第135期强子物理在线论坛

Efimov physics in three hadron systems and the New observable for hadronic molecules

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参考文献： JHEP(2025), (2026); PLB (2026); J.Phys.G (2025)

2026年7月

❖ Outlines

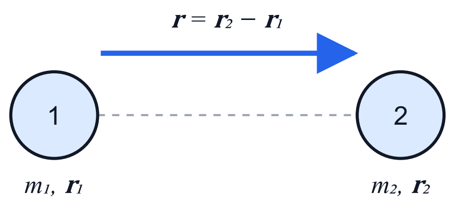
1. What is the Efimov effect?
2. Efimov study in three B mesons
3. Efimov study in three D mesons
4. Unparticle Phenomenon: the short-range production of three B mesons
5. Summary and outlook

❖ What is the Efimov effect?

- Solving bound states in three body system with only two-body potentials

Two-body system

$$\vec{r} \rightarrow (r, \theta, \phi)$$



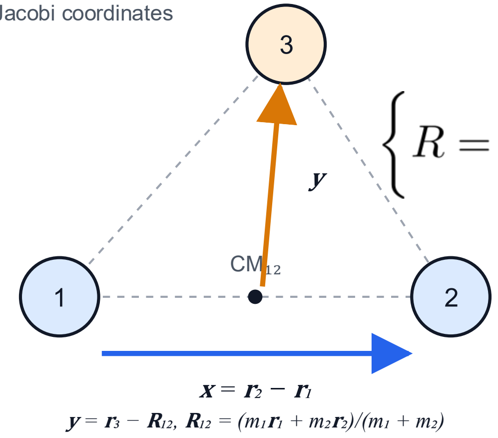
$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}-\frac{\hat{L}^2}{\hbar^2 r^2}\right)+V(r,\Omega)\right]\psi(r,\Omega)=E\psi(r,\Omega).$$

$$\psi(r,\Omega)=\sum_{lm}\frac{u_{lm}(r)}{r}Y_{lm}(\Omega).$$

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}+\frac{\hbar^2 l(l+1)}{2\mu r^2}-E\right]u_{lm}(r)+\sum_{l'm'}V_{lm,l'm'}(r)u_{l'm'}(r)=0,$$

Three-body system

Jacobi coordinates



$$\left\{R=\sqrt{|\vec{x}|^2+|\vec{y}|^2}, \Omega=(\alpha,\hat{x},\hat{y})\right\}$$

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial R^2}+\frac{5}{R}\frac{\partial}{\partial R}-\frac{\hat{\Lambda}^2}{R^2}\right)+V(R,\Omega)\right]\Psi(R,\Omega)=E\Psi(R,\Omega),$$

$$\Psi(R,\Omega)=R^{-5/2}\sum_n f_n(R)\Phi_n(R;\Omega).$$

$$\hat{H}_\Omega(R)\Phi_n(R;\Omega)=U_n(R)\Phi_n(R;\Omega).$$

$$\left[-\frac{\hbar^2}{2\mu_R}\frac{d^2}{dR^2}+U_n(R)-E\right]f_n(R)-\frac{\hbar^2}{2\mu_R}\sum_m\left[2P_{nm}(R)\frac{d}{dR}+Q_{nm}(R)\right]f_m(R)=0.$$

❖ What is the Efimov effect?

- when $r_0 \ll R \ll |a|$

$$U_n(R) \simeq \frac{\hbar^2}{2\mu R^2}(s_n^2 - \frac{1}{4}) \to \left[-\frac{d^2}{dR^2} + \frac{s_n^2 - 1/4}{R^2}\right] f_n(R) = \frac{2\mu E}{\hbar^2} f_n(R)$$

- s_n is real: $f_n(R) = AR^{1/2+s_n} + BR^{1/2-s_n}$
- $s_n = \pm i s_0$ is pure imaginary :

$$f_n(R) = AR^{1/2+is_0} + BR^{1/2-is_0} \propto \sqrt{R}\sin\left[s_0 \ln\left(\frac{R}{R_0}\right)\right]$$

- Discrete scaling symmetry

$$R \to \lambda R, \quad f(\lambda R) \propto \sin\left[s_0 \ln\left(\frac{R}{R_0}\right) + s_0 \ln\lambda\right]$$

- when taking $s_0 \ln\lambda_0 = \pi \to \lambda_0 = e^{\pi/s_0} \approx 22.7$

$$f(\lambda_0 R) = \pm f(R), \quad E_{n+1} = \lambda_0^2 E_n$$

$$\boxed{N_{\mathrm{node}} \sim \frac{s_0}{\pi} \ln\frac{|a|}{r_0}.}$$

❖ What is the Efimov effect?

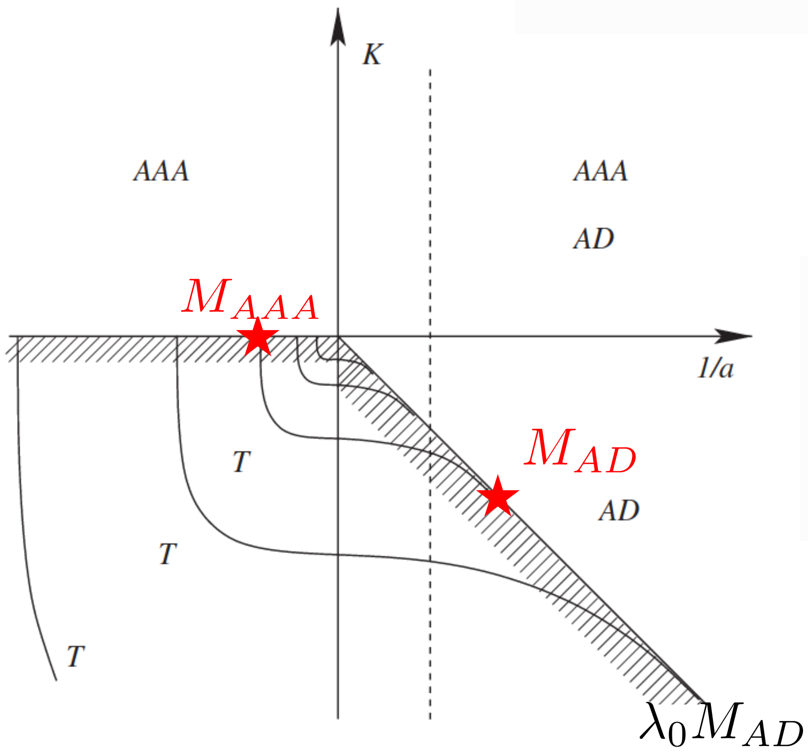
- Efimov plot

E. Braaten, HWH, Phys. Rept. 428 259 (2006)

$$K_n(a) = -\sqrt{\frac{mB_3^{(n)}(a)}{\hbar^2}}.$$

$$K = \mathrm{sgn}(E)\sqrt{\frac{m|E|}{\hbar^2}}.$$

a>0 for bound state



$$E_{AD}^{\mathrm{th}} = -B_2 = -\frac{\hbar^2}{ma^2}.$$

$$K_{AD}^{\mathrm{th}} = -\sqrt{\frac{mB_2}{\hbar^2}} = -\frac{1}{a}, \qquad a > 0.$$

例如：氚核(2n+1p)，锂11（锂9+2n）

✓ 冷原子系统实现 Nature 440, 315 (2016); Nat. Phys. 5, 227 (2009)

(通过 Feshbach 共振调节a时)

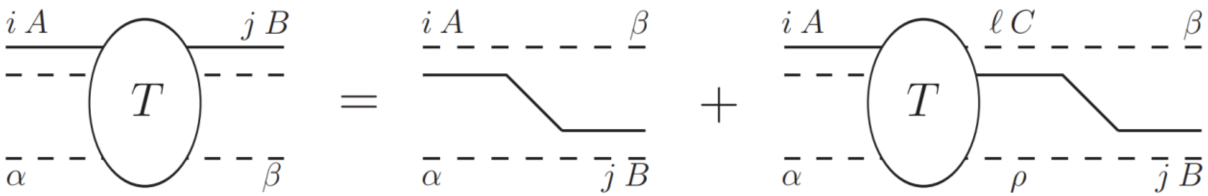
❖ Efimov effect in momentum space

- Scale separation

$$\gamma = \frac{1}{|a|} \ll p, q \ll \Lambda \simeq \frac{1}{r_0} \simeq M_\pi$$

Pionless EFT

- Dimer-particle scattering



$$A(p) = \frac{4}{\sqrt{3}\pi} \int_0^\infty dx \, \ln\left(\frac{1+x+x^2}{1-x+x^2}\right) A(px),$$ with $q = px$

considering $A(p) = p^{s-1}$ $\longrightarrow$ $$1 = \frac{4}{\sqrt{3}\pi} \int_0^\infty dx \, x^{s-1} \ln\left(\frac{1+x+x^2}{1-x+x^2}\right).$$ $\longrightarrow$ $$\boxed{1 = \frac{8}{\sqrt{3}} \frac{\sin(\pi s/6)}{s \cos(\pi s/2)}.}$$ $\longrightarrow$ $s = \pm i1.00624$

- RGI requires oscillating three-body force at LO

$$A(p) \propto \frac{1}{p} \sin\left[s_0 \ln\left(\frac{p}{\kappa_0}\right)\right] \longrightarrow g_3(\Lambda) = -\frac{9g_2^2}{\Lambda^2} H(\Lambda),$$

$$H(\Lambda) = \frac{\cos[s_0 \ln(\Lambda/\Lambda_*) + \arctan s_0]}{\cos[s_0 \ln(\Lambda/\Lambda_*) - \arctan s_0]}.$$

$$H(\lambda_0 \Lambda) = H(\Lambda), \qquad \lambda_0 = e^{\pi/s_0} \simeq 22.7.$$

❖ Hadronic Molecule: r/a → 0

- Successful in deuteron description: the Weinberg compositeness criterion

S. Weinberg, Phys. Rev. 137 B672 (1965)

$$\boxed{|\Psi\rangle = \begin{pmatrix} \lambda|\psi_0\rangle \\ \chi(\boldsymbol{k})|h_1h_2\rangle \end{pmatrix},} \quad T_{\rm NR}(E) = \frac{g_0^2}{E + E_B + g_0^2\mu/(2\pi)(ik+\gamma)}, \quad a = -2\frac{1-\lambda^2}{2-\lambda^2}\left(\frac{1}{\gamma}\right) + \mathcal{O}\left(\frac{1}{\beta}\right),$$

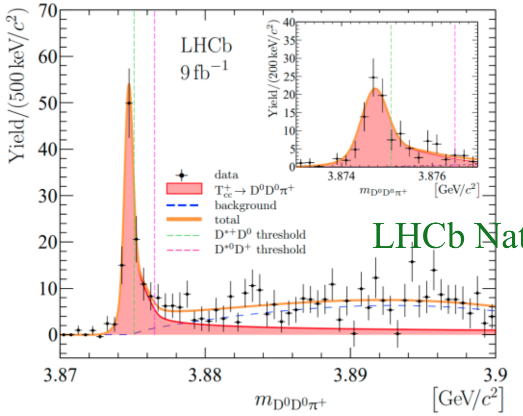
np scattering

a<0 for bound state

$$T_{\rm NR}(E) = -\frac{2\pi}{\mu}\frac{1}{1/a + (r/2)k^2 - ik}, \quad r = -\frac{\lambda^2}{1-\lambda^2}\left(\frac{1}{\gamma}\right) + \mathcal{O}\left(\frac{1}{\beta}\right).$$

$$\boxed{a = -4.3(14)\,\text{fm}[-5.419(7)] \qquad r = 0.0(14)\,\text{fm}[1.764(8)]}$$

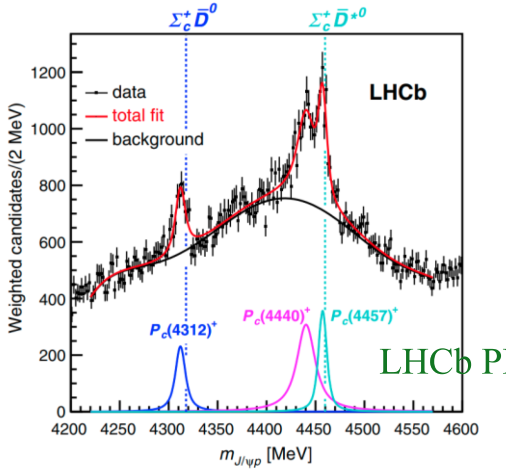
- Emergence of near-threshold structures



T_{cc} @2021

$D^{*+}D^0$-$D^{*0}D^+$

LHCb Nature Commun. 13 3351 (2022)



$P_{c\bar{c}}$ @2015

$\bar{D}^{(*)}\Sigma_c^{(*)}$

LHCb PRL 122 222001 (2019)

❖ Three-B-Mesons in Pionless EFT

- Zb(10610) and Zb ′ (10650) are weak bound states of two B mesons

$(\bar{B}B^* + B\bar{B}^*)/\sqrt{2}$ $\quad B^*\bar{B}^*$

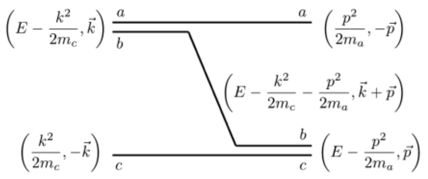
Exp. $\quad E_B^Z = -3(2)\,\mathrm{MeV}, E_B^{Z'} = -2.7(16)\,\mathrm{MeV}$ PDG, https://pdg.lbl.gov

HQEFT $\quad E_B^Z = 5.0(2.5)\,\mathrm{MeV}, E_B^{Z'} = 1.0(0.5)\,\mathrm{MeV}$ M. Cleven et al., EPJA 47 120 (2011)

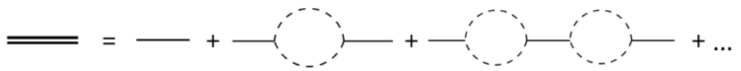
$$\boxed{a_Z = 1.2(3)\,\mathrm{fm}, a_{Z'} = 2.7(0.7)\,\mathrm{fm}, r \approx \frac{1}{M_\pi} = 1.4\,\mathrm{fm}}$$

- The leading order (LO) Lagrangian for three-B-meson dynamics

$$\begin{aligned}
\mathcal{L} = {} & B_\alpha^\dagger \left(i\partial_t + \frac{\nabla^2}{2M_B}\right) B_\alpha + \bar{B}_\alpha^\dagger \left(i\partial_t + \frac{\nabla^2}{2M_B}\right) \bar{B}_\alpha \\
& + B_{i\alpha}^{*\dagger} \left(i\partial_t + \frac{\nabla^2}{2M_{B^*}}\right) B_{i\alpha}^* + \bar{B}_{i\alpha}^{*\dagger} \left(i\partial_t + \frac{\nabla^2}{2M_{B^*}}\right) \bar{B}_{i\alpha}^* \\
& + Z_{iA}^\dagger \Delta Z_{iA} + Z_{iA}^{\prime\dagger} \Delta' Z_{iA}' \\
& - g\left[Z_{iA}^\dagger \left(\bar{B}_{j\alpha}^*\, \delta_{ij} (\tau_2\tau_A)_{\alpha\beta}\, B_\beta + \bar{B}_\alpha\, \delta_{ij} (\tau_2\tau_A)_{\alpha\beta}\, B_{j\beta}^*\right) + h.c.\right] \\
& - g'\left[Z_{iA}^{\prime\dagger}\, \bar{B}_{j\alpha}^*\, (U_i)_{jk} (\tau_2\tau_A)_{\alpha\beta}\, B_{k\beta}^* + h.c.\right] + \dots\,,
\end{aligned}$$



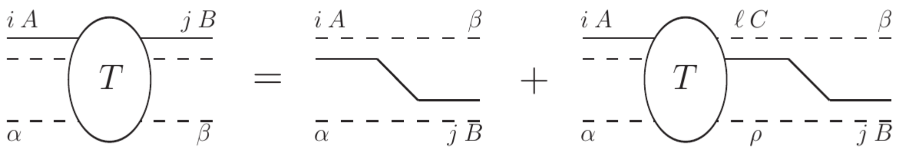
$\mathcal{M}_{ZB\to ZB}, \mathcal{M}_{ZB^*\to ZB^*},$
$\mathcal{M}_{ZB^*\to Z'B}, \mathcal{M}_{Z'B^*\to Z'B^*}$



$$iS_{ij}^D(p_0,\vec{p}) = -i\frac{2\pi}{g^2\mu_D S_1^D c_D}\frac{1}{-\gamma_D + \sqrt{-2\mu_D\left(p_0 - \frac{\vec{p}^2}{2(m_i+m_j)}\right) - i\varepsilon}}\,,$$

❖ Dimer-particle scattering

◻ The single-channel integral equations



$$T^{I,S}_{(0)}(E,k,p) = C_6^I C_6^S\, \mathcal{M}_0(k,p) + \int_0^{\Lambda} dq\, C_6^I C_6^S \mathcal{M}_1(p,q) T^{I,S}_{(0)}(E,k,q)$$

S-wave ZB : $(I,S) = (3/2,1), (1/2,1)$

S-wave ZB^* : $(I,S) = (3/2,0), (3/2,2)$

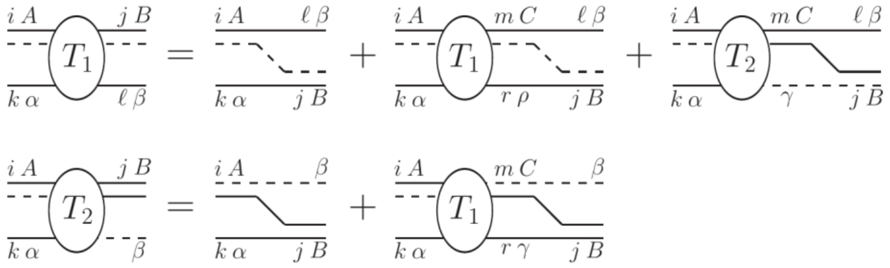
S-wave $Z'B^*$:

$(I,S) = (3/2,1), (3/2,1), (1/2,0)$

$(1/2,1), (1/2,2)$

$(3/2,0)$

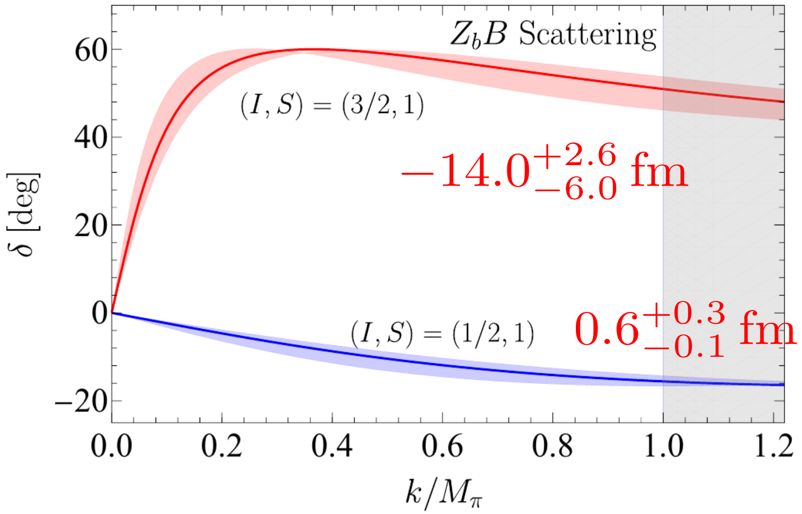
◻ The coupled-channel integral equations

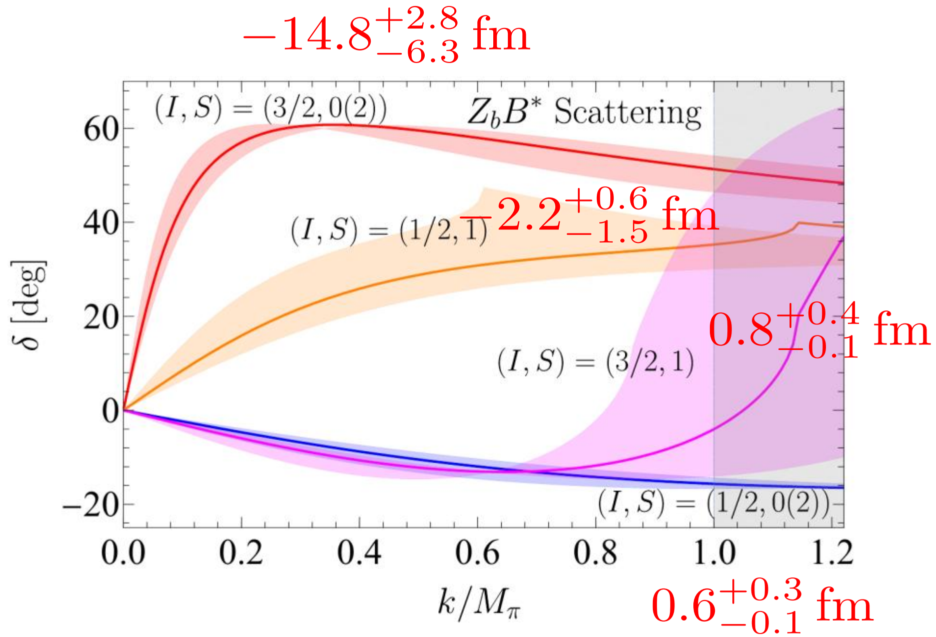


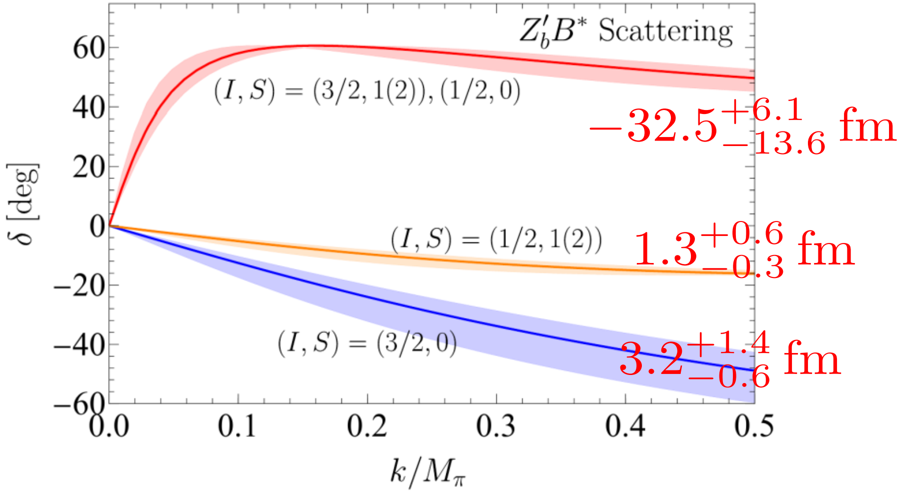
S-wave $ZB^* - Z'B$:

$(I,S) = (1/2,1), (3/2,1)$

❖ Dimer-particle scattering







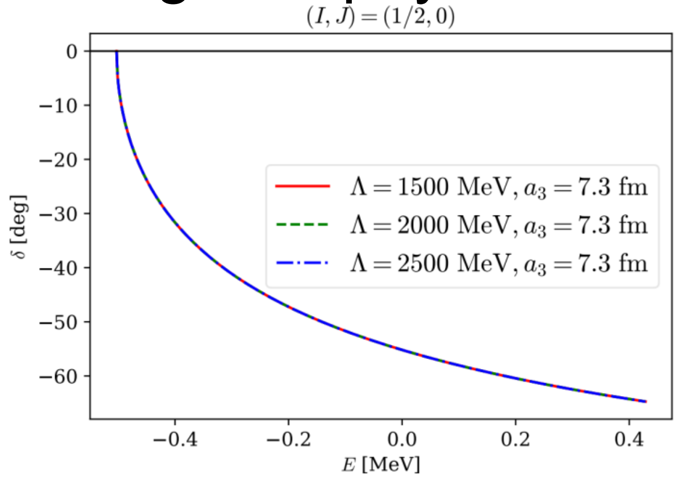
Hints that no Efimove state emerges in these ZB channels:

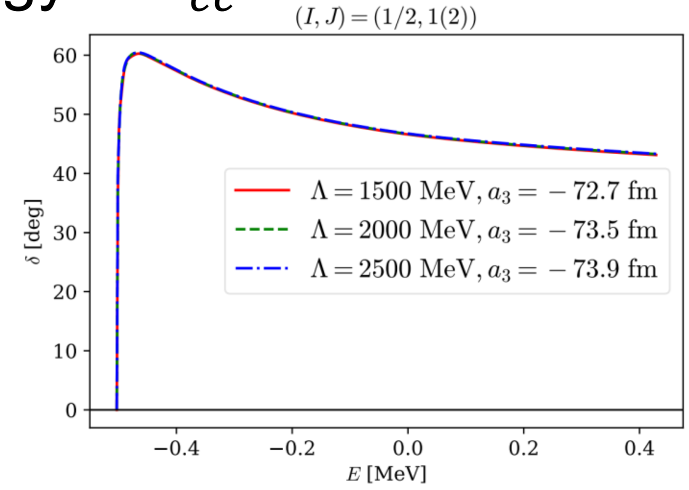
- Convergence achieves as cutoff grows to infinity
- Large positive phase shift but associated with negative scattering length

No three body force in LO pioneless EFT

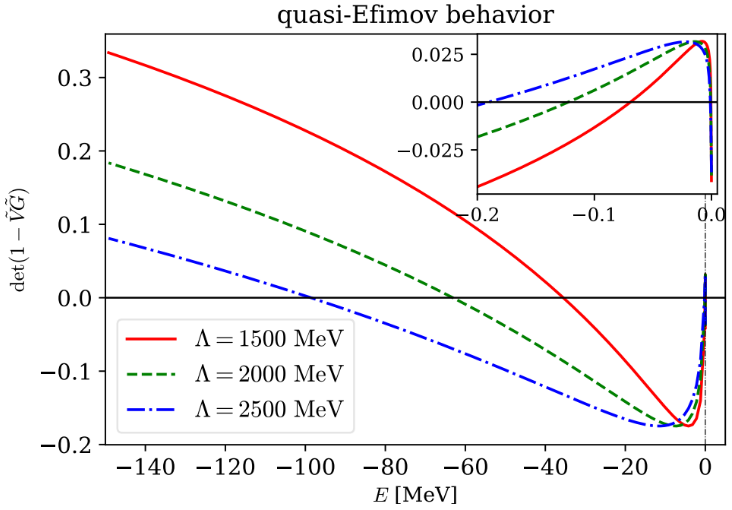
❖ Three-D-Mesons in Pionless EFT

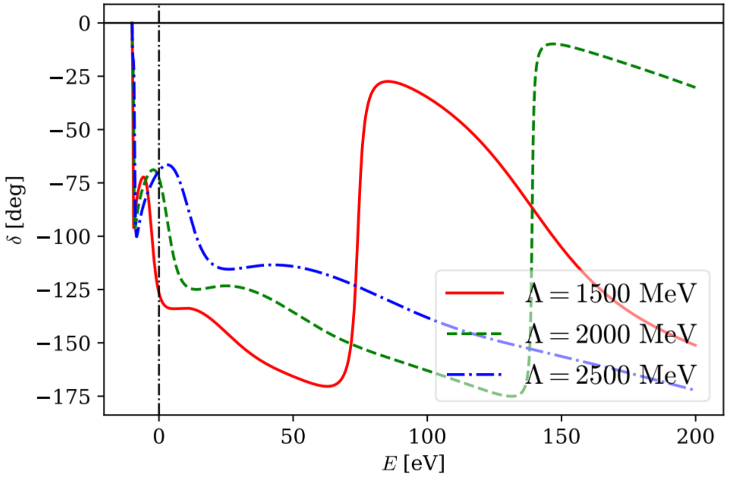
- $T_{cc}^{*}D^{*}$ scattering with physical binding energy of T_{cc}^{*}





- $T_{cc}^{*}D^{*}$ scattering with T_{cc}^{*} binding energy artificially set for illustration





❖ Three-D-Mesons in Pionless EFT

□ Coupled channel effects

$$1 = \frac{4\lambda_i}{\sqrt{3}} \frac{1}{s_i} \frac{\sin \frac{\pi}{6} s_i}{\cos \frac{\pi}{2} s_i}$$

has imaginary solution when $\lambda_i > \lambda_c = \frac{3\sqrt{3}}{2\pi} \approx 0.827$

(I, J)	$\mathcal{A}$	λ_i
$(1/2, 0)$	-1	-1
$(3/2, 0)$	-	-
$(1/2, 1)$	$\begin{pmatrix} -\frac{1}{3} & 1 & -\frac{\sqrt{5}}{3} \\ 1 & \frac{1}{2} & -\frac{\sqrt{5}}{2} \\ -\frac{\sqrt{5}}{3} & -\frac{\sqrt{5}}{2} & -\frac{1}{6} \end{pmatrix}$	$2, -1, -1$
$(3/2, 1)$	$\begin{pmatrix} \frac{2}{3} & \frac{2\sqrt{5}}{3} \\ \frac{2\sqrt{5}}{3} & \frac{1}{3} \end{pmatrix}$	$2, -1$
$(1/2, 2)$	$\begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}$	$2, -1$

Matrix index of various dimer with quant. Num. (1,0), (0,1), (1,2)

❖ Two-body interaction in three-body dynamics

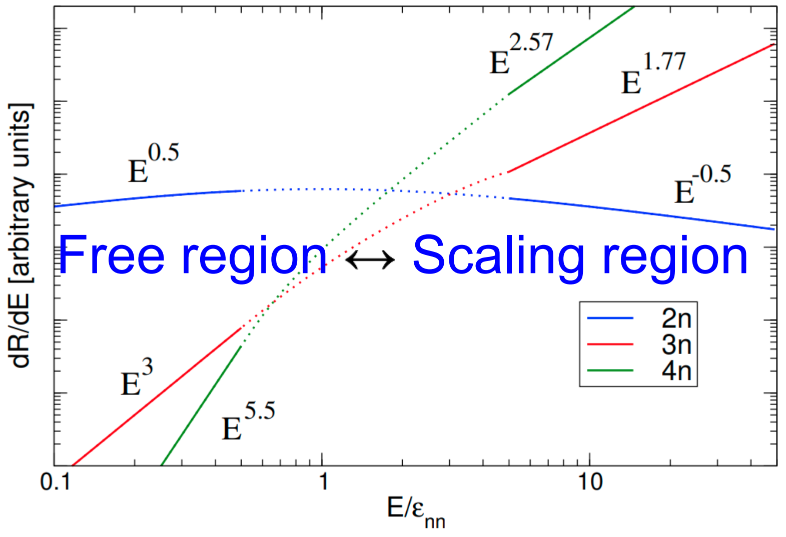
◻ Unparticle Phenomenon: Power-law behavior of short-range production rates

- Large scattering length: |a|>>r
- Approximate nonrelativistic conformal symmetry

$$\mathrm{Im}\, G_{\mathcal{U}}(\omega, \boldsymbol{p}) \sim \begin{cases} \delta\left(\omega - \frac{p^2}{2M_{\mathcal{U}}}\right), & \Delta = \frac{3}{2}, \\ \left(\omega - \frac{p^2}{2M_{\mathcal{U}}}\right)^{\Delta - \frac{5}{2}} \theta\left(\omega - \frac{p^2}{2M_{\mathcal{U}}}\right), & \Delta > \frac{3}{2}. \end{cases}$$

- Universal power-law behavior of cross section of short-range production

$$\frac{d\sigma}{dE} \sim (E_0 - E)^{\Delta - \frac{5}{2}}.$$



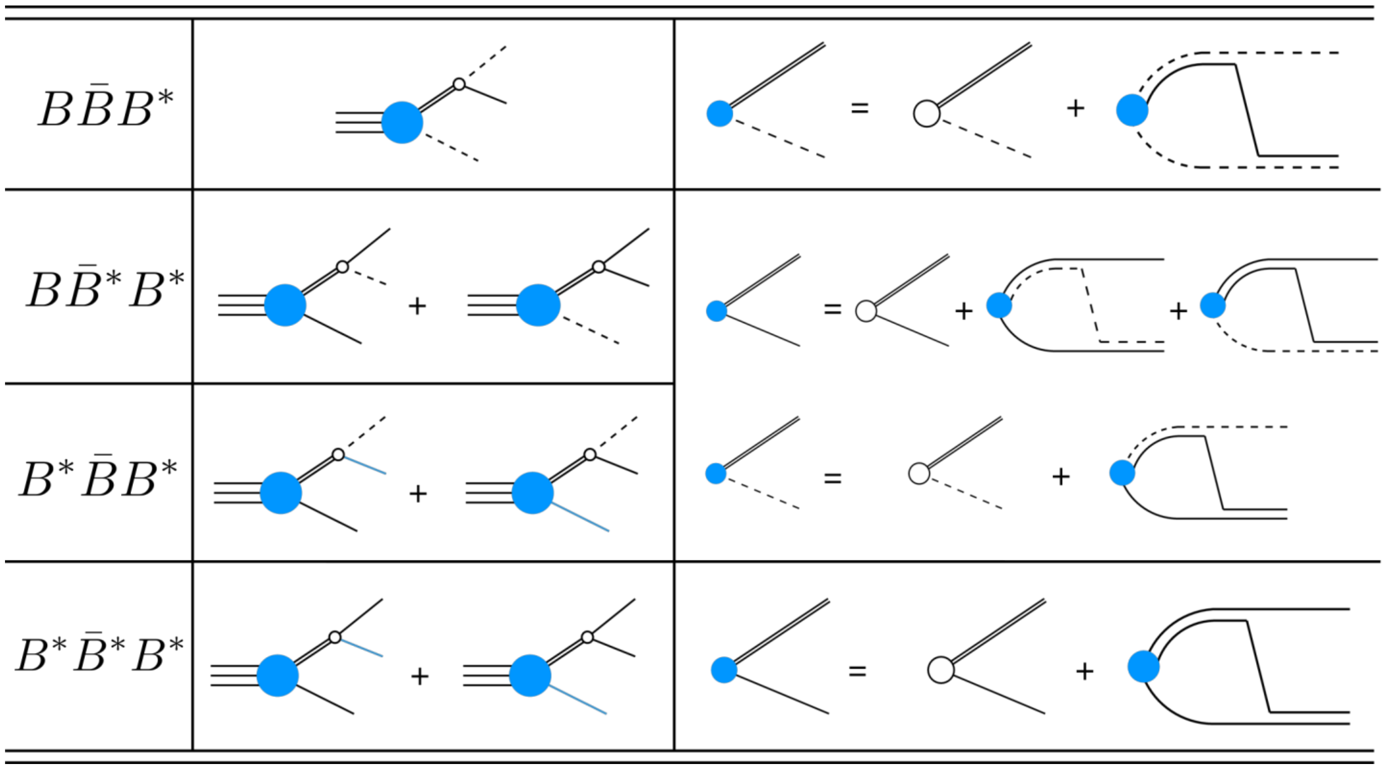
E. Braaten, HWH, PRD 107 034017 (2023)

- Example:
  - ✓ multi-neutron system HWH, D. T. Son, PNAS 118 (2021)
  - ✓ multi-neutral charm mesons E. Braaten, HWH, PRL 128 032002 (2022)

❖ Short-range production of three B mesons

- Short-range production of ZB dimer-particle and 3B channels

$(I,S) = (1/2,1),(3/2,1)$



Symmetrization procedure

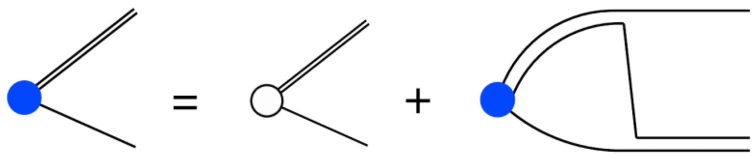
ZB

$ZB^* - Z'B$

$Z'B^*$

❖ Short-range production of three B mesons

☐ The single ZB dimer-particle channel



$$\Gamma_L(E,p) = A_L(E,p) + (-1)^L C_{SI} \int_0^\infty \frac{dq}{\pi} \frac{qM_h}{\mu p} \frac{\Gamma_L(E,q) Q_L\left(\frac{-2\mu E + q^2 + p^2}{2\mu p q / M_h}\right)}{-1/a + \sqrt{-2\mu E + (\mu/\tilde{\mu})\, q^2 - i\epsilon}} \,.$$

$$E \to \delta(x-1),$$

$$q = \sqrt{2\tilde{\mu}(E+\delta)} \to \sqrt{2\tilde{\mu}\delta}\sqrt{x}\,,$$

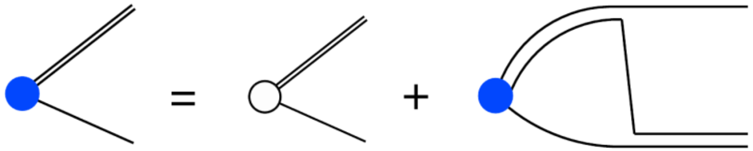
$$\Gamma_L(x,z) = g_L\,(2\tilde{\mu}\delta)^{L/2}\, z^{L/2} + (-1)^L C_{SI} \int_0^\infty \frac{dy}{2\pi\sqrt{z}} \frac{(1+r)^2}{r\sqrt{1+2r}} \frac{Q_L\left(\frac{(1+r)^2(y+z)-(1+2r)(x-1)}{2r(1+r)\sqrt{yz}}\right)}{-1+\sqrt{1-x+y-i\epsilon}}\, \Gamma_L(x,y)\,.$$

Line shape of Γ_L only depends on mass ratio *r* and partial-wave factor C_{SI}

$$r \equiv \frac{M_l}{M_h}, \mu = \frac{M_l M_h}{M_l + M_h} = \frac{r}{r+1} M_h, \frac{1}{\tilde{\mu}} = \frac{1}{M_l + M_h} + \frac{1}{M_l}, E = \frac{p^2}{2\tilde{\mu}} - \frac{\gamma^2}{2\mu} = \frac{p^2}{2\tilde{\mu}} - \delta$$

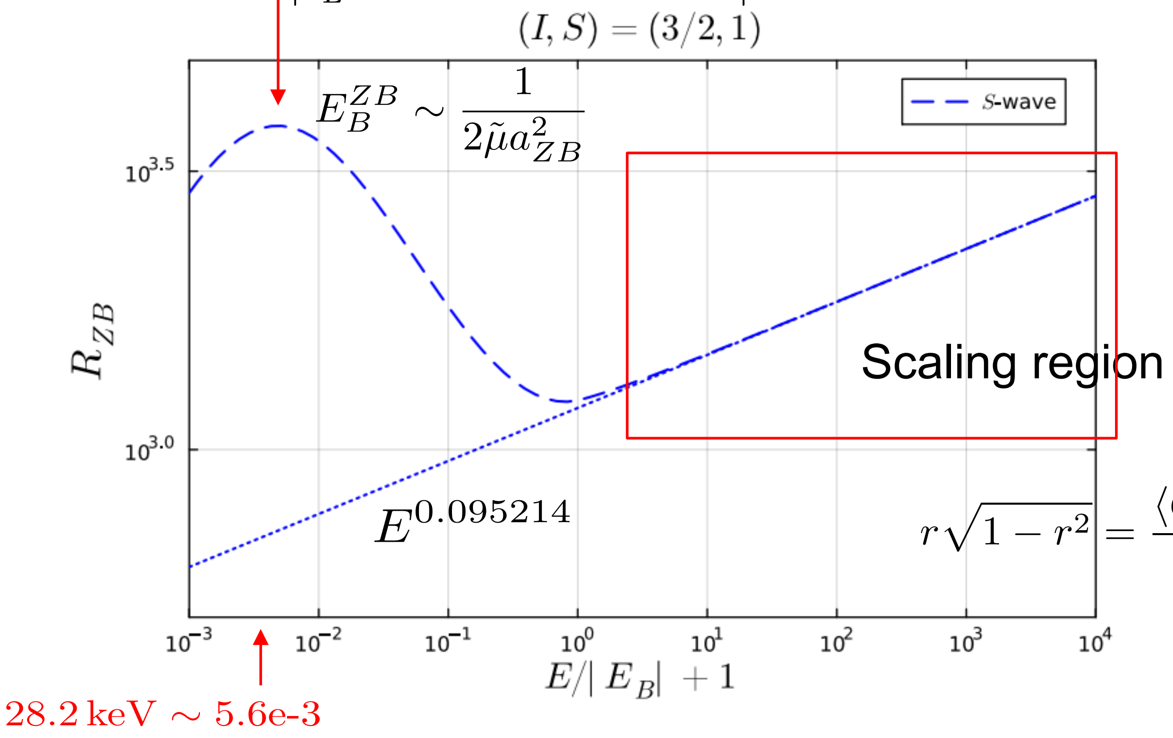
❖ Short-range production of three B mesons

◻ The single ZB dimer-particle channel



$$R(x) = \int_{-1}^{+1} d\hat{z}\, \frac{\tilde{\mu}\sqrt{\tilde{\mu}\delta}}{\sqrt{2}\pi} \left| \sum_L (2L+1) P_L(\hat{z}) \Gamma_L(x) \right|^2 \sqrt{x},$$

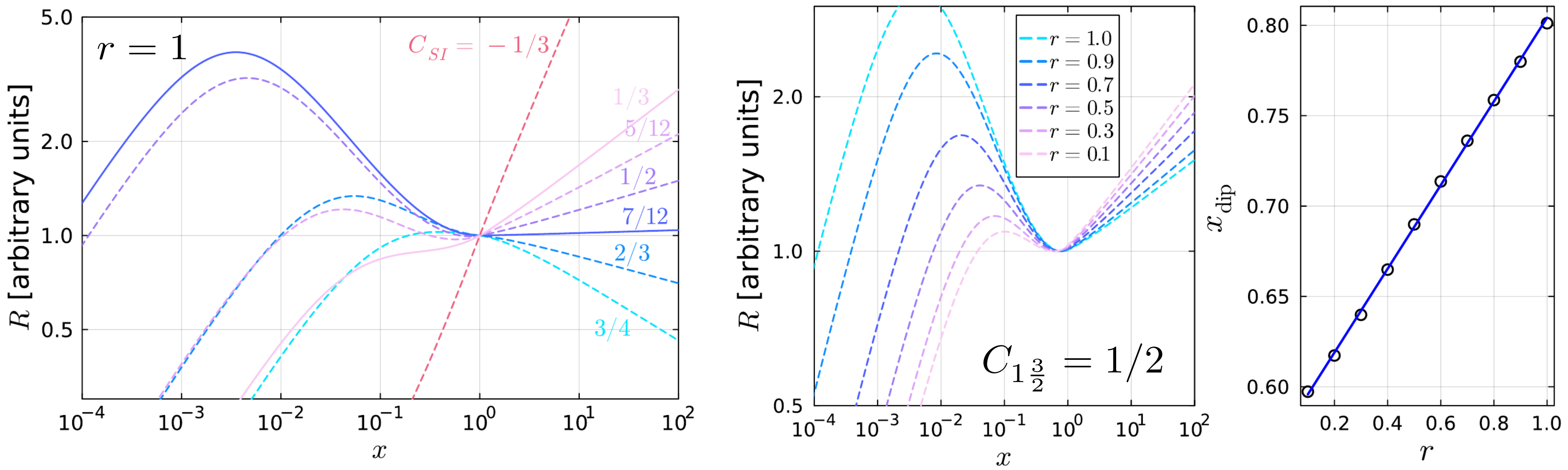
starts from ZB threshold



$$r\sqrt{1-r^2} = \frac{\langle \mathcal{O}_{11} \rangle}{4\pi} \frac{\pi}{s\cos(s\pi/2)} \sin(s \arcsin r)$$

❖ Short-range production of three B mesons

- A model-independent binding energy extraction by measuring the short-range production rate



$$E_{\rm dip}^{\rm exp} = \delta(x_{\rm dip} - 1)$$

$$\boxed{x_{\rm dip} = 0.2308r + 0.5729} \Rightarrow \delta = \frac{E_{\rm dip}^{\rm exp}}{-0.1983}$$

❖ Short-range production of three B mesons

☐ Finite-range corrections

$$\Gamma_L(x,z) = g_L\,(2\tilde{\mu}\delta)^{L/2}\,z^{L/2} + (-1)^L C_{SI}\int_0^\infty \frac{dy}{2\pi\sqrt{z}}\,\frac{(1+r)^2}{r\sqrt{1+2r}}\,\frac{Q_L\left(\frac{(1+r)^2(y+z)-(1+2r)(x-1)}{2r(1+r)\sqrt{yz}}\right)}{-1+\sqrt{1-x+y-i\epsilon}}\,\Gamma_L(x,y)\,.$$

$$k\cot\delta_0(k) = -\gamma + \frac{1}{2}\rho\left(k^2+\gamma^2\right)$$

$$\Gamma_L(x,z) = g_L\,(2\tilde{\mu}\delta)^{L/2}\,z^{L/2} + (-1)^L C_{SI}\int_0^\infty \frac{dy}{2\pi\sqrt{z}}Q_L\left(\frac{(1+r)^2(y+z)-(1+2r)(x-1)}{2r(1+r)\sqrt{yz}}\right)\Gamma_L(x,y)$$
$$\times\frac{1+r}{r}\,\frac{1+r}{\sqrt{2r+1}}\left(\frac{1}{-1+\sqrt{1-x+y-i\epsilon}+\frac{\rho\gamma}{2}(x-y)}\right)$$

☐ The troublesome unphysical pole when $r_e \equiv \rho > 0$

$$T(k=ip) = \frac{2\pi}{\mu}\,\frac{2/r_e}{(p-\frac{1}{a})(p-\frac{2}{r_e})},\qquad Z = \frac{dT^{-1}}{dE}\Big|_{E=E_{\rm pole}} = \frac{\mu^2}{2\pi}\left(\frac{r_e}{2a}\frac{1}{p_{\rm on}}+\frac{1}{p_{\rm on}}-r_e\right)$$

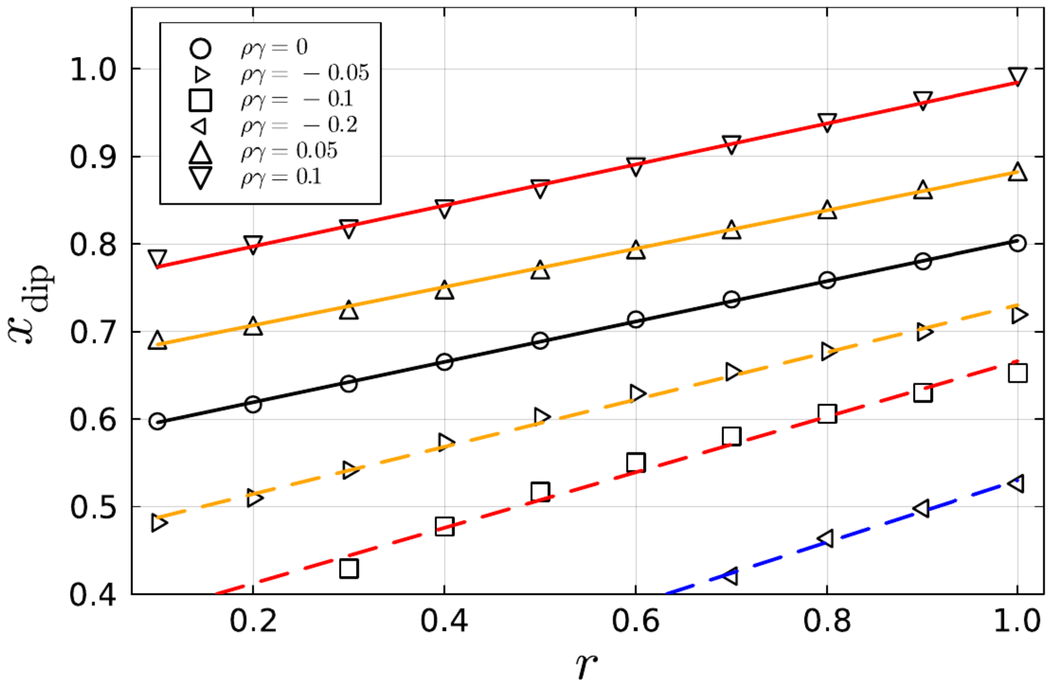
$$p^1_{\rm on} = \frac{1}{a},\quad p^2_{\rm on} = \frac{2}{r_e}$$

$$\xrightarrow{\text{positive } r_e}\begin{cases} Z^{-1}_{p^1_{\rm on}} = \frac{\mu^2}{2\pi}(a-r_e) > 0 \text{ for bound states} \\ Z^{-1}_{p^2_{\rm on}} = -\frac{\mu^2}{2\pi}\frac{r_e}{2} < 0 \text{ unphysical pole}\end{cases}\qquad \xrightarrow{\text{negative } r_e}\begin{cases} Z^{-1}_{p^1_{\rm on}} = \frac{\mu^2}{2\pi}(a-r_e) > 0 \text{ for bound states} \\ Z^{-1}_{p^2_{\rm on}} = -\frac{\mu^2}{2\pi}\frac{r_e}{2} > 0 \text{ for virtual states}\end{cases}$$

❖ Short-range production of three B mesons

☐ Finite-range corrections

$|\rho\gamma| \lesssim 0.3$

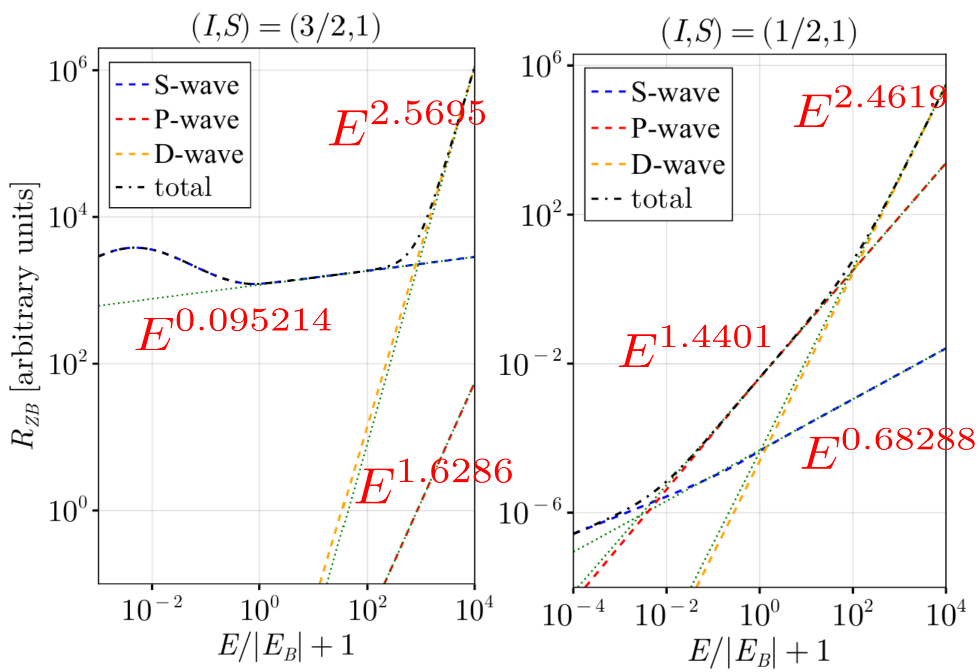


		Z_b/Z_b'	$X(3872)$
$\rho\gamma$	Linear relation	Variation of Eq. (7) of main text	Variation of Eq. (8) of main text
0	$0.2308r + 0.5729$	$-E_{\mathrm{dip}}^{\mathrm{exp}}/0.1983$	$-E_{\mathrm{dip}}^{\mathrm{exp}}/0.2126$
-0.05	$0.2699r + 0.4603$	$-E_{\mathrm{dip}}^{\mathrm{exp}}/0.2721$	$-E_{\mathrm{dip}}^{\mathrm{exp}}/0.2889$
-0.1	$0.3175r + 0.3487$	$-E_{\mathrm{dip}}^{\mathrm{exp}}/0.3365$	$-E_{\mathrm{dip}}^{\mathrm{exp}}/0.3563$
0.05	$0.2191r + 0.6631$	$-E_{\mathrm{dip}}^{\mathrm{exp}}/0.1197$	$-E_{\mathrm{dip}}^{\mathrm{exp}}/0.1333$
0.1	$0.2342r + 0.7502$	$-E_{\mathrm{dip}}^{\mathrm{exp}}/0.0176$	$-E_{\mathrm{dip}}^{\mathrm{exp}}/0.0322$

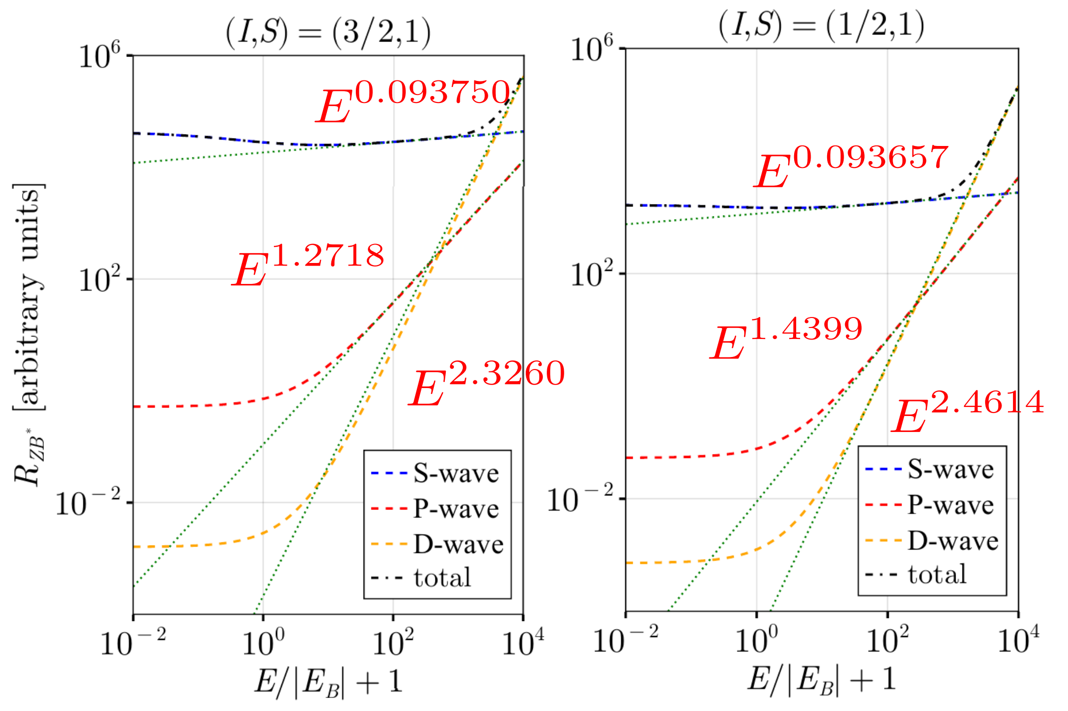
❖ Short-range production of three B mesons

- Short-range production of ZB dimer-particle channels $E^{s_i-\frac{1}{2}}$

ZB single channel



ZB*–Z$'$B coupled channel



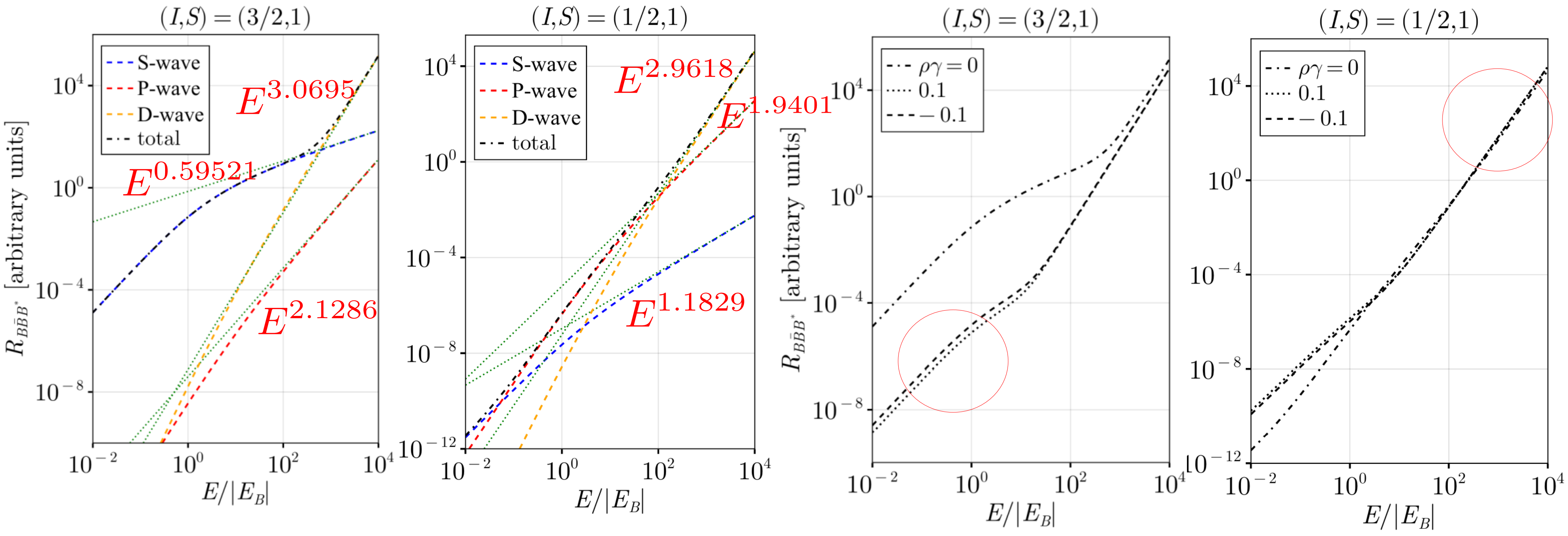
❖ Short-range production of three B mesons

☐ Short-range production of 3B mesons channels

$$E^{s_i}$$

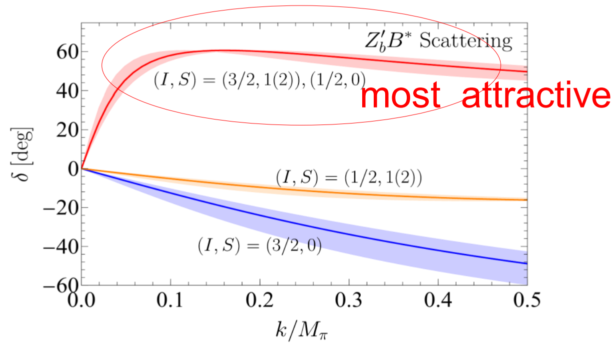
$$R^{abc}(E) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3} \left|\Gamma^{abc}(E,\vec{p},\vec{k})\right|^2 2\pi\delta\left(E - E_{\vec{p}} - E_{\vec{q}_1} - E_{\vec{q}_2}\right),$$

starts from 3B threshold



Summary

- Two-body hadronic molecule leaves imprints in corresponding 3body observables



  - Direct evidence by confirmation of Efimov effects

    No Efimove effects in ZB scattering

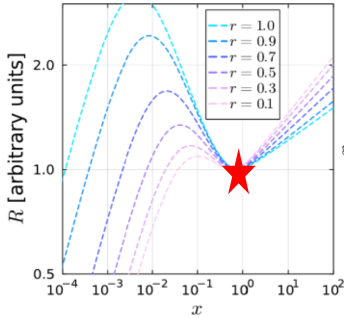
  - Observation of power-law behavior consistent with prediction of nonrelativistic conformal symmetry

    Predictions of the short-range production rates are given for both ZB and 3B channels

- A model independent mass determination for a wide range of near-threshold states from the short-range production of dimer-particle final states

$$\delta = E_{\rm dip}^{\rm exp}/-0.1983 \quad \text{for Z/Z'}$$

$$\delta = E_{\rm dip}^{\rm exp}/-0.2126 \quad \text{for X(3872)}$$



Outlook

- Difficulty in meauring 3body observables, especially for heavy quark sectors
  - To study the Efimov effects in ZB/XD scattering in a box

    If the pion mass dependence of dimer-particle scattering can serve as benchmark test for hadronic molecule picture?

- More elegant treatment for the finite range corrections

Thank you for your attention!