Workshop on Few-Body Problems in Nuclear and Hadron Physics

General Hamiltonian Approach to the N-Body Finite-Volume Formalism

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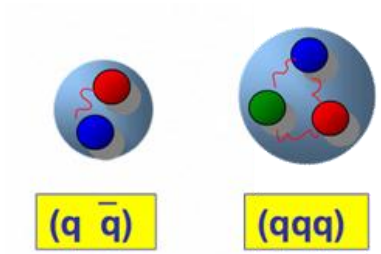
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Outline

- Background： Bare state + Coupled channel
- Non-Perturbative Hamiltonian Framework(NPHF)
- $\omega \sim \rho\pi \sim 3\pi$ and $\rho \sim 2\pi$
- $\sigma \sim 2\pi \sim 4\pi$
- Summary and Outlook

Background：Beyond Quark Model

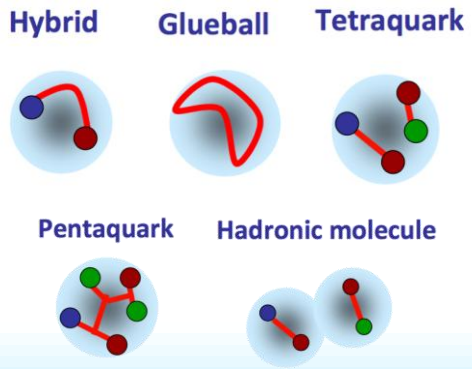
Hadron state:
meson, baryon//Exotic

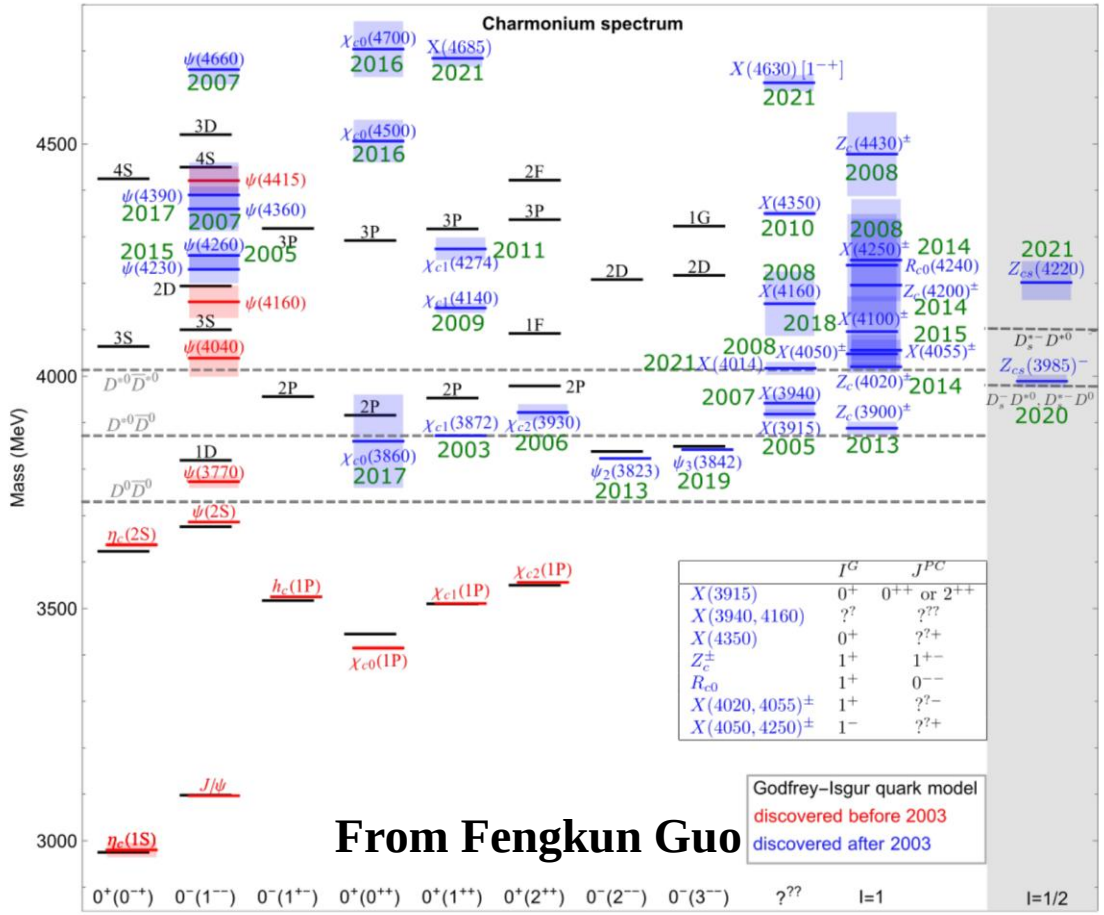


Quark model?

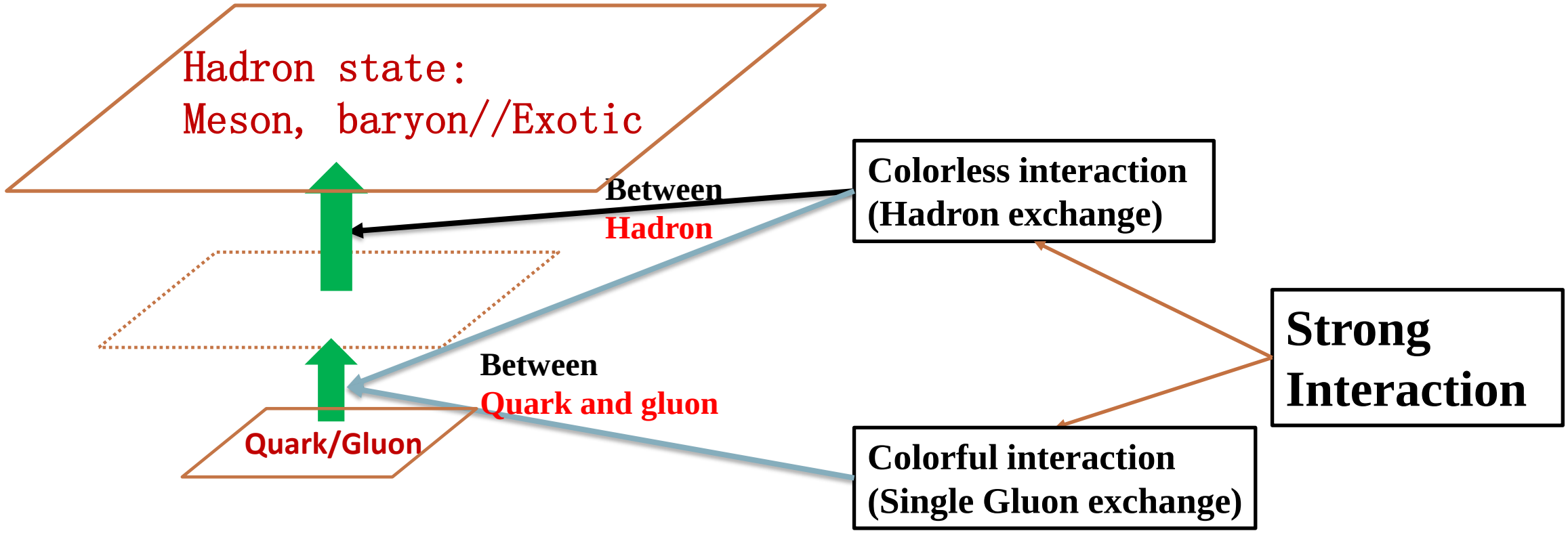
Quark/Gluon

Exotic

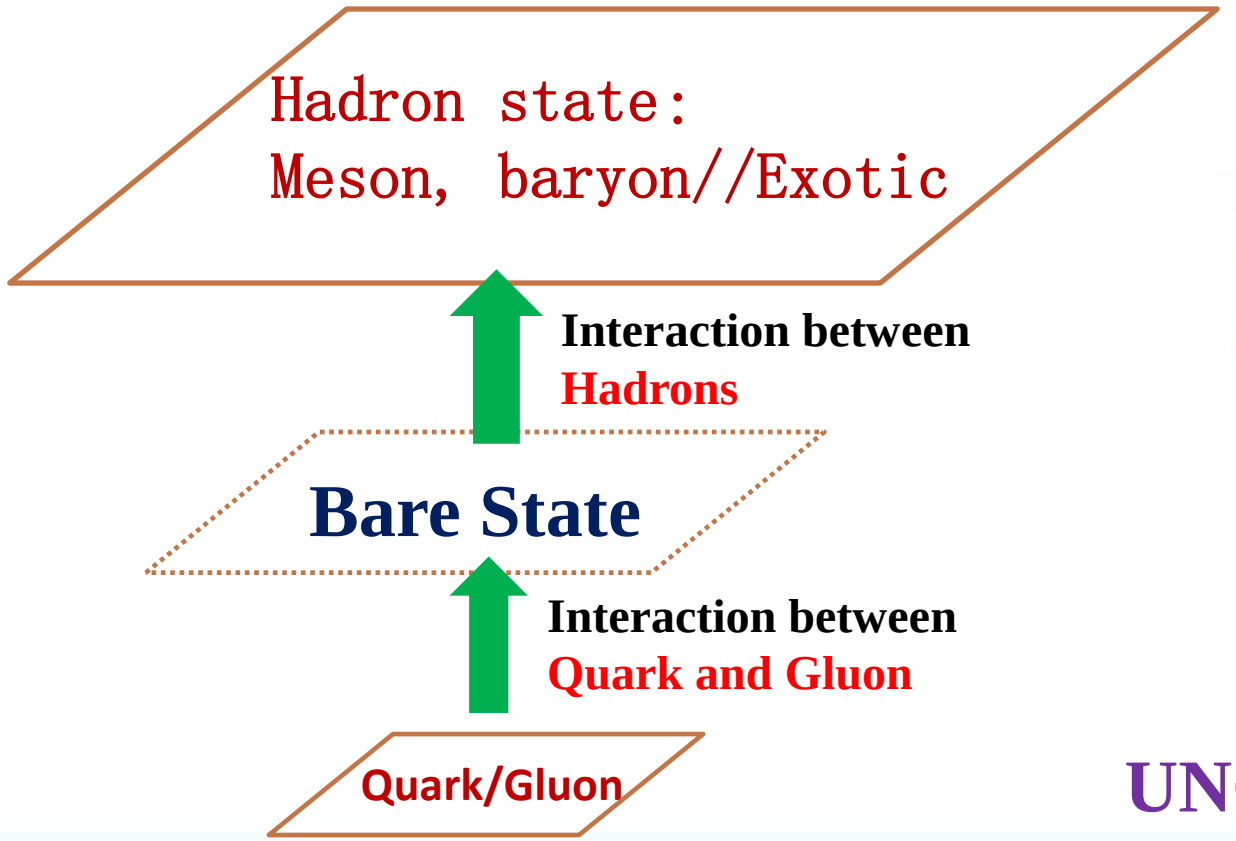


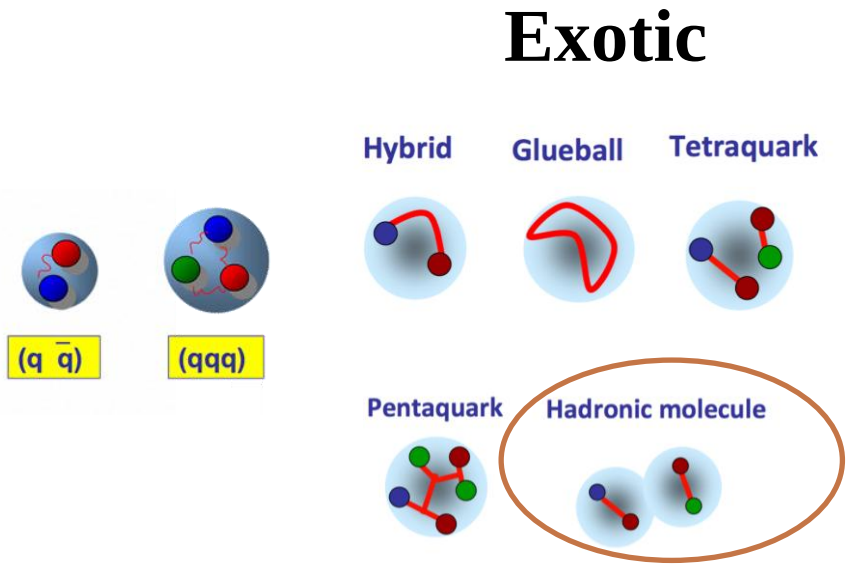


Background：Strong Interaction



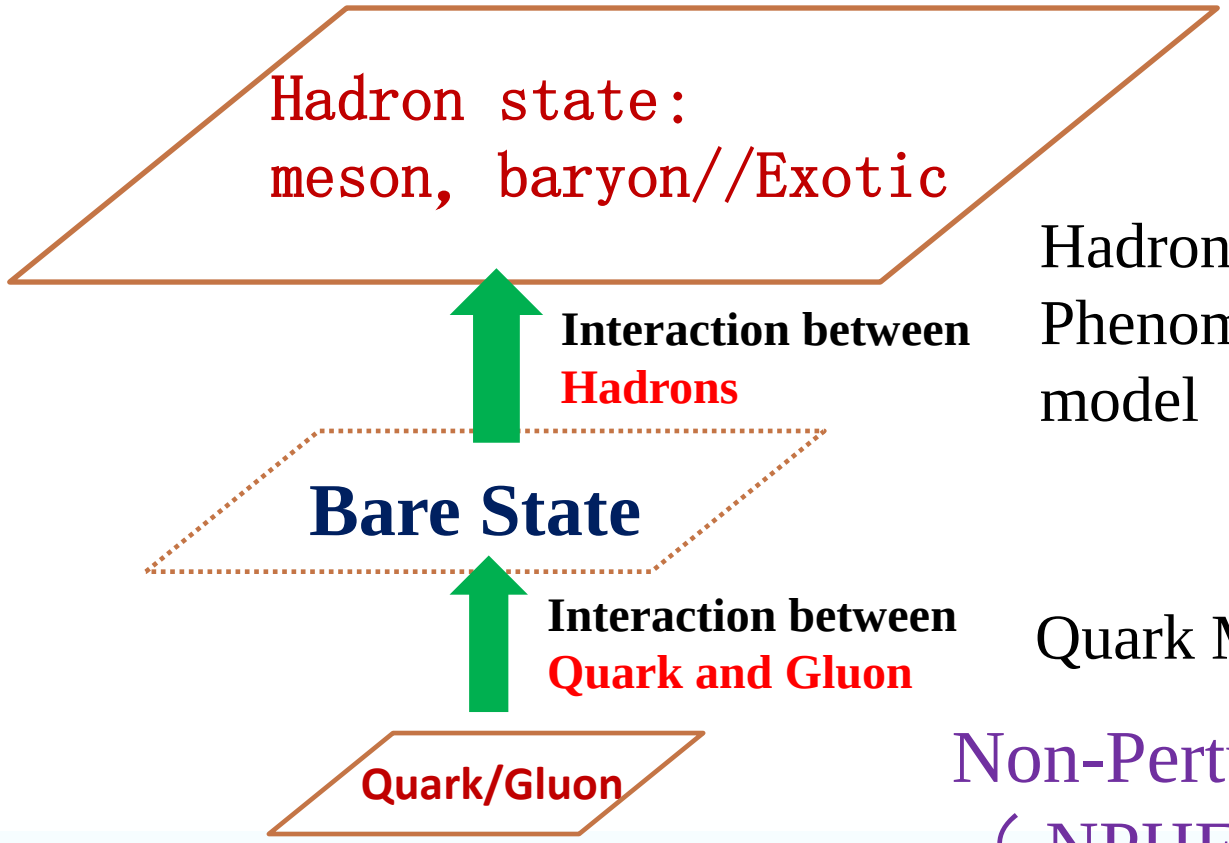
Background： Bare state + Coupled channel





UNQUENCHED quark model

Background： Bare state + Coupled channel



Hadron level
Phenomenological model

Quark Model

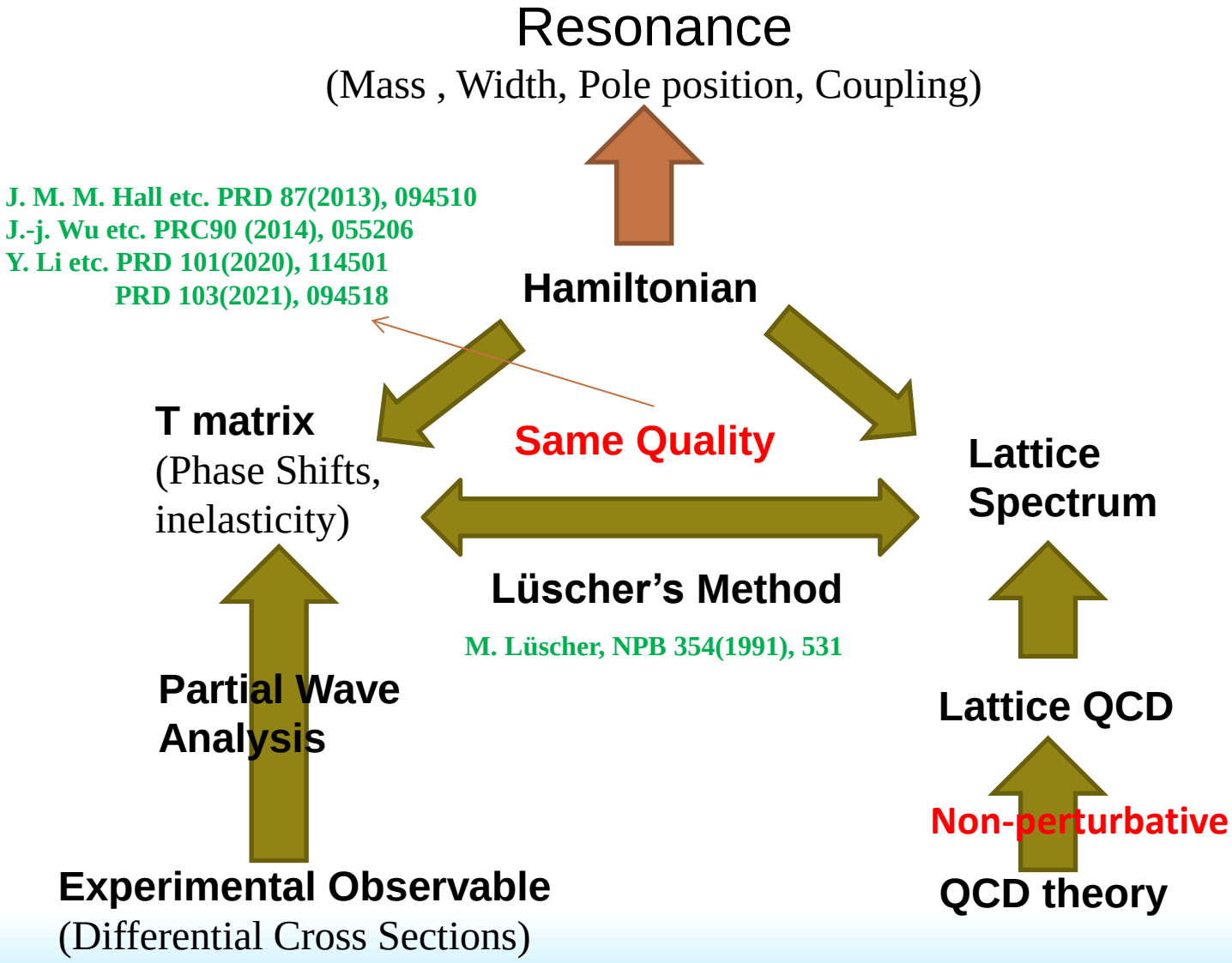
Coupled channel

Bare state (single)
+multi-hadron

The number of hadrons is **not conserved**!

Non-Perturbative Hamiltonian Framework
（NPHF）

Introduction of NPHF



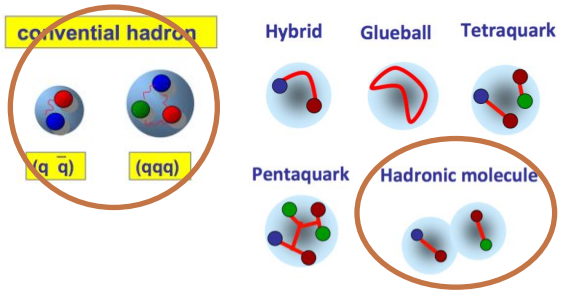
Three body

RFT: Hansen and Sharpe, PRD 90(2014), 116003; …

NREFT: Hammer, Pang, and Rusetsky, JHEP 09(2017), 109; ….

FVU: Mai and Döring, EPJA 53(2017), 240; ….

NPHF： bare state + hadron channel



$$H = H_0 + H_I$$

$$H_0 = \sum_{i=1,n} |B_i\rangle m_i \langle B_i| + \sum_{\alpha} |\alpha(k_\alpha)\rangle \left[\sqrt{m_{\alpha 1}^2 + k_\alpha^2} + \sqrt{m_{\alpha 2}^2 + k_\alpha^2} \right] \langle \alpha(k_\alpha)|$$

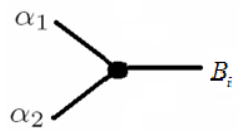
$|B_i\rangle$ bare state， bare mass m_i ⟷ Quark level

$|\alpha(k_\alpha)\rangle$ non-interaction channels ⟷ Hadron level

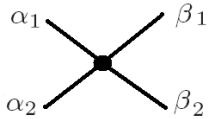
$$H_I = \hat{g} + \hat{v}$$

$$\hat{g} = \sum_{\alpha} \sum_{i=1,n} \left[|\alpha(k_\alpha)\rangle g_{i,\alpha}^{+} \langle B_i| + |B_i\rangle g_{i,\alpha} \langle \alpha(k_\alpha)| \right]$$

$$\hat{v} = \sum_{\alpha,\beta} |\alpha(k_\alpha)\rangle v_{\alpha,\beta} \langle \beta(k_\beta)|$$



From 3P0 model, and the wavefunction of quark model



One boson exchange (OBE)

Resonance
(Mass, width, pole position, coupling)

Hamiltonian

T matrix
(phase shift, inelasticity)

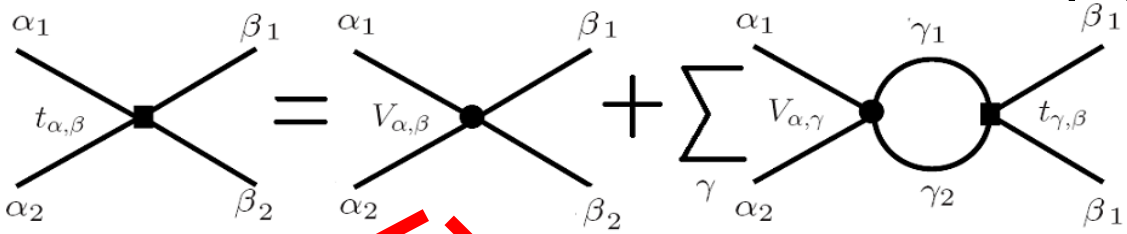
Lattice spectrum

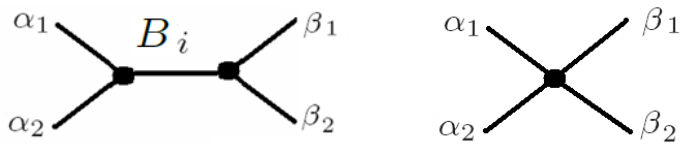
Introduction of HEFT

Argonne-Osaka Model

- **T Matrix:**

$$t_{\alpha,\beta}(k_\alpha,k_\beta,E)=V_{\alpha,\beta}(k_\alpha,k_\beta)+\sum_\gamma\int k_\gamma{}^2dk_\gamma\,\frac{V_{\alpha,\gamma}(k_\alpha,k_\gamma)t_{\gamma,\beta}(k_\gamma,k_\beta,E)}{E-\sqrt{m_{\gamma1}^2+k_\gamma^2}-\sqrt{m_{\gamma2}^2+k_\gamma^2}+i\varepsilon}$$





$$g_{i,\alpha}^*\frac{1}{E-m_i}g_{i,\beta}$$

$$V_{\alpha,\beta}$$

$$S_{\alpha,\beta}=1-i2\sqrt{\rho_\alpha}t_{\alpha,\beta}(k_{0\alpha},k_{0\beta},E)\sqrt{\rho_\beta}$$

$$\rho_\alpha=\frac{\pi k_{0\alpha}\sqrt{m_{\alpha1}^2+k_{0\alpha}^2}\sqrt{m_{\alpha1}^2+k_{0\alpha}^2}}{E}$$

$$\eta e^{2i\delta_\alpha}=S_{\alpha,\alpha}$$

Resonance
(Mass , Width, Pole position, Coupling)

Hamiltonian

T matrix
(Phase Shifts, inelasticity)

Lattice Spectrum

Introduction of NPHF

J. M. M. Hall etc. PRD 87(2013), 094510
J.-j. Wu etc. PRC90 (2014), 055206

- **Hamiltonian with discrete momentum**

Continuum → Discrete

$$\int d\vec{k} \quad \text{and} \quad \left|\alpha(\vec{k}_\alpha)\right\rangle \quad \text{and} \quad \left\langle\beta(\vec{k}_\beta)\middle|\alpha(\vec{k}_\alpha)\right\rangle = \delta_{\alpha\beta}\delta(\vec{k}_\alpha - \vec{k}_\beta)$$

$$\sum_{\mathrm{i}}\left(2\pi/\mathrm{L}\right)^3 \quad \text{and} \quad \left(2\pi/\mathrm{L}\right)^{-3/2}\left|\vec{k}_{\mathrm{i}}, -\vec{k}_{\mathrm{i}}\right\rangle_\alpha \quad \text{and} \quad {}_\beta\left\langle\vec{k}_{\mathrm{j}}, -\vec{k}_{\mathrm{j}}\middle|\vec{k}_{\mathrm{i}}, -\vec{k}_{\mathrm{i}}\right\rangle_\alpha = \delta_{\alpha\beta}\delta_{\mathrm{ij}}$$

$$H_0 = \sum_{i=1,n}\left|B_i\right\rangle m_i \left\langle B_i\right| + \sum_{\alpha,\mathrm{i}}\left|\vec{k}_{\mathrm{i}}, -\vec{k}_{\mathrm{i}}\right\rangle_\alpha \left[\sqrt{m_{\alpha_B}^2 + k_\alpha^2} + \sqrt{m_{\alpha_M}^2 + k_\alpha^2}\right]_\alpha \left\langle\vec{k}_{\mathrm{i}}, -\vec{k}_{\mathrm{i}}\right|$$

$$H_I = \sum_{\mathrm{j}}\left(2\pi/\mathrm{L}\right)^{3/2}\sum_\alpha\sum_{i=1,n}\left[\left|\vec{k}_{\mathrm{j}}, -\vec{k}_{\mathrm{j}}\right\rangle_\alpha g^+_{i,\alpha}\left\langle B_i\right| + \left|B_i\right\rangle g_{i,\alpha}\,{}_\alpha\left\langle\vec{k}_{\mathrm{j}}, -\vec{k}_{\mathrm{j}}\right|\right] + \sum_{\mathrm{i\,i}}\left(2\pi/\mathrm{L}\right)^3\sum_{\alpha\ \beta}\left|\vec{k}_{\mathrm{i}}, -\vec{k}_{\mathrm{i}}\right\rangle_\alpha v_{\alpha,\beta}\,{}_\beta\left\langle\vec{k}_{\mathrm{j}}, -\vec{k}_{\mathrm{j}}\right|$$

$$[H_0]_{N_c+1} = \begin{pmatrix} m_0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & \epsilon_1(k_0) & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & \epsilon_2(k_0) & \cdots & 0 & 0 & \cdots \\ 0 & 0 & 0 & \ddots & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & \epsilon_{n_c}(k_0) & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \epsilon_1(k_1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad [H_I]_{N_c+1} = \begin{pmatrix} 0 & g_1^V(k_0) & g_2^V(k_0) & \cdots & g_{n_c}^V(k_0) & g_1^V(k_1) & \cdots \\ g_1^V(k_0) & v_{1,1}^V(k_0,k_0) & v_{1,2}^V(k_0,k_0) & \cdots & v_{1,n_c}^V(k_0,k_0) & v_{1,1}^V(k_0,k_1) & \cdots \\ g_2^V(k_0) & v_{2,1}^V(k_0,k_0) & v_{2,2}^V(k_0,k_0) & \cdots & v_{2,n_c}^V(k_0,k_0) & v_{2,1}^V(k_0,k_1) & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots \\ g_{n_c}^V(k_0) & v_{n_c,1}^V(k_0,k_0) & v_{n_c,2}^V(k_0,k_0) & \cdots & v_{n_c,n_c}^V(k_0,k_0) & v_{n_c,1}^V(k_0,k_1) & \cdots \\ g_1^V(k_1) & v_{1,1}^V(k_1,k_0) & v_{1,2}^V(k_1,k_0) & \cdots & v_{1,n_c}^V(k_1,k_0) & v_{1,1}^V(k_1,k_1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$(H_0+H_I)\,|\Psi> = E\,|\Psi>$ Eigen-Value $\longleftrightarrow$ Lattice Spectrum

Eigen-Vector

Resonance
(Mass , Width, Pole position, Coupling)

Hamiltonian

T matrix
(Phase Shifts, inelasticity)

Lattice Spectrum

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1. Finite-volume matrix Hamiltonian model for a Δ→Nπ syste PRD 87 (2013) no.9, 094510
2. Finite-volume Hamiltonian method for coupled-channels interactions in lattice QCD PRC 90 (2014) no.5, 055206
3. Lattice QCD Evidence that the Λ(1405) Resonance is an Antikaon-Nucleon Molecule PRL 114 (2015) no.13, 132002
4. Hamiltonian effective field theory study of the N∗(1535) resonance in lattice QCD PRL 116 (2016) no.8, 082004
5. Hamiltonian effective field theory study of the N∗(1440) resonance in lattice QCD PRD 95 (2017) no.3, 034034
6. Structure of the Λ(1405) from Hamiltonian effective field theory PRD 95 (2017) no.1, 014506
7. Nucleon resonance structure in the finite volume of lattice QCD PRD 95 (2017) no.11, 114507
8. Structure of the Roper Resonance from Lattice QCD Constraints PRD 97(2018) no.9, 094509
9. Kaonic Hydrogen and Deuterium in Hamiltonian Effective Field Theory PLB 808(2020),135652
10. Partial Wave Mixing in Hamiltonian Effective Field Theory PRD 101(2020) no.11,114501
11. Hamiltonian effective field theory in elongated or moving finite volume PRD 103(2021) no.9, 094518
12. Regularization in Nonperturbative Extensions of Effective Field Theory PRD 106 (2022) no.3, 034506
13. Novel Coupled Channel Framework Connecting the Quark Model and Lattice QCD for the Near-threshold Ds States PRL128(2022),112001
14. The investigations of the P-wave Bs states combining quark model and lattice QCD in the coupled channel framework JHEP01 (2023) 058
15. Effects of multiple single-particle basis states in scattering systems Annals Phys. 459(2023) 169531
16. Low-lying odd-parity nucleon resonances as quark-model like states PRD 108 (2023) 9, 094519
17. Structure of the Λ(1670) resonance PRD 109 (2024) 5, 054025
18. Investigation on the bottom analogs Tbb of Tcc+ PRD 110 (2024) 7, 074007
19. Pion photoproduction of nucleon excited states with Hamiltonian effective field theory PRD 110 (2024) 9, 094015
20. Study of the pion-mass dependence of ρ-meson properties in lattice QCD PRD 109 (2024) 3, 034505
21. Searching for the first radial excitation of the Δ(1232) in lattice QCD JPG 51 (2024) 6, 065106
22. Three-coupled-channel analysis of Zc(3900) involving $DD^{\bar{}}*$, πJ/ψ, and ρηc PRD 110 (2024) 11, 114029
23. New insight into the exotic states strongly coupled with the $DD^{\bar{}}*$ from the Tcc+ Sci.Bull. 69 (2024) 3036-3042
24. Generalized boost transformations in finite volumes and application to Hamiltonian methods JHEP 08 (2024) 178
25. Spectral parameters of the ρ resonance from lattice QCD JHEP 08 (2025) 064
26. Finite volume Hamiltonian method for two-particle systems containing long-range potential on the lattice JHEP 04 (2025) 108
27. Structure of the Ω-(2012) with Hamiltonian effective field theory PRD 112 (2025) 5, L051503
28. Nucleon resonance structure up to 2 GeV and the nature of the Roper resonance PRD 111 (2025) 11, 116002
29. Exploring the Ω− spectrum in lattice QCD JPG 52 (2025) 6, 065103
30. The odd-parity strange baryons Σ (1/2−) below 1.8 GeV with Hamiltonian effective field theory 2509.17510

$N^*(1440), N^*(1535), N^*(1650), N^*(1710), \Delta(1232), \Delta(1600), \Lambda(1405), \Lambda(1670),$
$D_{s0}(2317), D_{s1}(2460), D_{s1}(2536), D_{s2}(2573), X(3872), T_{cc}, Z_c(3900), \ldots\ldots$

NPHF

Resonance
(Mass , Width, Pole position, Coupling)

Complex Continuum Momentum Space

Hamiltonian

Real Continuum Momentum Space

Real Discrete Momentum Space

T matrix
(Phase Shifts, inelasticity)

Lattice Spectrum

University of Chinese Academy of Sciences

The Hamiltonian in a Box

Jia-Jun Wu（UCAS）

Collaborator：T.-S. H. Lee， Derek B. Leinweber， Anthony W. Thomas ， Zhan-Wei Liu

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bctp

Bethe Forum Multihadron Dynamics in a Box
Bonn, Germany, 10/09/2019

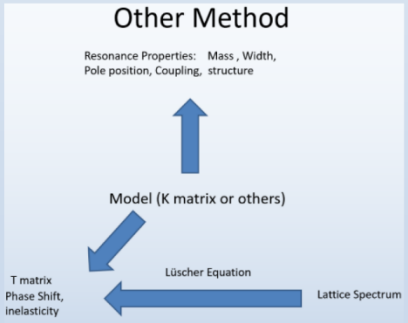
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Story from …

Akaki (阿卡基. 儒塞斯基) asked :

What is really benefit compare to other Process ?





I try to answer:

1. It is a different formalism — Exponential Suppressed Correction
2. EigenVector — Model dependent
3. Different parameters method — Also can be applied in other Method

…… 4. Long-rang force

Lu Meng, E. Epelbaum, *JHEP* 10 (2021) 051

I admit that this formalism can not provide more useful information.

Hamiltonian Approach can be extended to multi-hadron case

But, now I found

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The Hamiltonian in a Box

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bctp

Bethe Forum Multihadron Dynamics in a Box
Bonn, Germany, 10/09/2019

The Multihadron system

$H=$

	$\lvert B\rangle$	$\lvert \alpha_{ij}\beta_k\rangle$	$\lvert \beta_1\beta_2\beta_3\rangle$
$\lvert B\rangle$	diagonal √	$B \to \alpha_{ij}\beta_k$ decay √	0 √
$\lvert \alpha_{ij}\beta_k\rangle$		$\alpha_{ij}\beta_k \to \alpha'_{ij}\beta'_k$ √ Rest Frame two body	$\alpha_{ij} \to \beta_i\beta_j$? Boost Frame decay
$\lvert \beta_1\beta_2\beta_3\rangle$			$\beta_1\beta_2\beta_3 \to \beta_1\beta_2\beta_3$? Three bodies contact Term

Long way, Heavy work ……

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Introduction of NPHF

Kang, Leiweber, Thoams, Wang, Yang, Wu, 2603.07205

- **Hamiltonian with discrete momentum**

$$[H_0]_{N_c+1} = \begin{pmatrix} m_0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & \epsilon_1(k_0) & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & \epsilon_2(k_0) & \cdots & 0 & 0 & \cdots \\ 0 & 0 & 0 & \ddots & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & \epsilon_{n_c}(k_0) & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \epsilon_1(k_1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$[H_I]_{N_c+1} = \begin{pmatrix} 0 & g_1^V(k_0) & g_2^V(k_0) & \cdots & g_{n_c}^V(k_0) & g_1^V(k_1) & \cdots \\ g_1^V(k_0) & v_{1,1}^V(k_0,k_0) & v_{1,2}^V(k_0,k_0) & \cdots & v_{1,n_c}^V(k_0,k_0) & v_{1,1}^V(k_0,k_1) & \cdots \\ g_2^V(k_0) & v_{2,1}^V(k_0,k_0) & v_{2,2}^V(k_0,k_0) & \cdots & v_{2,n_c}^V(k_0,k_0) & v_{2,1}^V(k_0,k_1) & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots \\ g_{n_c}^V(k_0) & v_{n_c,1}^V(k_0,k_0) & v_{n_c,2}^V(k_0,k_0) & \cdots & v_{n_c,n_c}^V(k_0,k_0) & v_{n_c,1}^V(k_0,k_1) & \cdots \\ g_1^V(k_1) & v_{1,1}^V(k_1,k_0) & v_{1,2}^V(k_1,k_0) & \cdots & v_{1,n_c}^V(k_1,k_0) & v_{1,1}^V(k_1,k_1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$H =$

	$\|B\rangle$	$\|\alpha_{ij}\beta_k\rangle$	$\|\beta_1\beta_2\beta_3\rangle$
$\|B\rangle$	diagonal	B -> $\alpha_{ij}\beta_k$ decay	B -> $\beta_1\beta_2\beta_3$ decay
$\|\alpha_{ij}\beta_k\rangle$		$\alpha_{ij}\beta_k$ -> $\alpha'_{ij}\beta'_k$ Rest Frame two body	α_{ij} -> $\beta_i\beta_j$ Boost Frame decay
$\|\beta_1\beta_2\beta_3\rangle$			$\beta_1\beta_2\beta_3$ -> $\beta_1\beta_2\beta_3$ Three bodies Contact Term

Resonance
(Mass , Width, Pole position, Coupling)

Hamiltonian

T matrix
(Phase Shifts, inelasticity)

Lattice Spectrum

Introduction of NPHF

Kang, Leinweber, Thomas, Wang, Yang, Wu, 2603.07205

$$\mathbf{k}^r = \frac{\sqrt{(\omega_1(\mathbf{k}^*)+\omega_2(\mathbf{k}^*))^2+\mathbf{P}^2}}{\omega_1(\mathbf{k}^*)+\omega_2(\mathbf{k}^*)}\mathbf{k}^*_{\parallel} + \frac{\omega_1(\mathbf{k}^*)}{\omega_1(\mathbf{k}^*)+\omega_2(\mathbf{k}^*)}\mathbf{P} + \mathbf{k}^*_{\perp},$$

$$\mathbf{k}^* = \frac{\omega_1(\mathbf{k}^r)+\omega_2(\mathbf{P}-\mathbf{k}^r)}{\sqrt{(\omega_1(\mathbf{k}^r)+\omega_2(\mathbf{P}-\mathbf{k}^r))^2-\mathbf{P}^2}}\mathbf{k}^r_{\parallel} - \frac{\omega_1(\mathbf{k}^r)}{\sqrt{(\omega_1(\mathbf{k}^r)+\omega_2(\mathbf{P}-\mathbf{k}^r))^2-\mathbf{P}^2}}\mathbf{P} + \mathbf{k}^r_{\perp},$$

$$\mathcal{J}^r = \left|\frac{\partial \mathbf{k}^*}{\partial \mathbf{k}^r}\right| = \frac{\omega_1(\mathbf{k}^*)\omega_2(\mathbf{k}^*)}{\omega_1(\mathbf{k}^*)+\omega_2(\mathbf{k}^*)}\frac{\omega_1(\mathbf{k}^r)+\omega_2(\mathbf{P}-\mathbf{k}^r)}{\omega_1(\mathbf{k}^r)\omega_2(\mathbf{P}-\mathbf{k}^r)}.$$

$H=$

	$\lvert B\rangle$	$\lvert\alpha_{ij}\beta_k\rangle$	$\lvert\beta_1\beta_2\beta_3\rangle$
$\lvert B\rangle$	diagonal	B -> $\alpha_{ij}\beta_k$ decay	B -> $\beta_1\beta_2\beta_3$ decay
$\lvert\alpha_{ij}\beta_k\rangle$		$\alpha_{ij}\beta_k$ -> $\alpha'_{ij}\beta'_k$ Rest Frame two body	α_{ij} -> $\beta_i\beta_j$ Boost Frame decay
$\lvert\beta_1\beta_2\beta_3\rangle$	Li, Wu, Lee, Young JHEP08(2024), 178		$\beta_1\beta_2\beta_3$ -> $\beta_1\beta_2\beta_3$ Three bodies Contact Term

Resonance
(Mass , Width, Pole position, Coupling)

Hamiltonian

T matrix
(Phase Shifts, inelasticity)

Lattice Spectrum

Li, Wu, Lee, Young
JHEP08(2024), 178

Introduction of NPHF

Infinite - Finite connection

$$T = V + VG_2T \qquad T^L = T + TG_2^LT^L$$
$$T^L = V + VG_2^BT^L \qquad G_2^L \equiv G_2^B - G_2$$

Finite Volume： Rest frame -> Moving frame

$$T^L(p_f^*,p_i^*;P^*) = V(p_f^*,p_i^*;P^*) + \int \frac{d^4k^*}{(2\pi)^4} V(p_f^*,k^*;P^*)G_2^B(k^*,P^*)T^L(k^*,p_i^*;P^*)$$

$$G_2^B(k^*,P^*) := \sum_{\mathbf{k}=(2\pi/L)\mathbf{n}\ (\mathbf{n}\in\mathcal{Z}^3)} \left(\frac{2\pi}{L}\right)^3 \delta^3(\mathbf{k}^*-\mathbf{k})G_2(k,P^*)$$

$$T^L(p_f^*,p_i^*;P^*) = V(p_f^*,p_i^*;P^*) + \int \frac{dk_0^*}{2\pi}\frac{1}{L^3} \sum_{\mathbf{k}^*=\frac{2\pi\mathbf{n}}{L},\,\mathbf{n}\in\mathcal{Z}^3} V(p_f^*,k^*;P^*)G_2(k^*,P^*)T^L(k^*,p_i^*;P^*)$$

$$\int \frac{d^3k^*}{(2\pi)^3} \to \int \frac{d^3k^r}{(2\pi)^3}\mathcal{J}^r \to \frac{1}{L^3}\sum_{\mathbf{k}^r}\mathcal{J}^r$$

$$\int \frac{dk_0^*}{2\pi}\left(\frac{1}{L^3}\sum_{\mathbf{k}^r}\right)\mathcal{J}^r \to \int \frac{dk_0^*}{2\pi}\int \frac{d^3k^r}{(2\pi)^3}\mathcal{J}^r + \int \frac{dk_0^*}{2\pi}\left(\frac{1}{L^3}\sum_{\mathbf{k}^r} - \int \frac{d^3k^r}{(2\pi)^3}\right)\mathcal{J}^r$$

$$T^{r,L}(p_f^*,p_i^*;P^*) = T(p_f^*,p_i^*;P^*) + \int \frac{dk_0^*}{2\pi}\left(\frac{1}{L^3}\sum_{\mathbf{k}^r} - \int \frac{d^3k^r}{(2\pi)^3}\right)$$
$$\times \mathcal{J}^r T(p_f^*,k^*;P^*)G_2(k^*,P^*)T^{r,L}(k^*,p_i^*;P^*)$$

Infinite Volume： Three-dimensional reduction

$$T(p_f^*,p_i^*;P^*) = V(p_f^*,p_i^*;P^*) + \int \frac{d^4k^*}{(2\pi)^4} V(p_f^*,k^*;P^*)G_2(k^*;P^*)T(k^*,p_i^*;P^*)$$

$$G_2(k;P) = \frac{1}{(k^2-m_1^2+i\epsilon)}\frac{1}{((P-k)^2-m_2^2+i\epsilon)}$$

$$(1)\ k_0^* = -\omega_1(\mathbf{k}^*) + i\epsilon,\ (2)\ k_0^* = P_0^* - \omega_2(\mathbf{k}^*) + i\epsilon$$

$$G_2(k^*;P^*) \to \frac{1}{-2\omega_1(\mathbf{P}^*-\mathbf{k}^*)}\frac{1}{P_0^*+\omega_1(\mathbf{k}^*)-\omega_2(\mathbf{k}^*)}\frac{(2\pi)i\,\delta(k_0^*+\omega_1(\mathbf{k}^*))}{P_0^*+\omega_1(\mathbf{k}^*)+\omega_2(\mathbf{k}^*)}$$
$$+\frac{1}{2\omega_2(\mathbf{k}^*)}\frac{1}{P_0^*+\omega_1(\mathbf{k}^*)-\omega_2(\mathbf{k}^*)}\frac{(2\pi)i\,\delta(k_0^*-(P_0^*-\omega_2(\mathbf{k}^*)))}{P_0^*-\omega_1(\mathbf{k}^*)-\omega_2(\mathbf{k}^*)+i\epsilon}$$

$$T(\mathbf{p}_f^*,\mathbf{p}_i^*;E^*(q)) = V(\mathbf{p}_f^*,\mathbf{p}_i^*;E^*(q))$$
$$+i\int \frac{d^3k^*}{(2\pi)^3}\frac{V(\mathbf{p}_f^*,\mathbf{k}^*;E^*(q)}{4\omega_1(\mathbf{k}^*)\omega_2(\mathbf{k}^*)}\frac{T(\mathbf{k}^*,\mathbf{p}_f^*;E^*(q))}{E^*(q)-\omega_1(\mathbf{k}^*)-\omega_2(\mathbf{k}^*)+i\epsilon}$$

$$T^{r,L}(\mathbf{p}_f^*,\mathbf{p}_i^*;E^*(q)) = T(\mathbf{p}_f^*,\mathbf{p}_i^*;E^*(q)) + i\left(\frac{1}{L^3}\sum_{\mathbf{k}^r} - \int \frac{d^3k^r}{(2\pi)^3}\right)$$
$$\times \mathcal{J}^r \frac{T(\mathbf{p}_f^*,\mathbf{k}^*;E^*(q))}{4\omega_1(\mathbf{k}^*)\omega_2(\mathbf{k}^*)}\frac{T^{r,L}(\mathbf{k}^*,\mathbf{p}_i^*;E^*(q))}{E^*(q)-\omega_1(\mathbf{k}^*)-\omega_2(\mathbf{k}^*)+i\epsilon}$$

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Introduction of NPHF

$$\int \frac{d^3k^*}{(2\pi)^3} \to \int \frac{d^3k^r}{(2\pi)^3}\mathcal{J}^r \to \frac{1}{L^3}\sum_{\mathbf{k}^r}\mathcal{J}^r$$

$$\mathbf{k}^r = (k^r_{\parallel}, \mathbf{k}^r_{\perp}) = (\gamma\,\beta\,b^* + \gamma\,k^*_{\parallel}, \mathbf{k}^*_{\perp}) \equiv \mathcal{A}\,\mathbf{k}^*_{\parallel} + \mathcal{B}\,\mathbf{P} + \mathbf{k}^*_{\perp}\,,$$

$$\beta = \frac{|\mathbf{P}|}{\sqrt{a^{*\,2} + \mathbf{P}^2}}\,, \qquad \mathcal{A} = \gamma = \frac{\sqrt{a^{*\,2} + \mathbf{P}^2}}{a^*}\,, \qquad \mathcal{B} = \frac{b^*}{a^*}$$

$$\mathbf{k}^r = \frac{\sqrt{\left(\omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*)\right)^2 + \mathbf{P}^2}}{\omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*)}\mathbf{k}^*_{\parallel} + \frac{\omega_1(\mathbf{k}^*)}{\omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*)}\mathbf{P} + \mathbf{k}^*_{\perp}\,,$$

$$\mathbf{k}^* = \frac{\omega_1(\mathbf{k}^r) + \omega_2(\mathbf{P} - \mathbf{k}^r)}{\sqrt{\left(\omega_1(\mathbf{k}^r) + \omega_2(\mathbf{P} - \mathbf{k}^r)\right)^2 - \mathbf{P}^2}}\mathbf{k}^r_{\parallel} - \frac{\omega_1(\mathbf{k}^r)}{\sqrt{\left(\omega_1(\mathbf{k}^r) + \omega_2(\mathbf{P} - \mathbf{k}^r)\right)^2 - \mathbf{P}^2}}\mathbf{P} + \mathbf{k}^r_{\perp}\,,$$

$$\mathcal{J}^r = \left|\frac{\partial \mathbf{k}^*}{\partial \mathbf{k}^r}\right| = \frac{\omega_1(\mathbf{k}^*)\omega_2(\mathbf{k}^*)}{\omega_1(\mathbf{k}^*) + \omega_2(\mathbf{k}^*)}\frac{\omega_1(\mathbf{k}^r) + \omega_2(\mathbf{P} - \mathbf{k}^r)}{\omega_1(\mathbf{k}^r)\omega_2(\mathbf{P} - \mathbf{k}^r)}.$$

Reduce the dimensions of Hamiltonian

$H =$

	$\|B\rangle$	$\|\alpha_{ij}\beta_k\rangle$	$\|\beta_1\beta_2\beta_3\rangle$
$\|B\rangle$	diagonal	B -> $\alpha_{ij}\beta_k$ decay	B -> $\beta_1\beta_2\beta_3$ decay
$\|\alpha_{ij}\beta_k\rangle$		$\alpha_{ij}\beta_k$ -> $\alpha'_{ij}\beta'_k$ Rest Frame two body	α_{ij} -> $\beta_i\beta_j$ Boost Frame decay
$\|\beta_1\beta_2\beta_3\rangle$			$\beta_1\beta_2\beta_3$ -> $\beta_1\beta_2\beta_3$ Three bodies Contact Term

$$H_I = \begin{bmatrix} V^{\mathrm{F}}_{\{p_f,p_i\}} & V^{\mathrm{F}}_{\{p_f,p_i'\}} \\ V^{\mathrm{F}}_{\{p_f',p_i\}} & V^{\mathrm{F}}_{\{p_f',p_i'\}} \end{bmatrix} \rightarrow \begin{bmatrix} V^{\mathrm{F}}_{\{\xi,\xi\}} & V^{\mathrm{F}}_{\{\xi,\xi'\}} \\ V^{\mathrm{F}}_{\{\xi',\xi\}} & V^{\mathrm{F}}_{\{\xi',\xi'\}} \end{bmatrix} \rightarrow \begin{bmatrix} V^{\Gamma}_{\{\xi,\xi\}} & 0 & V^{\Gamma}_{\{\xi,\xi\}} & 0 \\ 0 & V^{\Gamma'}_{\{\xi,\xi\}} & 0 & V^{\Gamma'}_{\{\xi,\xi\}} \\ V^{\Gamma}_{\{\xi',\xi\}} & 0 & V^{\Gamma}_{\{\xi',\xi'\}} & 0 \\ 0 & V^{\Gamma'}_{\{\xi',\xi\}} & 0 & V^{\Gamma'}_{\{\xi',\xi'\}} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} V^{\Gamma}_{\{p_f,p_i\}} & 0 \\ 0 & V^{\Gamma'}_{\{p_f,p_i\}} \end{bmatrix}$$

1. Find a complete set of reference momenta (Isospin)
2. Solve the I-matrix eigenvalue equation
3. Write the finite volume potential for specific irrep
4. Solve the Hamiltonian eigenvalue problem

1. Find a complete set of reference momenta

$$\mathcal{H} = \oplus_{\vec{n}^o} \mathcal{S}(\vec{n}^o) \qquad \mathcal{S}(\vec{n}^o) \equiv \{\vec{n} | \vec{n} = g\vec{n}^o, g \in G\}$$

Two-body

$$|\vec{n}, \underline{\lambda_1}, \underline{\lambda_2}; \vec{d}\rangle = \sum_{\lambda_1,\lambda_2} D^{s_1}_{\lambda_1,\underline{\lambda_1}}(R_{\rm st}(\vec{n})) D^{s_2}_{\lambda_2,\underline{\lambda_2}}(R_{\rm st}(\vec{d}-\vec{n}))\, |\vec{n}, \lambda_1, \lambda_2\rangle$$

Helicity form

$$g\,|\vec{n}, \underline{\lambda_1}, \underline{\lambda_2}; \vec{d}\rangle = e^{-i\left(\underline{\lambda_1}\varphi_w(\vec{n},g) + \underline{\lambda_2}\varphi_w(\vec{d}-\vec{n},g)\right)}\, |g\vec{n}, \underline{\lambda_1}, \underline{\lambda_2}; g\vec{d}\rangle \quad , \forall g \in O_h^+$$

$$\mathbb{P}\,|\vec{n}, \underline{\lambda_1}, \underline{\lambda_2}; \vec{d}\rangle = \eta\, e^{i\pi(\Delta(\vec{n})s_1 + \Delta(\vec{d}-\vec{n})s_2)}\, |-\vec{n}, -\underline{\lambda_1}, -\underline{\lambda_2}; -\vec{d}\rangle$$

$$\mathcal{S}(\vec{n}^o) = \begin{cases} \sum_{\underline{\lambda_1}\geq 0}^{s_1} \sum_{\underline{\lambda_2}=-s_2}^{s_2} \oplus \mathcal{S}_{\underline{\lambda_1},\underline{\lambda_2}}(\vec{n}^o) = \sum_{\underline{\lambda_1}=-s_1}^{s_1} \sum_{\underline{\lambda_2}\geq 0}^{s_2} \oplus \mathcal{S}_{\underline{\lambda_1},\underline{\lambda_2}}(\vec{n}^o) & \text{if } \mathrm{LG}(\vec{n}^o)^- \neq \emptyset \\ \sum_{\underline{\lambda_1}=-s_1}^{s_1} \sum_{\underline{\lambda_2}=-s_2}^{s_2} \oplus \mathcal{S}_{\underline{\lambda_1},\underline{\lambda_2}}(\vec{n}^o) & \text{otherwise} \end{cases}$$

$$\mathcal{S}_{\underline{\lambda_1},\underline{\lambda_2}}(\vec{n}^o) = \mathrm{span}\left\{g\,|\vec{n}^o, \underline{\lambda_1}, \underline{\lambda_2}\rangle\,,\, g \in G\right\}$$

Here reference momentum $\vec{n}^o$ is for $(\vec{n}_1, \vec{d} - \vec{n}_1)$ only 7 types with $\vec{d} = 0$, (000), (00a), (0aa), (aaa), (0ab), (aab), (abc)

Three-body

Here reference momentum $\vec{n}^o$ now is for $(\vec{n}_1, \vec{n}_2, \vec{d} - \vec{n}_1 - \vec{n}_2)$ which are 54 types when $\vec{d} = 0$

$$\mathcal{S}(\{\vec{n}^o\}) = \mathrm{span}\left\{g\,|\{\vec{n}^o\}\rangle\,,\, g \in G\right\}$$

Therefore, instead of focusing on only few patterns as in the two-particle case, we must now perform calculation for each momentum.

1. Find a complete set of reference momenta

$$\mathcal{H} = \oplus_{\vec{n}^o} \mathcal{S}(\vec{n}^o)$$

+ Isospin space projection

Two-body

$$|\vec{n}, \lambda_1, \lambda_2\rangle \rightarrow |\vec{n}; I, I_z; \lambda_1, \lambda_2\rangle = \sum_{i,j} C^{II_z}_{I_1 i; I_1 j} |\pi^i(\vec{n}), \lambda_1; \pi^j(\vec{d}-\vec{n}), \lambda_2\rangle$$

$$|\vec{d}-\vec{n}; I, I_z; \lambda_2, \lambda_1\rangle = (-1)^{I-2I_1+2s} |\vec{n}; I, I_z; \lambda_1, \lambda_2\rangle \qquad \mathcal{S}_{\underline{\lambda_1},\underline{\lambda_2}}(\vec{n}^o) \rightarrow \text{span}\{g\,|\vec{n}^o; I, I_z; \underline{\lambda_1}, \underline{\lambda_2}\rangle\,,\, g \in G\}$$

$$\mathcal{U}(\vec{n}) = \{g \in G \mid g\vec{n} = \vec{d} - \vec{n}\}$$

Case 1: $\mathcal{U}(\vec{n}^o) = \emptyset$ — same as before

Case 2: $\mathcal{U}(\vec{n}^o)^+ = \emptyset \,\&\, \mathcal{U}(\vec{n}^o)^- \neq \emptyset$

$$\mathcal{S}(\vec{n}^o) = \sum_{\underline{\lambda_1}=-s}^{s} \sum_{\underline{\lambda_2}=-\underline{\lambda_1}}^{s} \bigoplus \mathcal{S}_{\underline{\lambda_1},\underline{\lambda_2}}(\vec{n}^o)$$

Case 3: $\mathcal{U}(\vec{n}^o)^+ \neq \emptyset \,\&\, \mathcal{U}(\vec{n}^o)^- = \emptyset$

$$\mathcal{S}(\vec{n}^o) = \sum_{\underline{\lambda_1}=-s}^{s} \sum_{\underline{\lambda_2}=-s}^{\underline{\lambda_1}} \bigoplus \mathcal{S}_{\underline{\lambda_1},\underline{\lambda_2}}(\vec{n}^o) = \sum_{\underline{\lambda_2}=-s}^{s} \sum_{\underline{\lambda_1}=-s}^{\underline{\lambda_2}} \bigoplus \mathcal{S}_{\underline{\lambda_1},\underline{\lambda_2}}(\vec{n}^o)$$

Case 4: $\mathcal{U}(\vec{n}^o)^+ \neq \emptyset \,\&\, \mathcal{U}(\vec{n}^o)^- \neq \emptyset$

$$\mathcal{S}(\vec{n}^o) = \sum_{\underline{\lambda_1}\geq 0}^{s} \sum_{\underline{\lambda_2}=-s}^{\underline{\lambda_1}} \bigoplus \mathcal{S}_{\underline{\lambda_1},\underline{\lambda_2}}(\vec{n}^o) = \sum_{\underline{\lambda_2}\geq 0}^{s} \sum_{\underline{\lambda_1}=-s}^{\underline{\lambda_2}} \bigoplus \mathcal{S}_{\underline{\lambda_1},\underline{\lambda_2}}(\vec{n}^o)$$

Three-body

$$|\{\vec{n}\}; I, I_z; \Gamma_S, a\rangle = \sum_{i_{1,2,3}} C^{I,I_z,\Gamma_S}_{a\,;\,i_1,i_2,i_3} |\pi^{i_1}(\vec{n}_1)\pi^{i_2}(\vec{n}_2)\pi^{i_3}(\vec{n}_3)\rangle \qquad \sum_{i_1,i_2,i_3} \left(C^{I,I_z,\Gamma'_S}_{b\,;\,i_1,i_2,i_3}\right)^* C^{I,I_z,\Gamma_S}_{a\,;\,i_{s_1},i_{s_2},i_{s_3}} = \delta_{\Gamma'_S\Gamma_S} \mathcal{R}^{\Gamma_S}_{ba}(s) \quad ,\forall s \in S_3$$

$$s\,|\pi^{i_1}(\vec{n}_1)\pi^{i_2}(\vec{n}_2)\pi^{i_3}(\vec{n}_3)\rangle := |\pi^{i_{\bar{s}_1}}(\vec{n}_1)\pi^{i_{\bar{s}_2}}(\vec{n}_2)\pi^{i_{\bar{s}_3}}(\vec{n}_3)\rangle = |\pi^{i_1}(\vec{n}_{s_1})\pi^{i_2}(\vec{n}_{s_2})\pi^{i_3}(\vec{n}_{s_3})\rangle$$

$$\mathcal{S}^{\Gamma_S}(\{\vec{n}^o\}) = \text{span}\,\{g\,|\{\vec{n}^o\}; I, I_z; \Gamma_S, a\rangle\,, g \in G, a = 1, \cdots, \dim\Gamma_S\}$$

2. Solve the I-matrix eigenvalue equation

$$\hat{P}^{\Gamma}_{\mu\nu} = \frac{\dim\Gamma}{|G|}\sum_{g\in G}\bar{D}^{\Gamma}_{\mu\nu}(g)g \qquad \{\hat{P}^{\Gamma}_{\mu\nu}\,|\vec{n}^o,\underline{\lambda_1},\underline{\lambda_2}\rangle\ ,\nu=1,\cdots,\dim\Gamma\}$$

If $\{|n_i>\}$ are dependent, we can extract independent state from $I_{ij}=<n_i|n_j>$, by find eigenvector of I-matrix of nonzero eigenvalue.

Two-body

$$I^{\Gamma}_{\underline{\lambda_1},\underline{\lambda_2}}(\vec{n}^o)=\sum_{g\in G}\bar{D}^{\Gamma}(g)\,\langle\vec{n}^o,\underline{\lambda_1},\underline{\lambda_2}|\,g\,|\vec{n}^o,\underline{\lambda_1},\underline{\lambda_2}\rangle=\sum_{g\in \mathrm{Per}(\vec{n})}\bar{D}^{\Gamma}(g)\,\langle\vec{n}^o,\underline{\lambda_1},\underline{\lambda_2}|\,g\,|\vec{n}^o,\underline{\lambda_1},\underline{\lambda_2}\rangle$$

$$=\begin{cases}\sum\limits_{g\in \mathrm{Per}(\vec{n}^o)}\bar{D}^{\Gamma}(g)\cdot\mathcal{R}\left(\Phi(g)\right)\cdot\mathrm{sign}(\vec{n}^o;g) & \text{if } \underline{\lambda_1}=\underline{\lambda_2}=0\\ \sum\limits_{g\in \mathrm{Per}(\vec{n}^o)^+}\bar{D}^{\Gamma}(g)\cdot\mathcal{R}\left(\Phi(g)\right)\cdot e^{-i\underline{\lambda}\left(\varphi_w(\vec{n},g)+\varphi_w(\vec{d}-\vec{n},g)\right)} & \text{if } \underline{\lambda_1}=\underline{\lambda_2}=\underline{\lambda}\neq 0\\ \sum\limits_{g\in \mathrm{Per}(\vec{n}^o)^P}\bar{D}^{\Gamma}(g)\cdot\mathcal{R}\left(\Phi(g)\right)\cdot e^{-i\underline{\lambda}\left(\varphi_w(\vec{n},g)-\varphi_w(\vec{d}-\vec{n},g)\right)}\cdot\mathrm{sign}(\vec{n}^o;g) & \text{if } \underline{\lambda_1}=-\underline{\lambda_2}=\underline{\lambda}\neq 0\\ \sum\limits_{g\in \mathrm{LG}(\vec{n})^+}\bar{D}^{\Gamma}(g)\cdot e^{-i\left(\underline{\lambda_1}\varphi_w(\vec{n},g)+\underline{\lambda_2}\varphi_w(\vec{d}-\vec{n},g)\right)} & \text{otherwise}\end{cases}$$

$$\mathrm{sign}(\vec{n};g)=\begin{cases}1 & \text{if } g\in G^+\\ e^{-i\pi\left(\Delta(\vec{n})s_1+\Delta(\vec{d}-\vec{n})s_2\right)} & \text{if } g\in G^-\end{cases}$$

Three-body

$$I^{\Gamma,\Gamma_S}_{b\nu',a\nu}(\{\vec{n}^o\})=\frac{|G|}{\dim\Gamma}\,\langle\{\vec{n}^o\};\Gamma_S,b|\,\hat{P}^{\Gamma\dagger}_{\mu\nu'}\hat{P}^{\Gamma}_{\mu\nu}\,|\{\vec{n}^o\};\Gamma_S,a\rangle\ =\sum_{g\in G}\langle\{\vec{n}^o\};\Gamma_S,b|g|\{\vec{n}^o\};\Gamma_S,a\rangle\bar{D}^{\Gamma}_{\nu'\nu}(g)$$

$$=\sum_{g\in \mathrm{Per}(\{\vec{n}^o\})}\langle\{\vec{n}^o\};\Gamma_S,b|g|\{\vec{n}^o\};\Gamma_S,a\rangle\bar{D}^{\Gamma}_{\nu'\nu}(g)$$

3. Write the finite volume potential for specific irrep

Two-body/ Three-body

I => eigen-vector => basis => potential

$$I_{b\nu',a\nu} = \sum_{g\in \mathrm{Per}(\{\vec{n}^r\})} \langle \alpha_b(\{\vec{n}^r,\,\lambda\}_N)|\,\hat{g}\,|\alpha_a(\{\vec{n}^r,\,\lambda\}_N)\rangle\, D^*_{\nu'\nu}(g)$$

$$I\,c^r = \lambda^r\,c^r$$

$$|\Gamma, r, \mu\rangle = \mathcal{N}\sum_{b,\nu}\sum_{g\in G}(c^r)_{b\nu}\, D^{\Gamma *}_{\mu\nu}(g)\,\hat{g}\;|\alpha_b(\{\vec{n}^r,\,\lambda\}_N)\rangle$$

$$\begin{aligned}
V^{\Gamma}_{\vec{n}'_o r',\vec{n}_o r} &= N_{\vec{n}_o} N_{\vec{n}'_o} \sum_{\nu,\nu'} [c^{r'}_{\vec{n}'_o}]^*_{\nu'} [c^{r}_{\vec{n}_o}]_{\nu} \langle \vec{n}'_o|\, P^{\Gamma\dagger}_{\mu\nu'} \hat{V}^L P^{\Gamma}_{\mu\nu}\,|\vec{n}_o\rangle \\
&= \sqrt{\frac{1}{|\mathrm{LG}(\vec{n}_o)||\mathrm{LG}(\vec{n}'_o)|}} \sum_{g\in G} \left([c^{r'}_{\vec{n}'_o}]^\dagger \cdot \bar{D}^{\Gamma}(g) \cdot [c^{r}_{\vec{n}_o}]\right) \langle \vec{n}'_o|\,\hat{V}^L g\,|\vec{n}_o\rangle \\
&= \sqrt{\frac{1}{|\mathrm{LG}(\vec{n}_o)||\mathrm{LG}(\vec{n}'_o)|}} \sum_{\substack{g\in \frac{G}{\mathrm{LG}(\vec{n}_o)} \\ h\in \mathrm{LG}(\vec{n}_o)}} \left([c^{r'}_{\vec{n}'_o}]^\dagger \cdot \bar{D}^{\Gamma}(g) \cdot \bar{D}^{\Gamma}(h) \cdot [c^{r}_{\vec{n}_o}]\right) \langle \vec{n}'_o|\,\hat{V}^L gh\,|\vec{n}_o\rangle \\
&= \frac{N_{\vec{n}'_o}}{N_{\vec{n}_o}} \sum_{g\in \mathrm{LC}(\vec{n}_o)} \left([c^{r'}_{\vec{n}'_o}]^\dagger \cdot \bar{D}^{\Gamma}(g) \cdot [c^{r}_{\vec{n}_o}]\right) \langle \vec{n}'_o|\,\hat{V}^L g\,|\vec{n}_o\rangle \\
&= \frac{N_{\vec{n}'_o}}{N_{\vec{n}_o}} \sum_{g\in \mathrm{LC}(\vec{n}_o)} \left([c^{r'}_{\vec{n}'_o}]^\dagger \cdot \bar{D}^{\Gamma_\eta}(g) \cdot [c^{r}_{\vec{n}_o}]\right) V^L\left(\vec{n}'_o, g\vec{n}_o\right),
\end{aligned}$$

$$V^{\mathrm{F}}_{\{\xi,\xi'\}} \to \begin{bmatrix} V^{\Gamma}_{\{\xi,\xi\}} & 0 \\ 0 & V^{\Gamma'}_{\{\xi,\xi\}} \end{bmatrix}$$

4. Solve the Hamiltonian eigenvalue problem

$\omega\sim\rho\pi\sim3\pi$ and $\rho\sim2\pi$

$H_I^{\rho}=$

	$\lvert\rho>$	$\lvert\pi\pi>$
$\lvert\rho>$	0	$\rho\to\pi\pi$
$\lvert\pi\pi>$		0

$$\langle\rho_0\pi(\vec{k},\sigma';-\vec{k})|\hat{V}|\omega_0,\sigma\rangle = \frac{-ig_{\omega\rho\pi}}{6\sqrt{2}\pi}C^{1\sigma}_{1\sigma-\sigma';1\sigma'}\mathcal{Y}_{1,\sigma-\sigma'}(\vec{k})\mathcal{J}^{\omega_0}_{\rho_0\pi}(0;k,k),$$

$$\langle\rho_0\pi(\vec{k}',\sigma';-\vec{k}')|\hat{V}|\rho_0\pi(\vec{k},\sigma;-\vec{k})\rangle = \frac{1}{4}c_1(\vec{k}'\cdot\vec{k})\delta_{\sigma\sigma'}\mathcal{J}_{\rho_0\pi\rho_0\pi}(k',k',k,k),$$

$$\langle\rho_0,\sigma|\hat{V}|2\pi(\vec{k}_1;\vec{k}_2)\rangle \equiv v_\sigma(\vec{k}_1,\vec{k}_2) \equiv v(\vec{k}_1,\vec{k}_2)Y^*_{1\sigma}(\vec{k}_1^*) = \frac{g_{\rho\pi\pi}}{\sqrt{6}\pi}\mathcal{Y}^*_{1\sigma}(\vec{k}_1^*)\mathcal{J}_{\rho_0\pi\pi}(|\vec{k}_1+\vec{k}_2|,k_1,k_2),$$

$$\langle\rho_0\pi(\vec{k}',\sigma;-\vec{k}')|\hat{V}|3\pi(\vec{k}_1;\vec{k}_2;\vec{k}_3)\rangle = \sum_{(i,j,l)}\delta^3(\vec{k}'+\vec{k}_i)\langle\rho_0(\vec{k}',\sigma)|\hat{V}|2\pi(\vec{k}_j,\vec{k}_l)\rangle.$$

$H_I^{\omega}=$

	$\lvert\omega>$	$\lvert\rho\pi>$	$\lvert 3\pi>$
$\lvert\omega>$	0	$\omega\to\rho\pi$	0
$\lvert\rho\pi>$		$\rho\pi\to\rho\pi$	$(\rho)\pi\to(\pi\pi)\pi$
$\lvert3\pi>$			0

$$F(k,\Lambda) = \left(\frac{\Lambda^2}{k^2+\Lambda^2}\right)^2$$

$\Lambda_{\rho\pi\pi},\Lambda_{\omega\rho\pi},\Lambda_{\rho\pi\rho\pi}$

fixed as 1 GeV, 1 GeV ,0.5GeV

$$\langle\vec{n}'|\hat{V}^L|\vec{n}\rangle = (2\pi/L)^m\langle p_{\vec{n}'}|\hat{V}|p_{\vec{n}}\rangle$$

parameters :

$m_\rho, m_\omega(=m_\rho+\delta m)$

$g_{\rho\pi\pi},\ g_{\omega\rho\pi},\ c_1$

Scheme (MeV)

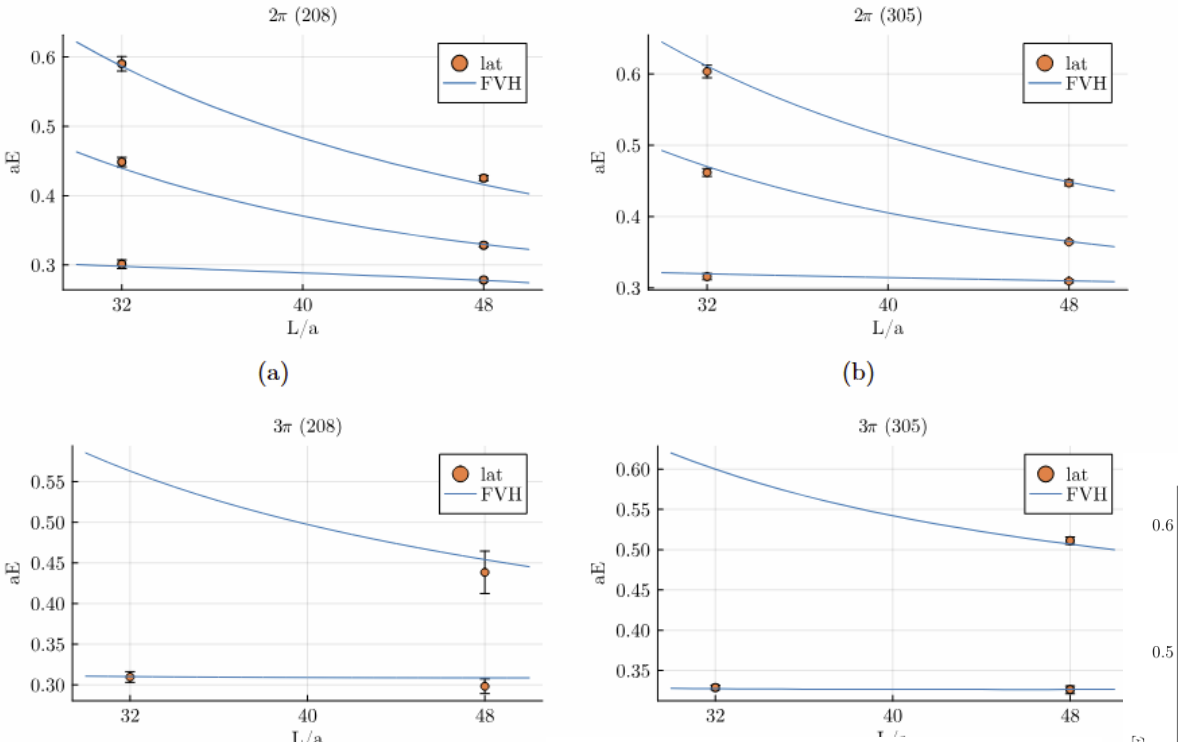
A: $\delta m=0,\quad c_1\neq0$

B1: $\delta m=0,\quad c_1=0$

B1: $\delta m=15,\quad c_1=0$

B1: $\delta m=30,\quad c_1=0$

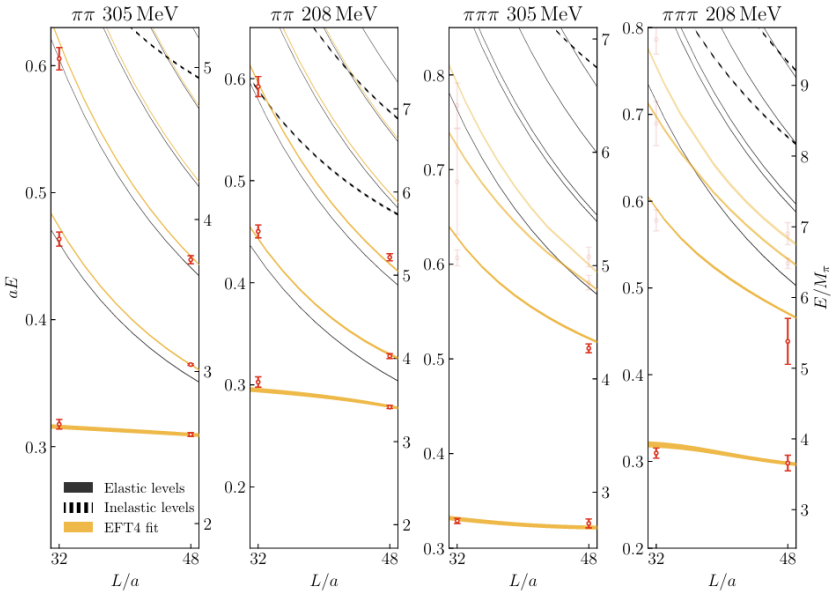
$\omega\sim\rho\pi\sim3\pi$ and $\rho\sim2\pi$



MeV	$\hat{\chi}^2$	$m_{\rho 0}^{208}$	$m_{\rho 0}^{305}$	$g_{\rho\pi\pi}$	$g_{\omega\rho\pi}$	$c_1/10^{-5}$	δm
A	1.2	831(13)	872(10)	7.70(0.45)	$29.0^{+12.2}_{-5.4}$	$1.3^{+4.4}_{-1.5}$	0(fixed)
B1	1.2	831(13)	872(10)	7.67(0.45)	26.4(3.5)	0(fixed)	0(fixed)
B2	1.1	831(13)	872(10)	7.69(0.45)	30.7(3.1)	0(fixed)	15(fixed)
B3	1.1	832(13)	872(10)	7.70(0.44)	34.4(3.0)	0(fixed)	30(fixed)

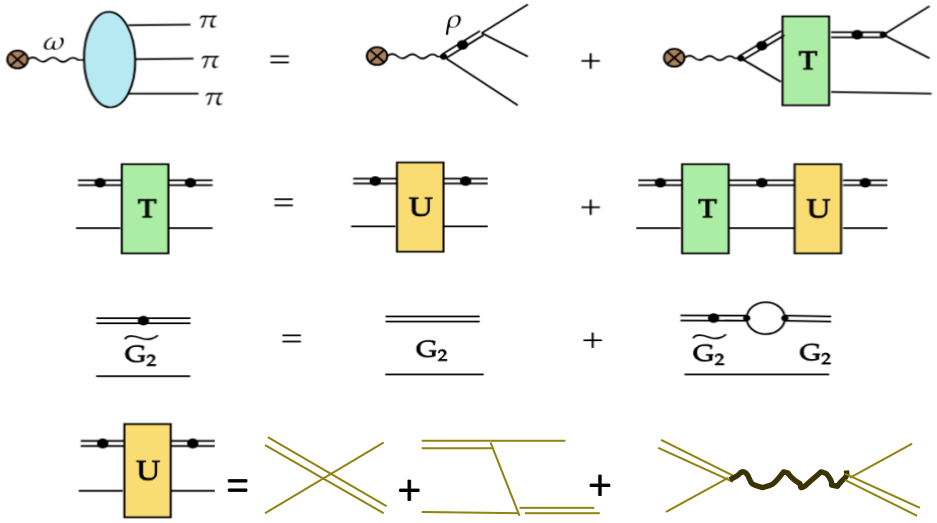
Yan, Mai, Garofalo, Meißner, Liu, Liu, and Urbach,
PRL 133, 211906(2024).

M_π/MeV	$\tilde{K}^{-1}$	c_{11}	$\chi^2_{\rm dof}$
305	$a_0 + a_1\sigma$	$\frac{c_0}{s-M_\omega^2}$	1.25
208	$a_0 + a_1\sigma$	$\frac{c_0}{s-M_\omega^2}$	1.57
305/208	$\frac{\sigma-M_\rho^2}{2g^2}$	$c_{11}^{\rm EFT}$	3.16
	$\frac{\sigma-M_\rho^2(M_V,a)}{2g^2}$	$c_{11}^{\rm EFT}$	2.29

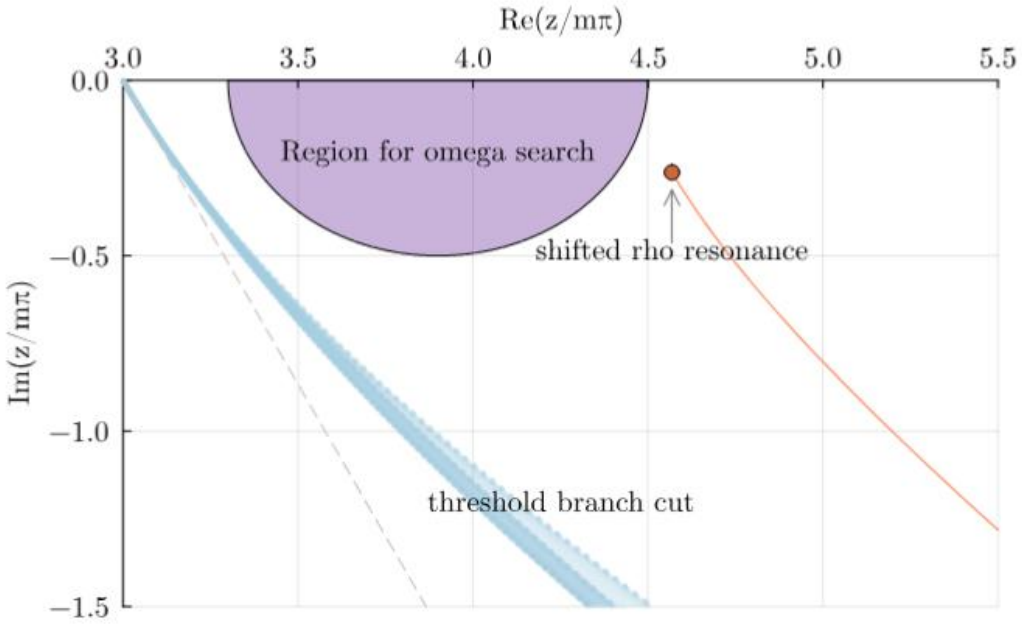


1. $m_\rho, g_{\rho\pi\pi}$ determined by ρ spectrum
2. c_1 and δm can be effectively absorbed by the coupling $g_{\omega\rho\pi}$, which mainly determined by the ω spectrum.

$\omega \sim \rho\pi \sim 3\pi$ and $\rho \sim 2\pi$



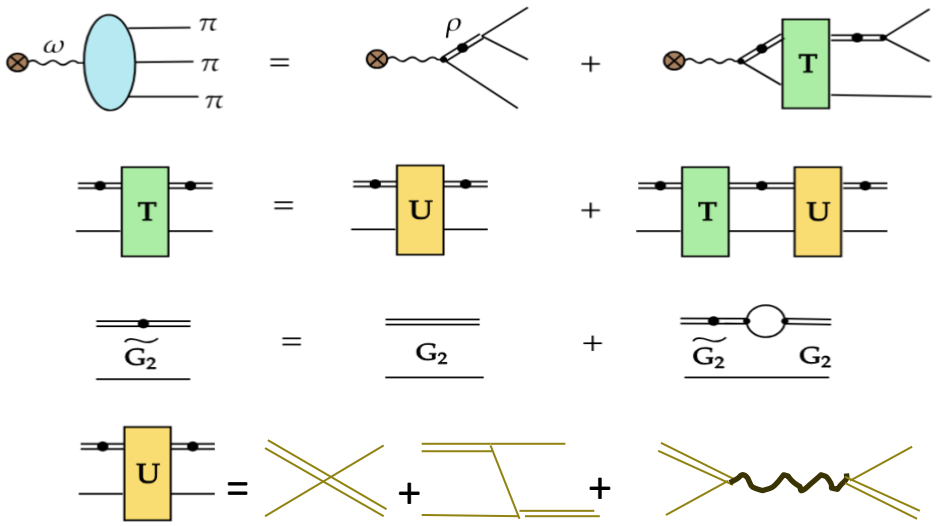
$$T(p,k;z) = U^{\text{in}}(p,k;z) + \int_0^\infty q^2 dq\; U^{\text{ker}}(p,q;z)\tilde{G}_2(q;z+i0^+)T(q,k;$$



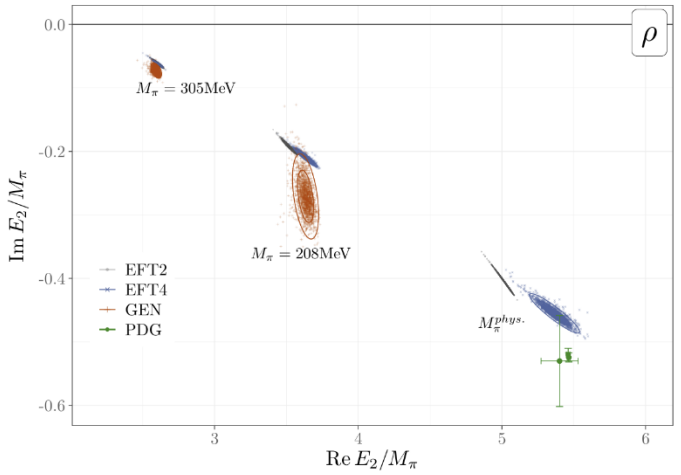
	$m_\pi = 208$ MeV		$m_\pi = 305$ MeV	
Scheme	z_ρ	z_ω	z_ρ	z_ω
A	$742.3(5.3) - i\,55.0(5.3)$	$777.9(14.0) - i\,0.31(0.09)$	$790.8(4.8) - i\,28.1(3.3)$	$823.3(5.8)$
B1	$744.8(5.5) - i\,55.3(5.6)$	$779.0(10.0) - i\,0.23(0.07)$	$792.0(4.8) - i\,27.7(3.3)$	$829.6(5.1)$
B2	$743.1(5.2) - i\,54.9(5.1)$	$776.7(10.2) - i\,0.30(0.09)$	$791.5(5.0) - i\,28.2(3.4)$	$827.2(5.4)$
B3	$743.3(5.3) - i\,55.1(5.4)$	$776.3(9.7) - i\,0.33(0.07)$	$791.0(4.9) - i\,28.2(3.5)$	$826.8(5.3)$

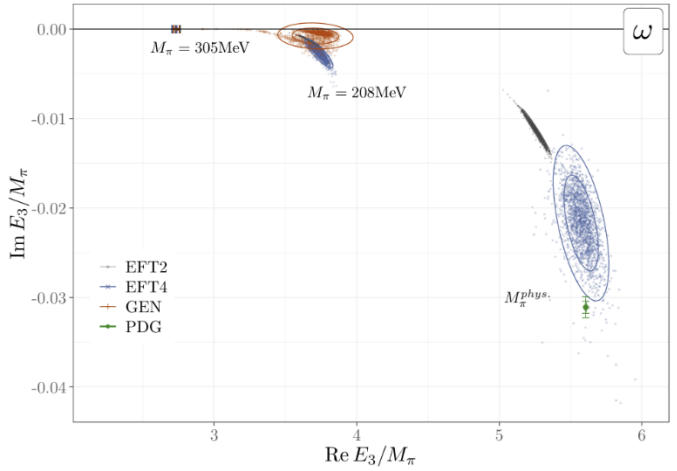
$\omega \sim \rho\pi \sim 3\pi$ and $\rho \sim 2\pi$

Yan, Mai, Garofalo, Meißner, Liu, Liu, and Urbach, PRL 133, 211906(2024).



$$T(p,k;z) = U^{\text{in}}(p,k;z) + \int_0^\infty q^2 dq\; U^{\text{ker}}(p,q;z)\tilde{G}_2(q;z+i0^+)T(q,k;z)$$





Spectral parameters of the ρ resonance from lattice QCD

CLQCD Collaboration • Zhengli Wang (UCAS, Beijing) et al. (Feb 5, 2025)

Published in: *JHEP* 08 (2025) 064 • e-Print: 2502.03700 [hep-lat]

Lattice spacing dependence ?

	$m_\pi = 208$ MeV		$m_\pi = 305$ MeV	
Scheme	z_ρ	z_ω	z_ρ	z_ω
A	$742.3(5.3) - i\,55.0(5.3)$	$777.9(14.0) - i\,0.31(0.09)$	$790.8(4.8) - i\,28.1(3.3)$	$823.3(5.8)$
B1	$744.8(5.5) - i\,55.3(5.6)$	$779.0(10.0) - i\,0.23(0.07)$	$792.0(4.8) - i\,27.7(3.3)$	$829.6(5.1)$
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B3	$743.3(5.3) - i\,55.1(5.4)$	$776.3(9.7) - i\,0.33(0.07)$	$791.0(4.9) - i\,28.2(3.5)$	$826.8(5.3)$

(3.57-i 0.26)m_π (3.73-i 0.0016)m_π (2.59 - i 0.09)m_π (2.71)m_π

$\sigma \sim 2\pi \sim 4\pi$

$H_I^{\sigma} =$

	$\lvert\sigma\rangle$	$\lvert 2\pi\rangle$	$\lvert 4\pi\rangle$
$\lvert\sigma\rangle$	0	$\boldsymbol{\sigma \to 2\pi}$	$\boldsymbol{\sigma \to 4\pi}$
$\lvert 2\pi\rangle$		0	0
$\lvert 4\pi\rangle$			0

$$\boldsymbol{\sigma \to 2\pi}: \frac{g_{\sigma\pi\pi}}{m_\pi}\left(k^2+\frac{7}{8}m_\pi^2\right)\mathcal{J}_{\pi\pi}(k,k)\,F(k,\Lambda_{2\pi})$$

$$\boldsymbol{\sigma \to (\rho\rho \to) 4\pi}$$

In S_4 group, only survive irrep is [2,2] which has two dimensions

$$-2g_{\sigma\to 4\pi} f_{\sigma\to 4\pi}(\vec{k}_1,\vec{k}_3,\vec{k}_2,\vec{k}_4)\mathcal{J}_{\sigma\pi\pi\pi\pi}(0,k_1,k_2,k_3,k_4)\prod_{i=1}^{4}F^{\frac{3}{2}}(k_i,\Lambda_{4\pi})$$

$$-2\sqrt{3}g_{\sigma\to 4\pi} f_{\sigma\to 4\pi}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\mathcal{J}_{\sigma\pi\pi\pi\pi}(0,k_1,k_2,k_3,k_4)\prod_{i=1}^{4}F^{\frac{3}{2}}(k_i,\Lambda_{4\pi})$$

$$f_{\sigma\to 4\pi}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)=(\vec{k}_1-\vec{k}_2)\cdot(\vec{k}_3-\vec{k}_4)+(\vec{k}_1-\vec{k}_4)\cdot(\vec{k}_2-\vec{k}_3)$$

parameters :

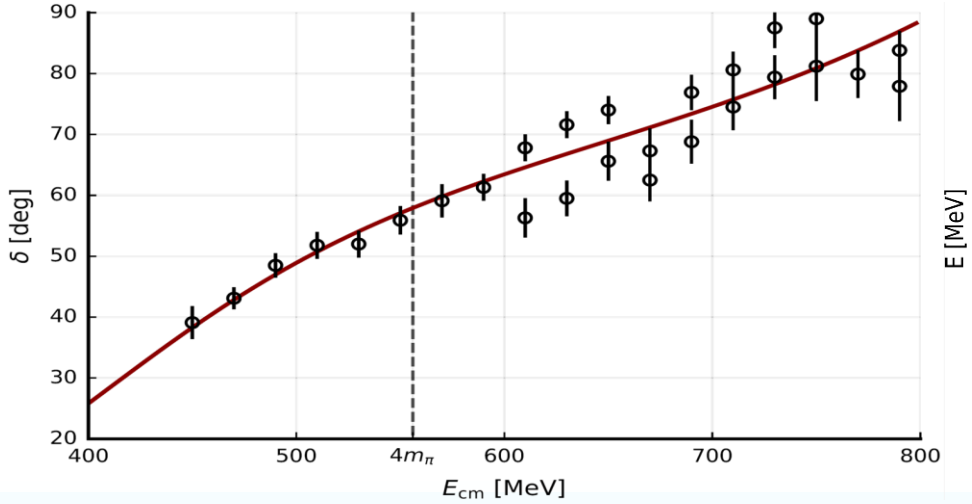
Fit: $m_\sigma = 798$ MeV

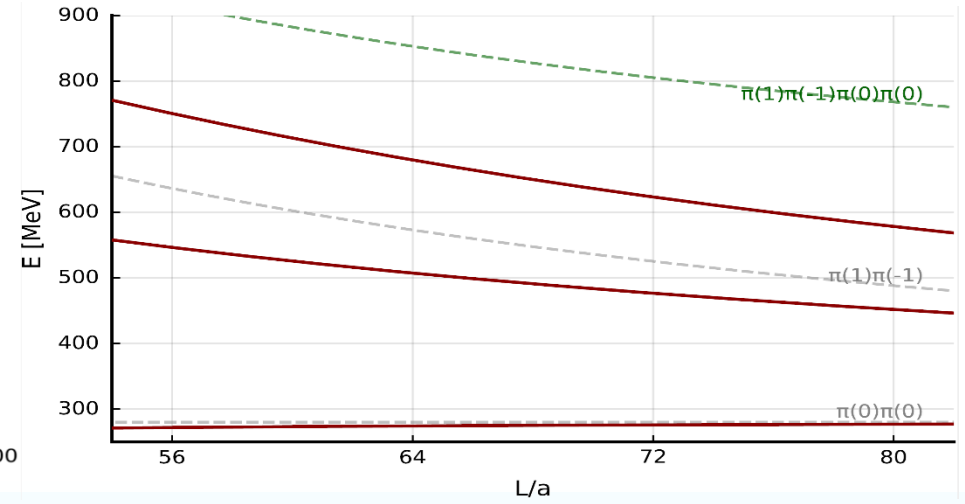
$g_{\sigma\pi\pi}$=66.9

$\Lambda_{2\pi}$=346 MeV

Fixed: $g_{\sigma\to 4\pi}$ by $\Gamma_{4\pi}$

$\Lambda_{4\pi}$=300 MeV





$\sigma \sim 2\pi \sim 4\pi$

$H_I^{\sigma} =$

	$\|\sigma>$	$\|2\pi>$	$\|4\pi>$
$\|\sigma>$	0	$\boldsymbol{\sigma \to 2\pi}$	$\boldsymbol{\sigma \to 4\pi}$
$\|2\pi>$		0	0
$\|4\pi>$			0

$$\boldsymbol{\sigma \to 2\pi}: \frac{g_{\sigma\pi\pi}}{m_\pi}\left(k^2+\frac{7}{8}m_\pi^2\right)\mathcal{J}_{\pi\pi}(k,k)\,F(k,\Lambda_{2\pi})$$

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In S_4 group, only survive irrep is [2,2] which has two dimensions

$$-2g_{\sigma\to4\pi}f_{\sigma\to4\pi}(\vec{k}_1,\vec{k}_3,\vec{k}_2,\vec{k}_4)\mathcal{J}_{\sigma\pi\pi\pi\pi}(0,k_1,k_2,k_3,k_4)\prod_{i=1}^{4}F^{\frac{3}{2}}(k_i,\Lambda_{4\pi})$$

$$-2\sqrt{3}g_{\sigma\to4\pi}f_{\sigma\to4\pi}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\mathcal{J}_{\sigma\pi\pi\pi\pi}(0,k_1,k_2,k_3,k_4)\prod_{i=1}^{4}F^{\frac{3}{2}}(k_i,\Lambda_{4\pi})$$

$$f_{\sigma\to4\pi}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)=(\vec{k}_1-\vec{k}_2)\cdot(\vec{k}_3-\vec{k}_4)+(\vec{k}_1-\vec{k}_4)\cdot(\vec{k}_2-\vec{k}_3)$$

parameters :

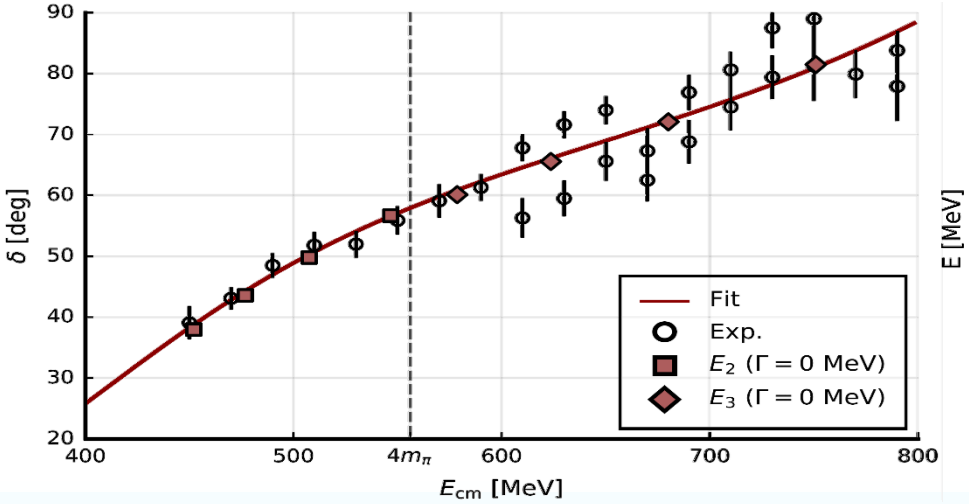
Fit: $m_\sigma = 798$ MeV

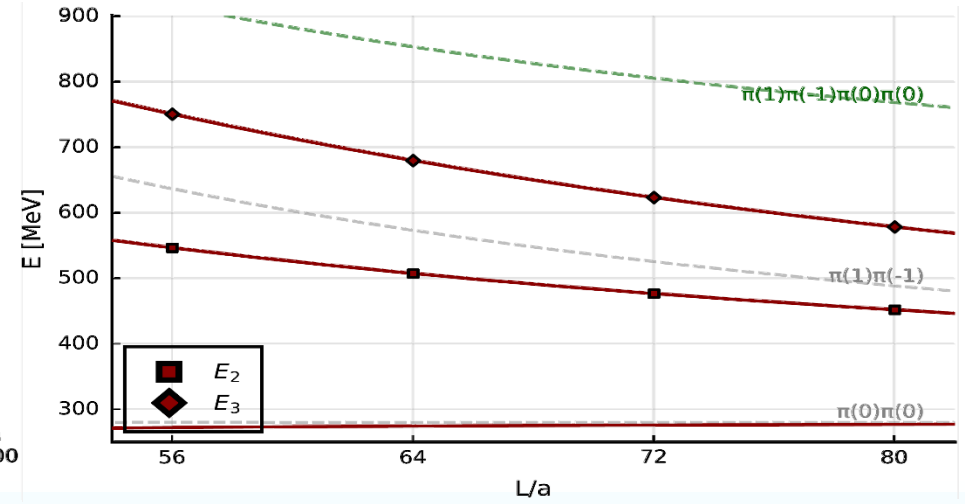
$g_{\sigma\pi\pi}$=66.9

$\Lambda_{2\pi}$=346 MeV

Fixed: $g_{\sigma\to4\pi}$ by $\Gamma_{4\pi}$

$\Lambda_{4\pi}$=300 MeV





$\sigma \sim 2\pi \sim 4\pi$

parameters :

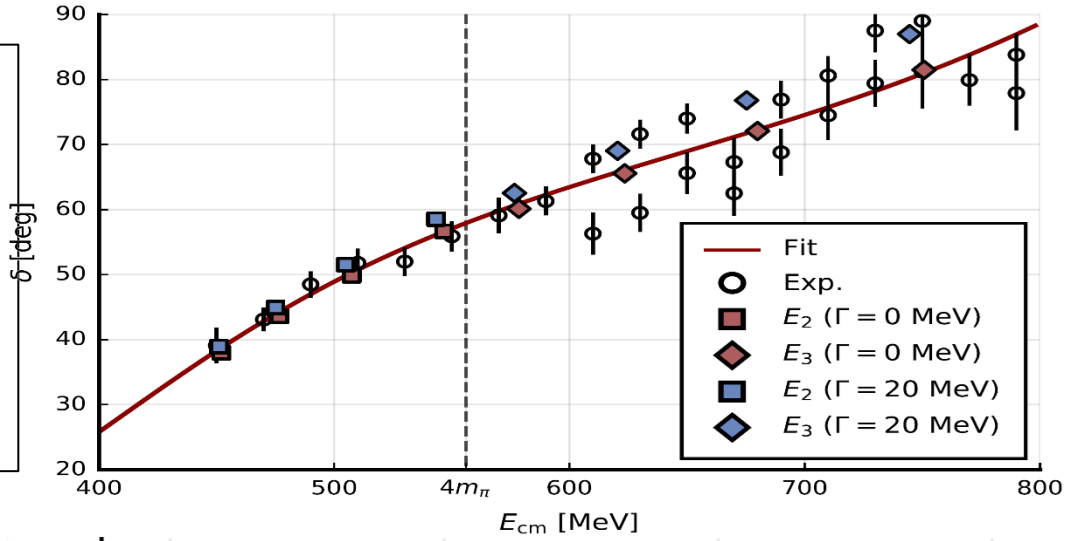
Fit: $m_\sigma = 798$ MeV

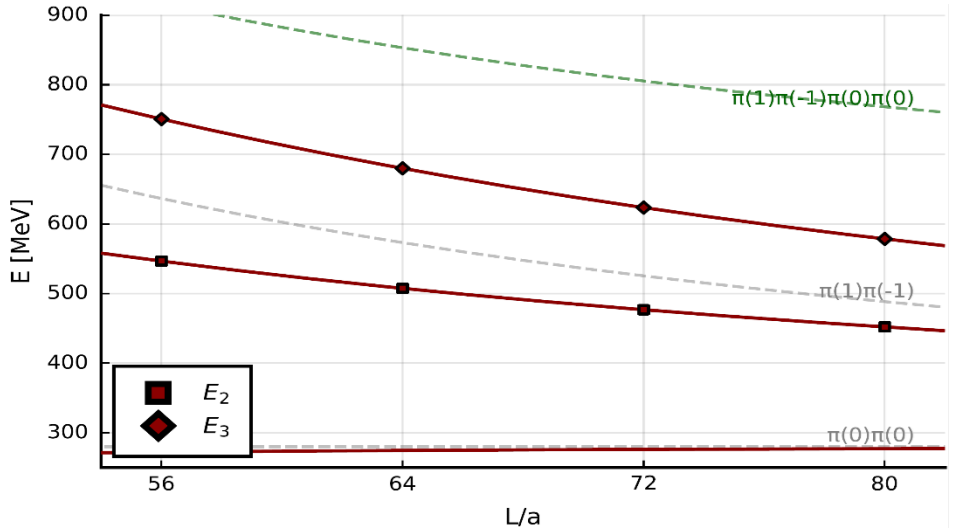
$g_{\sigma\pi\pi}$=66.9

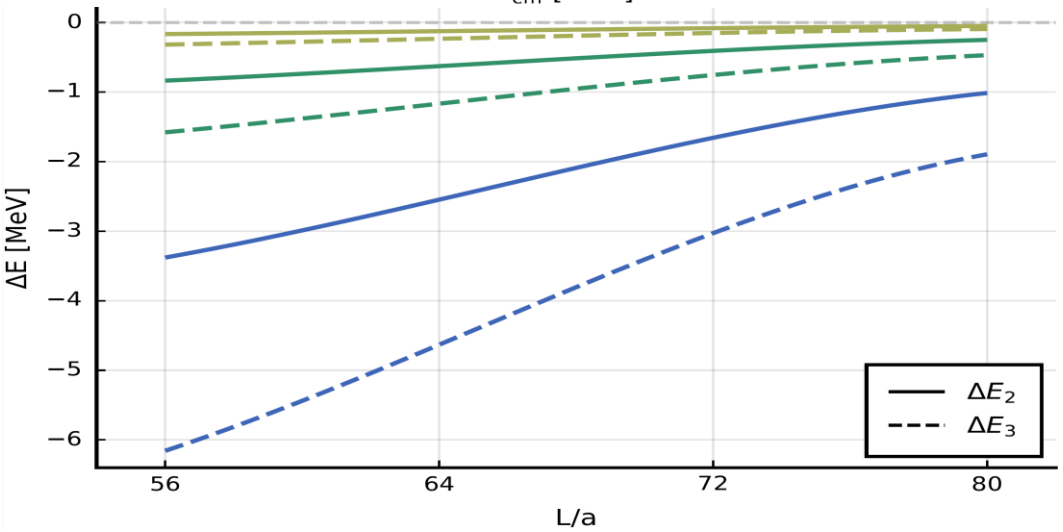
$\Lambda_{2\pi}$=346 MeV

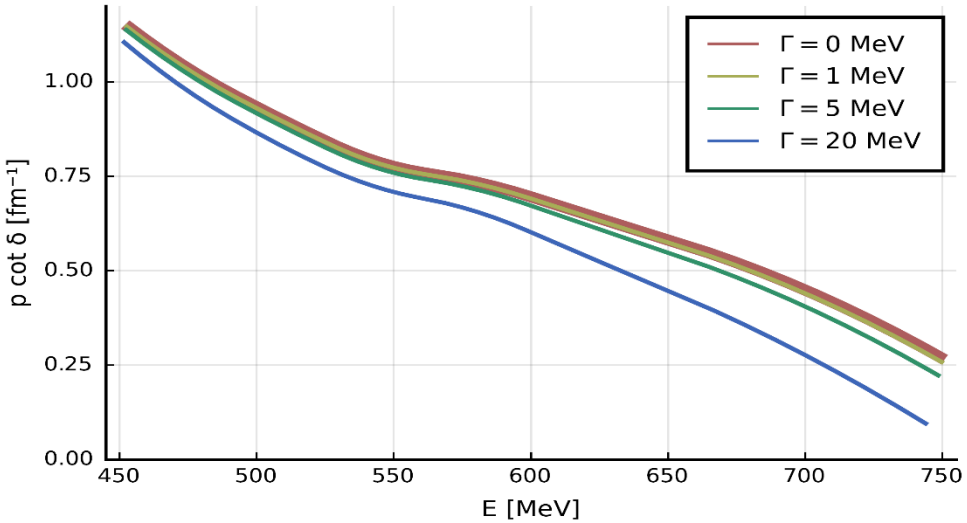
Fixed: $g_{\sigma\to 4\pi}$ by $\Gamma_{4\pi}$

$\Lambda_{4\pi}$=300 MeV









NPHFforFVE.jl: from interactions to finite-volume spectra

$N \leq 4$ · Julia 1.10+

A Julia package that turns a user-defined N-body Hamiltonian into symmetry-projected finite-volume spectra.

YOU PROVIDE

`FockChannel and Quantum Numbers`
particle contents, masses, spin, isospin,...

`Lattice Config Info`
Box size, lattice spacing (only for unit), irreps,...

`User-defined interaction`

→

THE PACKAGE HANDLES

Finite-volume Hamiltonian
Automatically constructs the many-body Hamiltonian from the supplied Fock sectors and interaction matrix elements.

Irrep projection
Projects the Hamiltonian to the selected finite-volume irreducible representation.

→

YOU OBTAIN

Low-lying levels
Ordered energy spectra for every requested irrep.

Fock-channel weights
FockChannel components of each eigenvector.

```
result = compute!(project, my_V, MyParams())
```

Given lattice spectra, the package can also **fit the parameters** of a user-defined Hamiltonian.

https://github.com/yukangUCAS/NPHFforFVE.jl (2026), gitHubrepository.

Summary

- We introduce the Non-Perturbative Hamiltonian Framework (NPHF) approach, which connects the quark-gluon level to the hadronic level via a bare state, establishing a coupled-channel model with varying numbers of hadrons to achieve a comprehensive understanding of hadron properties.
- We provide two examples: three- and four- body case

1. $\omega \sim \rho\pi \sim 3\pi$ and $\rho \sim 2\pi$, obtain the pole positions of ω and ρ from the real lattice spectrum.

2. $\sigma \sim 2\pi \sim 4\pi$, by fitting the $\pi\pi$ s-wave phase shifts and assuming $\Gamma_{4\pi}$, we provide the lattice spectrum of this four-body system.

Thank you for your attention!