Relativistic Effects in Proton-Deuteron Scattering

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Motivation

0. q–bit

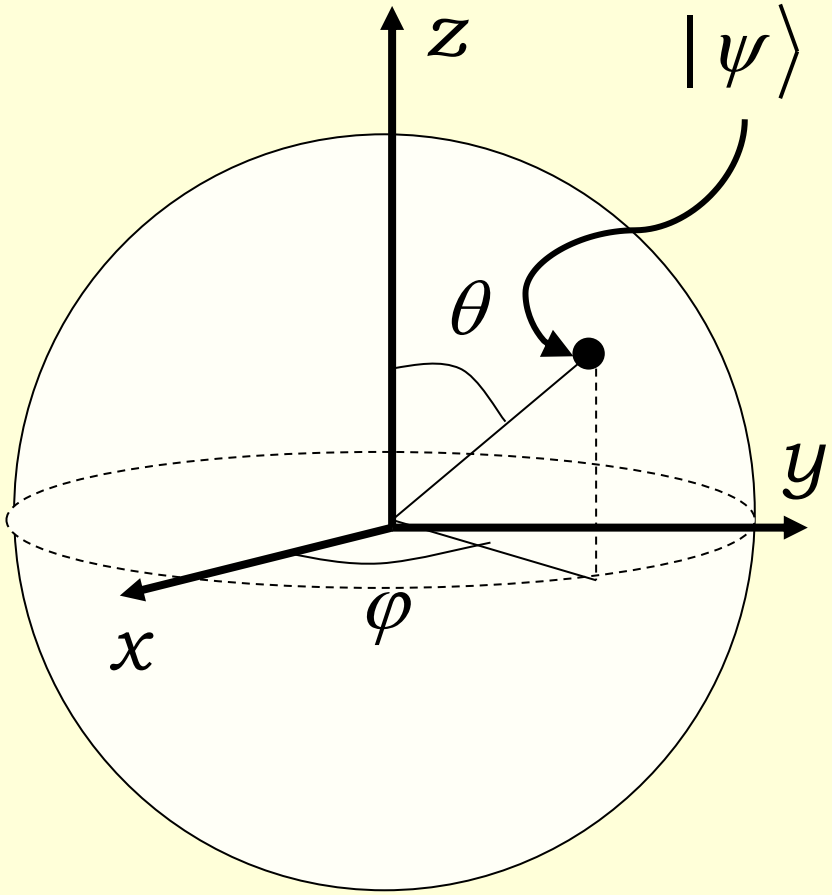
$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \alpha|0\rangle + \beta|1\rangle \qquad Spin = \frac{1}{2}$$

$$= \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

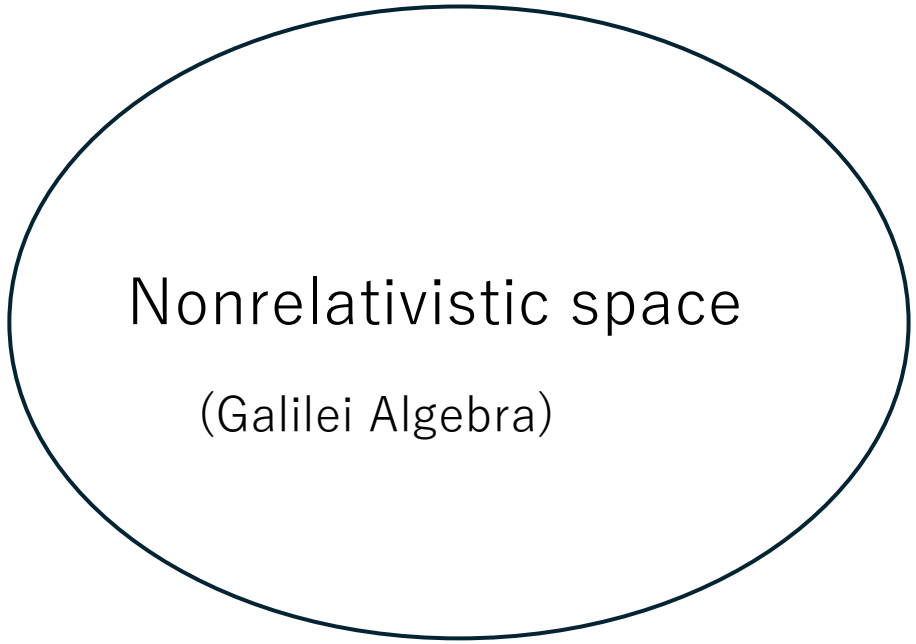
A single q-bit contains information about
the position on a sphere (latitude and longitude),
and because it is continuous,
it can provide an infinite amount of information.

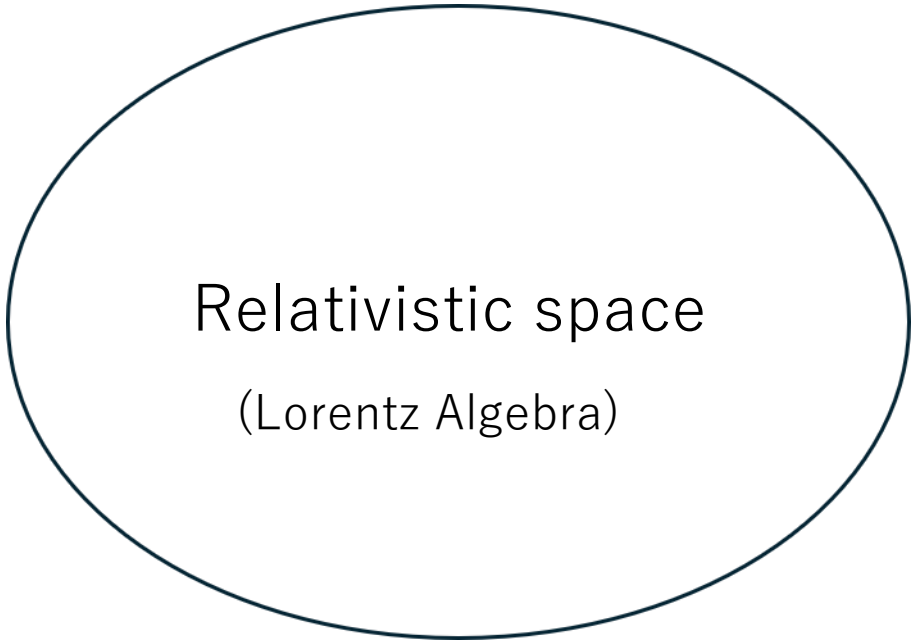
$$SU(2) \simeq S^3$$

isomorphism



Bloch globe





$$\mathfrak{se}(3) = \mathfrak{so}(3) \ltimes \mathbb{R}^3 \quad \not\simeq \quad \mathfrak{so}(1,3)$$

Mathematically totally different world !!!

Would like to make two bridges between them.



Nonrelativistic space

Relativistic space



1 Ten Generators

1.1 Classification and Fundamental Roles of the Ten Generators

These operators act as generators of infinitesimal spacetime transformations, defined and utilized as follows:

Generator	Symbol	Degrees of Freedom	Corresponding Spacetime Transformation & Physical Meaning
Time Translation	P^0 (Hamiltonian H)	1	Generates time evolution (Total Energy).
Spatial Translation	$\vec{P} = (P^1, P^2, P^3)$	3	Generates spatial shifts (Total Momentum).
Spatial Rotation	$\vec{J} = (J^1, J^2, J^3)$	3	Generates spatial axis rotations (Total Angular Momentum).
Lorentz Boost	$\vec{K} = (K^1, K^2, K^3)$	3	Generates transitions to inertial frames moving at different velocities.

Table 1: Classification of Poincaré Generators

By satisfying the commutation relations of the Poincaré group (Poincaré algebra), these 10 generators guarantee that physical laws remain identical regardless of the inertial frame of reference (Poincaré invariance / Lorentz invariance).

1.2 Specific Uses in Quantum Mechanics and Multi-Body Systems

1. **Construction of Finite Transformations (Unitary Operators):**
   A finite spacetime transformation operator $U(\Lambda, a)$ that shifts coordinates by a^μ and applies a Lorentz transformation Λ is constructed using the generators via the exponential map:

$$U(\Lambda, a) = \exp\left(-ia_\mu P^\mu - \frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}\right)$$

   (where $M^{\mu\nu}$ is an antisymmetric tensor unifying spatial rotations $\vec{J}$ and boosts $\vec{K}$).

   Assuming the Minkowski metric signature $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, the components of $M_{\mu\nu}$ are explicitly defined as follows:

   Explicitly written in matrix form:

$$M_{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & -J_3 & J_2 \\ -K_2 & J_3 & 0 & -J_1 \\ -K_3 & -J_2 & J_1 & 0 \end{pmatrix} \tag{1}$$

   The Poincaré algebra incorporates the 4-momentum generator $P_\mu = (P_0, \vec{P}) = (H, \vec{P})$, where $P_0 = H$ is the Hamiltonian and $\vec{P}$ is the spatial momentum.

2. **Classification of Particle States and Definition of Spin:**
   The invariant Casimir operators of the Poincaré group—specifically the squared mass operator $P^2 = P_\mu P^\mu = M^2$ and the Pauli-Lubanski vector W^2—are used to rigorously label and classify quantum states by their **mass** m and **spin** s eigenvalues.

3. **Incorporation of Interactions and Classification of Dynamics (Dirac Forms / BT Theory):**
   When extending from free-particle systems to interacting systems, the choice of which generators incorporate the interaction potential V plays a decisive role:

2 Rotation and Translation

The spacetime structure of "rotation" and "translation" in non-relativistic quantum mechanics is described by the Lie algebra of the **Galilei group**, rather than the Lorentz group.

In the Galilei algebra, the fundamental building blocks consist of the 3-dimensional spatial rotation generators $\vec{J}$, spatial translation generators $\vec{P}$, the time translation (Hamiltonian) generator H, and Galilean boost generators $\vec{K}$.

2.1 Fundamental Commutation Relations of the Galilei Lie Algebra

The primary commutation relations focusing on the rotation generators J_i and spatial translation generators P_i are as follows:

- **Commutation Relations Between Rotations:**

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k \tag{2}$$

  This reflects the algebraic structure of the rotation group $SO(3)$ (or its double cover $SU(2)$).

- **Commutation Relations Between Rotation and Spatial Translation:**

$$[J_i, P_j] = i\hbar\epsilon_{ijk}P_k \tag{3}$$

  This commutation relation signifies that the spatial translation generator $\vec{P}$ transforms as a 3-dimensional vector under spatial rotations.

- **Commutation Relations Between Spatial Translations:**

$$[P_i, P_j] = 0 \tag{4}$$

  Spatial translations in three orthogonal directions are **mutually commutative** (abelian group $\mathbb{R}^3$). Unlike Lorentz transformations, swapping the order of spatial shifts along different axes introduces no geometric discrepancy or curvature.

2.2 Algebraic Structure of Rotations and Translations: Semi-Direct Product

The combination of the rotation algebra $\mathfrak{so}(3)$ and the translation algebra $\mathbb{R}^3$ forms the Lie algebra of the **Euclidean group** $SE(3)$.

Because the right-hand side of $[J_i, P_j] = i\hbar\epsilon_{ijk}P_k$ yields the translation generator P_k once again, the set of translation generators $\vec{P}$ constitutes an ideal (a cyclic subalgebra) within the overall algebra.

As a Lie algebra structure, it takes the form of a **semi-direct product** rather than a direct product:

$$\mathfrak{se}(3) = \mathfrak{so}(3) \ltimes \mathbb{R}^3 \tag{5}$$

This accurately captures the non-commutative composition rule of geometric transformations, where a rotation is applied prior to a translation ($E(\vec{R}, \vec{a}) = \text{Translation}(\vec{a}) \cdot \text{Rotation}(\vec{R})$).

2.3 Comparison with Galilean Boosts ($\vec{K}$) and Relativistic Differences

The decisive structural distinction between non-relativistic (Galilei algebra) and relativistic (Lorentz algebra) mechanics manifests in the commutation relation between boost generators $\vec{K}$ and spatial translations $\vec{P}$:

Item	**Non-Relativistic (Galilei Algebra)**	**Relativistic (Lorentz Algebra $\mathfrak{so}(1,3)$)**
Translation Relations $[P_i, P_j]$	$= 0$ (Commutative)	$= 0$ (Commutative)
Rotation and Translation $[J_i, P_j]$	$= i\hbar\epsilon_{ijk}P_k$ (Vector)	$= i\hbar\epsilon_{ijk}P_k$ (Vector)
Boosts Between Themselves $[K_i, K_j]$	$= 0$ **(Commutative)**	$= -i\hbar\epsilon_{ijk}J_k$ **(Rotation generated)**

Table 2: Comparison of Commutation Structures Between Galilean and Lorentz Algebras

In relativity, combining boosts in two non-collinear directions generates a spatial rotation (Wigner rotation). In non-relativistic mechanics, however, boost generators commute with one another, allowing boost and rotation transformations to decouple far more simply.

In non-relativistic mechanics, "rotation and translation" possess a **semi-direct product structure** $SO(3) \ltimes \mathbb{R}^3$**, where the rotation group** $SO(3)$ **acts vectorially on the mutually commutative 3-dimensional translation group** $\mathbb{R}^3$.

Due to this structure, non-relativistic quantum mechanics can straightforwardly treat total momentum $\vec{P}$ and total angular momentum $\vec{J}$ as independent, interaction-free operators.

3. **Incorporation of Interactions and Classification of Dynamics (Dirac Forms / BT Theory):**
When extending from free-particle systems to interacting systems, the choice of which generators incorporate the interaction potential V plays a decisive role:

- **Instant Form:** Interactions are restricted to P^0 (Hamiltonian) and $\vec{K}$ (boosts), leaving $\vec{P}$ and $\vec{J}$ as free operators.
- **Front Form:** Formulated on a light-front surface ($x^+ = 0$), fixing seven of the generators as kinematic (free) operators.

3 Instant Form and Front Form e.t.c.

Beyond the **Instant Form** and **Front Form**—which are frequently employed in frameworks such as Bakamjian-Thomas theory—there exist other fundamental approaches for incorporating interactions into Poincaré generators.

Representative examples include the **Point Form**, which serves as the final entry in the three fundamental forms of relativistic dynamics proposed by Paul Dirac in 1949, as well as the **Covariant Form**, which is standard in quantum field theory.

Below is a concise summary of how interactions are incorporated into the generators across these different forms.

The method of incorporating interactions into Poincaré generators is categorized by Dirac's classification, depending on **which spacetime transformations (generators) are preserved as free geometric operations and which are required to carry the dynamics:**

Form	**Kinematic Generators (Non-interacting)**	**Dynamic Generators (Interacting)**	**Primary Applications & Advantages**
Instant Form	$\vec{P}, \vec{J}$ (6 generators)	$H, \vec{K}$ (4 generators)	Standard in few-body physics (BT theory, KG transformation, etc.).
Point Form	$\vec{J}, \vec{K}$ (6 generators)	$H, \vec{P}$ (4 generators)	Boost transformations are exact and geometrically simple.
Front Form	$P^+, \vec{P}_\perp, E^1, E^2, F^1, F^2, K^3$ (7 generators)	P^-, F^1, F^2 etc. (3 generators)	Hadronic structure calculations and light-front quantization.
Covariant Form	Fully covariant formulation without preferred axes	Interaction action / Lagrangian density $\mathcal{L}_{\text{int}}$	Quantum Field Theory and Bethe-Salpeter equations.

χEFT can be living in Covariant Form, no problem.

4 Primary motivation for adopting the Instant Form over the Covariant Form

The primary motivation for adopting the **Instant Form** (e.g., in Bakamjian-Thomas theory or Kamada-Glöckle transformations) over the **Covariant Form** (e.g., quantum field theory or Bethe-Salpeter equations) is that it enables high-precision numerical calculations while completely preserving the **quantum-mechanical state vector and probability interpretation within a 3-dimensional Hamiltonian framework**.

The specific advantages are detailed below.

4.1 Clear Probabilistic Interpretation and Hilbert Space Description

In the Covariant Form (such as the Bethe-Salpeter equation), four-dimensional spacetime coordinates x^μ are treated on an equal footing, which introduces the degree of freedom of "relative time." Consequently, an intuitive understanding of a 3-dimensional wave function (probability amplitude) becomes difficult, often giving rise to problems such as unphysical relative-time excitations or negative norms (probabilities).

In contrast, because the Instant Form defines states on a specific hypersurface of constant time ($t = 0$):

- The wave function $\psi(\vec{p}, \vec{q})$ can be defined as a **state vector in a standard Hilbert space**.
- Probability conservation and unitarity are manifestly preserved, yielding a transparent probabilistic interpretation for both bound and scattering states.

4.2 Direct Incorporation of Two-Body Potentials and T-Matrices

High-precision potentials used in nuclear physics (e.g., AV18, CD-Bonn, Chiral EFT) are constructed in 3-dimensional momentum space to reproduce two-body scattering data (phase shifts).

- **Drawback of Covariant Form:** Evaluating dynamics via four-dimensional Feynman diagrams makes it extremely difficult to directly incorporate non-relativistic, 3-dimensional potentials determined from experiment.

- **Advantage of Instant Form:** By applying Kamada-Glöckle (KG) or Coester-Pieper-Serduke (CPS) transformations, **high-precision two-body potentials and T-matrices determined non-relativistically can be directly introduced into a relativistic framework** simply by multiplying by off-shell boost factors.

4.3 High Compatibility with Multi-Body Scattering Theory (Faddeev / Lippmann-Schwinger Equations)

In few-body physics, integral equations such as the Faddeev or Lippmann-Schwinger equations serve as the standard approach for solving 3-body and 4-body scattering problems.

- **Simplicity of Kinematic Transformations:** In the Instant Form, total momentum $\vec{P}$ and total angular momentum $\vec{J}$ are free operators (containing no interactions). As a result, spatial rotations and transformations between 3-dimensional Jacobi momentum sets (channel permutations) remain geometrically straightforward.

- **Reduction of Computational Cost:** Compared to solving four-dimensional Bethe-Salpeter equations, the Faddeev equations in the Instant Form involve only 3-dimensional momentum variables. This avoids the curse of dimensionality and allows large-scale numerical computations (discretization of matrices and numerical quadrature) to be executed with high speed and precision.

4.4 Simplicity of Kinematic Generators

In the Instant Form, six of the ten Poincaré generators—namely, spatial translations $\vec{P}$ and spatial rotations $\vec{J}$—do not contain interaction terms.

As a result:

- Total momentum conservation holds as a simple sum: $\vec{P} = \sum \vec{p}_i$.
- Partial wave expansions and angular momentum algebra can be calculated using standard Racah algebra and Wigner $3j/6j$ symbols, identical to non-relativistic quantum mechanics.

While the Covariant Form excels at providing a manifestly Lorentz-covariant description, it incurs substantial complexity in the numerical calculation of wave functions and Hamiltonian dynamics in multi-body scattering.

By paying the price of **incorporating interactions into the boost generators K**, the **Instant Form gains major practical advantages: a 3-dimensional Hamiltonian framework, clear probability interpretation, and direct usability of high-precision two-body potentials.**

5 Bakamjian-Thomas (BT) theory B. Bakamjian, L. H. Thomas, *Phys. Rev.* **92**, 1300 (1953).

In nuclear physics and few-body problems involving strong interactions, the **Bakamjian-Thomas (BT) theory** and the **Kamada-Glöckle (KG) transformation** serve as the two foundational logical pillars for performing high-precision numerical calculations while preserving Poincaré invariance (Lorentz invariance).

While the BT theory provides a universal algebraic framework to construct the Poincaré algebra strictly from microscopic potentials, the KG transformation supplies a practical kinematic and geometric conversion scheme to incorporate that theoretical framework into multi-body scattering equations, such as the Faddeev equations.

Below is a comprehensive overview covering the structures, interrelationships, and mathematical formulations of both approaches.

5.1 Bakamjian-Thomas (BT) Theory

B. Bakamjian and L. H. Thomas, *Relativistic particle dynamics. II*, Phys. Rev. **92**, 1300 (1953), doi:10.1103/PhysRev.92.1300.

Formulated in 1953, the BT theory establishes a relativistic quantum-mechanical scheme for finite degrees of freedom that strictly satisfies the Poincaré algebra without relying on quantum field theory.

(1) Algebraic Requirements and Core Motivation

In relativistic quantum mechanics, the ten generators of the Poincaré group—energy (P^0), momentum ($\vec{P}$), angular momentum ($\vec{J}$), and boosts ($\vec{K}$)—must satisfy the standard Poincaré algebra:

$$[K_i, P_j] = i\hbar\delta_{ij}P^0, \quad [K_i, P^0] = i\hbar P_i, \quad [K_i, K_j] = -i\hbar\epsilon_{ijk}J_k, \quad \dots \tag{6}$$

In non-relativistic mechanics, one simply adds an interaction potential V to P^0. In the relativistic case, however, because P^0 appears on the right-hand side of $[K_i, P_j]$, **the boost generators $\vec{K}$ are algebraically required to contain interaction terms as well**.

(2) Formulation via the Mass Operator M

The BT framework explicitly separates the overall inertial motion (total momentum $\vec{P}$) from the internal dynamics (mass operator M):

1. **Incorporation of Interaction into Mass Operator:** An interaction operator v, which acts solely on internal variables, is added to the free mass operator M_0:

$$\boxed{M = M_0 + v} \tag{7}$$

2. **Construction of Total Hamiltonian and Boost Generators:** The total Hamiltonian $H = P^0$ and boost generators $\vec{K}$ are expressed as:

$$H = \sqrt{M^2 + \vec{P}^2} = \sqrt{(M_0 + v)^2 + \vec{P}^2} \tag{8}$$

$$\vec{K} = \frac{1}{2}\left\{\sqrt{M^2 + \vec{P}^2},\ \vec{R}\right\} - \frac{\vec{J} \times \vec{P}}{M + \sqrt{M^2 + \vec{P}^2}} \tag{9}$$

Through this prescription, the interaction v is consistently embedded into both H and $\vec{K}$, strictly closing the Poincaré algebra.

(3) Connection to Dirac Forms and Limitations

- **Dirac Forms:** Primarily implemented within the Instant Form or Front Form.
- **Advantages:** Enables the direct relativization of existing two-body potentials within a well-defined Hilbert space.
- **Limitation (Cluster Separability):** For systems with three or more particles, the BT framework breaks *cluster separability* (the property that interactions between clusters decouple as they are separated to spatial infinity). Resolving this for $N \geq 4$ body systems requires algebraic corrections such as the Pack-Polyzou transformation.

5.2 Kamada-Glöckle (KG) Transformation

The Kamada-Glöckle transformation is a practical kinematic and geometric algorithm designed to implement the fundamental concepts of BT theory into three-nucleon bound and scattering calculations based on the Faddeev equations.

(1) Background and Purpose

The non-relativistic three-body Faddeev equation is expressed in terms of the two-body T-matrix t and the permutation operator P:

$$T_3 = t + t\,G_0\,PT_3 \tag{10}$$

When relativizing this equation, simply substituting non-relativistic kinematics with relativistic momentum formulas violates Lorentz invariance and off-shell/on-shell consistency. The KG transformation **preserves the algorithmic structure of non-relativistic computational codes while correctly boosting the potential and momentum space in accordance with relativity**.

5.3 Quantitative Comparison and Interrelationship

The conceptual roles and properties of the two frameworks are summarized below:

Item	**Bakamjian-Thomas (BT) Theory**	**Kamada-Glöckle (KG) Transformation**
Role	**Universal Algebraic Framework**	**Computational & Geometric Algorithm**
Primary Objective	Exact realization of Poincaré algebra (10 generators) in interacting systems.	Relativistic boosting of potentials and geometric factors in Faddeev equations.
Scope	General few-body systems ($2, 3, 4 \ldots$ bodies, quark models).	Three-nucleon scattering (N-d) and bound states (^{3}H, ^{3}He).
Algebraic Treatment	Automatic construction of H and $\vec{K}$ from $M = M_0 + v$.	Jacobians, Wigner rotation matrices, and off-shell boost factors of T-matrices.
Numerical Advantage	Algebraic guarantee of exact Poincaré invariance.	Reuses existing non-relativistic Faddeev computational code structures.

5.4 Physical Implications and Applications

Combining these two theories has yielded prominent achievements in nuclear physics:

- **Relativistic Corrections to 3N Binding Energies:** It has been quantitatively demonstrated that relativistic effects yield a repulsive net correction (reducing binding energy) of tens to hundreds of keV for ^{3}H and ^{3}He.

- **Analysis of High-Energy N-d Scattering:** In nucleon-deuteron (p-d) elastic scattering at intermediate incident energies (100–300 MeV), this framework provides a benchmark for disentangling relativistic effects from three-nucleon force (3NF) contributions predicted by chiral effective field theory (χEFT), particularly around cross-section minima and polarization observables (such as A_y).

The ten **Poincaré generators** that constitute the Poincaré algebra serve as fundamental operators in relativistic quantum mechanics and quantum field theory for **executing spacetime symmetry transformations (transforming state vectors and physical observables between different observers' frames of reference)**.

Physically, they correspond to spacetime translations (4 spacetime shifts), spatial rotations (3 spatial rotations), and Lorentz boosts (3 velocity transformations).

6 KG transformation and CPS one

In relativistic few-body nuclear physics, both the **Kamada-Glöckle (KG) transformation** and the **Coester-Pieper-Serduke (CPS) transformation** serve as vital methodologies for embedding non-relativistic two-body interaction potentials into relativistic three-body (Faddeev) bound and scattering frameworks while preserving Poincaré invariance.

Although both approaches are rooted in Bakamjian-Thomas (BT) theory, they adopt distinct strategies regarding off-shell extensions, kinematic boost factors, and numerical implementation schemes.

Idenfication （Rel. ⇔ Nonrel.)

- **Type1** : Scale transformation
  [Kamada-Glöckle (**KG** transformation)]
  H. K, W. Glöckle, Phys. Rev. Lett. **80**, 2457 (1998)
  → Conservation: Deuteron binding energy, phase shifts
- **Type2**：Equivalent eigenvalue transformation
  [Coester-Pieper-Serduke form (**CPS** transformation)]
  Phys. Rev. C**11**, 1 (1975)
  → Conservation: Deuteron binding energy, phase shifts

Type1: *Kamada – Glöckle* (KG) transformation

"Scale-transform it from nonrelativity to relativity "

v : Relativistic potential

$$v(\vec{k},\vec{k}\,') = \frac{1}{h(k_{nr})} v^{(nr)}(\vec{k}_{nr},\vec{k}\,'_{nr}) \frac{1}{h(k'_{nr})}$$

Scale transformation

$$k_{nr} \equiv |\,\vec{k}_{nr}\,| = \sqrt{2m}\sqrt{\sqrt{\vec{k}^{\,2}+m^2}-m}$$

$$h(k_{nr}) = \sqrt{(1+\frac{k_{nr}^2}{2m^2})\sqrt{1+\frac{k_{nr}^2}{4m^2}}}.$$

Phys. Rev. Lett. **80,** 2457(1998)

Type2:
Coester-Pieper-Serduke (CPS) transformation

Phys. Rev. C11, 1 (1975)

$$\left(\frac{\hat{k}^2}{m} + V_{nr}\right)\Psi = \frac{k_0^2}{m}\Psi \qquad (1)$$

This can be related to an underlying relativistic NN

Schrödinger equation:

$$\left(2\sqrt{m^2 + \hat{k}^2} + v\right)\Psi = 2\sqrt{m^2 + k_0^2}\Psi \qquad (2)$$

$$V_{\mathrm{nr}} = \frac{1}{4m}\left(2\omega(\hat{k})v + 2v\omega(\hat{k}) + v^2\right)$$

On-shell properties (binding energy, phase shifts) are conserved.



Nonrelativistic space

Relativistic space

On-shell properties (binding energy, phase shifts) are conserved.



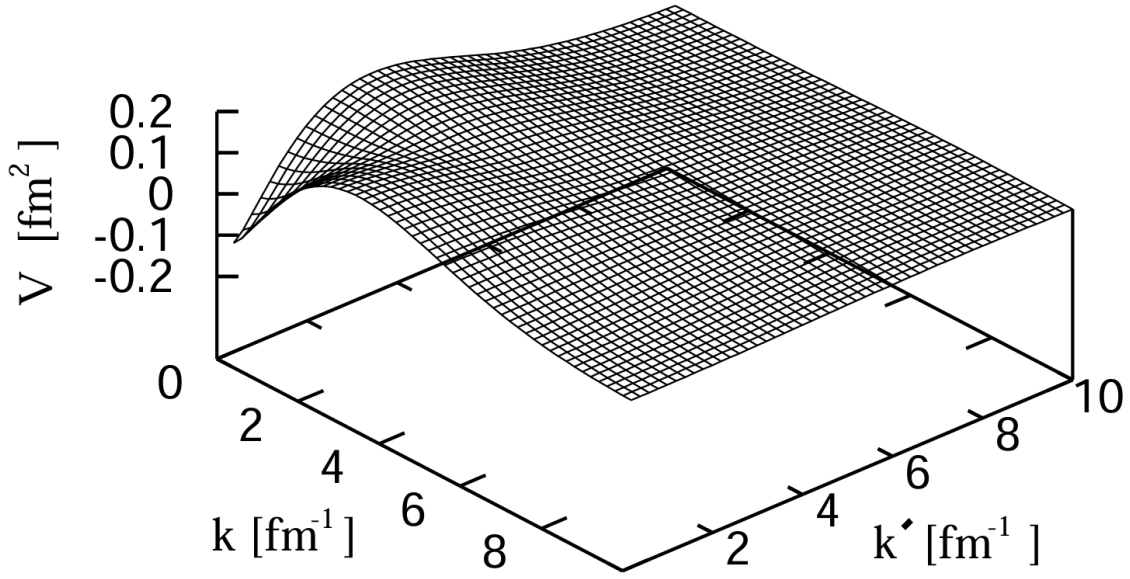
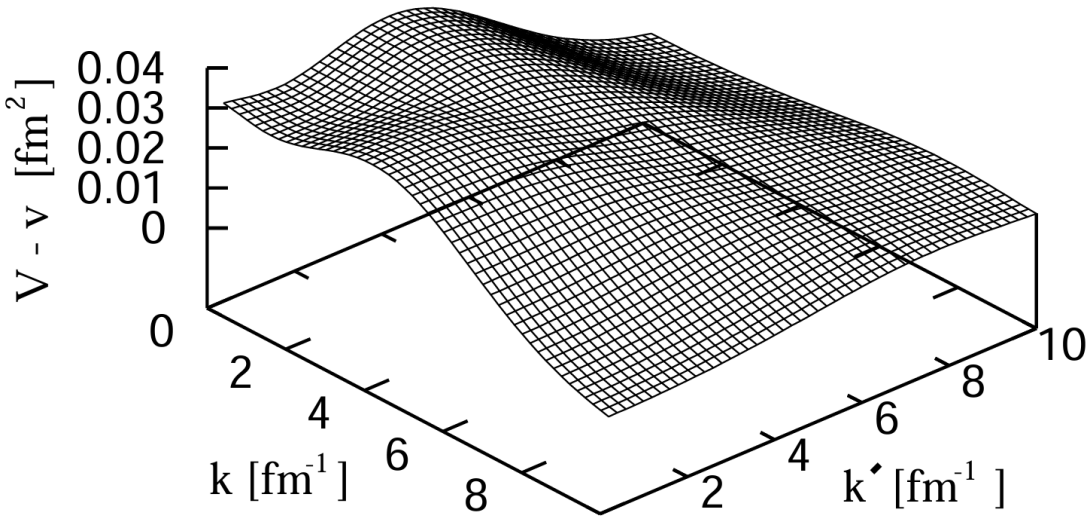
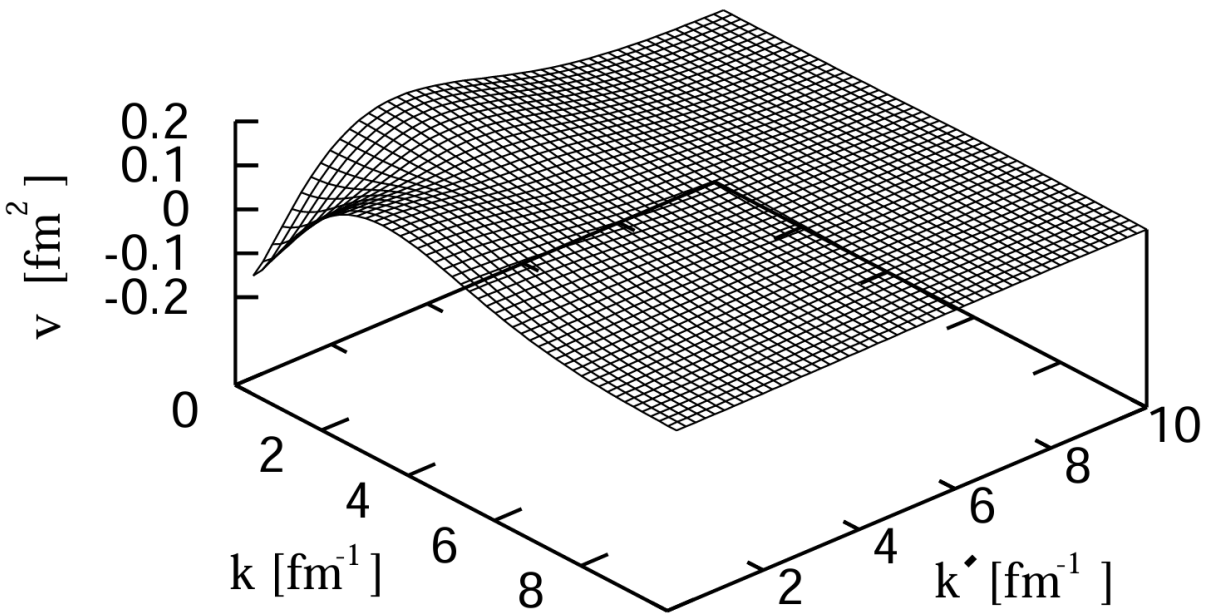


FIG. 1: The nonrelativistic potential $V(k, k')$ AV18 in the state $^1S_0$.

← **nonrelativistic potential**





← **transformed relativistic potential (CPS)**

Boost : Kamada – Glöckle Method transformation

H. K., W. Glöckle, Phys. Lett. **B655**, 119 (2007)

Mass operaor:

$$M_{23} = \sqrt{m_2^2 + \vec{k}^2} + \sqrt{m_3^2 + \vec{k}^2} + v_{23}$$

Bakamjian-Thomas (BT) Theory

Boosted Mass operaor: $M_{23}(\vec{q})^2 = M_{23}^2 + \vec{q}^2$

- **Non-Linear Energy Dependence:** Because M_{23}^2 appears inside square-root operators when defining the total three-body mass operator $M_{123} = \sqrt{M_{23}^2 + \vec{q}^2} + \sqrt{m_1^2 + \vec{q}^2} + V_{123}$, non-linear energy dependencies arise in the Faddeev kernel, complicating numerical iteration.

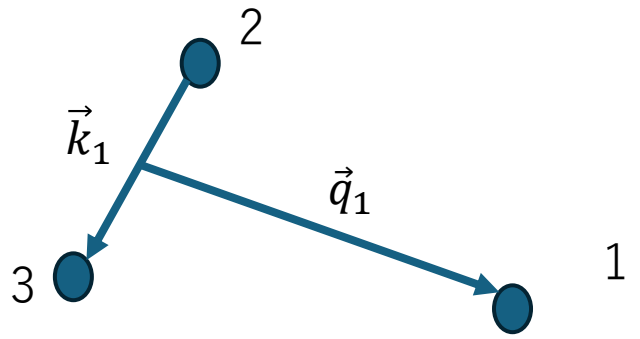
$$m_{123} = \sqrt{\left(\sqrt{m_2^2 + \vec{k}_1^2} + \sqrt{m_3^2 + \vec{k}_1^2} + v_{23}\right)^2 + \vec{q}_1^2} + \sqrt{m_1^2 + \vec{q}_1^2}$$

$$= \sqrt{\left(\sqrt{m_2^2 + \vec{k}_1^2} + \sqrt{m_3^2 + \vec{k}_1^2}\right)^2 + \vec{q}_1^2 + v_{23}^{(q_1)}} + \sqrt{m_1^2 + \vec{q}_1^2}$$

$$= \sqrt{m_1^2 + \vec{q}_1^2} + \sqrt{m_2^2 + \vec{q}_2^2} + \sqrt{m_3^2 + \vec{q}_3^2} + v_{23}^{(q_1)}$$

Boosted Potential

$$\vec{q}_1 + \vec{q}_2 + \vec{q}_3 = 0$$



Boosted 3-body mass operaor: $M_{123} = \sqrt{m_1^2+\vec{q}_1^{\,2}} + \sqrt{m_2^2+\vec{q}_2^{\,2}} + \sqrt{m_3^2+\vec{q}_3^{\,2}} + v_{23}^{(q_1)} + v_{31}^{(q_2)} + v_{12}^{(q_3)}$

H. K., W. Glöckle, Phys. Lett. **B655**, 119 (2007)

$$= \sqrt{m_1^2+\vec{q}_1^{\,2}} + \sqrt{\left(\sqrt{m_2^2+\vec{k}_1^{\,2}} + \sqrt{m_3^2+\vec{k}_1^{\,2}} + v_{23}\right)^2 + \vec{q}_1^{\,2}} + v_{31}^{(q_2)} + v_{12}^{(q_3)}$$

$$= \sqrt{m_2^2+\vec{q}_2^{\,2}} + \sqrt{\left(\sqrt{m_1^2+\vec{k}_2^{\,2}} + \sqrt{m_3^2+\vec{k}_2^{\,2}} + v_{31}\right)^2 + \vec{q}_2^{\,2}} + v_{12}^{(q_3)} + v_{23}^{(q_1)}$$

$$= \sqrt{m_3^2+\vec{q}_3^{\,2}} + \sqrt{\left(\sqrt{m_1^2+\vec{k}_3^{\,2}} + \sqrt{m_2^2+\vec{k}_3^{\,2}} + v_{12}\right)^2 + \vec{q}_2^{\,2}} + v_{23}^{(q_1)} + v_{31}^{(q_2)}$$

Relativistic Faddeev component ψ_i :

$$\psi_1 \equiv \frac{1}{M_{123} - \left(\sqrt{m_1^2+\vec{q}_1^{\,2}} + \sqrt{m_2^2+\vec{q}_2^{\,2}} + \sqrt{m_3^2+\vec{q}_3^{\,2}}\right)} v_{23}^{(q_1)}\ \Psi$$

$$\psi_2 \equiv \frac{1}{M_{123} - \left(\sqrt{m_1^2+\vec{q}_1^{\,2}} + \sqrt{m_2^2+\vec{q}_2^{\,2}} + \sqrt{m_3^2+\vec{q}_3^{\,2}}\right)} v_{31}^{(q_2)}\ \Psi$$

$$\psi_3 \equiv \frac{1}{M_{123} - \left(\sqrt{m_1^2+\vec{q}_1^{\,2}} + \sqrt{m_2^2+\vec{q}_2^{\,2}} + \sqrt{m_3^2+\vec{q}_3^{\,2}}\right)} v_{12}^{(q_3)}\Psi$$

$$\Psi = \psi_1 + \psi_2 + \psi_3$$

SciPost Phys. Proc. 3, 003 (2020)

7. Unitary Clothing Transformations Method

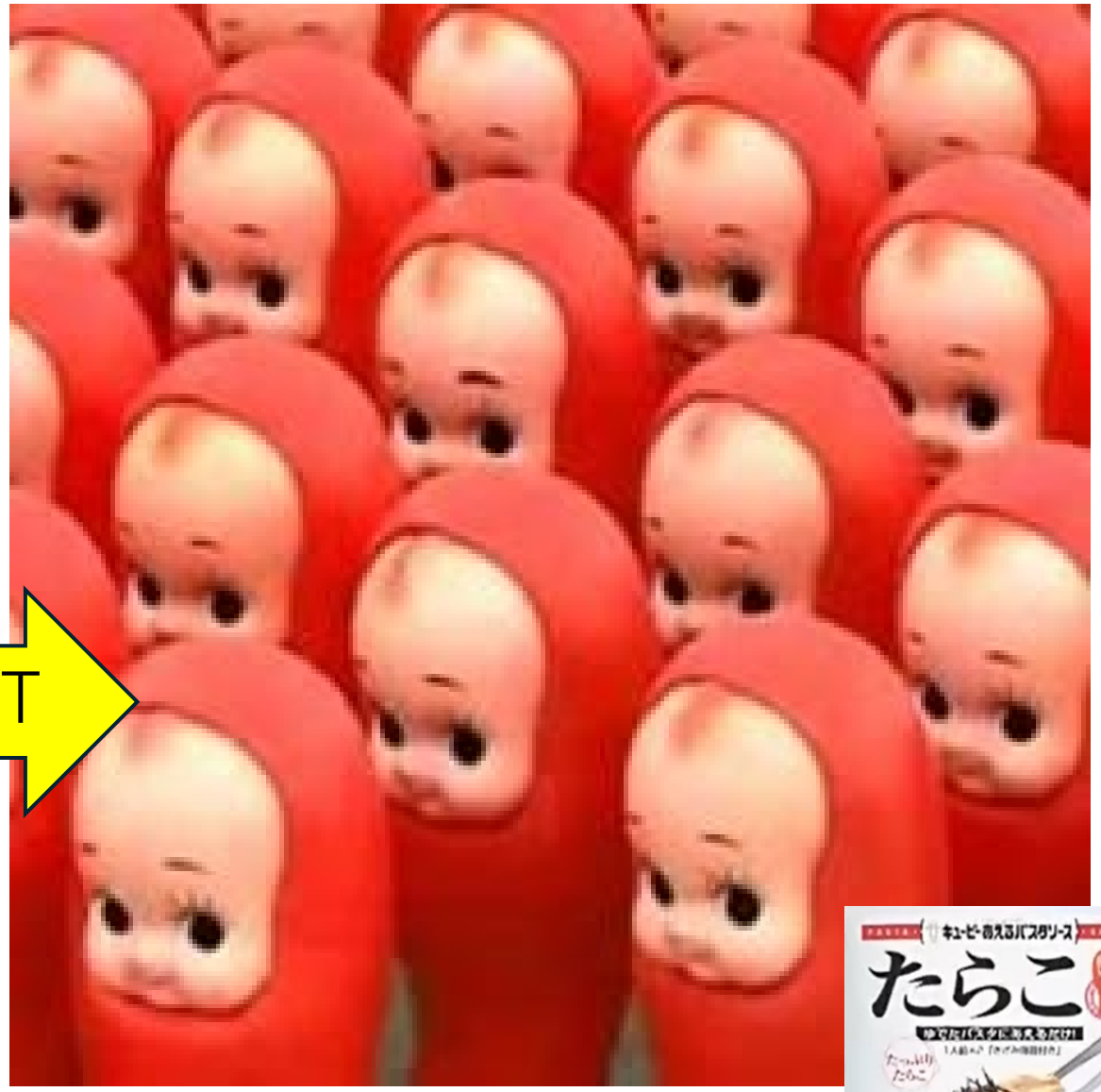
- The 'Unitary Closing Transformation Method' is a powerful technique used in dealing with quantum many-body problems.
- It involves transforming the system's Hamiltonian through a unitary transformation and introducing the concept of physically meaningful quasiparticles (particles dressed with an interaction layer), thereby making complex interactions more manageable.
- **Expressing the field Hamiltonian through the creation (annihilation) operators $\boldsymbol{\alpha_c}^\dagger$ (for example, $\boldsymbol{a_c}^\dagger$ [meson], $\boldsymbol{b_c}^\dagger$ [nucleon], $\boldsymbol{d_c}^\dagger$ [antinucleon]) that create (annihilate) particles 'dressed' with the so-called 'clothing'**
- UCT : $W(\alpha_c) = W(\alpha) = \exp(R), \quad R = -R^\dagger$
- H ≡ H(α) = $\mathrm{H_F}(\alpha)$ + $\mathrm{H_I}(\alpha)$ = $W(\alpha_c)\,\mathrm{H}(\alpha_c)\,W^\dagger(\alpha_c) \equiv \mathrm{K}(\alpha_c)$
- R: Lie algebra prepared to avoid phenomena that are not expected to occur the selected process: e.g.

  N+N $\nrightarrow$ N+N+π

Okubo Theory



UCT



$a^\dagger$ (a) [bear meson creation op.]、
$b^\dagger$ (b) [bear nucleon creation op.]

$b_c^\dagger$ (b_c) [clothed nucleon creation op.]



Kharkiv potential

- **A relativistic potential**； the state is described using a basis (Clothed Basis) that excludes the scattering states of the π-meson in quantum field theory.

(Application of Okubo‘s theory)

- *A. Arslanaliev, J. Golak, H. Kamada, A. V. Shebeko, R. Skibiński, M. M. Stepanova, H. Witala,* ISSN 1063-7796, Physics of Particles and Nuclei, **53**, 87 (2022).
- *A. Arslanaliev, P. Frolov, J. Golak, H. Kamada, A. Shebeko, R. Skibiński, M. Stepanova,H. Witala,* Few-Body Syst. **62**, 71 (2021).

8. Faddeev Calculations

Two-body t-matrix

Nonrelativistic LS eq.

$$t^{nr}(\vec{k},\vec{k}';\ E) = v^{nr}(\vec{k},\vec{k}'\quad) + \int d^3k'' \frac{v^{nr}(\vec{k},\vec{k}''\quad) t^{nr}(\vec{k}'',\vec{k}';\ E)}{E\ -\ k''^2/m\qquad\qquad + i\epsilon}.$$

Relativistic LS eq.

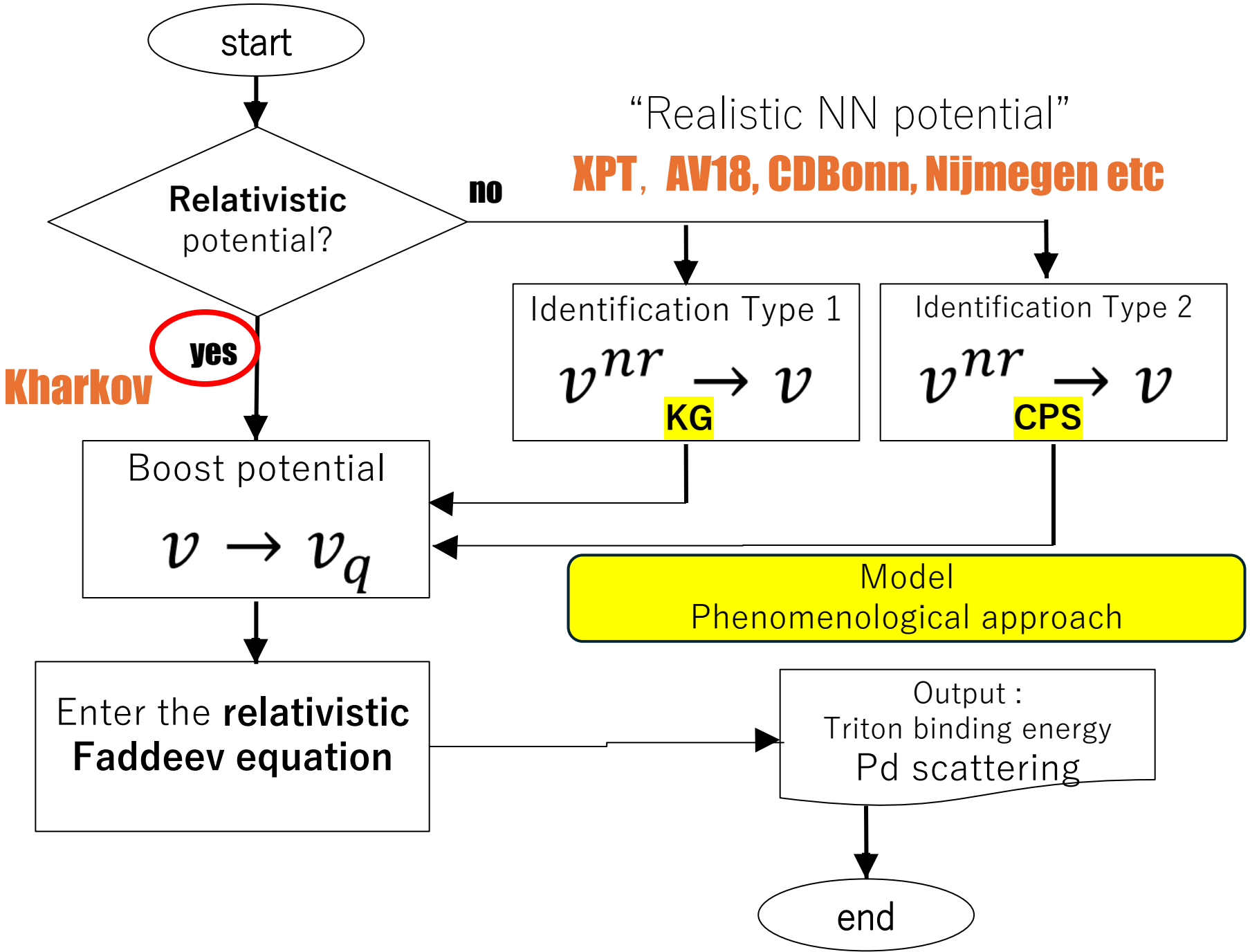
$$\hat{t}(\vec{k},\vec{k}';\vec{q}\,) = v\,(\vec{k},\vec{k}'\quad) + \int d^3k'' \frac{v\,(\vec{k},\vec{k}''\quad)\hat{t}(\vec{k}'',\vec{k}';\vec{q}\,)}{M_{23}\ -\ 2\omega(\vec{k}\,'')\qquad\qquad + i\epsilon}.$$

$$2\omega(\vec{k}) \equiv 2\sqrt{m^2 + \vec{k}^{\,2}}$$

$$M_{23} = \sqrt{(\sqrt{m_2^2 + \vec{\kappa}_1^2} + \sqrt{m_3^2 + \vec{\kappa}_1^2}\,)^2 + \vec{q}_1^{\,2}}$$

$$= \sqrt{m_2^2 + \vec{q}_2^{\,2}} + \sqrt{m_3^2 + \vec{q}_3^{\,2}}$$

$$= M_{123} - \sqrt{m_1^2 + \vec{q}_1^{\,2}}$$


start
Relativistic potential?
no
yes
Kharkov
"Realistic NN potential"
XPT, AV18, CDBonn, Nijmegen etc
Identification Type 1
$v^{nr} \rightarrow v$
KG
Identification Type 2
$v^{nr} \rightarrow v$
CPS
Boost potential
$v \rightarrow v_q$
Model
Phenomenological approach
Enter the relativistic Faddeev equation
Output :
Triton binding energy
Pd scattering
end

13MeV（Lab.）

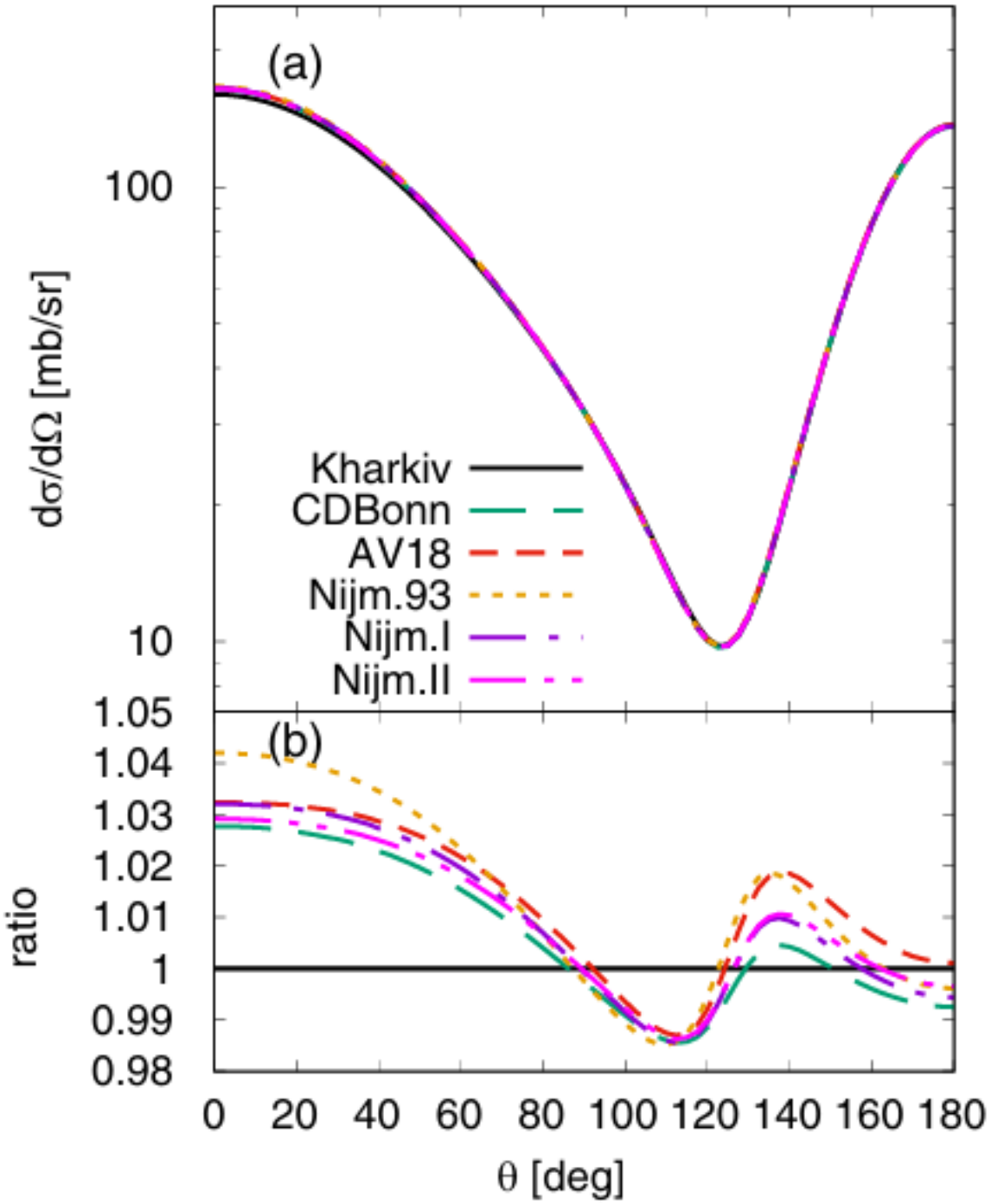


FIG. 1. Differential cross section of p-d elastic scattering at an incident proton laboratory energy of $E = 13$ MeV. (a) Results of nonrelativistic calculations performed using the CDBonn [6] (long-dashed green line), AV18 [7] (short-dashed red line), Nijmegen 93 [8] (dotted orange line), Nijmegen I (single dot-dashed violet line), and Nijmegen II (double dot-dashed magenta line) potentials are shown, together with the relativistic calculation based on the Kharkiv potential [20] (solid black line). (b) Relative differences of the non-relativistic results shown in (a) with respect to the predictions based on the Kharkiv potential for the differential cross section.

135MeV（Lab.）

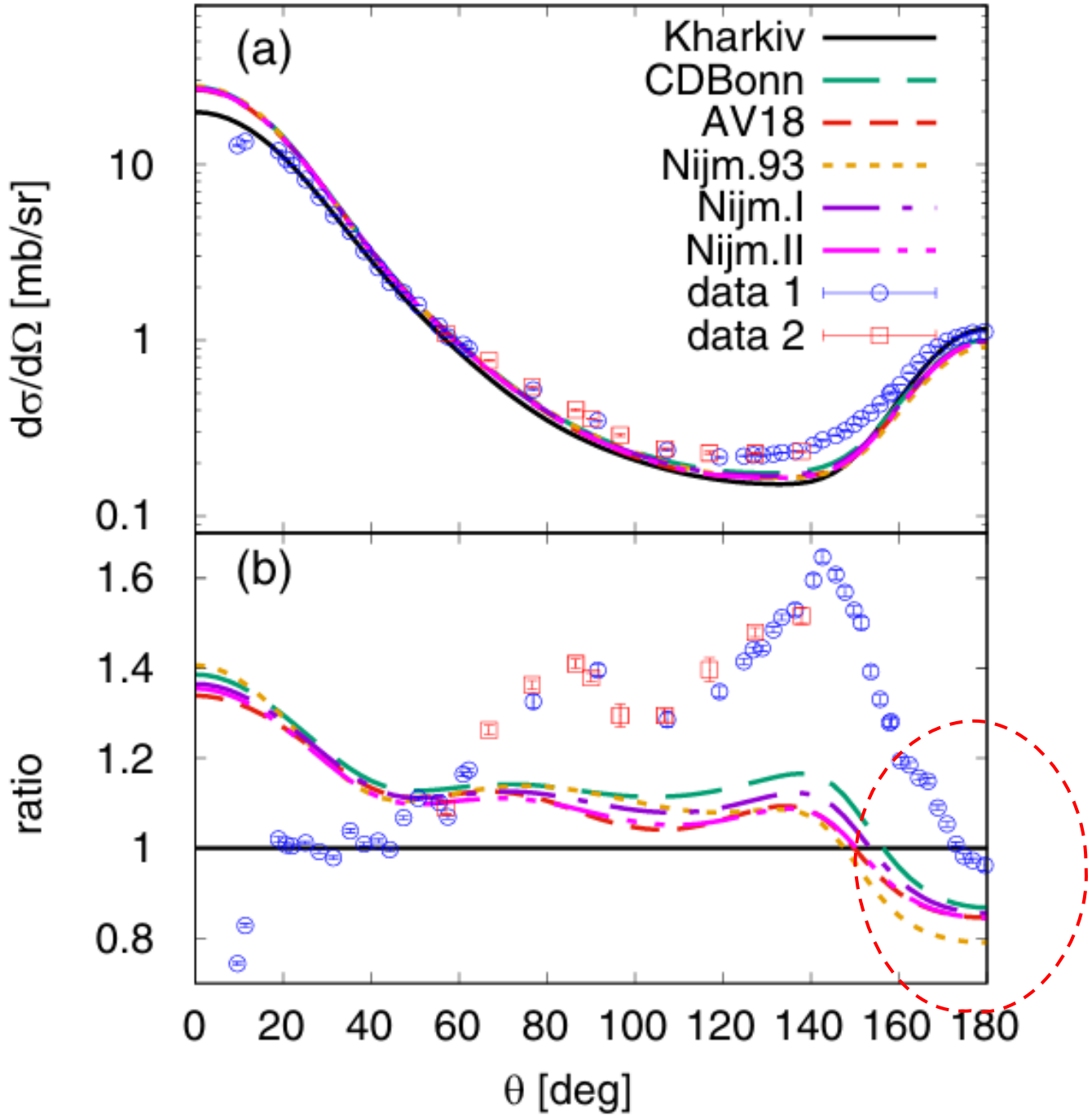


FIG. 2. Differential cross section of p-d elastic scattering at $E = 135$ MeV. The line colors are the same as in Fig. 1. Data sets 1 and 2 are taken from Refs. [32] and [33], respectively.

[32] H. Sakai, K. Sekiguchi, *et al.*, Precise Measurement of pd Elastic Scattering at 270 MeV and Three-Nucleon Force Effects, Phys. Rev. Lett. **84**, 5288 (2000); Polarization transfer measurements for dp scattering at 270 MeV, Nucl. Phys. **A684**, 577 (2001).

[33] N. Sakamoto *et al.*, Measurement of the vector and tensor analyzing powers for the d-p elastic scattering at $E_d = 270$ MeV, Phys. Lett. B **367**, 60 (1996).

135MeV (Lab.)

CDBonn
AV18
Nijm.93
Nijm.I
Nijm.II

Identification Type 2

$$v^{nr} \xrightarrow[\text{CPS}]{} v$$

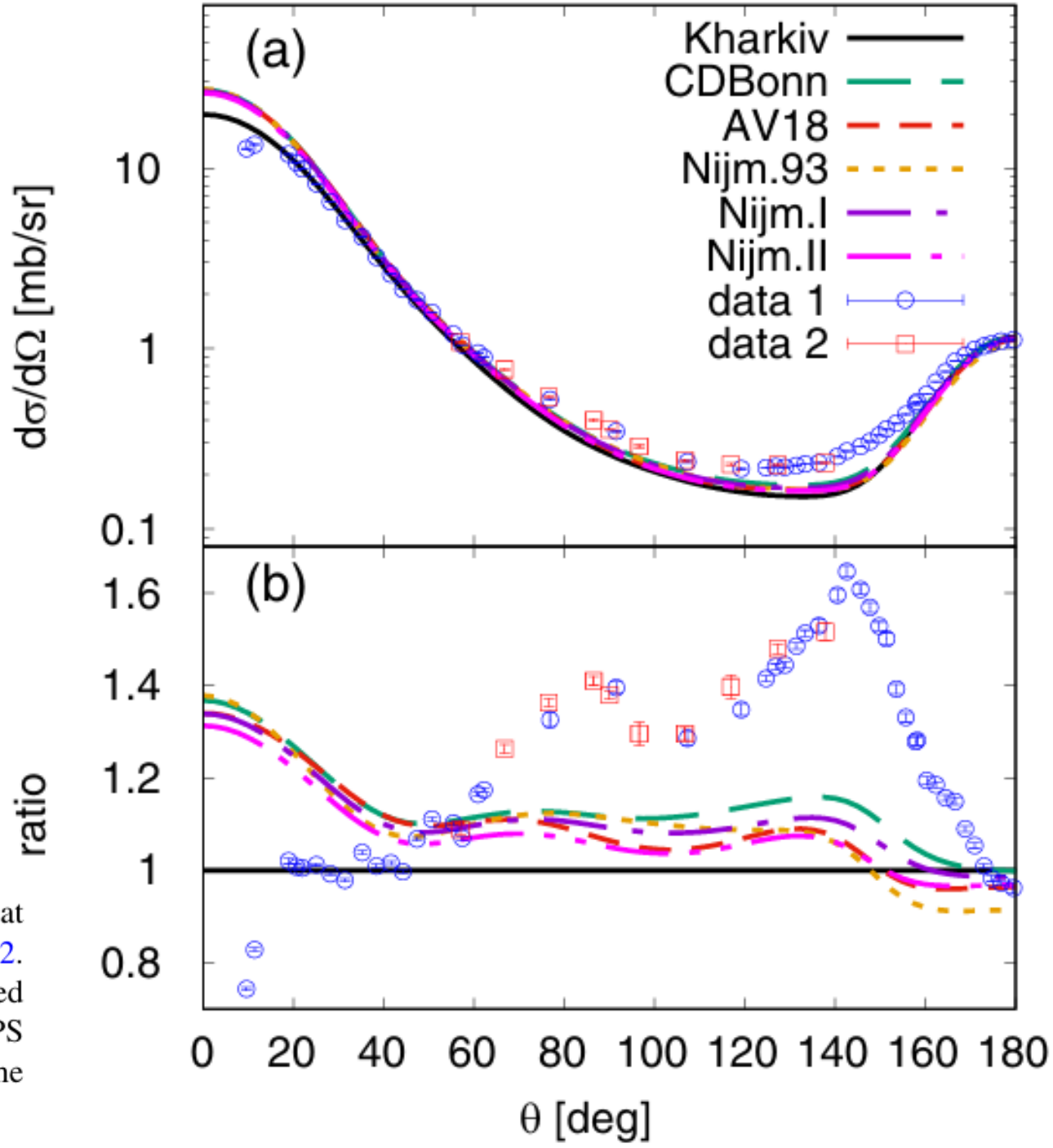
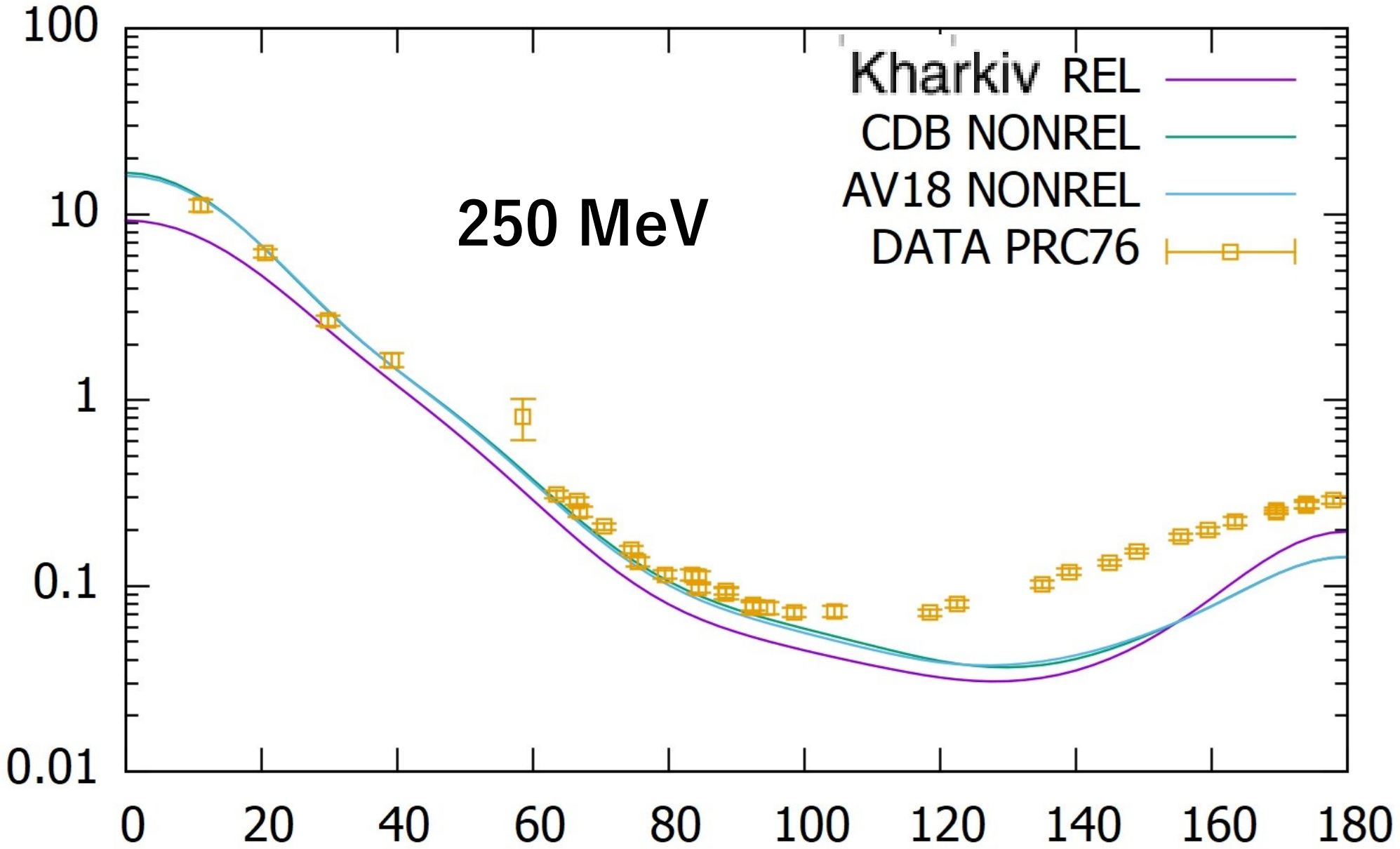
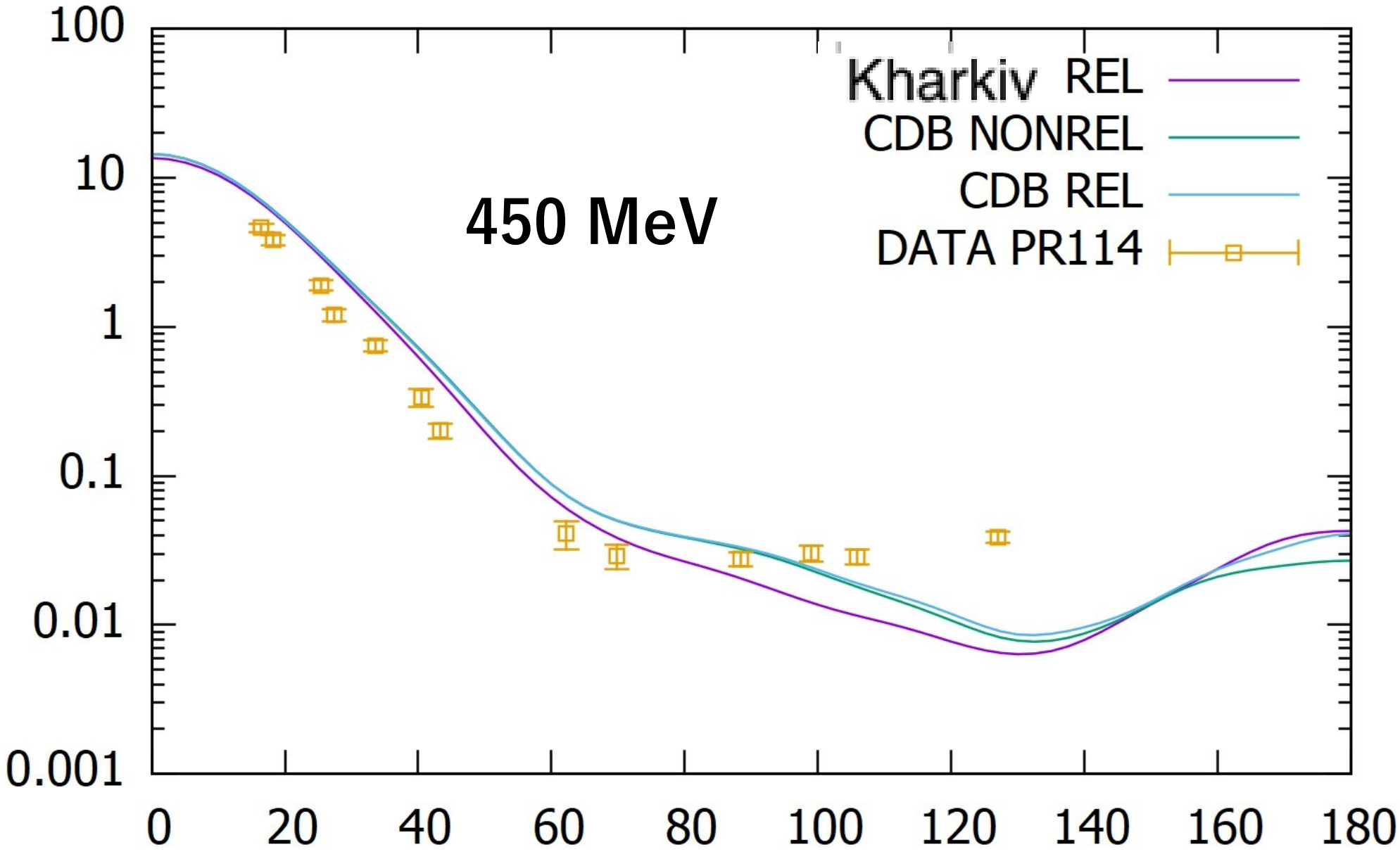


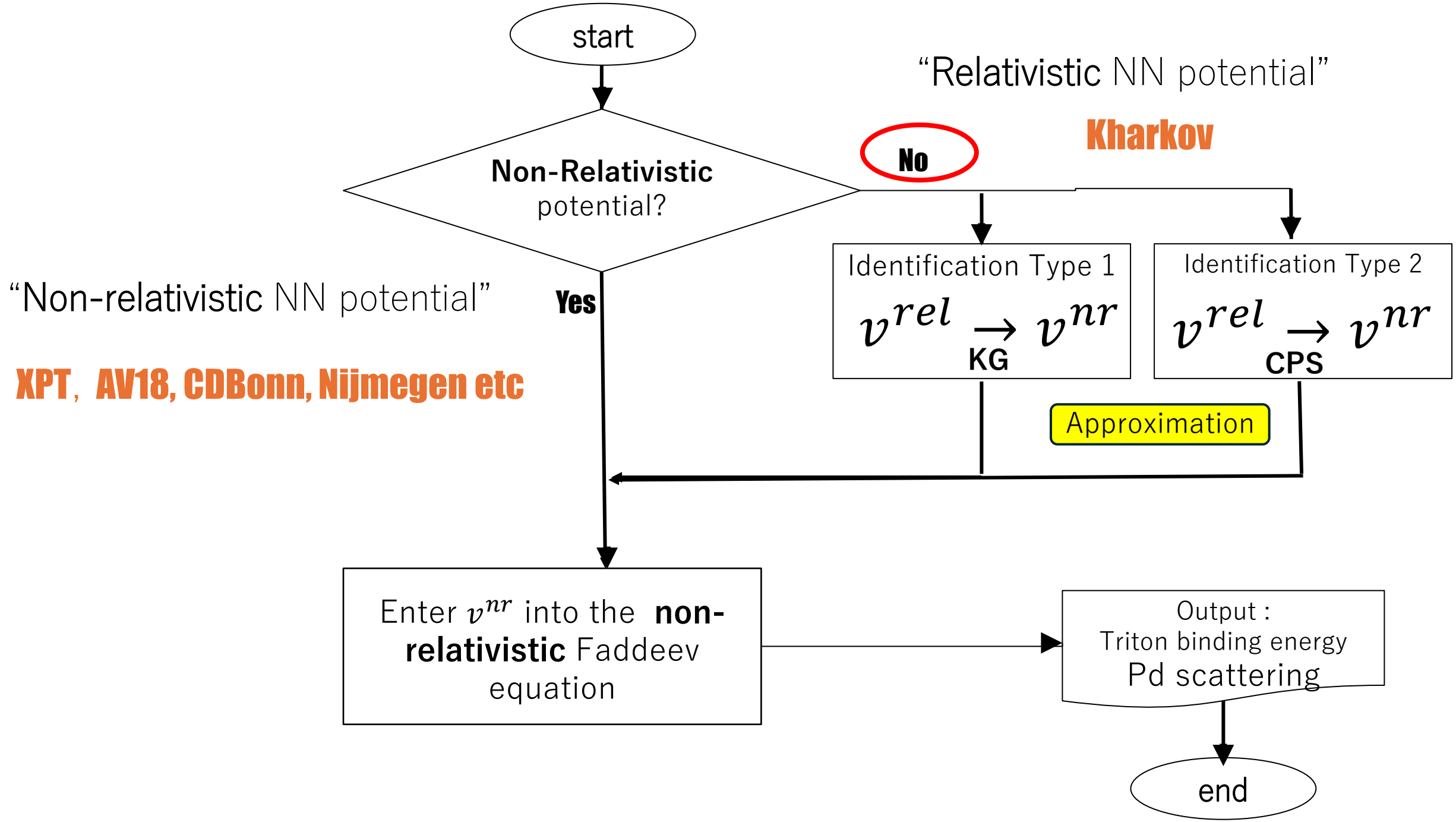
FIG. 3. Differential cross section of p-d elastic scattering at $E = 135$ MeV. The meaning of each line is the same as in Fig. 2. The realistic potentials (CDBonn, AV18, Nijmegen) are transformed into phenomenological relativistic potentials (PRP) using the CPS method and then employed in the relativistic Faddeev equations. The data and line types are the same as in Fig. 2.

Experimental Data

- 135MeV: K. Sekiguchi et al., Phys. Rev. C **65**, 034003 (2002).
- 250MeV: Y. Maeda et al., Phys. Rev. C **76** ,014004 (2007).
- 450MeV: A. V. Crewe et al., Phys. Rev. **114**, 1361 (1959).


250 MeV
Kharkiv REL
CDB NONREL
AV18 NONREL
DATA PRC76
100
10
1
0.1
0.01
0
20
40
60
80
100
120
140
160
180


100
10
1
0.1
0.01
0.001
0
20
40
60
80
100
120
140
160
180
450 MeV
Kharkiv REL
CDB NONREL
CDB REL
DATA PR114


start
Non-Relativistic potential?
No
Yes
"Relativistic NN potential"
Kharkov
"Non-relativistic NN potential"
XPT, AV18, CDBonn, Nijmegen etc
Identification Type 1
$v^{rel} \xrightarrow[KG]{} v^{nr}$
Identification Type 2
$v^{rel} \xrightarrow[CPS]{} v^{nr}$
Approximation
Enter v^{nr} into the **non-relativistic** Faddeev equation
Output :
Triton binding energy
Pd scattering
end

135MeV (Lab.)

Kharkiv

Identification Type 2

$$v^{rel} \underset{\text{CPS}}{\rightarrow} v^{nr}$$

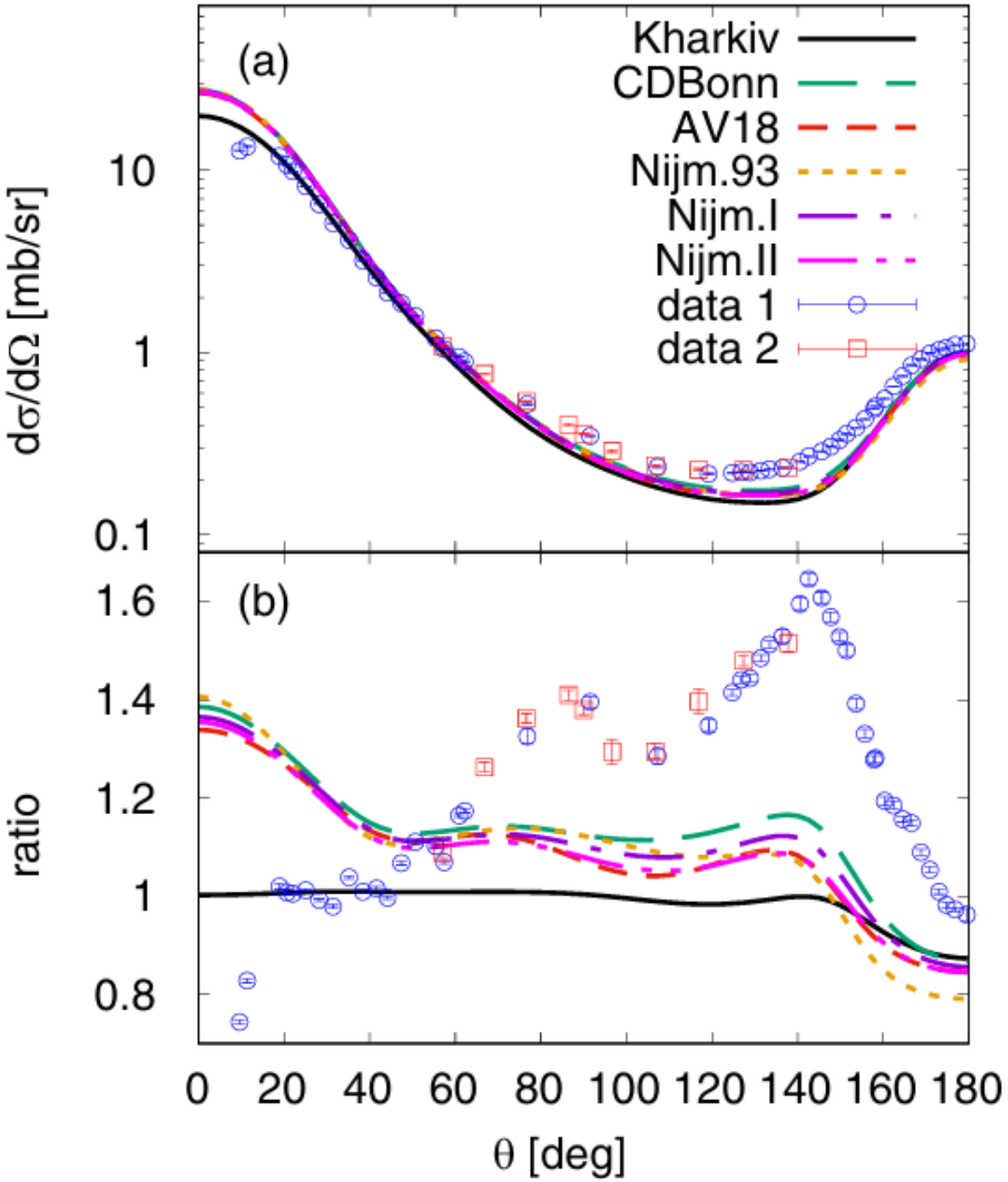
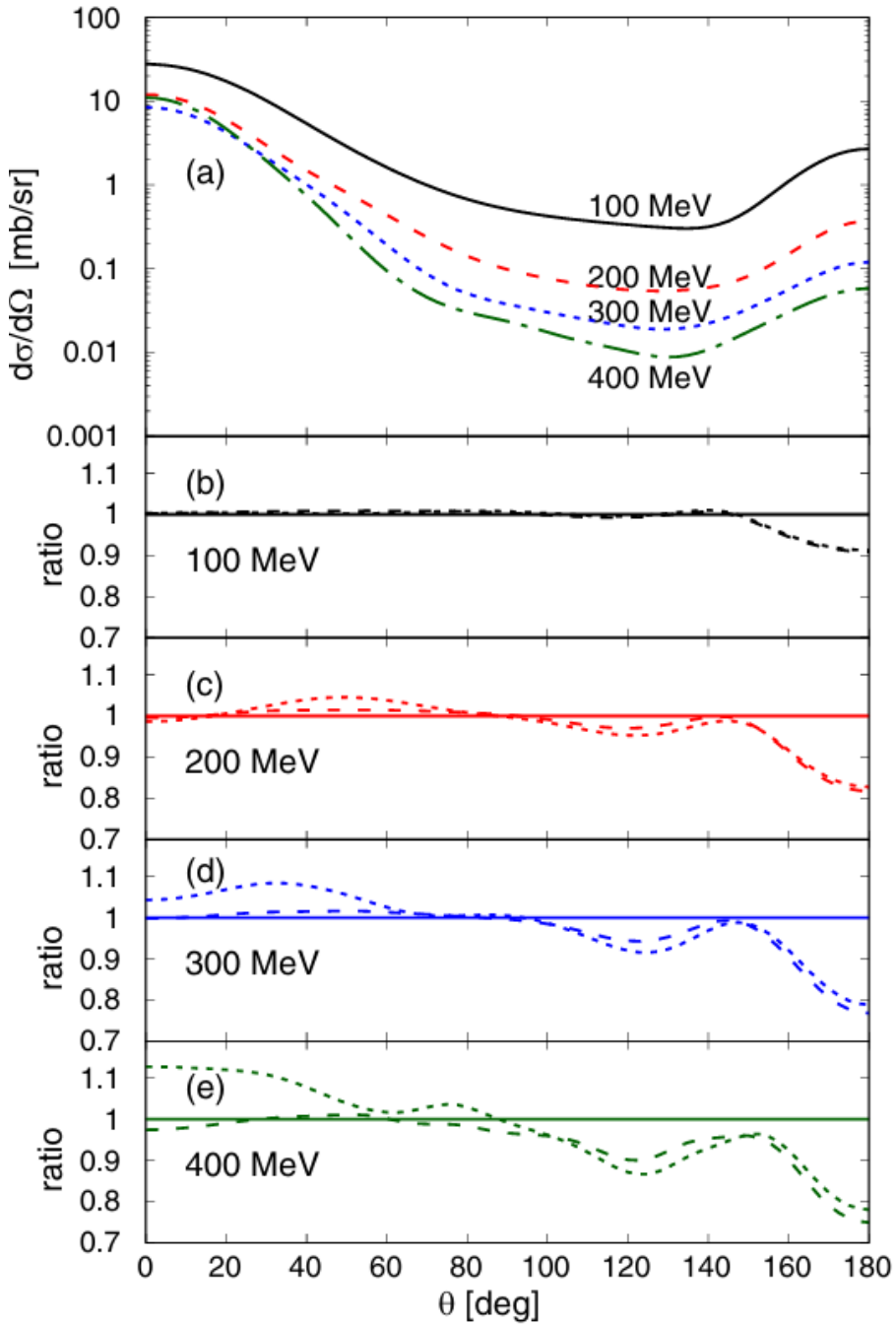
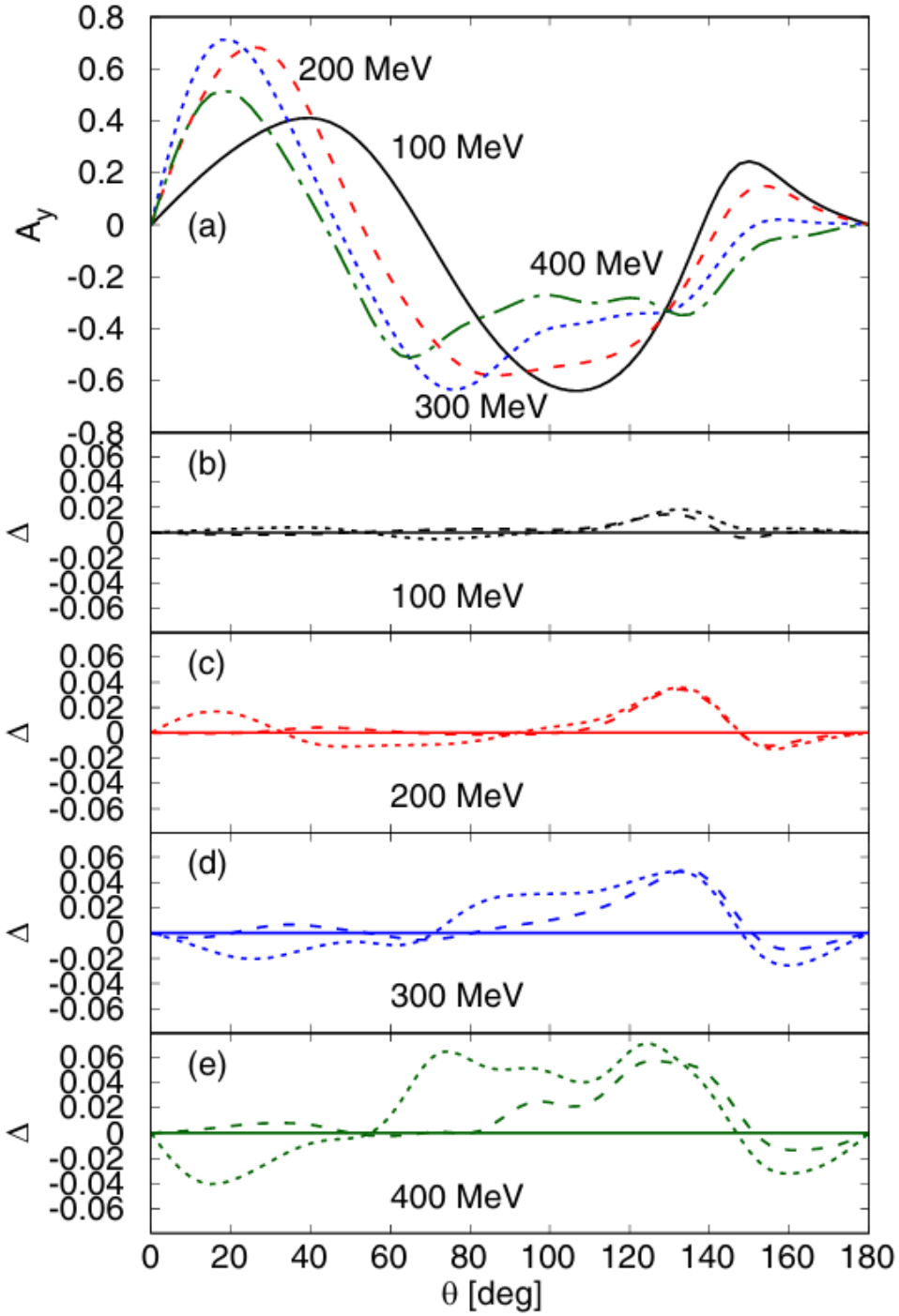


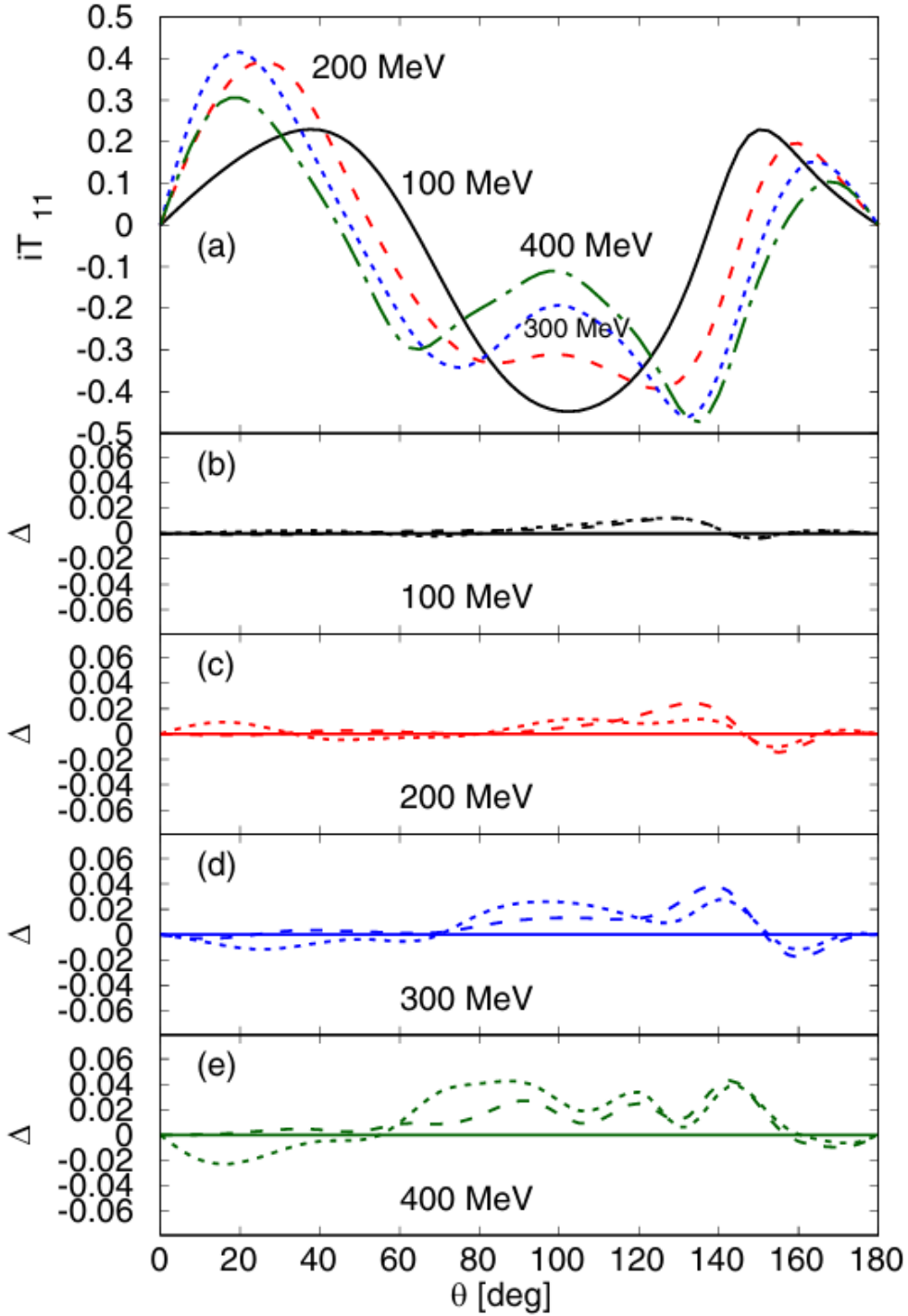
FIG. 4. Differential cross section of p-d elastic scattering at $E = 135$ MeV. The meaning of each line is the same as in Fig. 2, except that the Kharkiv potential has been replaced by its PNRP obtained via the inverse CPS transformation, and it is substituted into the nonrelativistic Faddeev equations. The experimental data are the same as in Fig. 2.

Including 3-body force: Tucson-Melbourne force

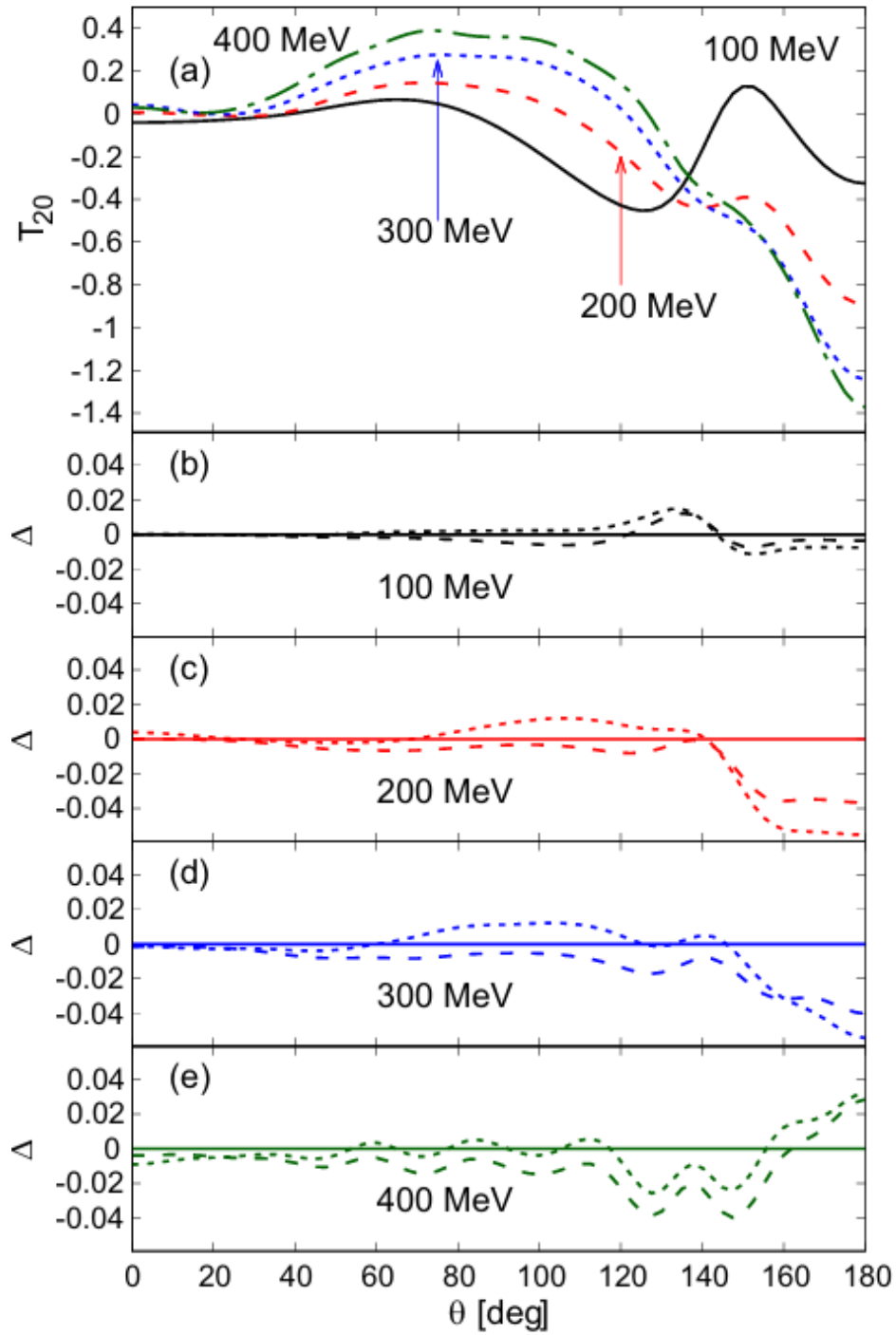


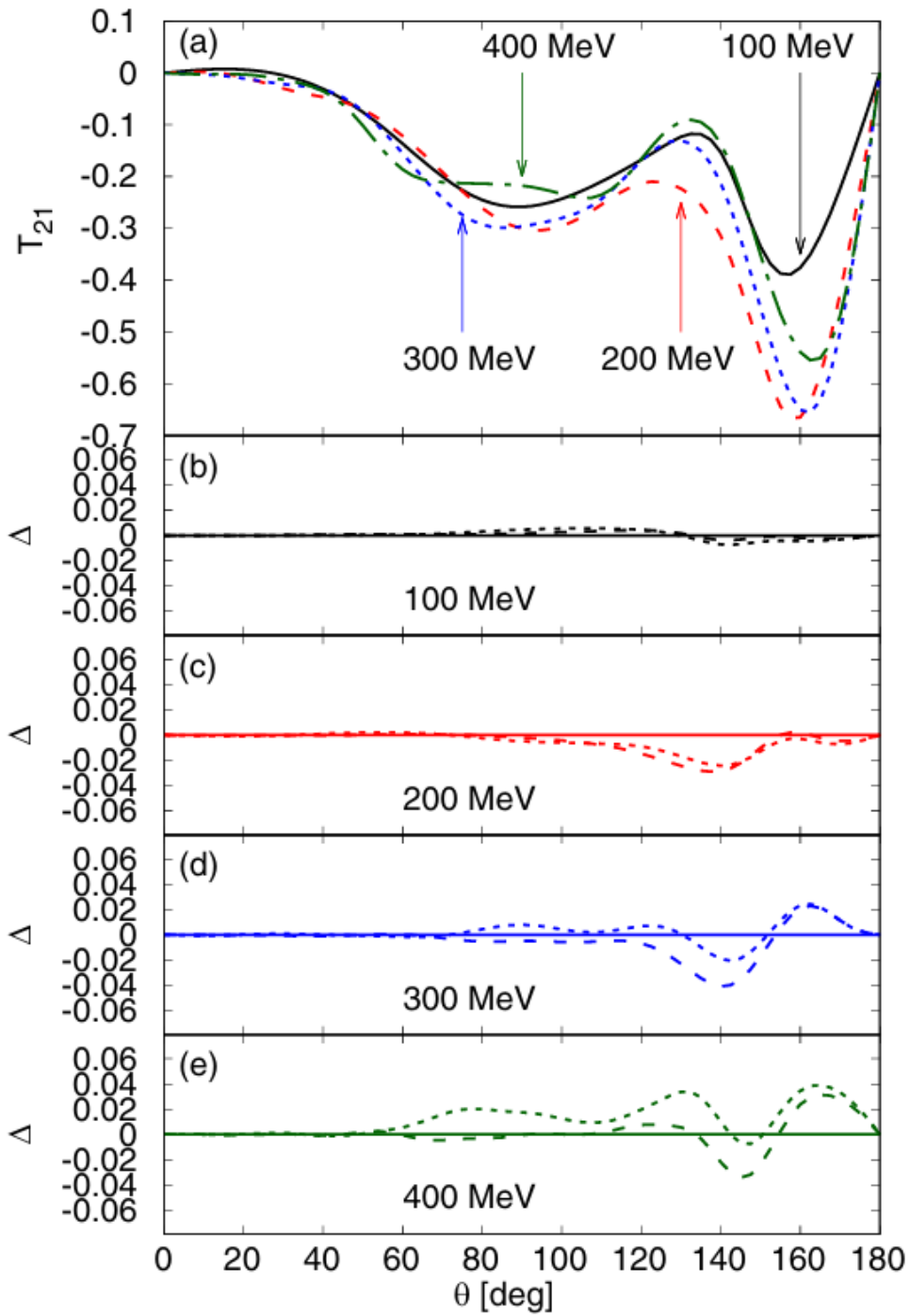
Long dashed: CPS
Short dashed: KG



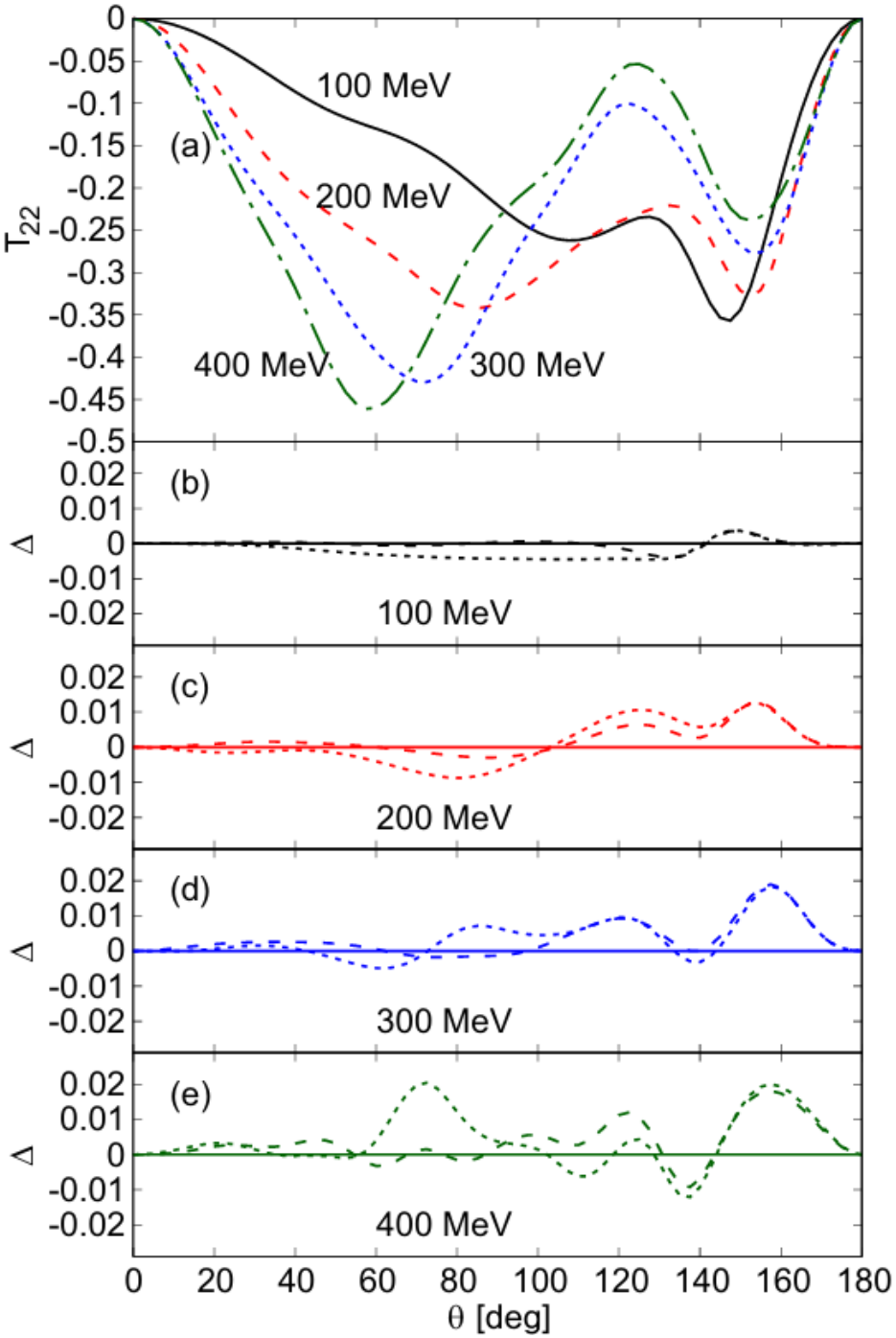


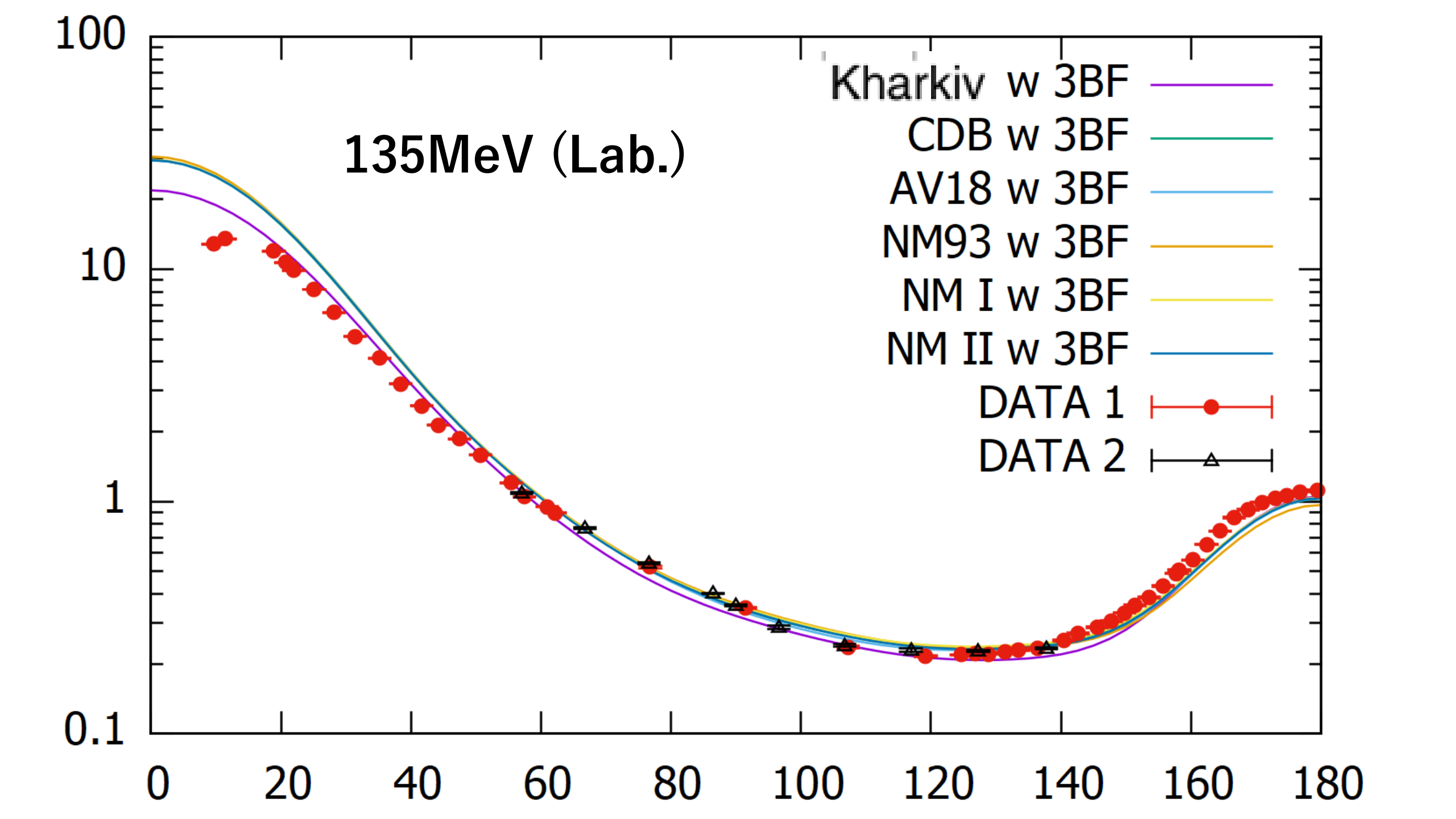
Long dashed: CPS
Short dashed: KG





Long dashed: CPS
Short dashed: KG




135MeV (Lab.)
Kharkiv w 3BF
CDB w 3BF
AV18 w 3BF
NM93 w 3BF
NM I w 3BF
NM II w 3BF
DATA 1
DATA 2
100
10
1
0.1
0
20
40
60
80
100
120
140
160
180

Triton Binding Energy

Potential	Relativistic	Nonrelativistic	DifferenceCD
Kharkiv	−7.799 MeV	−7.867 MeV	68keV
CDBonn	−8.150	−8.248	98

9. Conclusion and Outlook

- Our calculations is in the Instant Form (Bakamjian-Thomas Theory).
- There are two transformations between Relativistic potential and Non-relativistic one (KG and CPS) to conserve on-shell properties.
- We control the boosted potential by KG method.
- There are small differences between the Relativistic Faddeev calculations and Non-relativistic one (Energy range < 300MeV).
- The shift of the triton binding energy between rel. and nonrel. is about 50keV.

- Relativistic 3-body force are needed. (in progress)



北京盧溝橋：Marco Polo, who once visited China, praised the Lugou Bridge as the most beautiful bridge in the world.（1189, 1254）



盧溝橋/獅子

Because it was difficult to count the number of lions while crossing the bridge, in China, the phrase 'Lugou Bridge lions' is used to refer to something that cannot be counted. (499bodies?)



錦帯橋：The Kintai Bridge in Japan is beautiful with its arch, and its shape is a catenary curve, which provides the greatest strength.
Japanese carpenters assembled the bridge with this curve without any calculations. (1673)

$$y = a\cosh\left(\frac{x}{a}\right) + c$$

Thank you for your attention.