frac{1}{E_p \pm i\varepsilon - \hat{H}} \quad \text{or more generally} \quad \hat{G}^{(\pm)}(E) \equiv \frac{1}{E \pm i\varepsilon - \hat{H}} \quad \text{and} \quad \hat{G}_0^{(\pm)}(E) \equiv \frac{1}{E \pm i\varepsilon - \hat{H}_0}$$

With some algebra, it is straight forward to prove the **resolvent identity**

$$G^{(\pm)}(E) = G_0^{(\pm)}(E) + G^{(\pm)}(E) V G_0^{(\pm)}(E) = G_0^{(\pm)}(E) + G_0^{(\pm)}(E) V G^{(\pm)}(E)$$

note the general rule: V appearing here is the part removed from $G^{(\pm)}$.
No need to move the complete interaction!

1) Lippmann-Schwinger equation for the "in" and "out" states

$$|\vec{p}\rangle^{(\pm)} = \frac{\pm i\varepsilon}{E_p \pm i\varepsilon - H} |\vec{p}\rangle = \underbrace{\frac{\pm i\varepsilon}{E_p \pm i\varepsilon - H_0} |\vec{p}\rangle}_{=|\vec{p}\rangle} + \frac{1}{E_p \pm i\varepsilon - H_0} V \underbrace{\frac{\pm i\varepsilon}{E_p \pm i\varepsilon - H} |\vec{p}\rangle}_{=|\vec{p}\rangle^{(\pm)}}$$

2) Formal solution in terms of the resolvent

a. "in" and "out" states are eigenstates of $\hat{H}$ $(E_p - H) |\vec{p}\rangle^{(\pm)} = 0$

b. $|\vec{p}\rangle^{(\pm)} = \frac{\pm i\varepsilon}{E_p \pm i\varepsilon - H} |\vec{p}\rangle = \underbrace{\frac{\pm i\varepsilon}{E_p \pm i\varepsilon - H_0} |\vec{p}\rangle}_{=|\vec{p}\rangle} + \frac{1}{E_p \pm i\varepsilon - H} V \underbrace{\frac{\pm i\varepsilon}{E_p \pm i\varepsilon - H_0} |\vec{p}\rangle}_{=|\vec{p}\rangle}$ and ${}^{(\pm)}\langle \vec{p}| = \langle \vec{p}| + \langle \vec{p}| V \frac{1}{E_p \mp i\varepsilon - H}$

3) Introduction of the t-matrix

$$\begin{aligned} {}^{(-)}\langle \vec{p}' | \vec{p} \rangle^{(+)} &= \langle \vec{p}' | \vec{p} \rangle^{(+)} + \langle \vec{p}' | V \frac{1}{E_{p'} + i\varepsilon - H} |\vec{p}\rangle^{(+)} \\ &= \langle \vec{p}' | \vec{p} \rangle + \langle \vec{p}' | \frac{1}{E_p + i\varepsilon - H_0} V |\vec{p}\rangle^{(+)} + \langle \vec{p}' | V \frac{1}{E_{p'} + i\varepsilon - H} |\vec{p}\rangle^{(+)} \\ &= (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}') + \underbrace{\left(\frac{1}{E_{p'} + i\varepsilon - E_p} + \frac{1}{E_p + i\varepsilon - E_{p'}} \right)}_{=-2\pi i \delta(E_p - E_{p'})} \underbrace{\langle \vec{p}' | V |\vec{p}\rangle^{(+)}}_{\equiv t(\vec{p}', \vec{p})} \end{aligned}$$

Note the sign change!

use relation 2b)

use Lippmann-Schwinger equation

use eigenstates of $\hat{H}_0$ and $\hat{H}$

see next slide

S-matrix given by t-matrix



smooth function suitable for numerical calculation

t-matrix drives cross section:

$$d\sigma = (2\pi)^4 \delta(E_{p'} - E_p) \frac{1}{v_A} |t(\vec{p}', \vec{p})|^2$$

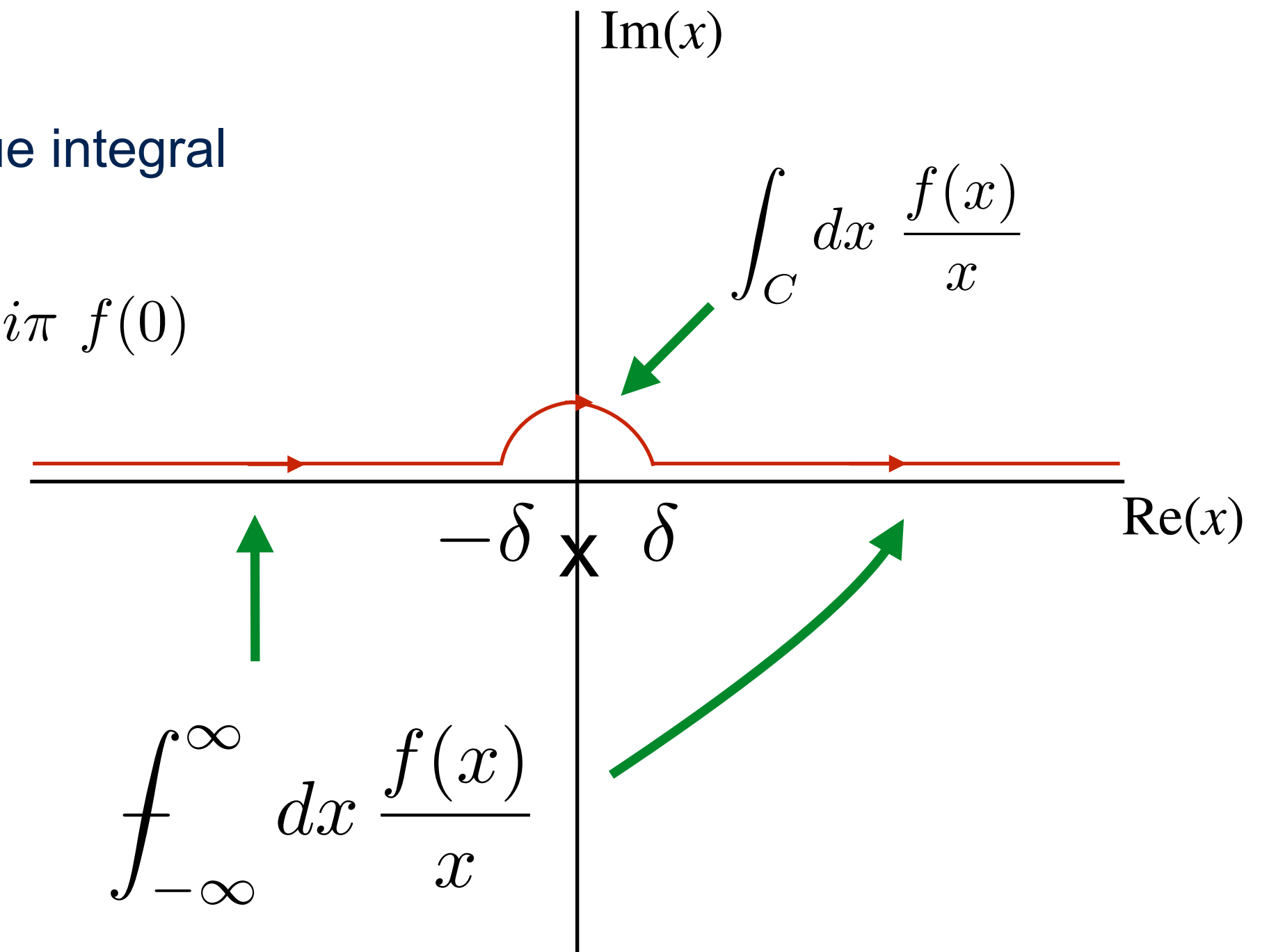
Treatment of the ε limits by integration in the complex plane

$$\int_{-\infty}^{\infty} dx \frac{f(x)}{x + i\varepsilon} = \int_{-\infty}^{\infty} dx \frac{f(x)}{x} - i\pi \int_{-\infty}^{\infty} dx f(x) \delta(x) = \int_{-\infty}^{\infty} dx \frac{f(x)}{x} - i\pi f(0)$$

principal value integral

or in shorter notation

$$\frac{1}{x + i\varepsilon} = \frac{\text{P}}{x} - i\pi\delta(x)$$



$$\int_C dx \frac{f(x)}{x} = \int_{\pi}^0 d\varphi \frac{i\delta \exp(i\varphi)}{\delta \exp(i\varphi)} f(\delta \exp(i\varphi)) = -i\pi f(0) \quad \checkmark$$

This relation was used to introduce the t-matrix and will be very important to implement the solution later on.

Numerical solution

Lippmann-Schwinger equation for the t-matrix (use directly partial wave basis $|p\alpha\rangle = |p(ls)jm\rangle$!)

$$t_{\alpha'\alpha}(p', p) = \langle p'\alpha' | V | p\alpha \rangle^{(+)} = \langle p'\alpha' | V | p\alpha \rangle + \langle p'\alpha' | V \frac{1}{E_p + i\varepsilon - H_0} V | p\alpha \rangle^{(+)}$$

energy parameter E

or

$$t_{\alpha'\alpha}(p', p) = V_{\alpha'\alpha}(p', p) + \sum_{\alpha''} \int dp'' p''^2 V_{\alpha'\alpha''}(p', p'') \frac{1}{E_p + i\varepsilon - E_{p''}} t_{\alpha''\alpha}(p'', p)$$

Two-body scattering observables: **on-shell t-matrix** ($E = E_{p'} = E_p$) is required.

Solution requires **half-off shell t-matrix**: $t_{\alpha'\alpha}(p', p)$ for $E = E_p = p^2/(2\mu)$.

Fully off-shell t-matrix ($E \neq E_p$) is important for **three-body problem**.

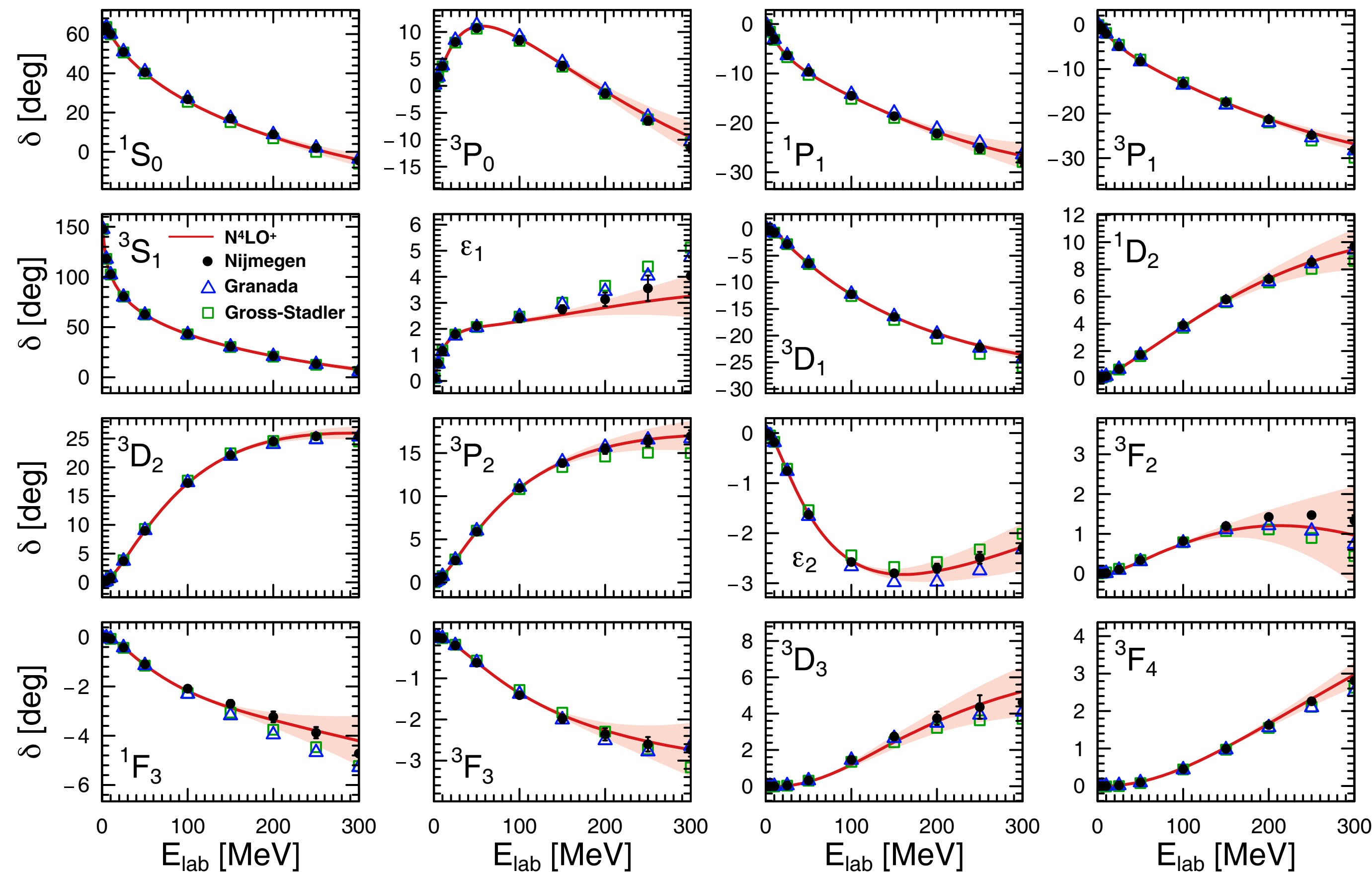
This is the central equation for numerical solutions of the scattering problem. **Singularity** can be treated with **subtraction**!

$$\begin{aligned} t_{\alpha'\alpha}(p', p) &= V_{\alpha'\alpha}(p', p) + \int_0^\infty dp'' p''^2 V_{\alpha'\alpha''}(p', p'') \frac{1}{E_q - E_{p''}} t_{\alpha''\alpha}(p'', p) - i\pi \int_0^\infty dp'' p''^2 V_{\alpha'\alpha''}(p', p'') \delta(E_q - E_{p''}) t_{\alpha''\alpha}(p'', p) \\ &= V_{\alpha'\alpha}(p', p) + \int_0^\infty dp'' \frac{p''^2 V_{\alpha'\alpha''}(p', p'') t_{\alpha''\alpha}(p'', p) - q^2 V_{\alpha'\alpha''}(p', q) t_{\alpha''\alpha}(q, p)}{E_q - E_{p''}} + \int_0^\infty dp'' \frac{q^2 V_{\alpha'\alpha''}(p', q) t_{\alpha''\alpha}(q, p)}{E_q - E_{p''}} - i\pi q \mu V_{\alpha'\alpha''}(p', q) t_{\alpha''\alpha}(q, p) \\ &= V_{\alpha'\alpha}(p', p) + \int_0^{p_{\max}} dp'' \frac{2\mu p''^2 V_{\alpha'\alpha''}(p', p'') t_{\alpha''\alpha}(p'', p) - 2\mu q^2 V_{\alpha'\alpha''}(p', q) t_{\alpha''\alpha}(q, p)}{q^2 - p''^2} + q^2 V_{\alpha'\alpha''}(p', q) t_{\alpha''\alpha}(q, p) \frac{\mu}{q} \ln \left(\frac{p_{\max} + q}{p_{\max} - q} \right) - i\pi q \mu V_{\alpha'\alpha''}(p', q) t_{\alpha''\alpha}(q, p) \end{aligned}$$

Example: solutions of the Lippmann-Schwinger Eq. for the chiral SMS NN interactions

High orders are necessary to obtain a good description of NN data. Description of data at order Q^5 is perfect. Uncertainty estimates small.

By construction (fit), $E_d = -2.225$ MeV agrees with experiment.



(Reinert et al. Eur. Phys. J. A 54, 86 (2018))

order	NN
Q^0	
Q^2	
Q^3	
Q^4	

FADDEEV EQUATIONS

3-body bound state problem

The aim is the implementation of a **three-body bound** state problem using so-called **Faddeev equations**

Schrödinger equation:

$$\hat{H} |\Psi\rangle = (\hat{H}_0 + \hat{V}_{12} + \hat{V}_{23} + \hat{V}_{31} + \hat{V}_{123}) |\Psi\rangle = E |\Psi\rangle$$

Three **pair interactions** V_{ij} contribute.

$\hat{H}_0$ is the **internal kinetic energy** (without CM motion)

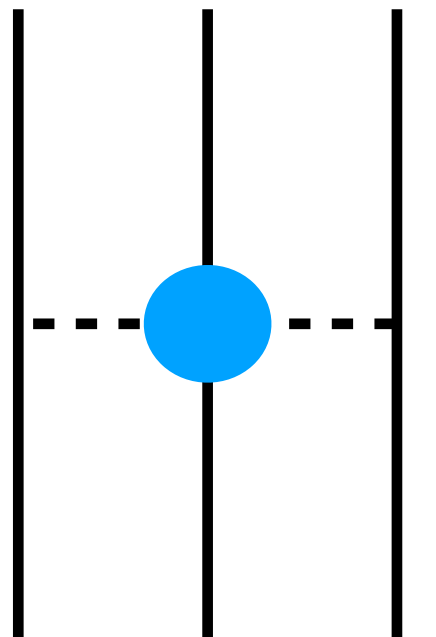
$\hat{V}_{123}$ is a possible three-body interaction

(note that this can always be decomposed in $\hat{V}_{123} = V_{123}^{(1)} + V_{123}^{(2)} + V_{123}^{(3)}$)

Here: three particles are **identical nucleons** (later assume one distinguishable particle) .



Enforce Pauli principle & relate the subsystems to each other!



Jacobi coordinates and permutation operators (here only spatial coordinates)

Essential new aspect of more than two-body systems: relative coordinates for the 3-body system

Start with single particle momenta: $\vec{k}_1 \quad \vec{k}_2 \quad \vec{k}_3$

Define **Jacobi momenta**:

$$\vec{p}_{12} = \frac{1}{m_1 + m_2} (m_2 \vec{k}_1 - m_1 \vec{k}_2) \quad \vec{p}_3 = \frac{1}{m_1 + m_2 + m_3} \left((m_1 + m_2) \vec{k}_3 - m_3 (\vec{k}_1 + \vec{k}_2) \right) \quad \vec{P} = \vec{k}_1 + \vec{k}_2 + \vec{k}_3$$

(compare to page 16)

or permutations thereof (here only cyclical and anticyclical)

$$\vec{p}_{23} = \frac{1}{m_2 + m_3} (m_3 \vec{k}_2 - m_2 \vec{k}_3) \quad \vec{p}_1 = \frac{1}{m_1 + m_2 + m_3} \left((m_2 + m_3) \vec{k}_1 - m_1 (\vec{k}_2 + \vec{k}_3) \right)$$

and

$$\vec{p}_{31} = \frac{1}{m_3 + m_1} (m_1 \vec{k}_3 - m_3 \vec{k}_1) \quad \vec{p}_2 = \frac{1}{m_1 + m_2 + m_3} \left((m_3 + m_1) \vec{k}_2 - m_2 (\vec{k}_3 + \vec{k}_1) \right)$$

Express in terms of single particle momenta and relate with each other

β

$$\vec{p}_{23} = -\frac{m_3}{m_2 + m_3} \vec{p}_{12} - \frac{m_2(m_1 + m_2 + m_3)}{(m_1 + m_2)(m_2 + m_3)} \vec{p}_3$$

γ

$$\vec{p}_1 = \vec{p}_{12} - \frac{m_1}{m_1 + m_2} \vec{p}_3$$

α

and

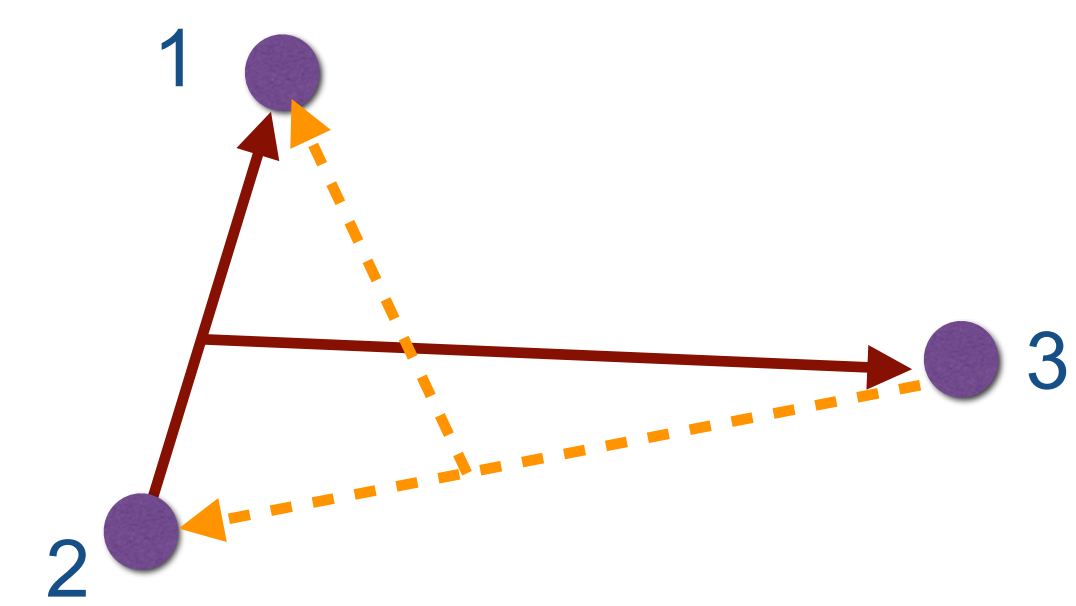
$$\vec{p}_{31} = -\frac{m_3}{m_1 + m_2} \vec{p}_{12} + \frac{m_1(m_1 + m_2 + m_3)}{(m_1 + m_2)(m_1 + m_3)} \vec{p}_3$$

$$\vec{p}_2 = -\vec{p}_{12} - \frac{m_2}{m_1 + m_2} \vec{p}_3$$

matrix elements of **coordinate transformations**

$${}_{(12)3} \langle \vec{p}'_{12} \vec{p}'_3 | \vec{p}_{23} \vec{p}_1 \rangle_{{}_{(23)1}} = \delta^{(3)} (\vec{p}_{23} + \beta \vec{p}'_{12} + \gamma \vec{p}'_3) \delta^{(3)} (\vec{p}_1 - \vec{p}'_{12} + \alpha \vec{p}'_3)$$

Indices indicate the kind of Jacobi coordinate (often skipped for convenience)



Partial wave representation of the coordinate transformation (only spatial)

Relation of momentum vector states to partial wave states

$$\langle \vec{p}_{12} | p'_{12} l'_{12} m'_{12} \rangle = \frac{\delta(p_{12} - p'_{12})}{p_{12} p'_{12}} Y_{l'_{12} m'_{12}}(\hat{p}_{12}) \quad \langle \vec{p}_3 | p'_3 l'_3 m'_3 \rangle = \frac{\delta(p_3 - p'_3)}{p_3 p'_3} Y_{l'_3 m'_3}(\hat{p}_3)$$

Combine to total orbital angular momentum $\vec{\hat{L}} = \vec{\hat{l}}_{12} + \vec{\hat{l}}_3$ Clebsh-Gordan coefficient

$$| p_{12} p_3 (l_{12} l_3) L M_L \rangle = \sum_{m_{12} m_3} (l_{12} l_3 L, m_{12} m_3 M_L) | p_{12} l_{12} m_{12} \rangle | p_3 l_3 m_3 \rangle$$

Completeness relation reads

$$\mathbb{I} = \sum_{l_{12} l_3 L M_L} \int dp_{12} p_{12}^2 dp_3 p_3^2 | p_{12} p_3 (l_{12} l_3) L M_L \rangle \langle p_{12} p_3 \underbrace{(l_{12} l_3) L M_L}_{\equiv \alpha} |$$

Wanted: $\langle p'_{12} p'_3 \alpha' \tilde{M}_L | p_{23} p_1 \alpha \tilde{M}_L \rangle_{(23)1} = \delta_{LL'} \frac{1}{2L+1} \sum_{M_L} {}_{(12)3} \langle p'_{12} p'_3 \alpha' M_L | p_{23} p_1 \alpha M_L \rangle_{(23)1}$ M independence because of rotational invariance!

$$= \delta_{LL'} \frac{1}{2L+1} \sum_{M_L} \int d^3 \tilde{p}'_{12} d^3 \tilde{p}'_3 d^3 \tilde{p}_{23} d^3 \tilde{p}_1 {}_{(12)3} \langle p'_{12} p'_3 \alpha' M_L | \vec{\tilde{p}}'_{12} \vec{\tilde{p}}'_3 \rangle_{(12)3} {}_{(12)3} \langle \vec{\tilde{p}}'_{12} \vec{\tilde{p}}'_3 | \vec{\tilde{p}}_{23} \vec{\tilde{p}}_1 \rangle_{(23)1} {}_{(23)1} \langle \vec{\tilde{p}}_{23} \vec{\tilde{p}}_1 | p_{23} p_1 \alpha M_L \rangle_{(23)1}$$

Averaging removes preference
for quantization axis

Spatial part reduces to one dimensional angular integration

$$\frac{1}{2L+1} \sum_{M_L} \int d^3\tilde{p}'_{12} d^3\tilde{p}'_3 d^3\tilde{p}_{23} d^3\tilde{p}_1 \frac{\delta(p'_{12} - \tilde{p}'_{12}) \delta(p'_3 - \tilde{p}'_3)}{\tilde{p}'_{12} p'_{12} \tilde{p}'_3 p'_3} \mathcal{Y}_{l'_{12} l'_3}^{*LM_L}(\hat{\tilde{p}}'_{12} \hat{\tilde{p}}'_3) \delta^{(3)}(\vec{\tilde{p}}_{23} + \beta \vec{\tilde{p}}'_{12} + \gamma \vec{\tilde{p}}'_3) \delta^{(3)}(\vec{\tilde{p}}_1 - \vec{\tilde{p}}'_{12} + \alpha \vec{\tilde{p}}'_3) \frac{\delta(p_{23} - \tilde{p}_{23}) \delta(p_1 - \tilde{p}_1)}{\tilde{p}_{23} p_{23} \tilde{p}_1 p_1} \mathcal{Y}_{l_{23} l_1}^{LM_L}(\hat{\tilde{p}}_{23} \hat{\tilde{p}}_1)$$

Due to δ -functions the 12 dimensional integration reduces to a 4 dimensional. Two δ -functions cannot be removed!

Several choices are possible! Here one example:

$$\frac{1}{2L+1} \sum_{M_L} \int d\hat{\tilde{p}}'_3 d\hat{\tilde{p}}_1 \frac{\delta\left(p'_{12} - \left|\vec{\tilde{p}}_1 + \alpha \vec{\tilde{p}}'_3\right|\right)}{p_{12}'^2} \mathcal{Y}_{l'_{12} l'_3}^{*LM_L}\left(\widehat{\vec{\tilde{p}}_1 + \alpha \vec{\tilde{p}}'_3}\right) \frac{\delta\left(p_{23} - \left|-\beta \vec{\tilde{p}}_1 - \vec{\tilde{p}}'_3\right|\right)}{p_{23}^2} \mathcal{Y}_{l_{23} l_1}^{LM_L}\left(-\widehat{\alpha \vec{\tilde{p}}_1 - \vec{\tilde{p}}'_3}\right)$$

$$= \int dx \frac{\delta\left(p'_{12} - \pi_{12}(p'_3 p_1 x)\right)}{p_{12}'^2} \frac{\delta\left(p_{23} - \pi_{23}(p'_3 p_1 x)\right)}{p_{23}^2} G_{\alpha'\alpha}(p'_3 p_1 x)$$

with $G_{\alpha'\alpha}(p'_3 p_1 x) = \frac{1}{2L+1} \sum_{M_L} 8\pi^2 \mathcal{Y}_{l'_{12} l'_3}^{*LM_L}\left(\widehat{\vec{\tilde{p}}_1 + \alpha \vec{\tilde{p}}'_3}\right) \mathcal{Y}_{l_{23} l_1}^{LM_L}\left(-\widehat{\alpha \vec{\tilde{p}}_1 - \vec{\tilde{p}}'_3}\right)$

Using $\hat{\tilde{p}}'_3 = \hat{e}_z$ and $\hat{\tilde{p}}_1$ in x-z-plane

$$\int d\hat{\tilde{p}}'_3 d\hat{\tilde{p}}_1 = 4\pi \cdot 2\pi \int_{-1}^1 dx$$

$$\vec{\tilde{p}}'_3 = \begin{pmatrix} 0 \\ 0 \\ p'_3 \end{pmatrix} \quad \vec{\tilde{p}}_1 = \begin{pmatrix} p_1 \sqrt{1-x^2} \\ 0 \\ p_1 x \end{pmatrix}$$

and coupled spherical harmonics

$$\mathcal{Y}_{l_{12} l_3}^{LM_L}(\hat{p}_{12} \hat{p}_3) = \sum_{m_{12} m_3} (l_{12} l_3 L, m_{12} m_3 M_L) Y_{l_{12} m_{12}}(\hat{p}_{12}) Y_{l_3 m_3}(\hat{p}_3)$$

For the nucleon/baryon state: spin and isospin transformations have to be done, too

$${}_{(12)3} \left\langle ((t_1 t_2) t'_{12} t_3) T' M'_T \middle| ((t_2 t_3) t_{23} t_1) T M_T \right\rangle_{(23)1} = \delta_{M_T M'_T} \delta_{T T'} (-)^{t_{23}+2t_1+t_2+t_3} \sqrt{\hat{t}'_{12} \hat{t}_{23}} \begin{Bmatrix} t_1 & t_2 & t'_{12} \\ t_3 & T & t_{23} \end{Bmatrix}$$

6j coefficient because of reordering the coupling of isospins t_1, t_2, t_3

The exactly same expression holds for spin.

short hand notation $\hat{k} = 2k + 1$

In the following, we use the following basis states for three nucleons (or replacing one by a hyperons)

$$\left| p_{12} p_3 \left[\left(l_{12} (s_1 s_2) s_{12} \right) j_{12} (l_3 s_3) I_3 \right] J M ((t_1 t_2) t_{12} t_3) T M_T \right\rangle_{(12)3} \equiv |p_{12} p_3 \alpha M\rangle_{(12)3}$$

The coordinate transformations can then be separated after recoupling of jj to LS coupling using $9j$ symbols

with

$$G_{\alpha'\alpha}(p'_3 p_1 x) = \sum_{LS} \hat{S} \sqrt{\hat{j}'_{12} \hat{I}'_3 \hat{j}_{23} \hat{I}_1} \begin{Bmatrix} l'_{12} & s'_{12} & j'_{12} \\ l'_3 & s_3 & I'_3 \\ L & S & J \end{Bmatrix} \begin{Bmatrix} l_{23} & s_{23} & j_{23} \\ l_1 & s_1 & I_1 \\ L & S & J \end{Bmatrix} \sum_{M_L} 8\pi^2 \mathcal{Y}_{l'_{12} l'_3}^{*LM_L} \left(\widehat{\vec{p}_1 + \alpha \vec{p}'_3 \vec{p}'_3} \right) \mathcal{Y}_{l_{23} l_1}^{LM_L} \left(-\widehat{\alpha \vec{p}_1 - \vec{p}'_3 \vec{p}_1} \right)$$

$$\times (-)^{s_{23}+2s_1+s_2+s_3} \sqrt{\hat{s}'_{12} \hat{s}_{23}} \begin{Bmatrix} s_1 & s_2 & s'_{12} \\ s_3 & S & s_{23} \end{Bmatrix} \times \delta_{M_T M'_T} \delta_{T T'} (-)^{t_{23}+2t_1+t_2+t_3} \sqrt{\hat{t}'_{12} \hat{t}_{23}} \begin{Bmatrix} t_1 & t_2 & t'_{12} \\ t_3 & T & t_{23} \end{Bmatrix}$$

Definition of permutation operators for identical particles

Relation of Interactions using **permutation operators** P_{ij} (interchange coordinates and quantum numbers,)

$$H = \underbrace{T_1 + T_2 + T_3}_{H_0} + \underbrace{V_{12} + V_{23} + V_{31}}_V$$

Identity of particles: interchange coordinates relate interaction in different subsystems $V_{23} = P_{12} P_{23} V_{12} P_{23} P_{12}$

$$V_{31} = P_{13} P_{23} V_{12} P_{23} P_{13}$$

Consider single particle coordinates (momenta and spin/isospin) to derive these relations:

$$\langle \vec{a}' \vec{b}' \vec{c}' | V_{12} | \vec{a} \vec{b} \vec{c} \rangle = \delta^{(3)}(\vec{c}' - \vec{c}) V(\vec{a}' \vec{b}', \vec{a} \vec{b})$$

Therefore, for pair (23):

particle 3 is not affected

V is the same for each pair

$$\langle \vec{a}' \vec{b}' \vec{c}' | V_{23} | \vec{a} \vec{b} \vec{c} \rangle = \delta^{(3)}(\vec{a}' - \vec{a}) V(\vec{b}' \vec{c}', \vec{b} \vec{c})$$

$$\langle \vec{a}' \vec{b}' \vec{c}' | P_{12} P_{23} V_{12} P_{23} P_{12} | \vec{a} \vec{b} \vec{c} \rangle = \langle \vec{b}' \vec{a}' \vec{c}' | P_{23} V_{12} P_{23} | \vec{b} \vec{a} \vec{c} \rangle = \langle \vec{b}' \vec{c}' \vec{a}' | V_{12} | \vec{b} \vec{c} \vec{a} \rangle$$

transpositions are hermitian

$$= \delta^{(3)}(\vec{a}' - \vec{a}) V(\vec{b}' \vec{c}', \vec{b} \vec{c}) = \langle \vec{a}' \vec{b}' \vec{c}' | V_{23} | \vec{a} \vec{b} \vec{c} \rangle$$

Partial waves allowed and permutation operators

Identical particles:

momentum/configuration space

(12) transposition in PW:

$$\begin{aligned}
 P_{12} | \vec{p}_{12} \vec{p}_3 \rangle &= | - \vec{p}_{12} \vec{p}_3 \rangle \\
 P_{12} | p_{12} p_3 (l_{12} l_3) L M_L \rangle &= \sum_{m_{12} m_3} \int d\hat{p}_{12} d\hat{p}_3 (l_{12} l_3 L, m_{12} m_3 M_L) Y_{l_{12} m_{12}}(\hat{p}_{12}) Y_{l_3 m_3}(\hat{p}_3) P_{12} | \vec{p}_{12} \vec{p}_3 \rangle \\
 &= \sum_{m_{12} m_3} \int d\hat{p}_{12} d\hat{p}_3 (l_{12} l_3 L, m_{12} m_3 M_L) (-)^{l_{12}} Y_{l_{12} m_{12}}(-\hat{p}_{12}) Y_{l_3 m_3}(\hat{p}_3) | - \vec{p}_{12} \vec{p}_3 \rangle \\
 &= (-)^{l_{12}} | p_{12} p_3 (l_{12} l_3) L M_L \rangle
 \end{aligned}$$

symmetry property of spherical harmonics

spin/isospin space

$$P_{12} \left| (s_1 s_2) s_{12} m_{12} \right\rangle = (-)^{s_1 + s_2 - s_{12}} \left| (s_1 s_2) s_{12} m_{12} \right\rangle$$

symmetry property of Clebsch-Gordan coefficients

for nucleons together

$$P_{12} | p_{12} p_3 \alpha \rangle = (-)^{\underbrace{l_{12} + s_1 + s_2 - s_{12} + t_1 + t_2 - t_{12}}_{=-1 \text{ Pauli!}}} | p_{12} p_3 \alpha \rangle$$

physical states have restrictions on partial waves!

Combinations of permutation operators simplify

$$\begin{aligned}
 \langle p'_{12} p'_3 \alpha' | P_{12} P_{23} + P_{13} P_{23} | p_{12} p_3 \alpha \rangle &= \langle p'_{12} p'_3 \alpha' | P_{12} P_{23} + P_{12} P_{13} P_{23} P_{12} | p_{12} p_3 \alpha \rangle \\
 &= \langle p'_{12} p'_3 \alpha' | P_{12} P_{23} + P_{12} P_{23} \underbrace{P_{12} P_{12}}_{=I} | p_{12} p_3 \alpha \rangle = 2 \langle p'_{12} p'_3 \alpha' | P_{12} P_{23} | p_{12} p_3 \alpha \rangle
 \end{aligned}$$

Jacobi coordinates and permutation operators

In case of identical particles, we need antisymmetrized/symmetrized states!

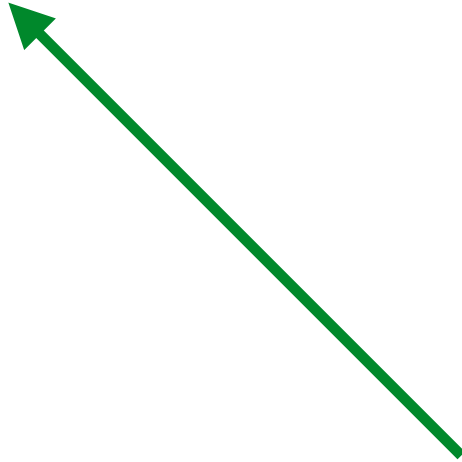
Coordinate transformations  permutation operators

$$\begin{aligned} & {}_{(12)3} \langle \vec{p}'_{12} \vec{p}'_3 \vec{P}' | P_{12} P_{23} | \vec{p}_{12} \vec{p}_3 \vec{P} \rangle_{(12)3} \\ &= \int d^3 k_1 d^3 k_2 d^3 k_3 {}_{(12)3} \langle \vec{p}'_{12} \vec{p}'_3 \vec{P}' | \vec{k}_1 \vec{k}_2 \vec{k}_3 \rangle \langle \vec{k}_1 \vec{k}_2 \vec{k}_3 | P_{12} P_{23} | \vec{p}_{12} \vec{p}_3 \vec{P} \rangle_{(12)3} \\ &= \int d^3 k_1 d^3 k_2 d^3 k_3 {}_{(12)3} \langle \vec{p}'_{12} \vec{p}'_3 \vec{P}' | \vec{k}_1 \vec{k}_2 \vec{k}_3 \rangle \underbrace{\langle \vec{k}_2 \vec{k}_3 \vec{k}_1 | \vec{p}_{12} \vec{p}_3 \vec{P} \rangle_{(12)3}}_{= \langle \vec{k}_1 \vec{k}_2 \vec{k}_3 | \vec{p}_{12} \vec{p}_3 \vec{P} \rangle_{(23)1}} = {}_{(12)3} \langle \vec{p}'_{12} \vec{p}'_3 \vec{P}' | \vec{p}_{12} \vec{p}_3 \vec{P} \rangle_{(23)1} \end{aligned}$$

Permutation operators and coordinate transformations are equivalent!

Same argument for spin/isospin degrees of freedom!

by the change of single particle momenta, effectively, p_{23} is set to p_{12} and p_1 to p_3



Rewriting the Schrödinger equation

assuming 3 identical fermions!

Bound state problem $\longrightarrow$ assume that $E < 0$ (even more $E < E_{2b}$)

Schrödinger equation for bound state wave function Ψ

anti- symmetry (Pauli principle)

$$(E - H_0) |\Psi\rangle = (V_{12} + V_{123}^{(3)}) |\Psi\rangle + P_{12}P_{23}(V_{12} + V_{123}^{(3)})P_{23}P_{12} |\Psi\rangle + P_{13}P_{23}(V_{12} + V_{123}^{(3)})P_{23}P_{13} |\Psi\rangle$$
$$= \left(\mathbb{I} + \underbrace{P_{12}P_{23} + P_{13}P_{23}}_{\equiv P} \right) (V_{12} + V_{123}^{(3)}) |\Psi\rangle$$

abbreviation P

This allows to define a Faddeev component and relate it to the wave function:

invertible since E negative

symmetric operator!

$$|\Psi\rangle = \frac{1}{E - H_0} (\mathbb{I} + P)(V_{12} + V_{123}^{(3)}) |\Psi\rangle = (\mathbb{I} + P) \underbrace{\frac{1}{E - H_0} (V_{12} + V_{123}^{(3)}) |\Psi\rangle}_{\equiv |\psi_{12}\rangle} = (\mathbb{I} + P) |\psi_{12}\rangle$$

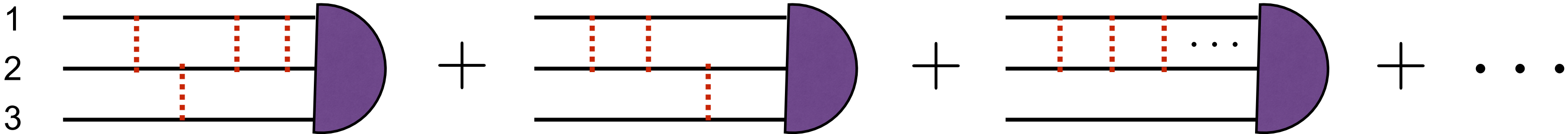
Faddeev component

Faddeev component fulfills:

$$|\psi_{12}\rangle = \frac{1}{E - H_0} (V_{12} + V_{123}^{(3)}) |\Psi\rangle = \frac{1}{E - H_0} (V_{12} + V_{123}^{(3)}) (\mathbb{I} + P) |\psi_{12}\rangle$$

many contributions by reinserting eq.
first interaction is in (12)

one part only consists of interactions in (12)
particle 3 is not affected here



Treatment of the not-connected term

The last piece is troublesome: when in part of the operator one of the particles is not affect (unit operator in this coordinate)

- *eigenvalue spectrum will have more than the accumulation point at zero*
- *numerical representation requires delta-function*
- *cannot be approximated in a finite dimensional space*

Fortunately, analytical treatment is possible:

$$\longrightarrow \underbrace{\left(\mathbb{I} - \frac{1}{E - H_0} V_{12} \right)}_{\text{Are we able to invert this?}} |\psi_{12}\rangle = \frac{1}{E - H_0} V_{12} P |\psi_{12}\rangle + \frac{1}{E - H_0} V_{123}^{(3)} (\mathbb{I} + P) |\psi_{12}\rangle$$

yes, use **Lippmann-Schwinger equation** (see two-body scattering!)

$\longrightarrow$ **Faddeev equations**

$$\longrightarrow \left(\mathbb{I} + \frac{1}{E - H_0} t_{12} \right) \left(\mathbb{I} - \frac{1}{E - H_0} V_{12} \right) = \mathbb{I}$$

$$\begin{aligned} |\psi_{12}\rangle &= \frac{1}{E - H_0} \left(V_{12} + t_{12} \frac{1}{E - H_0} V_{12} \right) P |\psi_{12}\rangle + \left(\mathbb{I} + \frac{1}{E - H_0} t_{12} \right) \frac{1}{E - H_0} V_{123}^{(3)} (\mathbb{I} + P) |\psi_{12}\rangle \\ &= \frac{1}{E - H_0} t_{12} P |\psi_{12}\rangle + \left(\mathbb{I} + \frac{1}{E - H_0} t_{12} \right) \frac{1}{E - H_0} V_{123}^{(3)} (\mathbb{I} + P) |\psi_{12}\rangle \end{aligned}$$

together with the **Lippmann-Schwinger equation** $t_{12} = V_{12} + V_{12} \frac{1}{E - H_0} t_{12}$ 

Discretized representation of the Faddeev equation

All parts together result in

$$\psi_{12}^{\alpha'}(p'_{12}p'_3) = \frac{1}{E - \frac{p'^2_{12}}{m} - \frac{3p'^2_3}{4m}} \int_{-1}^1 dx \int_0^\infty dp_1 p_1^2 \sum_{\alpha\alpha''} t_{\alpha'\alpha''}(p'_{12}\pi_{12}(p'_3p_1x); p'_3) G_{\alpha''\alpha}(p'_3p_1x) \psi_{12}^\alpha(\pi_{23}(p'_3p_1x)p_1)$$

for simplicity, I omitted the 3NF part here!

Eigenvalue equation depends on E

Needs to be varied until eigen value 1 is in the spectrum

A few notes on the discretization:

momenta need to be discretized using N_p and N_q momenta

Faddeev components can be label by one common index

$$\psi_{12}^\alpha(p_{12}p_3) \longrightarrow \psi(p_i q_j \alpha) \equiv \psi(i + j N_p + \alpha N_p N_q)$$

for the shifted momenta an interpolation is necessary

use representation as sum of function values

$$f(\pi) = \sum_{i=0}^{N_p-1} S_i(\pi) f(p_i)$$

use, e.g., cubic basis splines, cubic hermitian splines

grid points and corresponding integration weights: p_i, ω_i^p , q_j, ω_j^q and x_k, ω_k^x

This leads to the matrix equation (depending on 3-index i, k, α ..

$$\psi(ij\alpha) = \sum_{i'j'\alpha'} K(ij\alpha, i'j'\alpha') \psi(i'j'\alpha')$$

Discretized representation of the Faddeev equation

K can be read off

$$K(ij\alpha, i'j'\alpha') = \frac{1}{E - \frac{p_i^2}{m} - \frac{3q_j^2}{4m}} \sum_k \omega_k^x \omega_{j'}^q q_{j'}^2 \sum_m S_m(\pi'_{12}(q_j q_{j'} x_k)) \sum_{\alpha''} t_{\alpha'\alpha''}(p_i p_m; q_j) G_{\alpha''\alpha'}(q_j q_{j'} x_k) S_{i'}(\pi_{12}(q_j q_{j'} x_k))$$

Note that there are no integration weights for p required (just interpolations).

Also note that the matrix equation introduces an additional sum over $i'j'\alpha'$

Iterative solution

- Usual eigenvalue/-vector solvers scale with dimension N^3 $\longrightarrow$ performance is poor even for 3-body problem
- we only need one eigenvalue (for the ground state)

$\longrightarrow$ Iterative solution much more performant

Basic idea: interested in largest eigenvalue
spectrum of linear operator has only zero as accumulation point.

generate low dimensional basis by application of linear operator to arbitrary starting vector v_0

$$\{v_0, K v_0, K^2 v_0, K^3 v_0, \dots, K^n v_0\} \quad \textbf{(Krylov subspace)}$$

scaling with dimension better than $\propto n N^2$ (Why better than this scaling?)

also works for homogeneous equations when v_0 is appropriately chosen

How to avoid collecting almost collinear vectors in basis?

FADDEEV EQUATIONS

3-body scattering problem

Turn again to scattering problem. Again for three identical nucleons.

sets the poles of the t -matrix!

Assume: the system has only the **deuteron** as two-body bound state ($J_d = 1$).

Asymptotically incoming state: two-body bound state scattering on the third nucleon

Relative momentum

$$\vec{q}_0 = q_0 \hat{e}_z$$

Corresponding state

$$|b\rangle \equiv |\varphi_{12} M_d \vec{q}_0 m_3^s m_3^t\rangle \text{ or } |\varphi_{23} M_d \vec{q}_0 m_1^s m_1^t\rangle \text{ or } |\varphi_{31} M_d \vec{q}_0 m_2^s m_2^t\rangle$$

Also required (for outgoing states): asymptotic state of three non-interacting particles

$$|a\rangle \equiv |\vec{p}_0 \vec{q}_0 m_1^s m_2^s m_3^s m_1^t m_2^t m_3^t\rangle$$

Define "in" and "out" states

Similar to the two-body scattering problem: define "in"- and "out"-states corresponding to $|a\rangle$ and $|b\rangle$, i.e.

$$|b\rangle^{(\pm)} \equiv \lim_{\varepsilon \rightarrow 0^+} \frac{\pm i\varepsilon}{E \pm i\varepsilon - H} |b\rangle$$

and correspondingly for $|a\rangle$.

The Pauli principle requires **anti-symmetric states**. Assuming that the two-body subsystem is already anti-symmetrized (choose allowed partial waves!), the anti-symmetrizer only requires three parts

$$\{|b\rangle\} = \frac{1}{\sqrt{3}} (1 + P) |b\rangle$$

Directly translates to scattering states

$$\{|b\rangle^{(\pm)}\} = \frac{\pm i\varepsilon}{E \pm i\varepsilon - H} \{|b\rangle\} \quad (\text{limes } \varepsilon \rightarrow 0^+ \text{ is implied as usual})$$

Definition of the S-matrix and scattering amplitude

Based on "in"- and "out"-states: define the **S-matrix** and **read off the scattering amplitude**.

Two possible cases:

$$\frac{i\varepsilon}{E + i\varepsilon - H_0} |a\rangle = |a\rangle$$

1) breakup scattering

$$S_{ab} = \{^{(-)}\langle a|\} \{|b\rangle^{(+)}\} = ^{(-)}\langle a|(1+P)|b\rangle^{(+)} \quad (\text{note cancelation of factor 3})$$

Need to use **resolvent identity**

$$|a\rangle^{(\pm)} = \frac{\pm i\varepsilon}{E \pm i\varepsilon - H} |a\rangle = |a\rangle + \frac{1}{E \pm i\varepsilon - H} \left(V_{12} + V_{123}^{(3)} + V_{23} + V_{123}^{(1)} + V_{31} + V_{123}^{(2)} \right) |a\rangle$$

or

$$|a\rangle^{(-)} - |a\rangle^{(+)} = 2\pi i \delta(E - H) \left(V_{12} + V_{123}^{(3)} + V_{23} + V_{123}^{(1)} + V_{31} + V_{123}^{(2)} \right) |a\rangle$$

Again derive the standard relation of the scattering amplitude (t-matrix) and **S-matrix (for breakup)**

$$S_{ab} = ^{(+)}\langle a| \underbrace{(\cancel{1+P})}_{M_{ab}} |b\rangle^{(+)} - 2\pi i \delta(E_b - E_a) \langle a| \left(V_{12} + V_{123}^{(3)} + V_{23} + V_{123}^{(1)} + V_{31} + V_{123}^{(2)} \right) (1+P) |b\rangle^{(+)}$$

First part: spectator momenta cannot match!

Defines amplitude M_{ab} .

It is straightforward to obtain the cross section for break up as (phase space integral)

$$\frac{d^5\sigma}{d\hat{p} d\hat{q} dq} = (2\pi)^4 \frac{m}{3q_0} q^2 p |M_{\vec{p}\vec{q},q_0}|^2$$

2) elastic scattering

$$S_{b'b} = \left\{ {}^{(-)}\langle b' | \right\} \left\{ | b \rangle^{(+)} \right\} = {}^{(-)}\langle b' | (1 + P) | b \rangle^{(+)}$$

$$\frac{\pm i\varepsilon}{E_{b'} \pm i\varepsilon - H_0 - V_{12}} | b' \rangle = | b' \rangle$$

Again use **resolvent identity** (to single out **channel Hamiltonian**)

$$| b' \rangle^{(\pm)} = \frac{\pm i\varepsilon}{E \pm i\varepsilon - H} | b' \rangle = | b' \rangle + \frac{1}{E \pm i\varepsilon - H} \left(V_{123}^{(3)} + V_{23} + V_{123}^{(1)} + V_{31} + V_{123}^{(2)} \right) | b' \rangle$$

or

$$| b' \rangle^{(-)} - | b' \rangle^{(+)} = 2\pi i \delta(E - H) \left(V_{123}^{(3)} + V_{23} + V_{123}^{(1)} + V_{31} + V_{123}^{(2)} \right) | b' \rangle$$

Standard relation for the scattering amplitude (t-matrix) and S-matrix (for elastic scattering)

$$S_{b'b} = {}^{(+)}\langle b' | (1 + P) | b \rangle^{(+)} - 2\pi i \delta(E_b - E_{b'}) \underbrace{\langle b' | \left(V_{123}^{(3)} + V_{23} + V_{123}^{(1)} + V_{31} + V_{123}^{(2)} \right) (1 + P) | b \rangle^{(+)}}_{M_{b'b}}$$

This defines the amplitude $M_{b'b}$ for elastic scattering.

Enters cross section for elastic scattering (phase space integral)

$$\frac{d^2\sigma}{d\hat{q}} = (2\pi)^4 \frac{2m^2}{3} |M_{\vec{q}', q_0}|^2$$

Faddeev equations for the breakup amplitude

Formulate **Faddeev equations for the break up amplitude** and use the result **also for elastic scattering!**

Faddeev components of the scattering state

$$|\psi_{ij}\rangle \equiv \left(V_{ij} + V_{123}^{(k)} \right) \frac{i\varepsilon}{E + i\varepsilon - H} (1 + P) |b\rangle \quad \longrightarrow \quad |\psi_{23}\rangle + |\psi_{31}\rangle = P |\psi_{12}\rangle \quad \text{and} \quad M_{ab} = \langle a | (1 + P) | \psi_{12} \rangle$$

The scattering state is an eigenstate of the Schrödinger equation

$$\frac{1}{E - H_0} (1 + P) |\psi_{12}\rangle \equiv \frac{1}{E - H_0} \left(V_{12} + V_{123}^{(3)} + V_{23} + V_{123}^{(1)} + V_{31} + V_{123}^{(2)} \right) \frac{i\varepsilon}{E + i\varepsilon - H} (1 + P) |b\rangle = \frac{i\varepsilon}{E + i\varepsilon - H} (1 + P) |b\rangle$$

Definition of Faddeev component $\longrightarrow$ Faddeev equations (again using resolvent identities)

$$\begin{aligned} |\psi_{12}\rangle &= V_{12} \frac{i\varepsilon}{E + i\varepsilon - H_{12}} (1 + P) |b\rangle + V_{12} \frac{1}{E + i\varepsilon - H_{12}} \left(V_{123}^{(3)} + V_{23} + V_{123}^{(1)} + V_{31} + V_{123}^{(2)} \right) \frac{i\varepsilon}{E + i\varepsilon - H} (1 + P) |b\rangle + V_{123}^{(3)} \frac{i\varepsilon}{E + i\varepsilon - H} (1 + P) |b\rangle \\ &= V_{12} |b\rangle + \left(\mathbb{I} + V_{12} \frac{1}{E + i\varepsilon - H_{12}} \right) V_{123}^{(3)} \frac{1}{E + i\varepsilon - H_0} (1 + P) |\psi_{12}\rangle + V_{12} \frac{1}{E + i\varepsilon - H_{12}} P |\psi_{12}\rangle \end{aligned}$$

this uses $\frac{i\varepsilon}{E + i\varepsilon - H_0 - V_{12}} (1 + P) |b\rangle = |b\rangle$ ($P |b\rangle$ is not eigenstate of $H_0 + V_{12}$)

Faddeev equations for the breakup amplitude

Further simplifications use the Lippmann-Schwinger Eq.

$$V_{12} \frac{1}{E + i\varepsilon - H_0 - V_{12}} = t_{12} \frac{1}{E + i\varepsilon - H_0} \quad (\text{Lippmann-Schwinger equation for the two-body t-matrix in 3-body space})$$

$$|\psi_{12}\rangle = (E - H_0) |b\rangle + \underbrace{\left(\mathbb{I} + t_{12} \frac{1}{E + i\varepsilon - H_0} \right) V_{123} \frac{1}{E + i\varepsilon - H_0} (1 + P) |\psi_{12}\rangle + t_{12} \frac{1}{E + i\varepsilon - H_0} P |\psi_{12}\rangle}_{\equiv |T_{12}\rangle}$$

With the definition of $|T_{12}\rangle$, one gets immediately the corresponding Faddeev equations

$$|T_{12}\rangle = \left(\mathbb{I} + t_{12} \frac{1}{E + i\varepsilon - H_0} \right) V_{123}^{(3)} (1 + P) |b\rangle + t_{12} P |b\rangle \\ + \left(\mathbb{I} + t_{12} \frac{1}{E + i\varepsilon - H_0} \right) V_{123}^{(3)} \frac{1}{E + i\varepsilon - H_0} (1 + P) |T_{12}\rangle + t_{12} \frac{1}{E + i\varepsilon - H_0} P |T_{12}\rangle$$

... needs to be numerically implemented (see below)!

$|T_{12}\rangle$ directly gives the breakup and elastic amplitude!

$$M_{ab} = \langle a | (1 + P) (E_b - H_0) | b \rangle + \langle a | (1 + P) | T_{12} \rangle = \langle a | (1 + P) | T_{12} \rangle$$

$$M_{b'b} = \langle b' | P (E_b - H_0) | b \rangle + \langle b' | V_{123}^{(3)} (1 + P) | b \rangle + \langle b' | P | T_{12} \rangle + \langle b' | V_{123}^{(3)} \frac{1}{E_b + i\varepsilon - H_0} (1 + P) | T_{12} \rangle$$

Numerical implementation requires a careful **treatment of the singularities and poles**.

This needs to include the **singularity of the free resolvent** $\frac{1}{E + i\varepsilon - H_0}$

and also the **pole of the t-matrix** at two-body subsystem energies $E_{12} \approx E_d < 0$ (two-body binding energy):

$$t_{12} \approx (E_{12} - H_0) | \varphi_{12} \rangle \frac{1}{E_{12} + i\varepsilon - E_d} \langle \varphi_{12} | (E_{12} - H_0)$$

In the following, I will restrict myself to the NN interaction terms only.

3NF terms are usually easier to implement since more integrations allow for an easier removal of singularities.

Representation in Jacobi momentum basis

As for bound state problem, choose basis as for bound states : $|p q \alpha M\rangle$ (*often* (not always!) M will be dropped)

Energy is given by $E = E_d + \frac{3q_0^2}{4m}$ (E_d is two-body binding energy)

Incoming state in partial wave representation:

$$\langle p q \alpha M | b \rangle = \varphi_{l_{12}}(p) \frac{\delta(q - q_0)}{q q_0} Y_{l_{30}}^*(\hat{q}_0) (j_{12} I_3 J, m_d M - m_d M) (l_3 s_3 I_3, 0 M - m_d M - m_d) \delta_{t_{12}0}$$

(z -direction of $\vec{q}_0$ breaks rotational invariance, selects $M = m_d + m_3$)

Lippmann-Schwinger equation defines **off-shell t-matrix** (including **bound state pole**):

$$\langle p q \alpha | t_{12} | p' q' \alpha' \rangle = t_{\alpha\alpha'} \left(p, p'; E - \frac{3q^2}{4m} \right) \frac{\delta(q - q')}{q q'} \equiv \frac{\tilde{t}_{\alpha\alpha'} \left(p, p'; E - \frac{3q^2}{4m} \right)}{E_d + \frac{3q_0^2}{4m} + i\varepsilon - \frac{3q^2}{4m} - E_d} \frac{\delta(q - q')}{q q'}$$

Define $\left(\frac{3q_0^2}{4m} + i\varepsilon - \frac{3q^2}{4m} \right) \langle p q \alpha | T_{12} \rangle \equiv \tilde{T}_\alpha(pq)$ and use rewrite Faddeev equations in terms of $\tilde{t}_{\alpha\alpha'}$ and $\tilde{T}_\alpha$

We explicitly see all poles:

$$\begin{aligned}\tilde{T}_\alpha(pq) = & \sum_{\alpha''} \int dp'' p''^2 dq'' q''^2 \langle p q \alpha | \tilde{t}_{12} P | p'' q'' \alpha'' \rangle \langle p'' q'' \alpha'' | b \rangle \\ & + \sum_{\alpha''} \int dp'' p''^2 dq'' q''^2 \langle p q \alpha | \tilde{t}_{12} P | p'' q'' \alpha'' \rangle \frac{1}{E + i\varepsilon - \frac{3q''^2}{4m} - \frac{p''^2}{m}} \frac{\tilde{T}_{\alpha''}(p''q'')}{\frac{3q_0^2}{4m} + i\varepsilon - \frac{3q''^2}{4m}}\end{aligned}$$

The direct numerical treatment of both poles is difficult !

Permutation operator induces **angular dependence of the pole positions** (cannot be avoid for both poles at the same time!)

Trick: use **partial fraction decomposition** to separate the two poles depending on q_0 :

$$\frac{1}{E + i\varepsilon - \frac{3q''^2}{4m} - \frac{p''^2}{m}} \frac{1}{\frac{3q_0^2}{4m} + i\varepsilon - \frac{3q''^2}{4m}} = \left(\frac{1}{\frac{3q_0^2}{4m} + i\varepsilon - \frac{3q''^2}{4m}} - \frac{1}{E + i\varepsilon - \frac{3q''^2}{4m} - \frac{p''^2}{m}} \right) \frac{m}{mE_d - p''^2}$$

The coefficient is independent of q_0 and does not have any singularity ($E_d < 0$)!

1) **kernel of the equation** (inhomogeneity can be treated later (easier))

$$\begin{aligned}
&= \sum_{\alpha''} \int dp'' p''^2 dq'' q''^2 \langle p q \alpha | \tilde{t}_{12} P | p'' q'' \alpha'' \rangle \frac{1}{\frac{3q_0^2}{4m} + i\varepsilon - \frac{3q''^2}{4m}} \frac{m\tilde{T}_{\alpha''}(p''q'')}{mE_d - p''^2} \\
&\quad - \sum_{\alpha''} \int dp'' p''^2 dq'' q''^2 \langle p q \alpha | \tilde{t}_{12} P | p'' q'' \alpha'' \rangle \frac{1}{E + i\varepsilon - \frac{3q''^2}{4m} - \frac{p''^2}{m}} \frac{m\tilde{T}_{\alpha''}(p''q'')}{mE_d - p''^2}
\end{aligned}$$

Commute kinetic energy operator in second term with P

$$\begin{aligned}
&= \sum_{\alpha'\alpha''} \int dp' p'^2 \int dp'' p''^2 dq'' q''^2 \tilde{t}_{\alpha\alpha'} \left(p, p'; E - \frac{3q^2}{4m} \right) \langle p' q \alpha' | P | p'' q'' \alpha'' \rangle \frac{1}{\frac{3q_0^2}{4m} + i\varepsilon - \frac{3q''^2}{4m}} \frac{m\tilde{T}_{\alpha''}(p''q'')}{mE_d - p''^2} \\
&\quad - \sum_{\alpha'\alpha''} \int dp' p'^2 \int dp'' p''^2 dq'' q''^2 \tilde{t}_{\alpha\alpha'} \left(p, p'; E - \frac{3q^2}{4m} \right) \frac{1}{E + i\varepsilon - \frac{3q^2}{4m} - \frac{p'^2}{m}} \langle p' q \alpha' | P | p'' q'' \alpha'' \rangle \frac{m\tilde{T}_{\alpha''}(p''q'')}{mE_d - p''^2}
\end{aligned}$$

Two different ways to represent the permutation operator:

- keep q'' integration in the first term
- keep p' integration in the second term

As mentioned before (see page 32), there are several choices for representing the coordinate trafos/permutation operators

$$\langle p' q \alpha' | P | p'' q'' \alpha'' \rangle = \int_{-1}^1 dx \frac{\delta(p' - \pi(qq''x))}{p'^2} \frac{\delta(p'' - \pi'(qq''x))}{p''^2} 2G_{\alpha'\alpha''}(qq''x)$$

$$\langle p' q \alpha' | P | p'' q'' \alpha'' \rangle = \int_{-1}^1 dx \frac{\delta(p'' - \tilde{\pi}(p'qx))}{p''^2} \frac{\delta(q'' - \tilde{\chi}(p'qx))}{q''^2} 2\tilde{G}_{\alpha'\alpha''}(p'qx) \quad (\text{note the factors 2})$$

resulting in

$$= \underbrace{\int dq'' \frac{1}{\frac{3q_0^2}{4m} + i\varepsilon - \frac{3q''^2}{4m}} q''^2 \int_{-1}^1 dx \sum_{\alpha'\alpha''} \tilde{t}_{\alpha\alpha'} \left(p, \pi(qq''x); E - \frac{3q^2}{4m} \right) 2G_{\alpha'\alpha''}(qq''x) \frac{m\tilde{T}_{\alpha}(\pi'(qq''x)q'')}{mE_d - \pi'(qq''x)^2}}_{\equiv \mathcal{F}_{\alpha}(pq q'')}$$

$$- \underbrace{\int dp' \frac{1}{E + i\varepsilon - \frac{3q^2}{4m} - \frac{p'^2}{m}} p'^2 \sum_{\alpha'\alpha''} \tilde{t}_{\alpha\alpha'} \left(p, p'; E - \frac{3q^2}{4m} \right) \int_{-1}^1 dx 2\tilde{G}_{\alpha'\alpha''}(p'qx) \frac{m\tilde{T}_{\alpha''}(\tilde{\pi}(p'qx)\tilde{\chi}(p'qx))}{mE_d - \tilde{\pi}(p'qx)^2}}_{\mathcal{G}_{\alpha}(pp'q)}$$

Thereby $\mathcal{F}_{\alpha}$ and $\mathcal{G}_{\alpha}$ are smooth functions (no singularities!).

Poles treated as in the two-body case:

$$\begin{aligned}
 &= \int dq'' \frac{\frac{4m}{3} \mathcal{F}_\alpha(pqq'')}{q_0^2 + i\varepsilon - q''^2} - \int dp' \frac{m \mathcal{G}_\alpha(pp'q)}{\underbrace{mE - \frac{3q^2}{4} + i\varepsilon - p'^2}_{\equiv p_0^2(q)}} \quad (p_0^2 \text{ can be negative!}) \\
 &= \int_0^{q_{\max}} dq'' \frac{\frac{4m}{3} (\mathcal{F}_\alpha(pqq'') - \mathcal{F}_\alpha(pqq_0))}{q_0^2 + i\varepsilon - q''^2} + \left[\ln \left(\frac{q_{\max} + q_0}{q_{\max} - q_0} \right) - i\pi \right] \frac{4m}{6q_0} \mathcal{F}_\alpha(pqq_0) \\
 &\quad - \int_0^{p_{\max}} dp' \frac{m (\mathcal{G}_\alpha(pp'q) - \mathcal{G}_\alpha(pp_0q))}{\underbrace{mE - \frac{3q^2}{4} + i\varepsilon - p'^2}_{\equiv p_0^2(q)}} - \left[\ln \left(\frac{p_{\max} + p_0}{p_{\max} - p_0} \right) - i\pi \right] \frac{m}{2p_0} \mathcal{G}_\alpha(pp_0q)
 \end{aligned}$$

p_0 terms only contribute for $p_0^2 > 0$.

Set up equations for $T_\alpha(p_i q_j)$ for $i = 0, \dots, N_p - 1$ and $j = 0, \dots, N_q$ where $q_{N_q} = q_0$.

Using usual spline interpolation $f(p) = \sum_{m=0}^{N_p-1} S_m(p) f(p_m)$ and $f(q) = \sum_{n=0}^{N_q-1} \tilde{S}_n(q) f(q_n)$

→ discretized kernel $\mathcal{K}_{\alpha\alpha''}(p_i q_j, p_m q_l) = \mathcal{K}_{\alpha\alpha''}^{(1)}(p_i q_j, p_m q_l) + \mathcal{K}_{\alpha\alpha''}^{(2)}(p_i q_j, p_m q_l)$

With

$$\mathcal{K}_{\alpha\alpha''}^{(1)}(p_i q_j, p_m q_l) = \begin{cases} \omega_l^q \frac{\frac{4m}{3} \mathcal{F}_{\alpha\alpha''}^m(p_i q_j q_l)}{q_0^2 + i\varepsilon - q_l^2} & l \neq N_q \\ \left[\frac{1}{2q_0} \ln \left(\frac{q_{max} + q_0}{q_{max} - q_0} \right) - i\pi \frac{1}{2q_0} - \sum_{\tilde{l}=0}^{N_q-1} \omega_{\tilde{l}}^q \frac{1}{q_0^2 + i\varepsilon - q_{\tilde{l}}^2} \right] \frac{4m}{3} \mathcal{F}_{\alpha\alpha''}^m(p_i q_j q_0) & l = N_q \end{cases}$$

and

$$\mathcal{K}_{\alpha\alpha''}^{(2)}(p_i q_j, p_m q_l) = - \sum_{n=0}^{N_p-1} \omega_n^p \frac{m \mathcal{G}_{\alpha}^{ml}(p_i p_n q_j)}{p_0^2(q_j) + i\varepsilon - p_n^2} - \left[\frac{1}{2p_0} \left(\ln \left(\frac{p_{max} + p_0}{p_{max} - p_0} \right) - i\pi \right) - \sum_{n=0}^{N_p-1} \omega_n^p \frac{1}{p_0^2(q_j) + i\varepsilon - p_n^2} \right] m \mathcal{G}_{\alpha\alpha''}^{ml}(p_i p_0 q_j) \quad l \neq N_q$$

Two smooth functions now include appropriate interpolations:

$$\mathcal{F}_{\alpha\alpha''}^m(p_i q_j q_l) = q_l^2 \sum_{n=0}^{N_p-1} \sum_{\alpha'} \tilde{t}_{\alpha\alpha'} \left(p_i, p_n; E - \frac{3q_j^2}{4m} \right) \sum_k \omega_k^x S_n(\pi(q_j q_l x_k)) \frac{2G_{\alpha'\alpha''}(q_j q_l x_k)}{mE_d - \pi'(q_j q_l x_k)^2} \frac{mS_m(\pi'(q_j q_l x_k))}{mE_d - \pi'(q_j q_l x_k)^2}$$

$$\mathcal{G}_{\alpha\alpha''}^{ml}(p_i p_n q_j) = p_n^2 \sum_{\alpha'} \tilde{t}_{\alpha\alpha'} \left(p_i, p_n; E - \frac{3q_j^2}{4m} \right) \sum_{k=0}^{N_x} \omega_k^x \frac{2\tilde{G}_{\alpha'\alpha''}(p_n q_j x_k)}{mE_d - \tilde{\pi}(p_n q_j x_k)^2} \frac{m S_m(\tilde{\pi}(p_n q_j x_k)) \tilde{S}_l(\tilde{\chi}(p_n q_j x_k))}{mE_d - \tilde{\pi}(p_n q_j x_k)^2}$$

($\tilde{t}$ is also available at $p_0(q_j)$!)

2) inhomogeneity

$$\sum_{\alpha''} \int dp'' p''^2 dq'' q''^2 \langle p q \alpha | \tilde{t}_{12} P | p'' q'' \alpha'' \rangle \langle p'' q'' \alpha'' | b \rangle$$

$$= \int_{-1}^1 dx \sum_{\alpha' \alpha''} \tilde{t}_{\alpha \alpha'} \left(p, \pi(q q_0 x); E - \frac{3q^2}{4m} \right) 2G_{\alpha' \alpha''}(q q_0 x) \varphi_{l_{12}''}(\pi(q q_0 x)') Y_{l_3'' 0}^*(\hat{q}_0) (j_{12}'' I_3'' J, m_d M - m_d M) (l_3'' s_3 I_3'', 0 M - m_d M - m_d) \delta_{t_{12} 0}$$

or in discretized form

$$\mathcal{H}_{\alpha}(p_i q_j) = \sum_k \omega_k^x \sum_{\alpha' \alpha''} \tilde{t}_{\alpha \alpha'} \left(p_i, \pi(q_j q_0 x_k); E - \frac{3q_j^2}{4m} \right) 2G_{\alpha' \alpha''}(q_j q_0 x_k) \sum_n S_n(\pi'(q_j q_0 x_k)) \varphi_{l_{12}''}(p_n) Y_{l_3'' 0}^*(\hat{q}_0) (j_{12}'' I_3'' J, m_d M - m_d M) (l_3'' s_3 I_3'', 0 M - m_d M - m_d) \delta_{t_{12} 0}$$

The example code for solving the resulting set of linear equations (iterative solution is also possible)

$$\sum_{ml \alpha''} \left[\delta_{\alpha \alpha''} \delta_{im} \delta_{jl} - \mathcal{K}_{\alpha \alpha''}^{(1)}(p_i q_j, p_m q_l) - \mathcal{K}_{\alpha \alpha''}^{(2)}(p_i q_j, p_m q_l) \right] \tilde{T}_{\alpha''}(p_m q_l) = \mathcal{H}_{\alpha}(p_i q_j)$$

To be solve independently for each conserved JM . The inhomogeneity introduces dependence on m_d and m_3 !

Several J blocks contribute to the cross sections.

$$M_{b'b} = \langle b' | P (E_b - H_0) | b \rangle + \langle b' | P | T_{12} \rangle$$

The **first part** can be calculated **without using a partial wave decomposition** (advisable!)

$$\begin{aligned} \langle b' | P (E_b - H_0) | b \rangle = & \sum_{SM_S, l_{12} l'_{12}} \varphi_{l'_{12}} \left(\left| \vec{q}_0 + \frac{1}{2} \vec{q}'_0 \right| \right) Y_{l'_{12} m'_d + m'_3 - M_S}^* \left(\widehat{\vec{q}_0 + \frac{1}{2} \vec{q}'_0} \right) \varphi_{l_{12}} \left(\left| -\vec{q}_0' - \frac{1}{2} \vec{q}_0 \right| \right) Y_{l_{12} m_d + m_3 - M_S} \left(\widehat{-\vec{q}_0' - \frac{1}{2} \vec{q}_0} \right) \left(E_d - \frac{1}{m} \left| -\vec{q}_0' - \frac{1}{2} \vec{q}_0 \right|^2 \right) \\ & \times (l'_{12} s_{12} j_{12}, m'_d + m'_3 - M_S, M_S - m'_3 m_d) (s_{12} s_3 S, M_S - m'_3 m_3 M_S) \delta_{t'_{12} 0} \delta_{s'_{12} 1} \delta_{j'_{12} 1} \\ & \times (l_{12} s_{12} j_{12}, m_d + m_3 - M_S, M_S - m_3 m_3 M_S) \delta_{t_{12} 0} \delta_{s_{12} 1} \delta_{j_{12} 1} \\ & \times (-)^{s_{12}} \sqrt{\hat{s}_{12} \hat{s}_{12}} \begin{Bmatrix} 1/2 & 1/2 & s_{12} \\ 1/2 & S & s_{12} \end{Bmatrix} (-)^{t_{12}} \sqrt{\hat{t}_{12} \hat{t}_{12}} \begin{Bmatrix} 1/2 & 1/2 & t_{12} \\ 1/2 & T & t_{12} \end{Bmatrix} \end{aligned}$$

relation from page 33

For elastic scattering $|\vec{q}_0| = |\vec{q}'_0| = q_0$. The vectors only depend on the scattering angle ϑ : $\vec{q}_0 = \begin{pmatrix} 0 \\ 0 \\ q_0 \end{pmatrix}$ and $\vec{q}'_0 = \begin{pmatrix} q_0 \sin \vartheta \\ 0 \\ q_0 \cos \vartheta \end{pmatrix}$

The **second part** requires the treatment of the bound state pole:

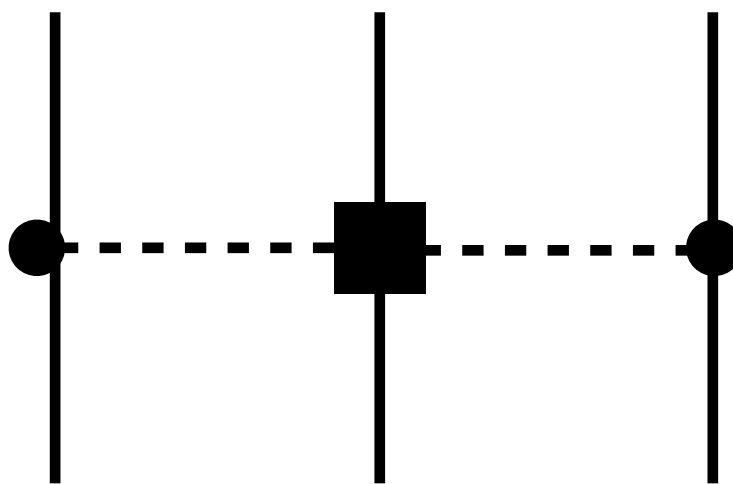
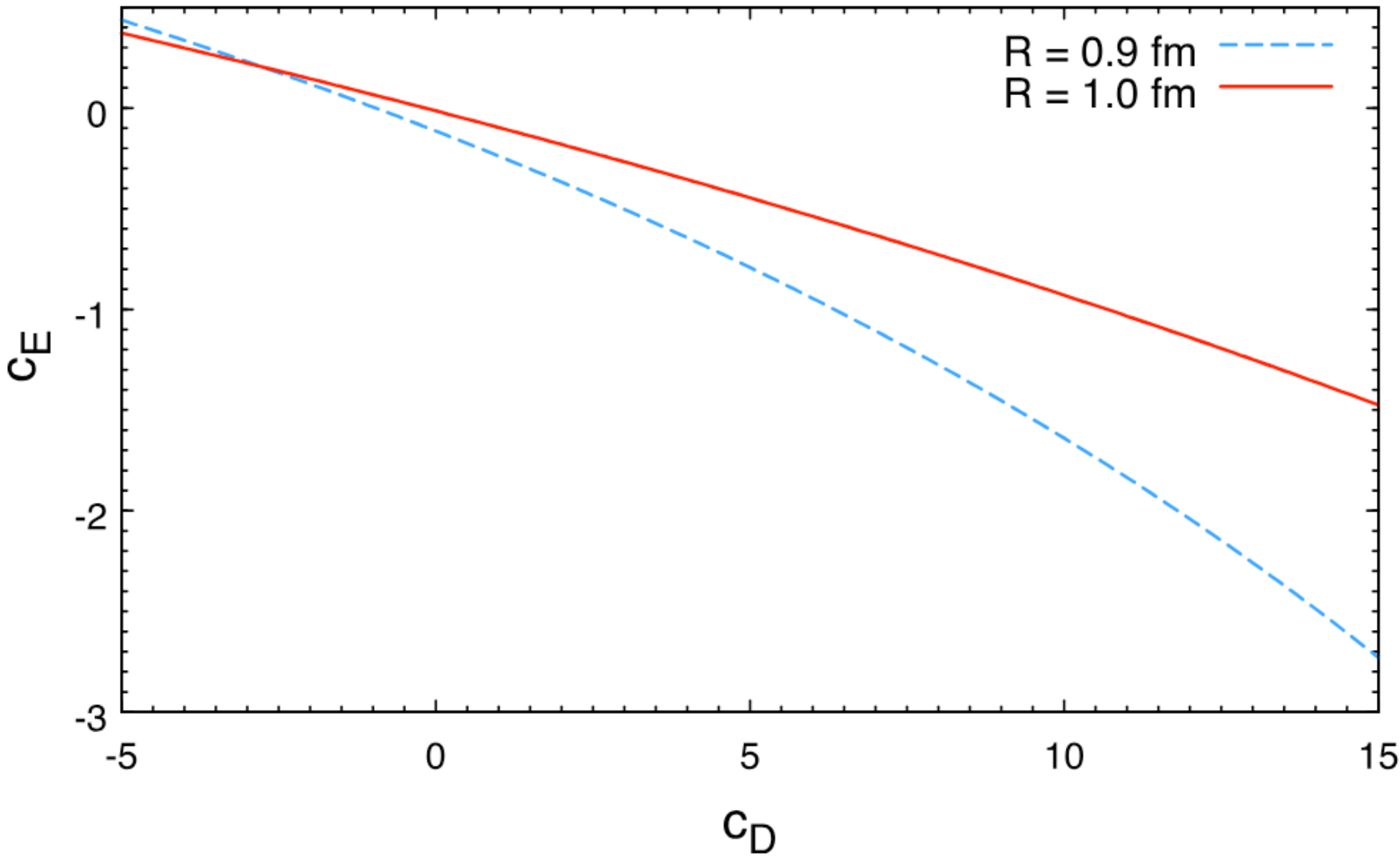
$$\langle b' | P | T_{12} \rangle = \int_0^\infty dq'' \frac{\frac{4m}{3} I(q_0 q'')}{q_0^2 + i\varepsilon - q''^2} = \int_0^{q_{max}} \frac{4m}{3} \frac{I(q_0 q'') - I(q_0 q_0)}{q_0^2 - q''^2} + \left[\ln \frac{q_{max} + q_0}{q_{max} - q_0} - i\pi \right] \frac{4m}{6q_0} I(q_0 q_0)$$

where

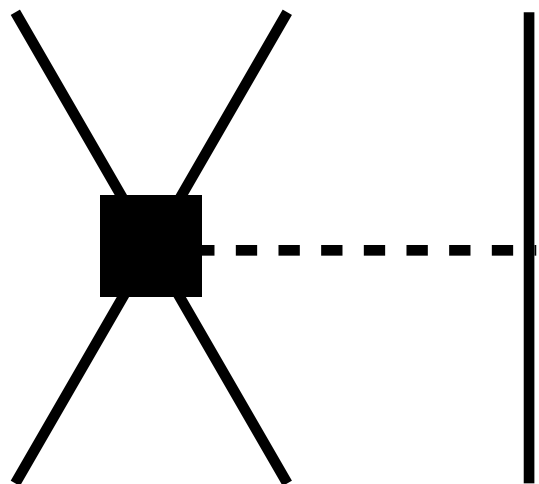
$$I(q_0 q'') = q''^2 \sum_{\alpha \alpha''} \delta_{s_{12} 1} \delta_{t_{12} 0} \delta_{j_{12} 1} \int_{-1}^1 dx \varphi_{l'_{12}}(\pi(q_0 q'' x)) Y_{l'_3 M - m'_d - m'_3}(\hat{q}'_0) (j'_{12} l'_3 J, m'_d M - m'_d M) (l'_{12} s'_{12} j'_{12}, M - m'_d - m'_3 m'_3 M - m'_d) 2G_{\alpha \alpha''}(q_0 q'' x) \tilde{T}_{\alpha''}(\pi'(q_0 q'' x) q'')$$

DETERMINATION OF 3N LECS

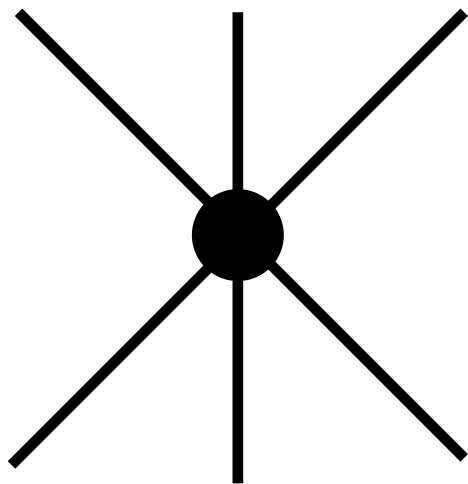
see Epelbaum et al. PRC 99, 024313 (2019)



$\propto c_1, c_3, c_4$

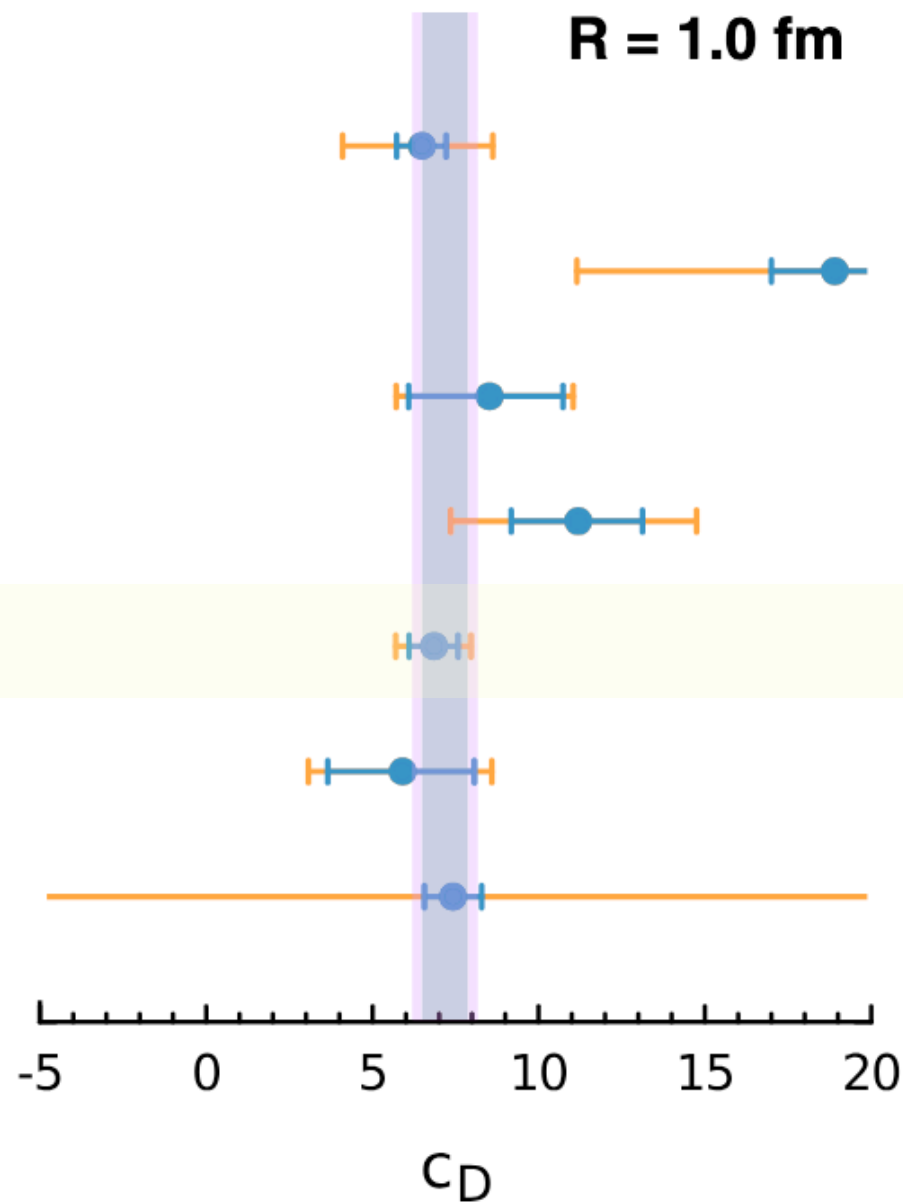
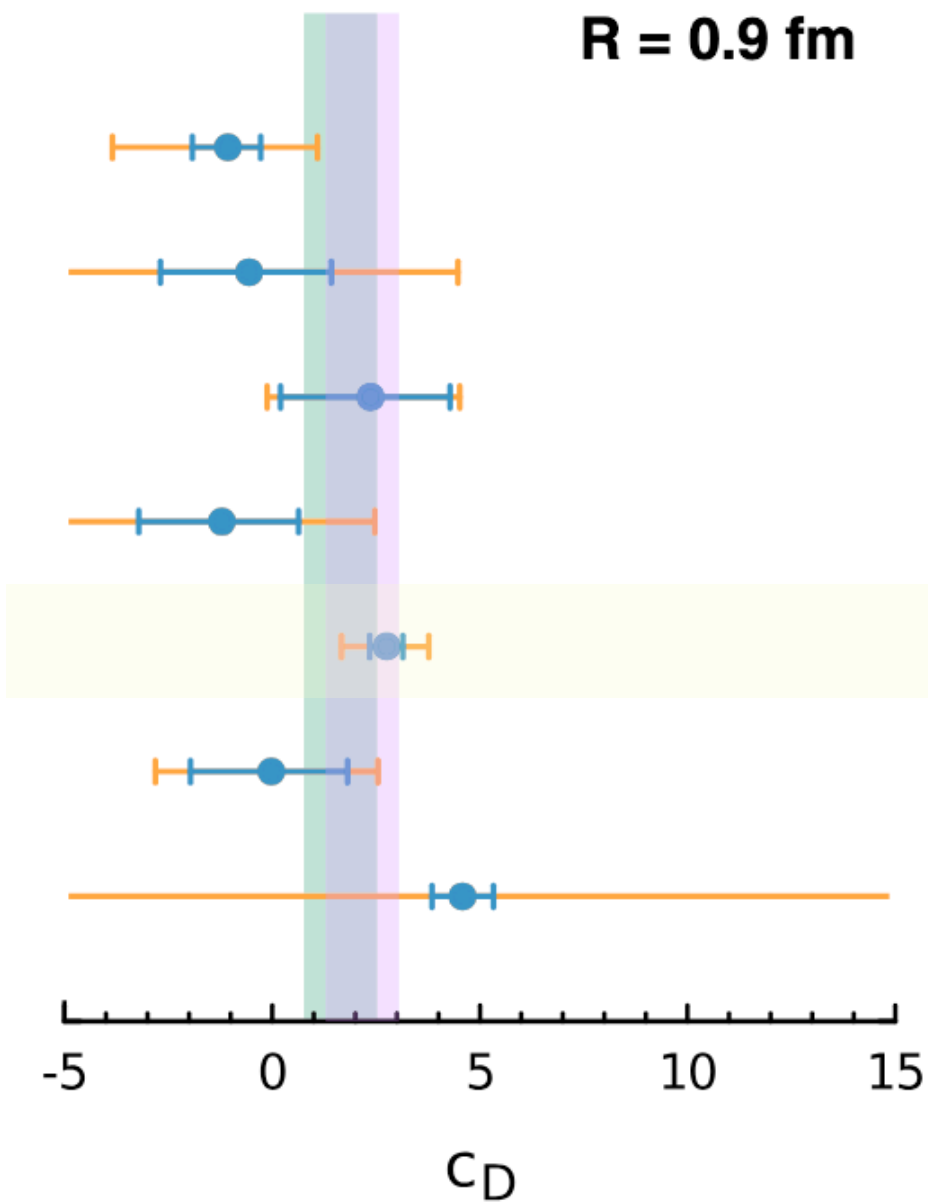


$\propto c_D$



$\propto c_E$

from πN



pd minimum of $d\sigma/d\theta$ at 135 MeV

nd σ_{tot} at 135 MeV

pd minimum of $d\sigma/d\theta$ at 108 MeV

nd σ_{tot} at 108 MeV

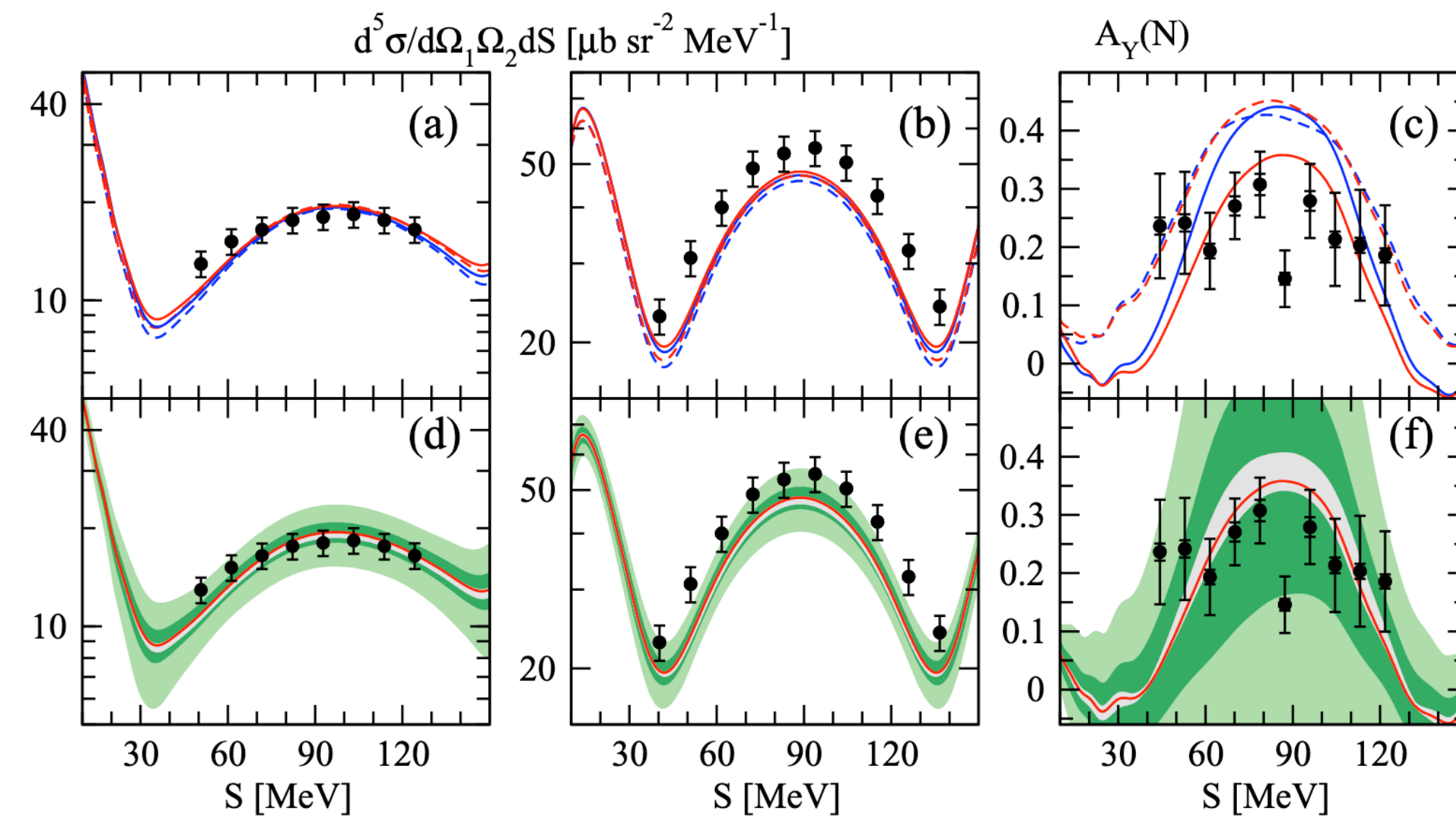
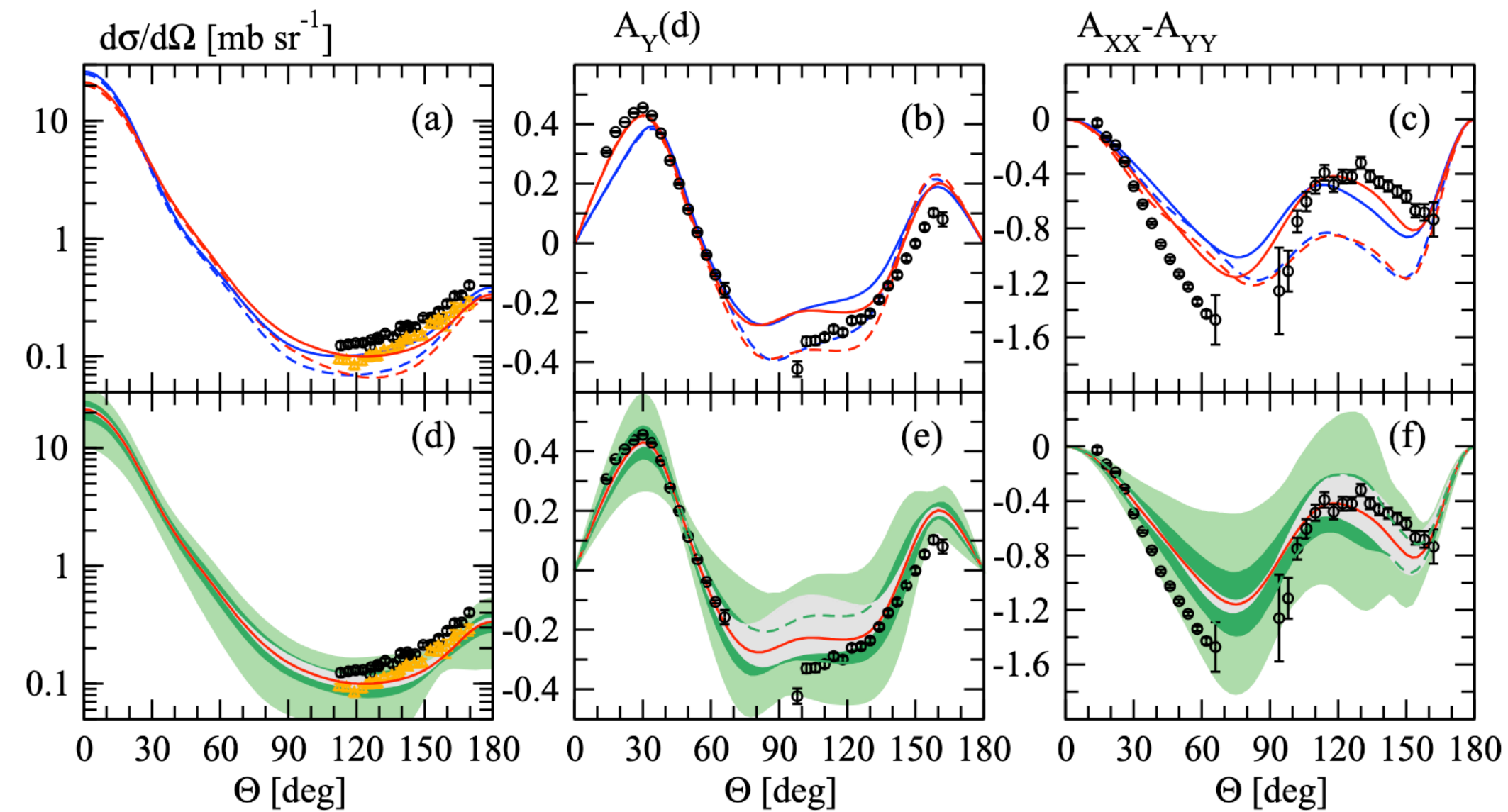
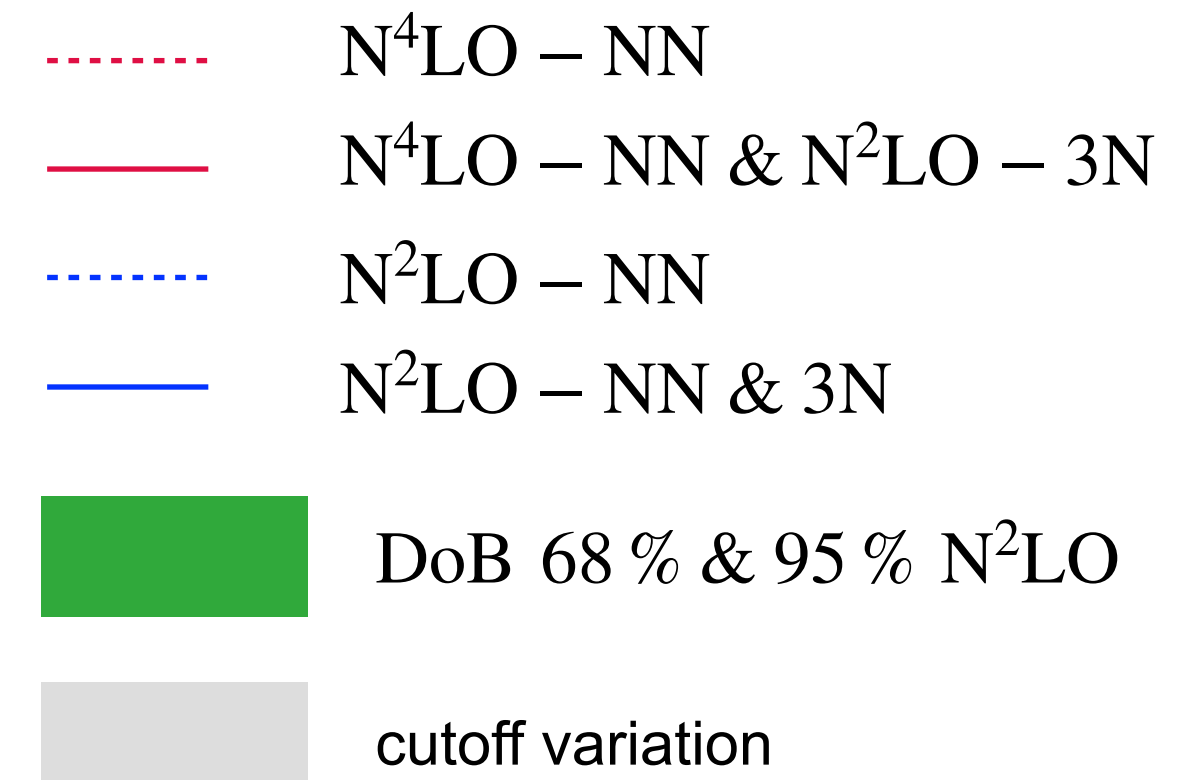
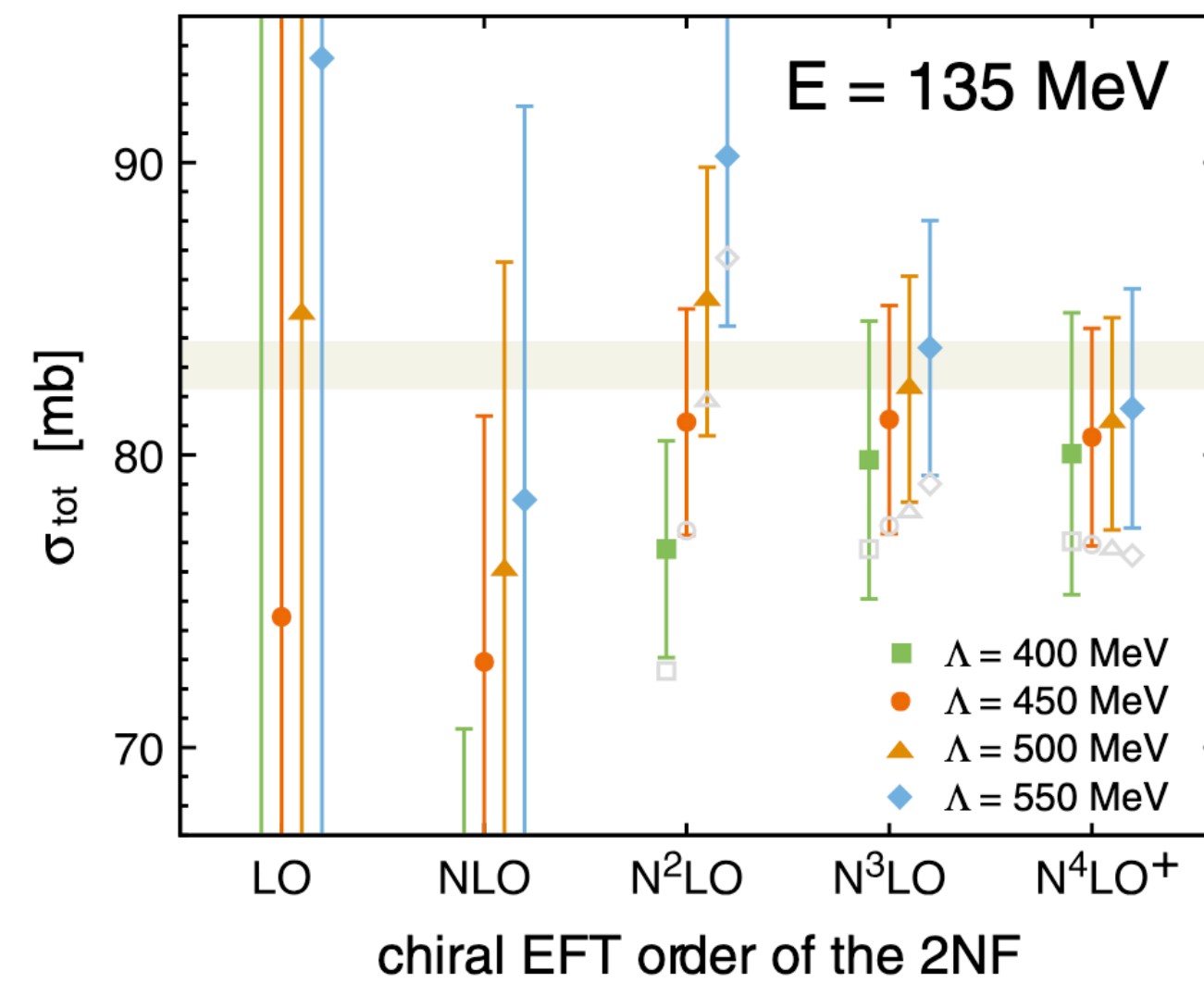
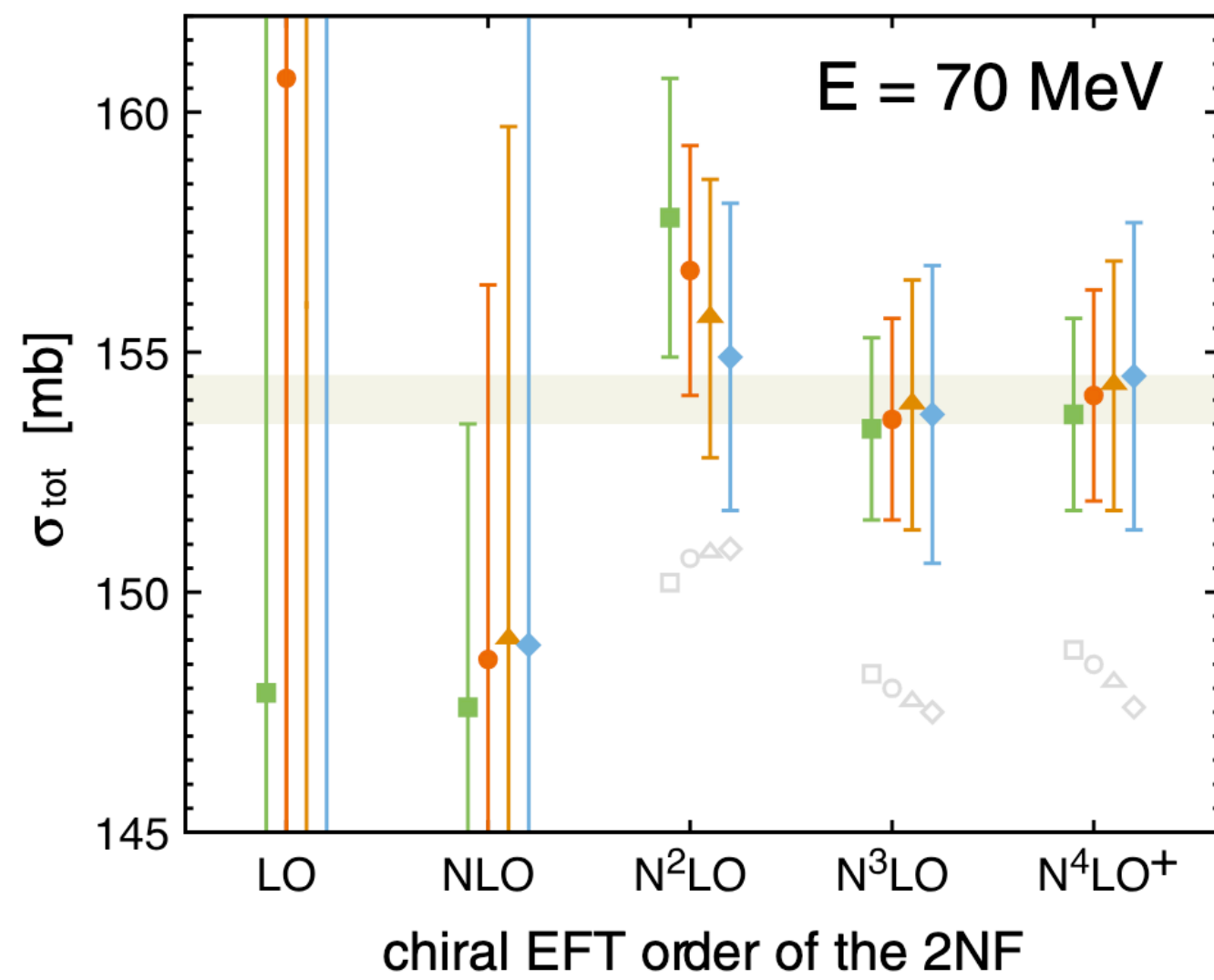
pd minimum of $d\sigma/d\theta$ at 70 MeV

nd σ_{tot} at 70 MeV

nd scattering length 2a

PREDICTIONS FOR 3N SCATTERING

Maris et al. PRC 106, 064002 (2022)



JACOBI-NCSM

Solve the Schrödinger equation using **HO states**

Two ingredients are necessary:

- **cfp** — antisymmetrized states for nucleons
- **transition coefficients** to separate off NN, YN, 3N and YNN

Note

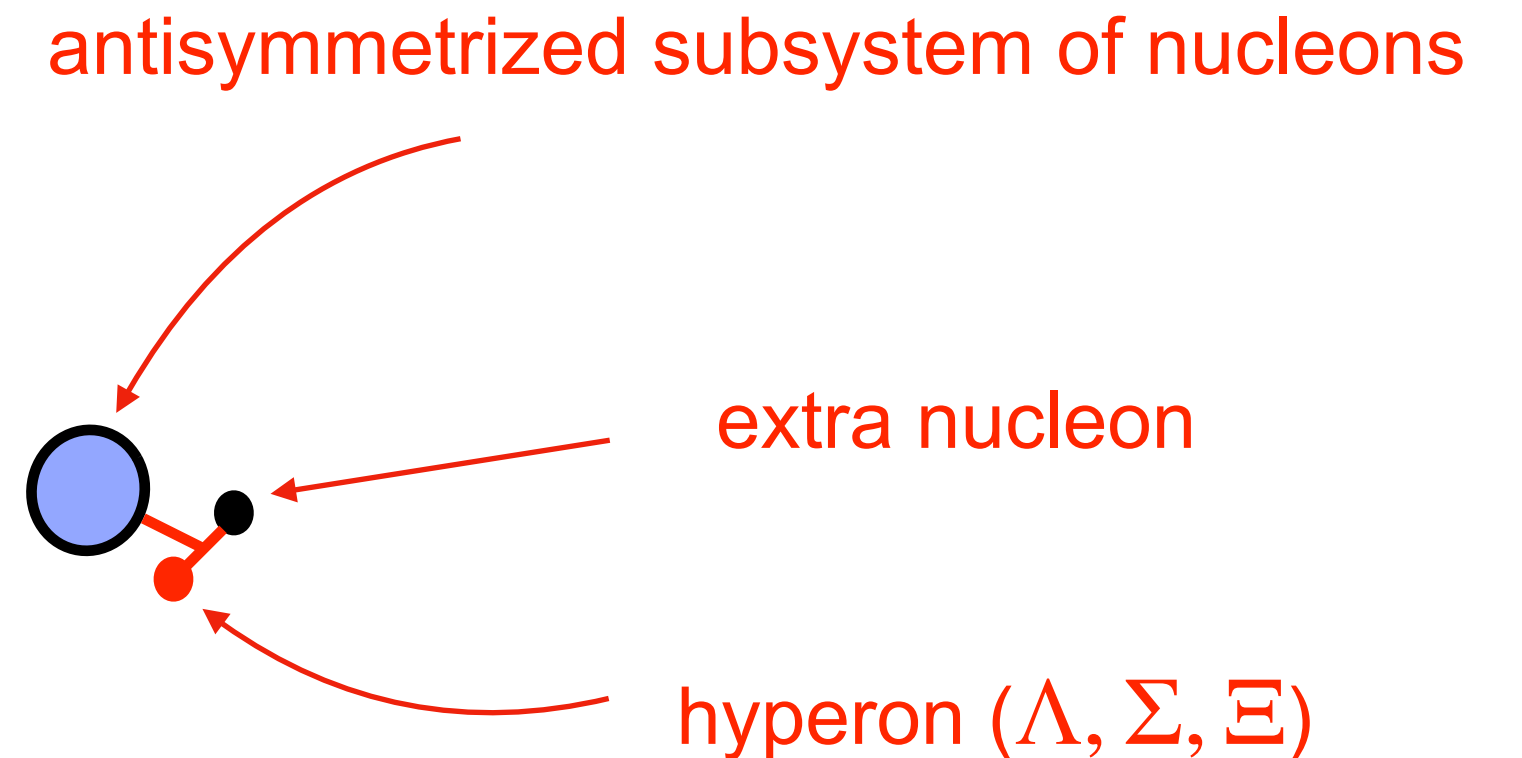
- extra hyperon does not need antisymmetrization
- extra pair of hyperons can be trivially symmetrized

Schrödinger equation (here for hypernucleus)

$$\langle \text{blue circle with red dot} | H | \text{blue circle with red dot} \rangle \langle \text{blue circle with red dot} | \Psi \rangle = E \langle \text{blue circle with red dot} | \Psi \rangle$$

e.g. for YN interaction

$$\langle \text{blue circle with red dot} | V_{YN} | \text{blue circle with red dot} \rangle = \langle \text{blue circle with red dot} | \text{blue circle with red dot and black dot} \rangle \langle \text{blue circle with red dot and black dot} | V_{YN} | \text{blue circle with red dot and black dot} \rangle \langle \text{blue circle with red dot and black dot} | \text{blue circle with red dot} \rangle$$



Application of to NN, YN, 3N and YNN interactions require the representation of particle transitions.

(see Liebig et al. EPJ A 52,103 (2016), Le et al. EPJ A 56, 301 (2020)
for combinatorial factors see Le et al. EPJ A 57, 217 (2021))

NEED COORDINATE TRAFOS AGAIN!

Liebig et al. EPJ A 52,103 (2016)

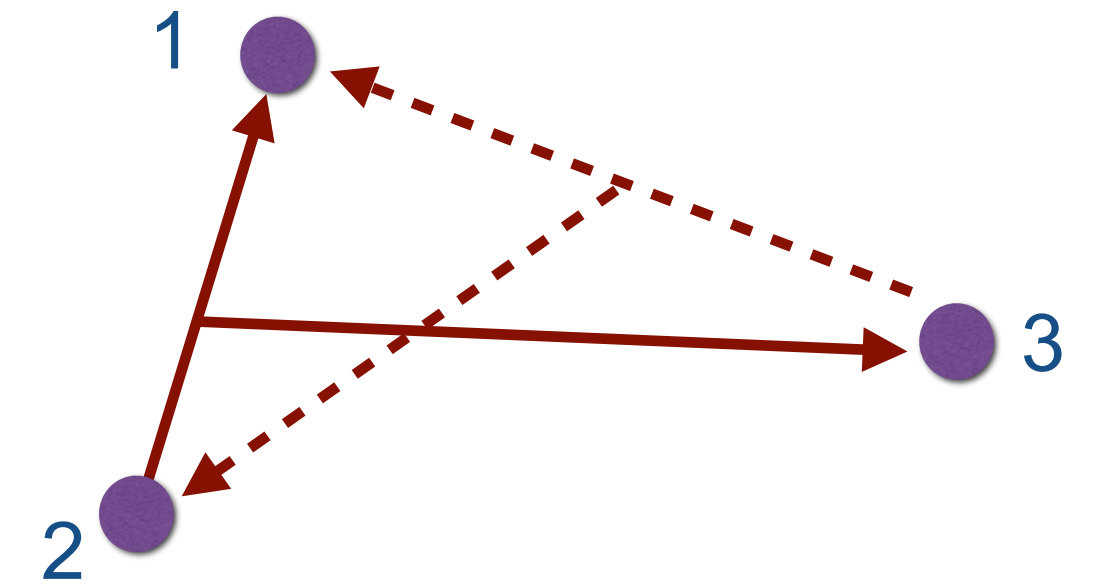
see also Navratil et al. PRC 61, 044001 (2000)

"particles" 1,2,3 more general: cluster, hyperon, nucleon characterized by s_i , t_i and mass m_i

spin/isospin — see slide 33

spatial part — HO state, all coordinates use same frequency but oscillator length scales with mass

dimensionless coordinates for oscillators



$$\rho_{12} = r_{12}/b_{12} = \sqrt{\mu_{12}\omega}(r_1 - r_2)$$

$$\rho_{13} = r_{13}/b_{12} = \sqrt{\mu_{13}\omega}(r_1 - r_3)$$

$$\rho_3 = R_3/b_3 = \sqrt{\mu_{12,3}\omega}\left(r_3 - \frac{1}{m_1 + m_2}(m_1 r_1 + m_2 r_2)\right)$$

$$\rho_2 = R_2/b_2 = \sqrt{\mu_{13,2}\omega}\left(r_2 - \frac{1}{m_1 + m_3}(m_1 r_1 + m_3 r_3)\right)$$

using reduced masses

$$\mu_{ij} = \frac{m_i m_j}{m_i + m_j} \quad \mu_{ij,k} = \frac{m_k(m_i + m_j)}{m_i + m_j + m_k}$$

with $d = m_2 m_3 / (m_1(m_1 + m_2 + m_3))$ both coordinates are related by

$$\begin{pmatrix} \rho_{13} \\ -\rho_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{d}{1+d}} & \sqrt{\frac{1}{1+d}} \\ \sqrt{\frac{1}{1+d}} & -\sqrt{\frac{d}{1+d}} \end{pmatrix} \begin{pmatrix} \rho_{12} \\ -\rho_3 \end{pmatrix}$$



Talmi-Moshinsky bracket:

$$(-)^{l_2+l_3} \left\langle n_{13} l_{13}, n_2 l_2 : L \middle| n_{12} l_{12}, n_3 l_3 : L \right\rangle_d$$

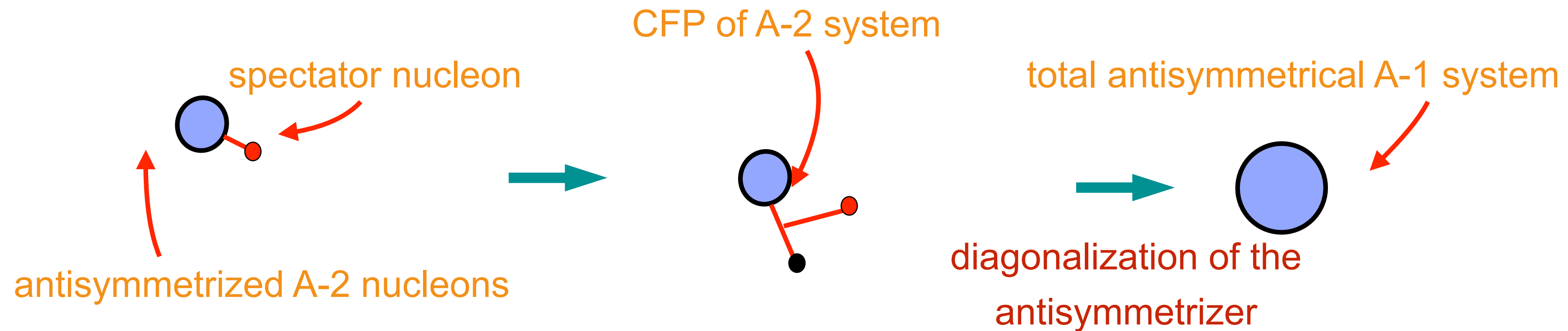
MASTER FORMULA

Master formula

$$\begin{aligned}
 & \langle n_{13} n_2 \left((l_{13}(s_1 s_3) S_{13}) J_{13} (l_2 s_2) I_2 \right) J \left((t_1 t_3) T_{13} t_2 \right) T \mid n_{12} n_3 \left((l_{12}(s_1 s_2) S_{12}) J_{12} (l_3 s_3) I_3 \right) J \left((t_1 t_2) T_{12} t_3 \right) T \rangle \\
 &= \hat{J}_{13} \hat{I}_2 \hat{J}_{12} \hat{I}_3 \sum_{LS} \hat{L}^2 \hat{S}^2 \begin{Bmatrix} l_{13} & S_{13} & J_{13} \\ l_2 & s_2 & I_2 \\ L & S & J \end{Bmatrix} \begin{Bmatrix} l_{12} & S_{12} & J_{12} \\ l_3 & s_3 & I_3 \\ L & S & J \end{Bmatrix} (-1)^{l_2+l_3} \langle n_{13} l_{13}, n_2 l_2 : L \mid n_{12} l_{12}, n_3 l_3 : L \rangle_d \\
 &\times (-1)^{S_{13}+s_2+S_{12}+s_3} \hat{S}_{13} \hat{S}_{12} \begin{Bmatrix} s_2 & s_1 & S_{12} \\ s_3 & S & S_{13} \end{Bmatrix} (-1)^{T_{13}+t_2+T_{12}+t_3} \hat{T}_{13} \hat{T}_{12} \begin{Bmatrix} t_2 & t_1 & T_{12} \\ t_3 & T & T_{13} \end{Bmatrix}.
 \end{aligned}$$

It is possible to match all required transitions to this 3-body transformation!

First, generate **antisymmetrized states** for the A-1 nucleon system



$$\langle \text{antisymmetrized A-2 nucleons} | \text{CFP of A-2 system} \rangle \quad \text{antisymmetrizer is equivalent to coordinate trafo with } d = 1/(A(A-2))$$

The CFP coefficients $\langle \text{antisymmetrized A-2 nucleons} | \text{total antisymmetrical A-1 system} \rangle$ are obtained by (numerical) diagonalization of the antisymmetrizer.

HO states guarantee:

- complete separation of antisymmetrized and other states
- **independence** of HO length/frequency

These coefficients are openly accessible as **HDF5** data files (*database under construction!*)

(download server reachable using python tool at <https://jugit.fz-juelich.de/a.nogga/jncsmcoeff>)

JACOBI-NCSM — TRANSITION COEFFICIENTS

Also need to couple out 2- and 3-nucleon subsystems: $\langle \text{core} \mid \text{1n} \rangle$ and $\langle \text{core} \mid \text{2n} \rangle$

CFP of A CFP of A-1 master formula

$$\langle \text{core} \mid \text{1n} \rangle = \langle \text{core} \mid \text{1n} \rangle \langle \text{1n} \mid \text{1n} \rangle \langle \text{1n} \mid \text{1n} \rangle$$

2N coefficient CFP of A-2 master formula CFP of A = 3

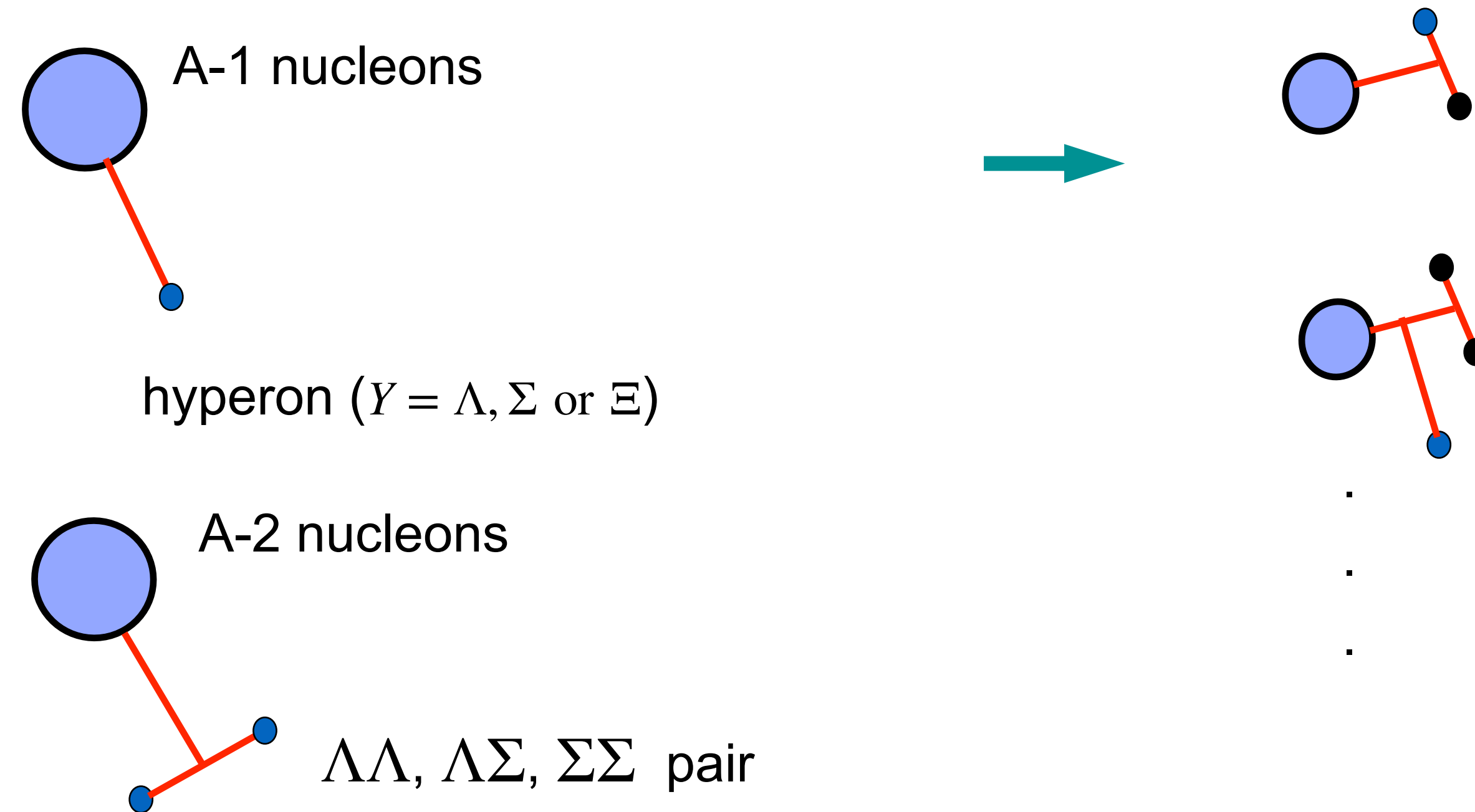
$$\langle \text{core} \mid \text{2n} \rangle = \langle \text{core} \mid \text{2n} \rangle \langle \text{2n} \mid \text{2n} \rangle \langle \text{2n} \mid \text{2n} \rangle \langle \text{2n} \mid \text{2n} \rangle$$

HO states guarantee independence of HO length/frequency

JACOBI-NCSM STATES FOR $S = -1$ AND -2

A-body hypernuclei state (no antisymmetrization with respect to nucleons required)

Third, rearrange baryons for the application of interactions, ...



Again HO states guarantee the independence of HO length/frequency.

Transition coefficients are also accessible as **HDF5** data files to anyone interested.

Converged results feasible for "**soft**" interactions.

(Le, Haidenbauer, Meißner, AN, 2020 & 2021)

Similarity renormalization group is by now a **standard tool** to obtain soft effective interactions for various many-body approaches (NCSM, coupled-cluster, MBPT, ...)

Idea: perform a unitary transformation of the NN (and YN interaction) using a cleverly defined "generator"

(Bogner et al. PRC 75,061001 (2007))

$$\frac{dH_s}{ds} = \left[\underbrace{[T, H(s)]}_{\equiv \eta(s)}, H(s) \right] \quad H(s) = T + V(s) \quad \lambda = \left(\frac{4\mu_{BN}^2}{s} \right)^{1/4}$$

define range of momenta that remain coupled to each other

this choice of generator drives $V(s)$ into a diagonal form in momentum space

- $V(s)$ will be **phase equivalent** to original interaction
- short range $V(s)$ will change towards **softer interactions**
- Evolution can be restricted to **2-,3-, ... body level** (approximation)
- **dependence** of results on λ or s is a measure for **missing terms**

The evolution naturally separates in 2- and 3-body,... parts.

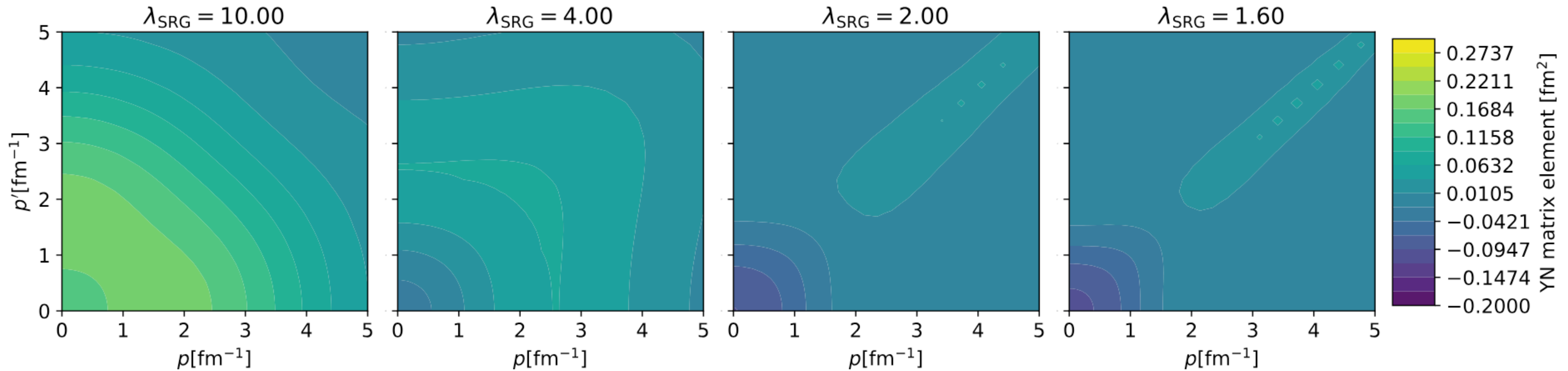
$$\frac{dV_{ij}(s)}{ds} = \left[\left[T_{ij}, V_{ij}(s) \right], T_{ij} + V_{ij}(s) \right] \quad \text{easily done — we use momentum space}$$

$$\begin{aligned} \frac{dV_{ijk}(s)}{ds} = & \left[\left[T_{ij}, V_{ij}(s) \right], V_{ki}(s) + V_{jk}(s) + V_{ijk}(s) \right] + \left[\left[T_{jk}, V_{jk}(s) \right], V_{ki}(s) + V_{ij}(s) + V_{ijk}(s) \right] \\ & + \left[\left[T_{ki}, V_{ki}(s) \right], V_{ij}(s) + V_{jk}(s) + V_{ijk}(s) \right] \quad \text{more involved but implemented} \\ & + \left[\left[T_{ij} + T_k, V_{ijk}(s) \right], T_{ij} + T_k + V_{ij}(s) + V_{ij}(s) + V_{ki}(s) + V_{ijk}(s) \right] \end{aligned}$$

The implementation uses again our discretization of the coordinate transformation/permutation operators!
Coordinates need to be adapted to fit interacting pairs or chosen coordinate for V_{ijk}

SRG INTERACTIONS (YN)

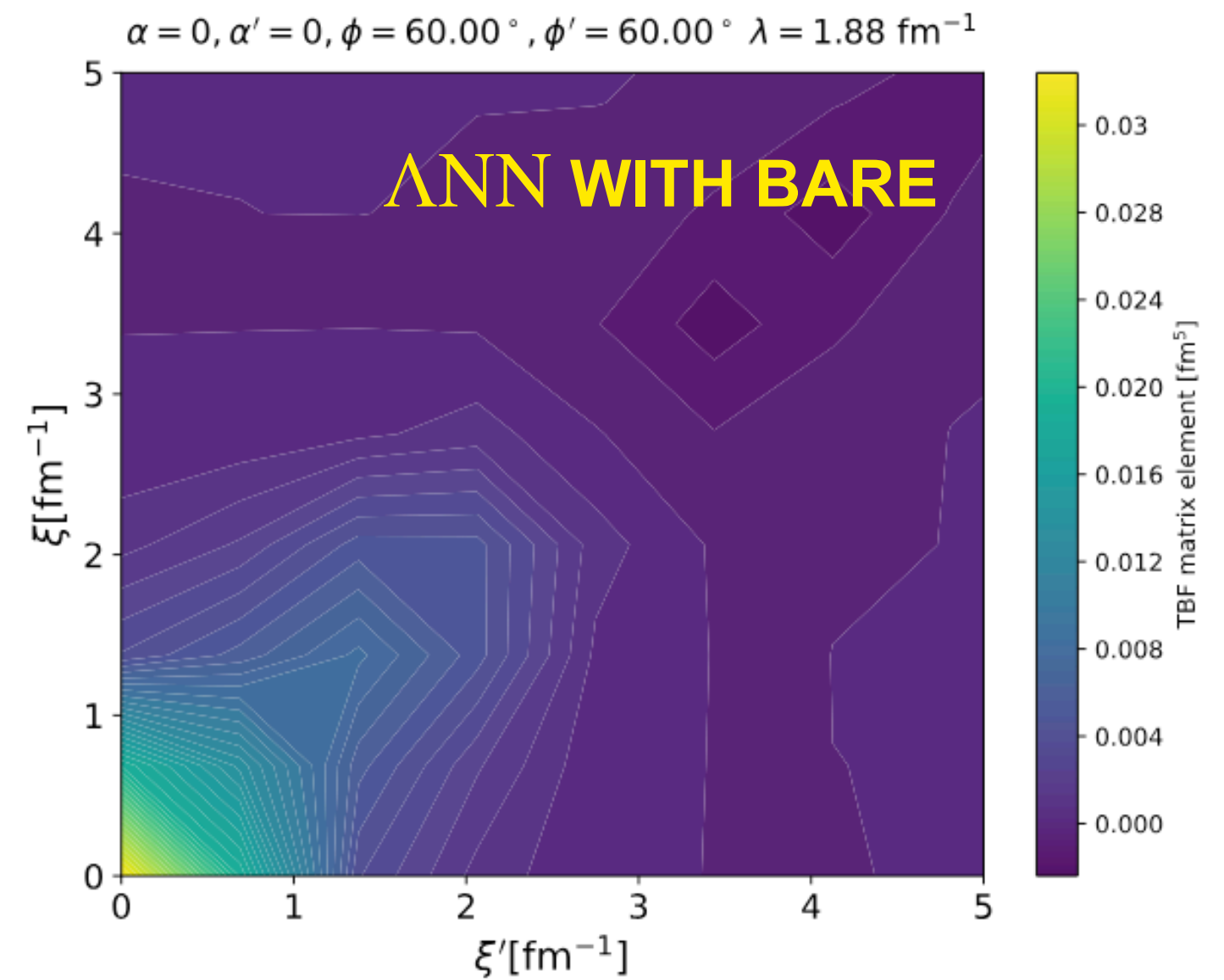
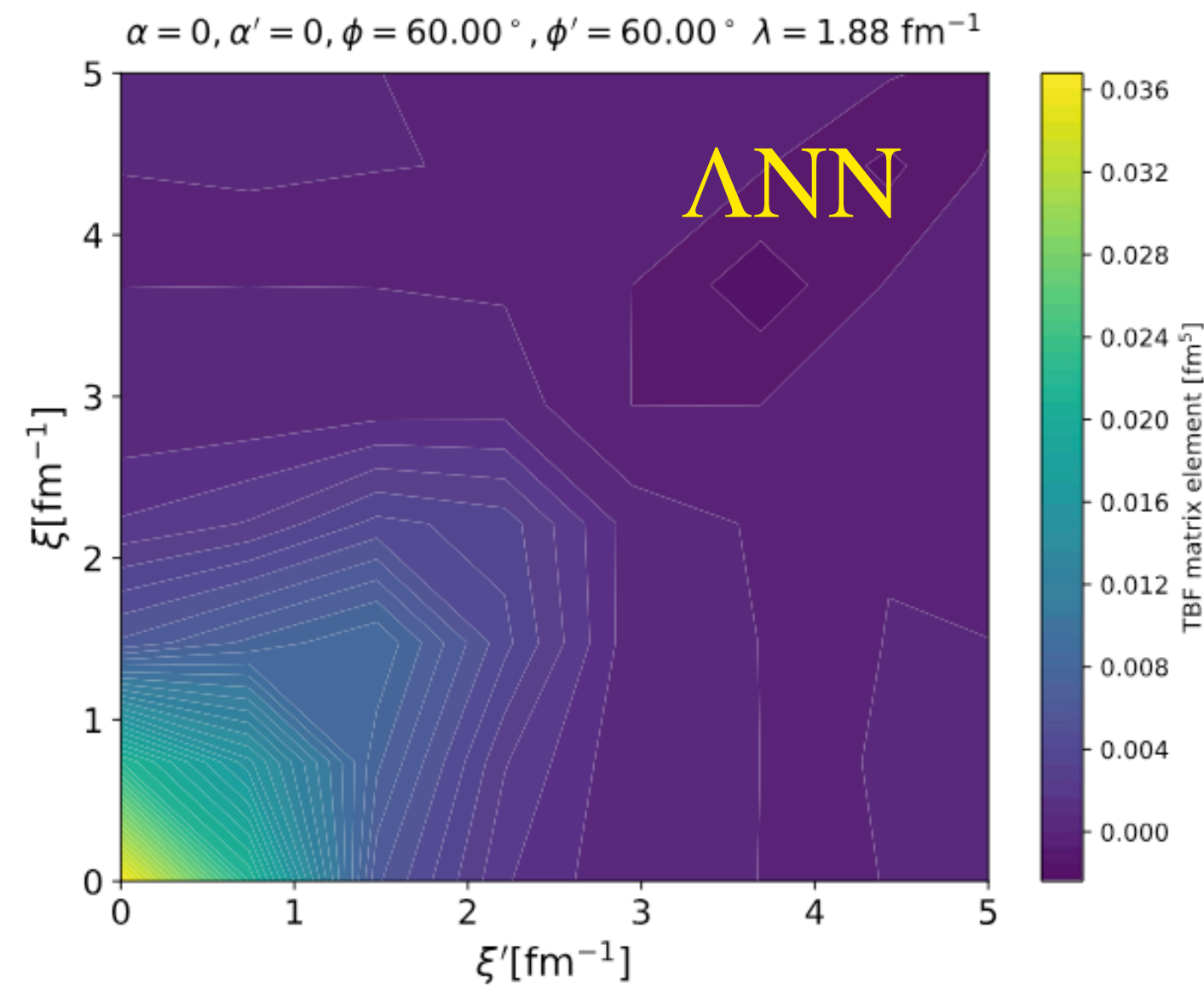
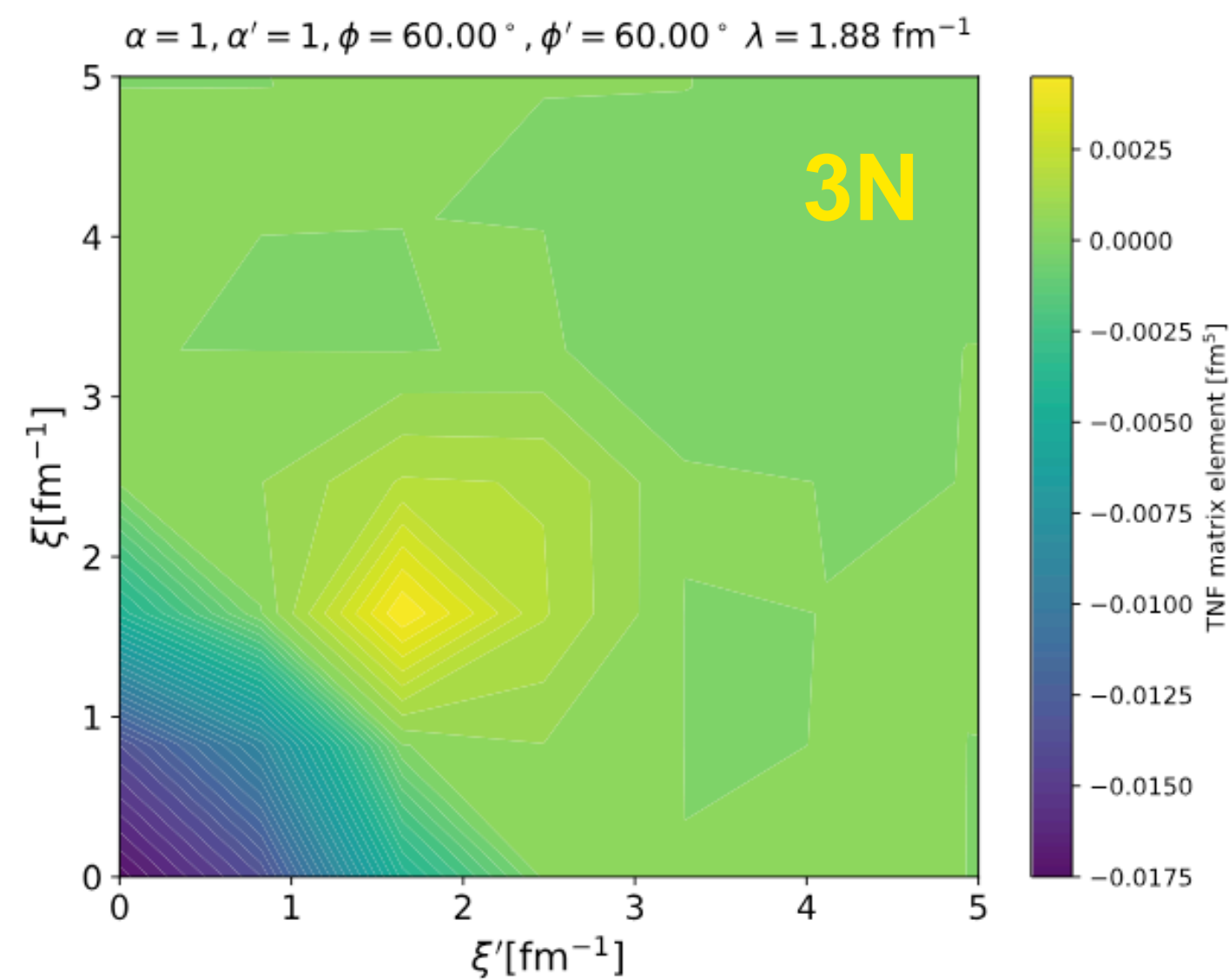
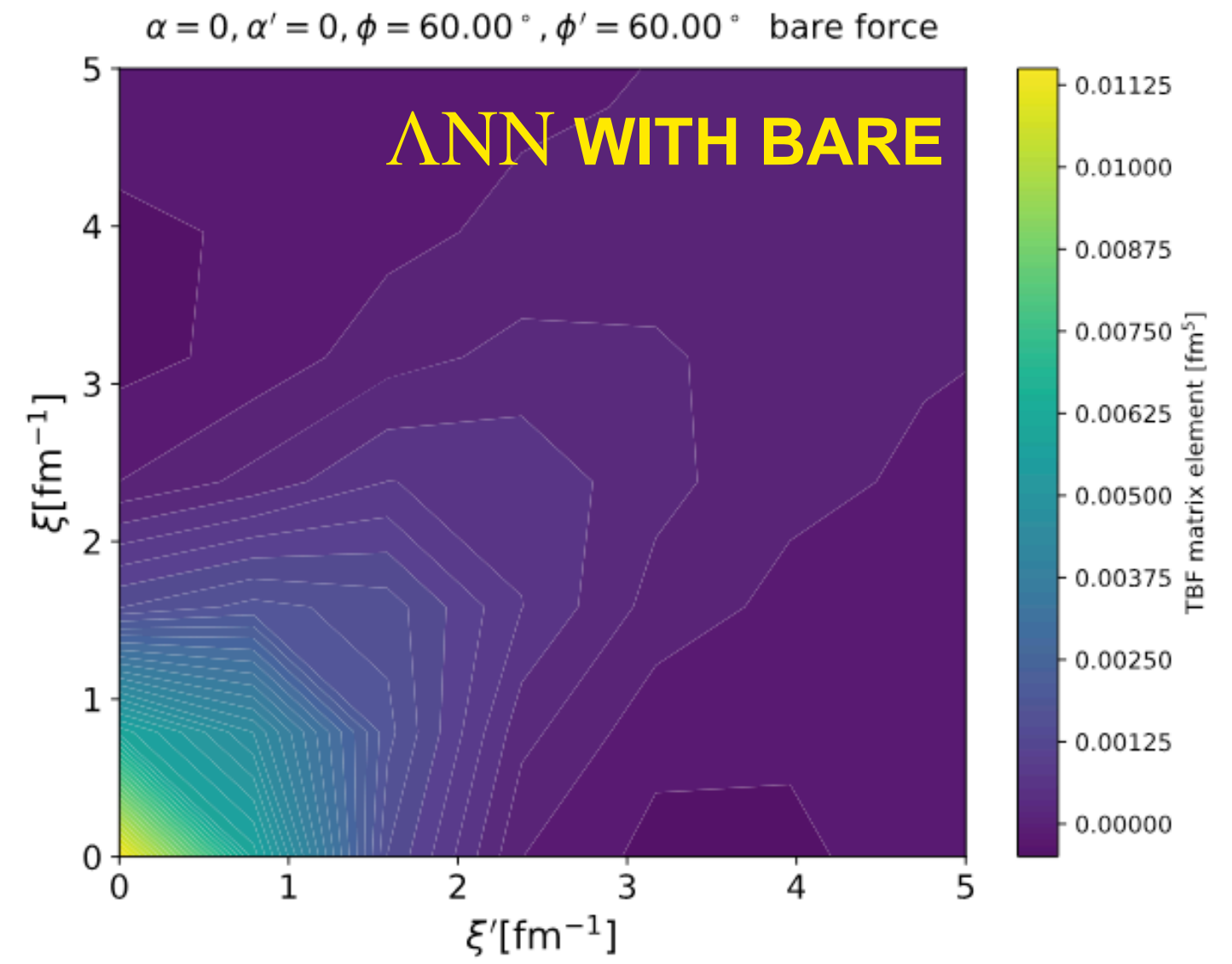
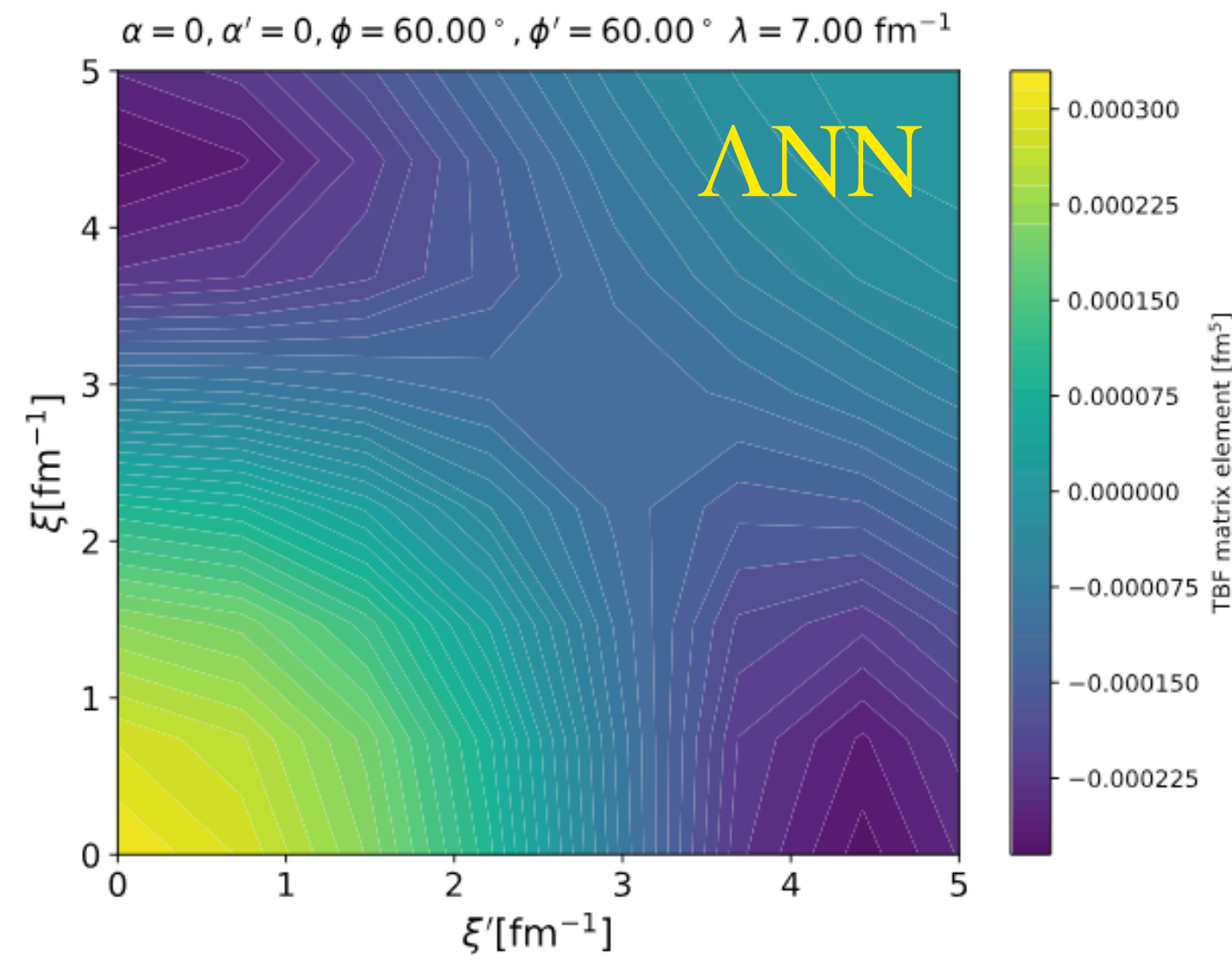
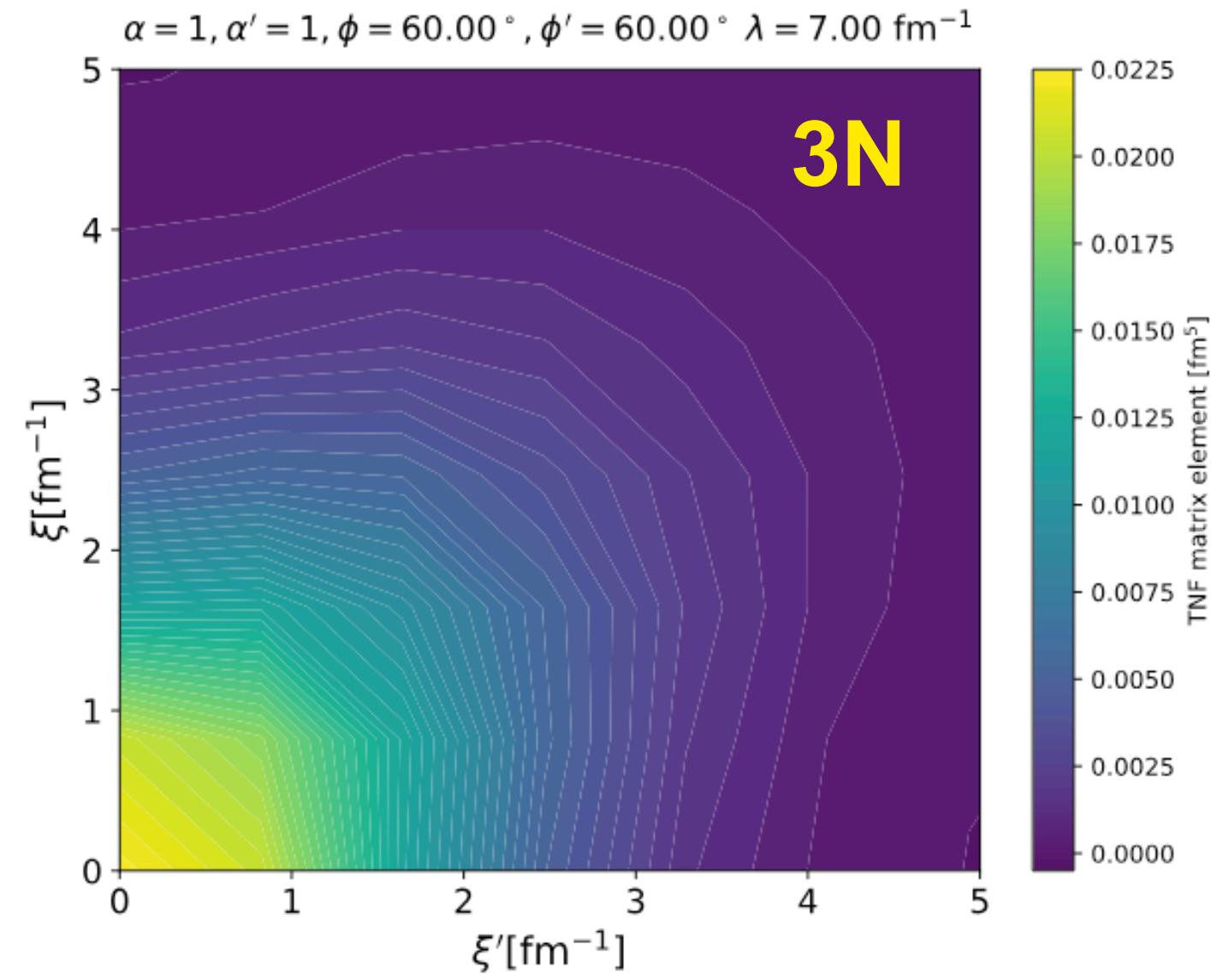
Λp - Λp matrix element for the 3S_1 depending on incoming and outgoing momenta



Haidenbauer et al. PPNP 149,104242 (2026)

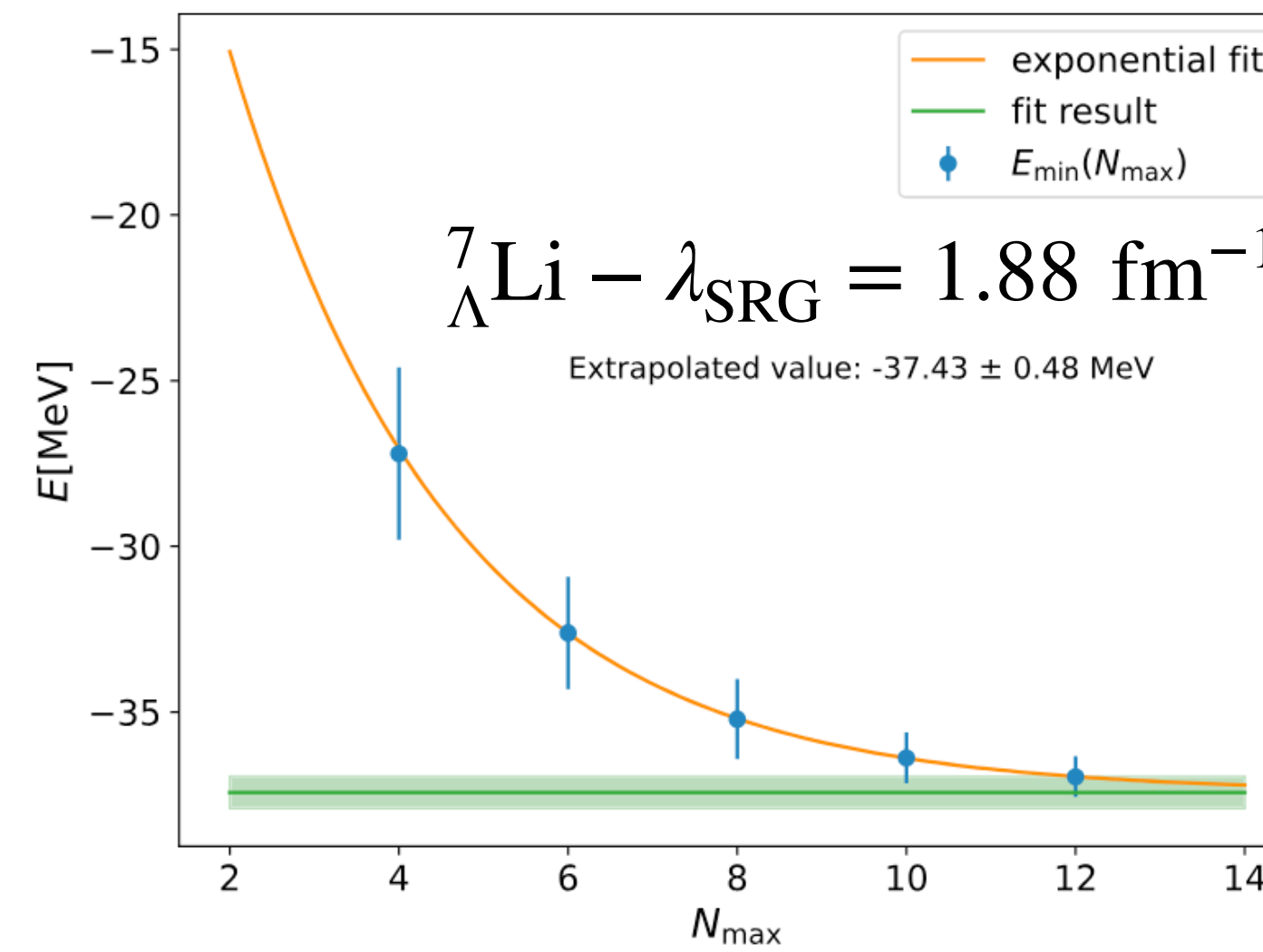
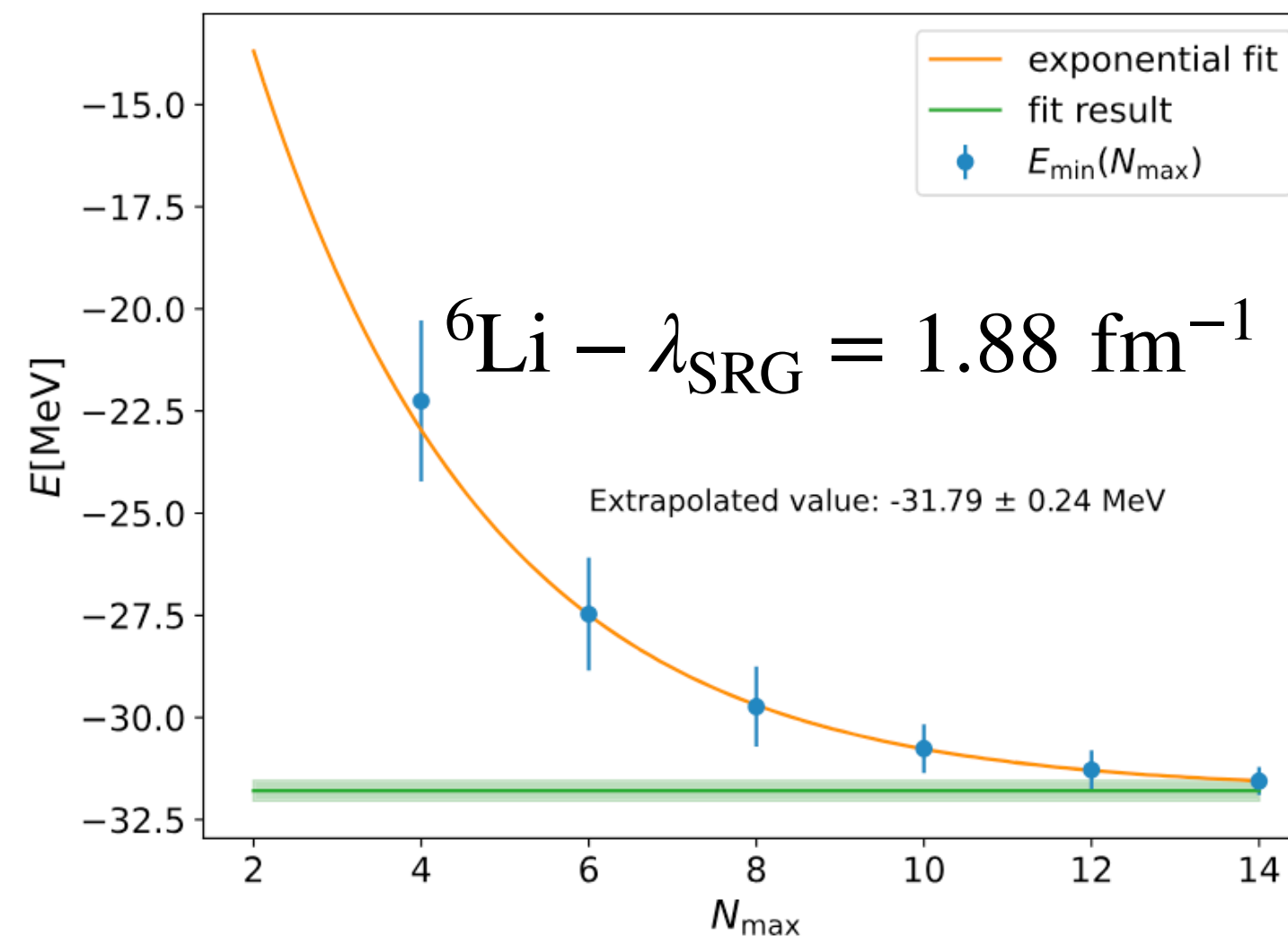
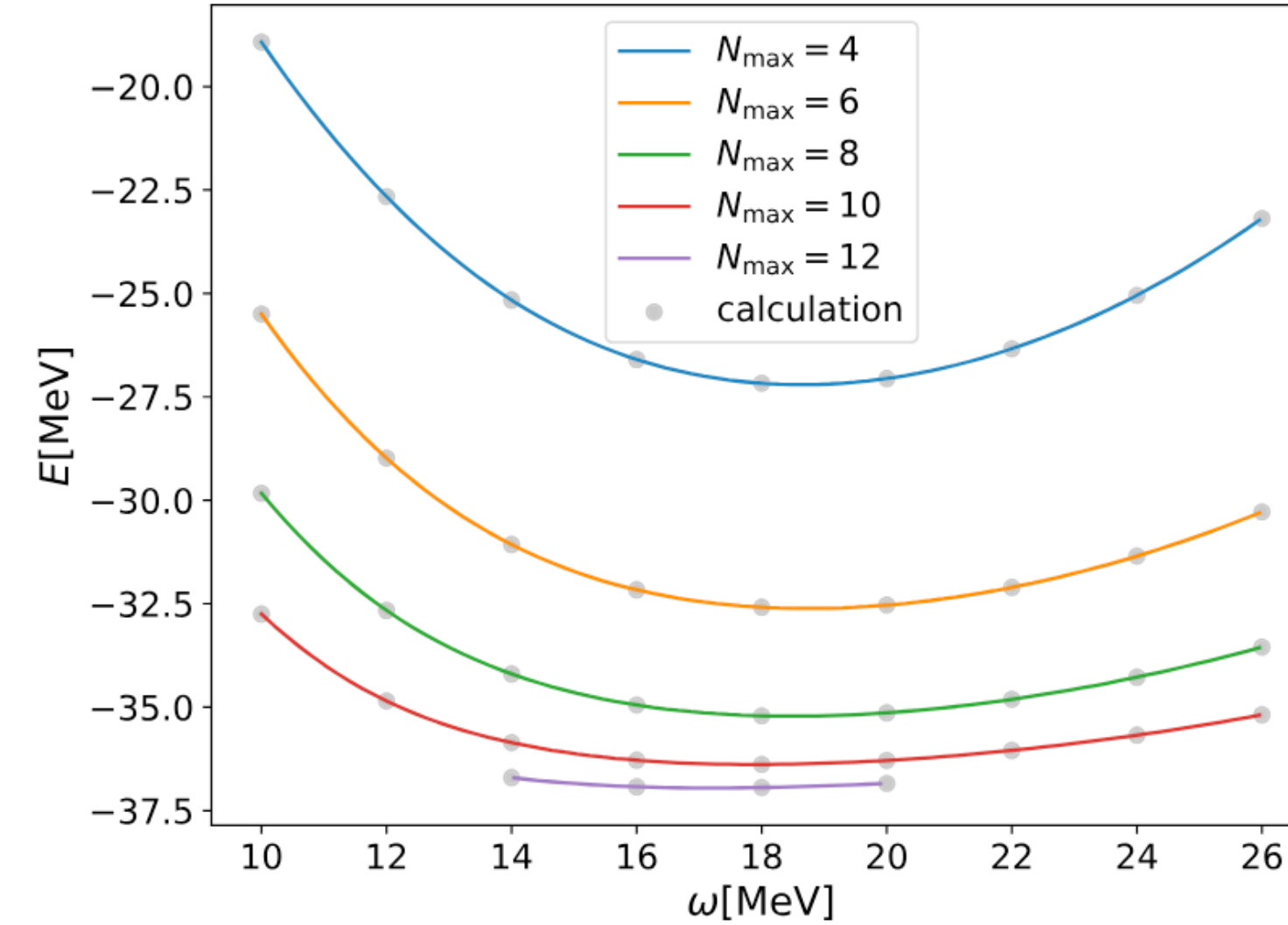
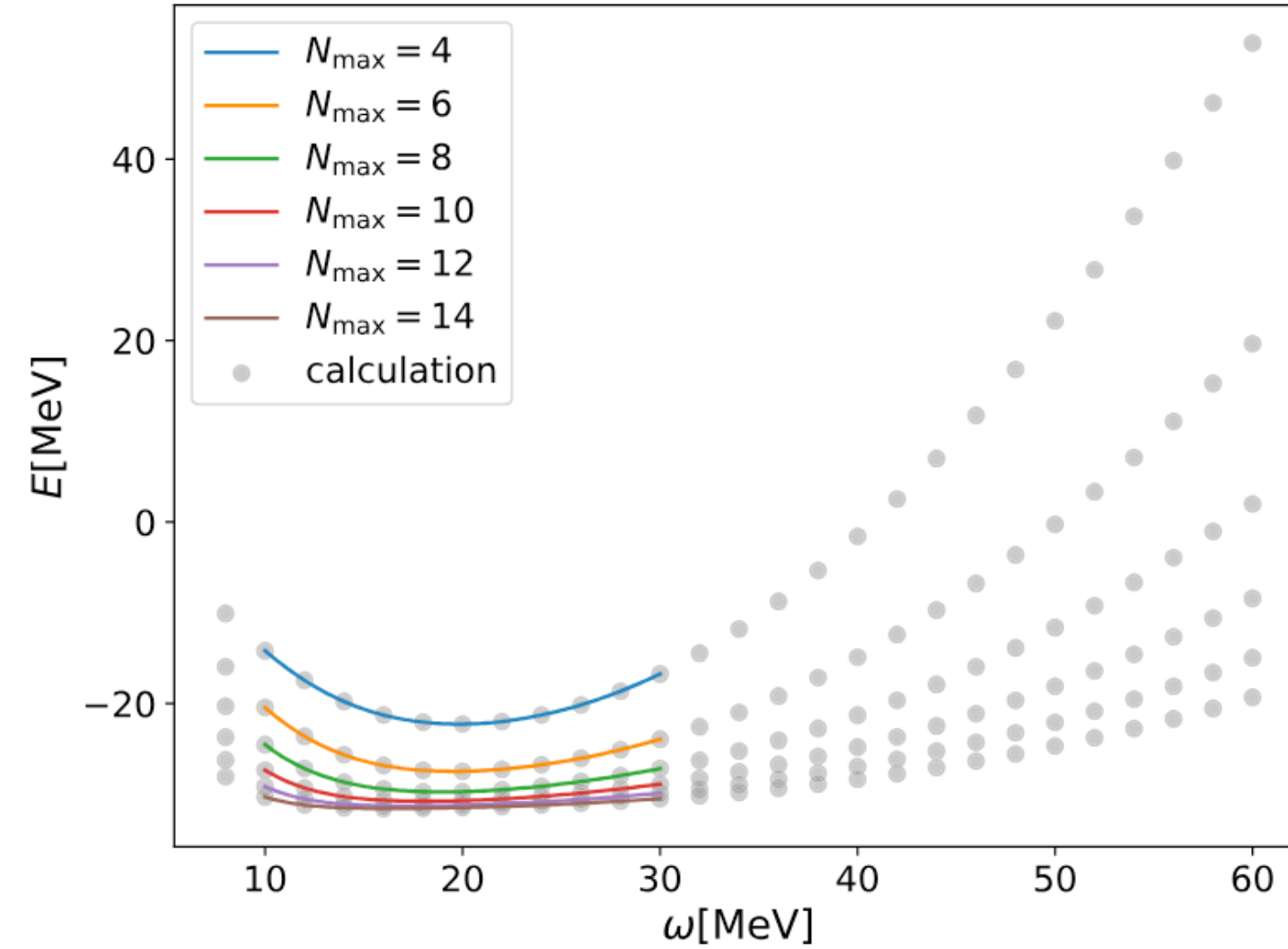
SRG 3BF

$$\xi, \xi' = p^2 + \alpha q^2 \quad J^\pi = \frac{1}{2}^+$$

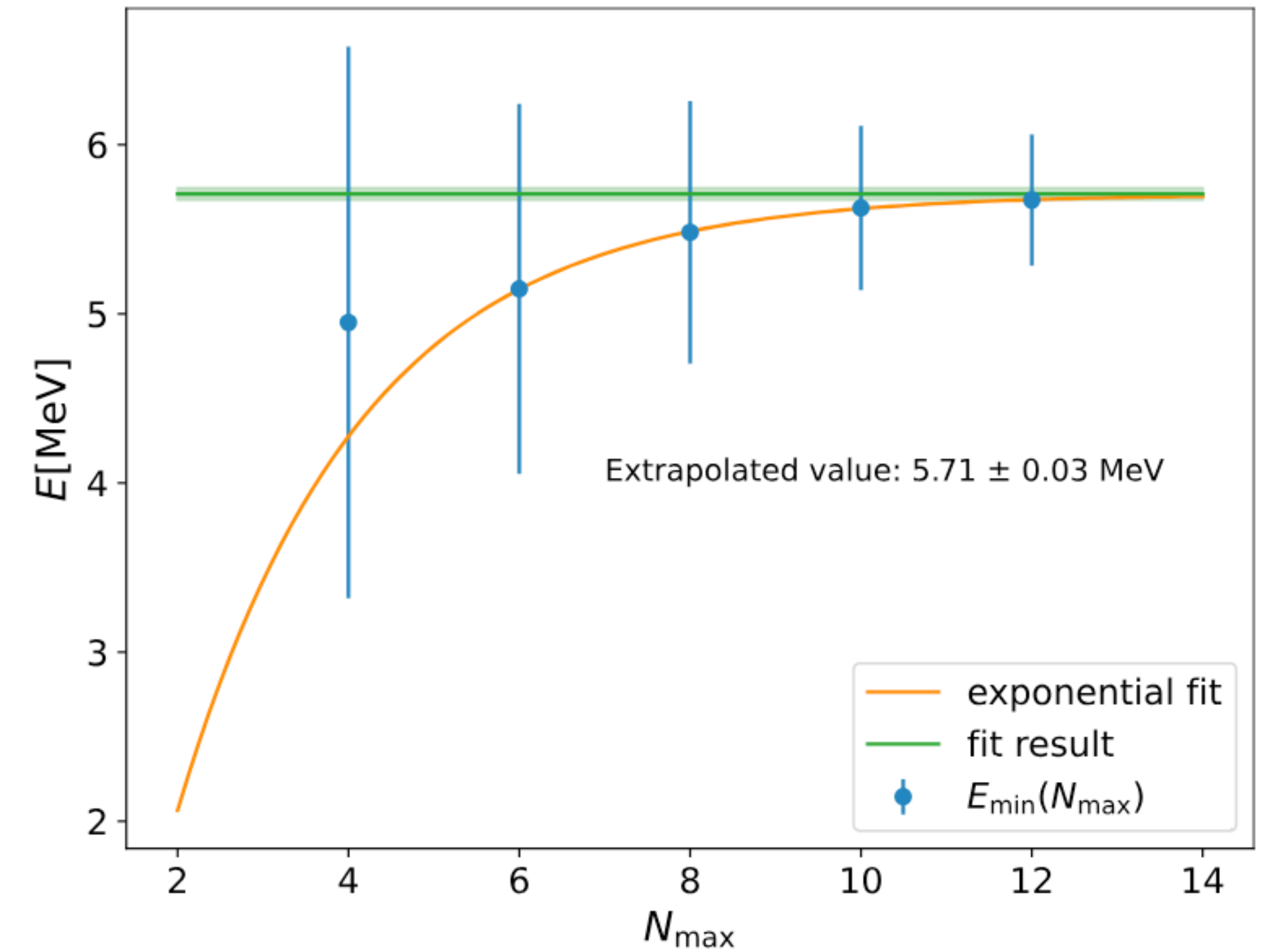
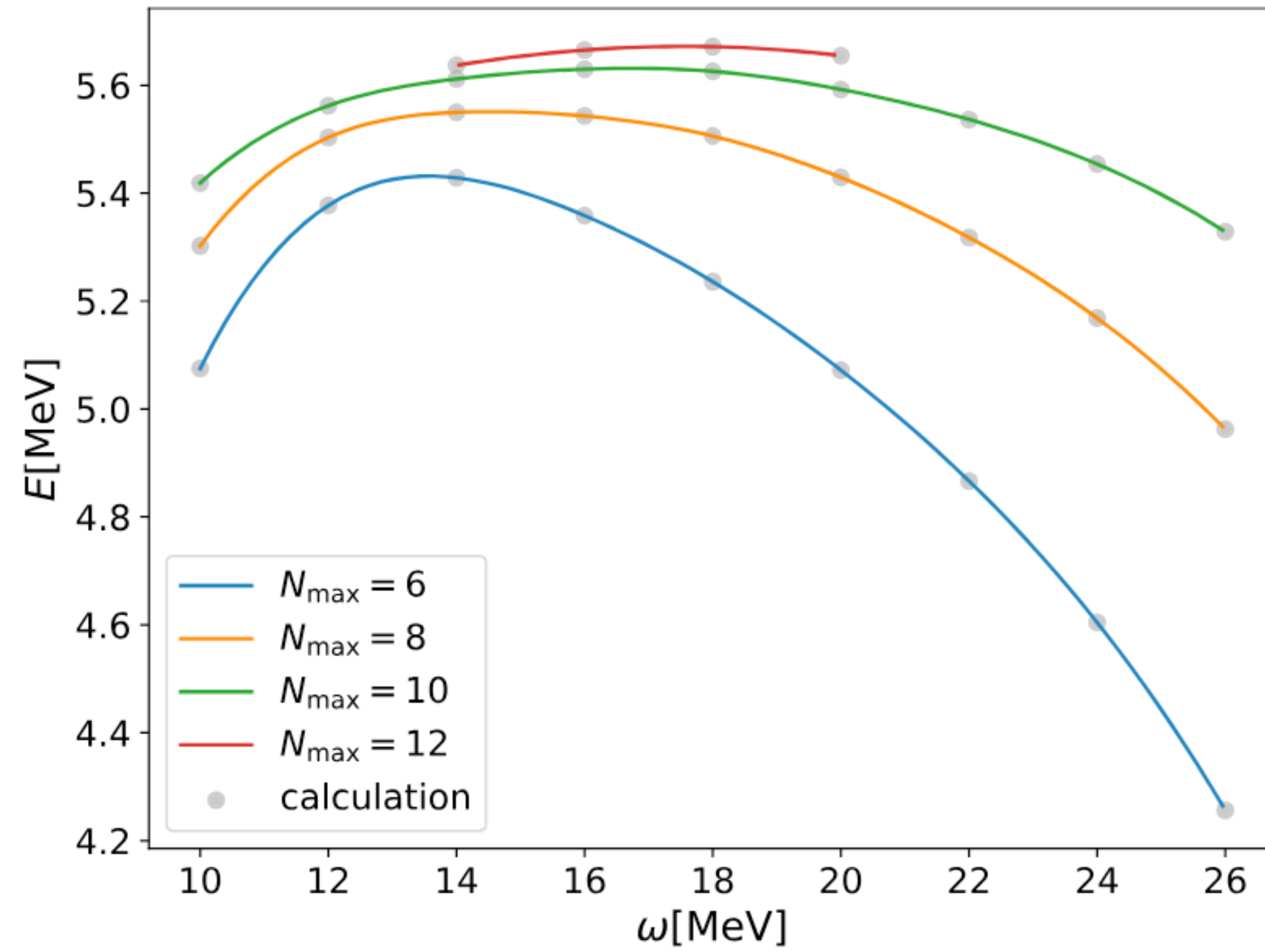


Haiderbauer et al. PPNP 149,104242 (2026)

CONVERGENCE OF BINDING ENERGY

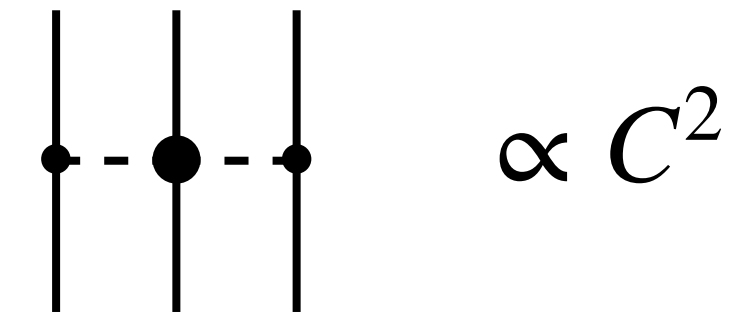
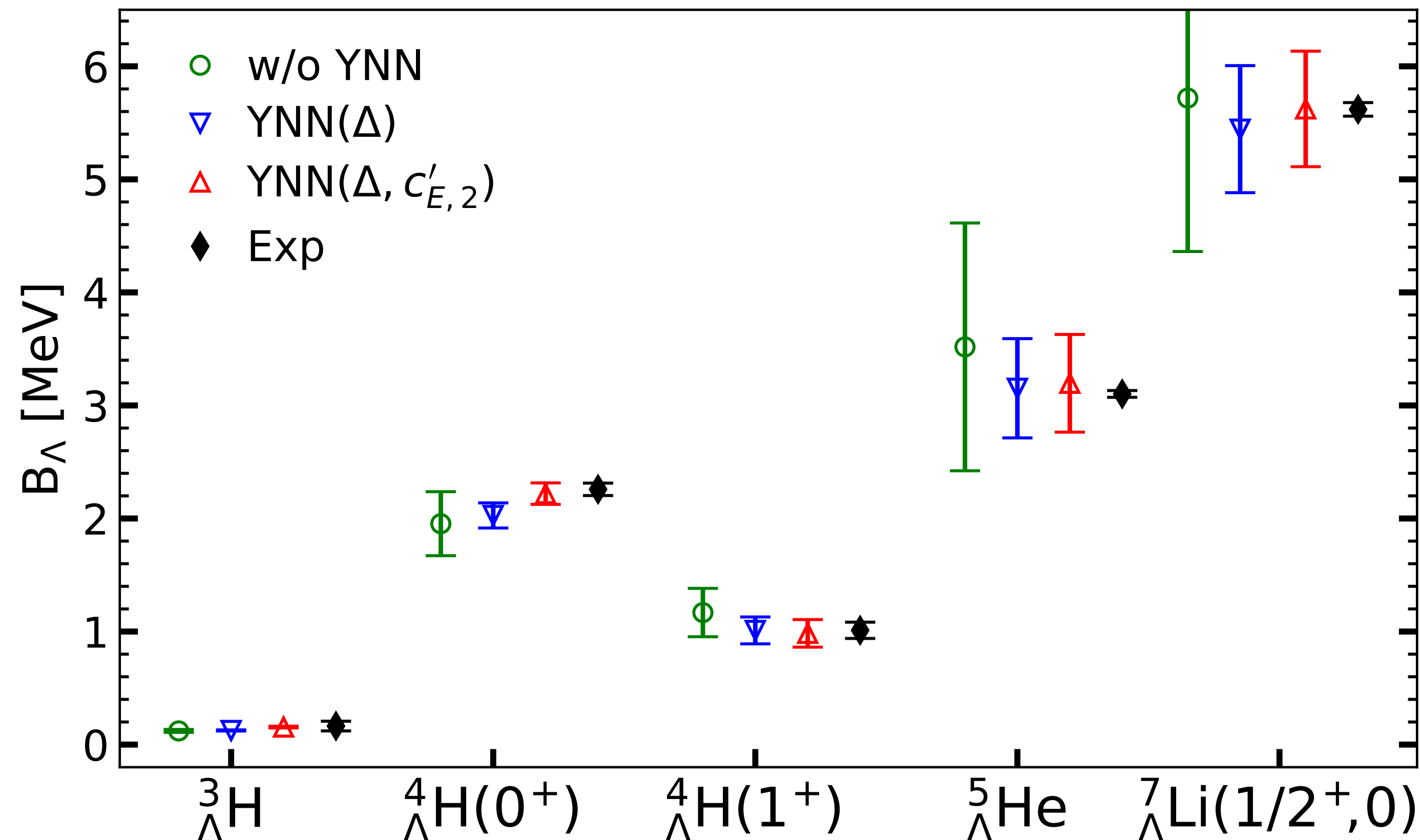


CONVERGENCE OF SEPARATION ENERGY

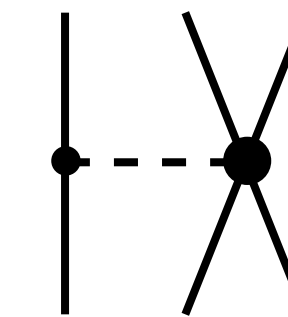


3BF CONTRIBUTION TO SEPARATION ENERGIES

- Fit to 0^+ and 1^+ state of ${}^4_{\Lambda}\text{He}$ and/or ${}^5_{\Lambda}\text{He}$
- spin-dependence in $A = 4$ not well explained by decuplet saturation
- C'_2 term improves 0^+ of ${}^4_{\Lambda}\text{He}$ and $1/2^+$ of ${}^7_{\Lambda}\text{Li}$
-

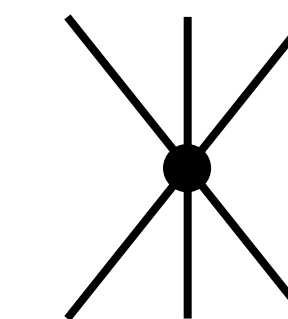


$$\propto C^2$$



$$\propto CG_1, CG_2$$

$$\propto C(G_1 + 3G_2)$$



$$\propto (G_1)^2, (G_2)^2, G_1G_2$$

$$\propto (G_1 + 3G_2)^2$$

for ΛNN : 1 LEC

add spin dependent YNN force

$$V_{\Lambda\text{NN}} = C'_2 \vec{\sigma}_1 \cdot (\vec{\sigma}_2 + \vec{\sigma}_3) (1 - \vec{\tau}_2 \cdot \vec{\tau}_3)$$