Amplitude Structure of $3 \to 3$ Scattering in a Mandelstam Variable Representation

Xu Zhang

Institute of Theoretical Physics, Chinese Academy of Sciences

Based on Xu Zhang and Feng-Kun Guo, arXiv:2609.00985

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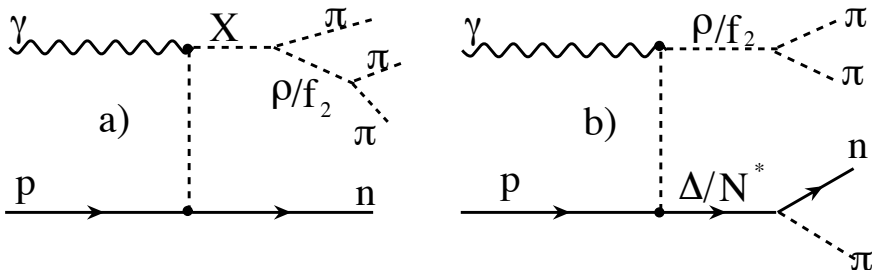
Workshop on Few-Body Problems in Nuclear and Hadron Physics

Outline

- **Background and Motivation**
  - Hadrons in three-body processes
  - Nonrelativistic Faddeev equations
  - Three-body amplitude with unitarity constraints
- **Dispersive approach**
  - Kinematics and partial waves
  - Two and three-body discontinuities
  - Two-body energy as dispersion variable
  - Separable interaction and simplification
  - Analytic continuation
- **Summary**

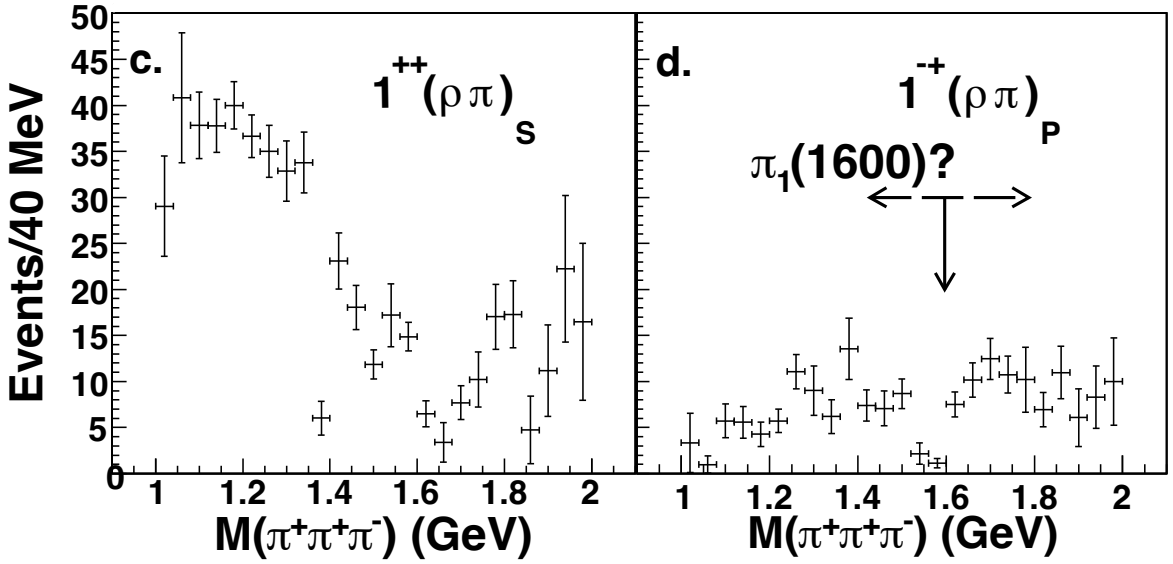
Hadron in three-body process

- **Photon-, meson-induced reactions**



Mina Nozar *et al.*, Phys.Rev.Lett. 102, 102002 (2009)

Analysing the 3π final state to search for $a_1(1260)$ and $\pi_1(1600)$

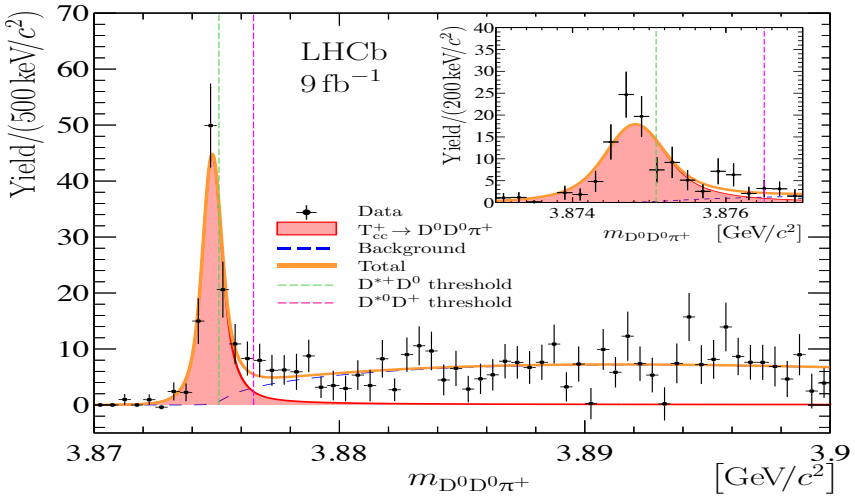


- **High-energy proton–proton collisions**

$pp \to T_{cc}^{+} X \to D^0 D^0 \pi^+ X$

Roel Aaij *et al.*, Nature Phys. 18 , 751 (2022)

The doubly charmed exotic state T_{cc} was observed in the $DD\pi$ final state



Hadron in three-body process

Many hadrons are observed in processes with multiparticle final states, where final-state interactions play important roles.

Light-hadron sector

- ω: vector meson decaying into 3π
- $a_1(1260)$: axial-vector meson in three-pion final states
- $a_1(1420)$: exotic candidate
- $\eta(1405/1475)$: isoscalar states
- $\pi_1(1600)$: hybrid-meson candidate

Heavy-hadron sector

- $X(3872)$: near $D\bar{D}\pi$ threshold
- T_{cc}^{+}: near $DD\pi$ threshold

- ➢ Understanding the nature of above hadrons calls for a careful treatment of three-body dynamics
- ➢ Scattering amplitudes are the theoretical bridge between experimental data and hadron properties

Faddeev equations

The amplitude is decomposed into three Faddeev components and obtained as the solution of a system of coupled linear integral equations

Ludwig D. Faddeev, Sov. Phys. JETP 12, 1014 (1961)

$T = \sum_{i=1}^{3} T^{(i)}, T^{(i)} = t_i + t_i\, G_0 \sum_{j \neq i} T^{(j)}$

t_i : two-body t-matrix of pair i in the three-particle space

G_0: free three-body Green function

- Rescattering series including all sequential two-body interactions
- Two-body unitarity of each t_i and the coupled-channel structure generate the three-body unitarity
- The equations are Fredholm-type and uniquely solvable
- Nonrelativistic

Faddeev equations

Faddeev equations are formulated in the nonrelativistic Schrödinger framework.

Kinematics

- Nonrelativistic kinetics violate Lorentz covariance
- Inapplicable when particle momenta are comparable to their masses

Dynamics

- Crossing symmetry of the S-matrix is absent
- No systematic studies of the analytic structure

- ➢ Nonrelativistic treatment is not adequate in the domain of high-energy particle reactions
- ➢ Amplitude may be built on the S-matrix principles — unitarity and analyticity — rather than on a Hamiltonian

Relation between $3 \to 3$ amplitude and $1 \to 3$ amplitude

The $3 \to 3$ amplitude appears implicitly in the $1 \to 3$ process

$1 \to 3$ decay processes

Nicola N. Khuri *et al.*, Phys. Rev. 119, 1115 (1960).

- Khuri–Treiman dispersive amplitudes
- The lowest-order nontrivial term corresponds to one real-particle exchange
- Kernel of Khuri–Treiman equations is related to that of Faddeev equation with a zero-range potential

> ➢ Khuri–Treiman amplitudes respect two- and three-body unitarity, analyticity, and crossing symmetry
> ➢ This is precisely what a dispersive construction aims to guarantee

Constructing 3 → 3 amplitudes from S-matrix constraints

Relativistic amplitude is constructed directly from the S-matrix constraints of unitarity and analyticity

Mandelstam's N/D formulation

Stanley Mandelstam, Phys. Rev. 140, B375 (1965)

- Five independent energy variables: s, two initial and two final subenergies
- Subenergies replaced by dimensionless $\sigma_i^2/(\sigma_1^2+\sigma_2^2+\sigma_3^2)$, with s as the dispersion variable
- Amplitudes are multivalued functions of s → kinematical branch points
- Kinematical branch points persist even for spinless particles after partial-wave decomposition
- The N/D function lives on a Riemann surface divided by kinematical cut and the discontinuity across a kinematical cut is unknown.

Three-body with s as dispersion variable: the physical amplitude is entangled with kinematical cuts, and the N/D function is defined on a multi-sheeted surface.

Subenergy as dispersion variable

- **Using a two-particle subenergy σ' instead of the total energy s as the dispersive integration variable**
- **No kinematical cut appears on the amplitude**

Motivation: the Khuri–Treiman equation

- The Khuri–Treiman equation for $1 \to 3$ decays
- Builds the amplitude on two-body unitarity relations
- Inhomogeneous term guarantees three-body unitarity

$3 \to 3$ amplitude

- Separating the two-body subenergy cuts from the three-body cut
- Solution satisfies three-body unitarity — generated automatically by two-body unitarity, analyticity and crossing symmetry

Kinematics

Three identical spinless particles of mass m_π

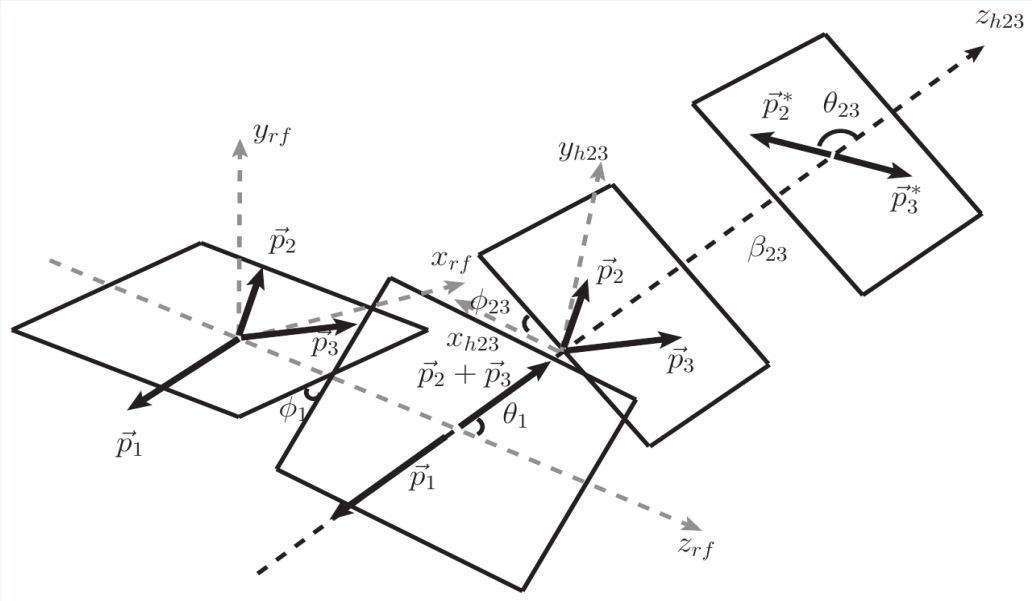
Only the S-wave projection $j = l = 0$ is retained

Mandelstam variables

Initial-state subenergy $\sigma \equiv q^2 = (p_2 + p_3)^2$

Final-state subenergy $\sigma' \equiv q'^2 = (p_2' + p_3')^2$

Total four-momentum squared $s = P^2$



Definition of angles that parametrize three-body kinematics with respect to the given reference frame

Connected / disconnected decomposition

The full amplitude decomposes as $T = T_d + T_c$:

- T_d: one particle propagates as a spectator while the other two scatter
- T_c: the connected three-body part

The S-wave connected amplitude $T(\sigma', s, \sigma)$ depends only on three Mandelstam variables

Three-particle states

One-particle states are normalized as $\langle \vec{p}\,' | \vec{p}\,\rangle = 2E(2\pi)^3\delta^3(\vec{p}\,' - \vec{p}\,) \equiv \tilde{\delta}(\vec{p}\,' - \vec{p}\,)$.

$$|\vec{p}_1\vec{p}_2\rangle = \frac{1}{2}(|\vec{p}_1\rangle|\vec{p}_2\rangle + |\vec{p}_2\rangle|\vec{p}_1\rangle)$$

$$|\vec{p}_1\vec{p}_2\vec{p}_3\rangle = \frac{1}{3!}(|\vec{p}_1\rangle|\vec{p}_2\rangle|\vec{p}_3\rangle + symm.) = \frac{1}{3}(|\vec{p}_1(\vec{p}_2\vec{p}_3)\rangle + cyclic)$$

$$I_3 = \int d\vec{p}\ |\vec{p}_1\vec{p}_2\vec{p}_3\rangle\langle\vec{p}_1\vec{p}_2\vec{p}_3| = \frac{1}{3}\int d\vec{p}\ |\vec{p}_1(\vec{p}_2\vec{p}_3)\rangle\langle\vec{p}_1(\vec{p}_2\vec{p}_3)|$$

$$+\frac{2}{3}\int d\vec{p}\ |\vec{p}_1(\vec{p}_2\vec{p}_3)\rangle\langle\vec{p}_2(\vec{p}_3\vec{p}_1)| \qquad d\vec{p} \equiv \prod_i d^3p_i/[2E_i(2\pi)^3]$$

Symmetrized amplitude

$$\langle \vec{p}\,'_1\vec{p}\,'_2\vec{p}\,'_3|T|\vec{p}_1\vec{p}_2\vec{p}_3\rangle =$$

$$\frac{1}{9}\sum_{i,j}\langle(\vec{p}\,'_{j1}\vec{p}\,'_{j2})\vec{p}\,'_{j3}|T|(\vec{p}_{i1}\vec{p}_{i2})\vec{p}_{i3}\rangle$$

- The fully symmetrized three-particle space contains nine terms
- Since the measure $d\vec{p}$ is invariant under relabeling of the dummy integration variables, the three diagonal terms are equal and the six off-diagonal terms are equal.

Partial-wave expansion

The expansion proceeds in two stages, both using Wigner D functions: first within the (23) pair rest frame, then for the pair-plus-spectator system in the overall reference frame

Step 1 • pair (23) in its own rest frame

$|\vec{p}_2\vec{p}_3\rangle = \sum_{l\lambda} \sqrt{2l+1}\ D^{l}_{\lambda 0}(\Omega_{23})\ |\vec{q}_1\ l\lambda\rangle,\ \vec{q}_1 = \vec{p}_2 + \vec{p}_3$

The z_{h23} axis is chosen such that the spin projection λ is conserved and becomes the helicity when boosted to the production frame

Step 2 • pair + spectator in the reference frame

$|\vec{q}_1\ l\lambda\rangle|\vec{p}_1\rangle = \sum_{jm} \sqrt{2j+1}\, D^{j}_{m\lambda}(\Omega_1)\ |P, jml\lambda\rangle_1$

$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3$; j, m: total orbital angular momentum and its projection on the z_{rf} axis; Ω_1: direction of $\vec{q}_1$

Result: partial-wave amplitudes

- connected: $\langle P', 0000|T_c|P, 0000\rangle = (2\pi)^4\delta^4(P'-P)\ T(\sigma', s, \sigma)$
- disconnected :$\langle P', 0000|T_d|P, 0000\rangle = (2\pi)^4\delta^4(P'-P)\delta(\sigma'-\sigma)\ t(\sigma)/\rho_s(\sigma)$

Only the S –wave $j = l = 0$ is kept

Phase space

With s and the subenergies $\sigma^{(\prime)}$ as variables, the unitarity condition takes a particularly simple form. The three-body phase space has two equivalent parametrizations

$$d\Phi_3 = \frac{d\sigma_1''}{2\pi}\,\rho(\sigma_1'')\,\rho_s(\sigma_1'')\,\frac{d\Omega_1}{4\pi}\frac{d\Omega_{23}}{4\pi}\,\theta^+(\phi(\sigma_2'',s,\sigma_3'')) = \frac{d\sigma_2''\,d\sigma_3''}{2\pi(8\pi)^2 s}\,\frac{d\Omega_2}{4\pi}\frac{d\phi_{31}}{2\pi}\,\theta^+(\phi)$$

Phase-space factors

$\rho(\sigma) = \lambda^{1/2}(\sigma, m_\pi^2, m_\pi^2)/(8\pi\sigma)$

$\rho_s(\sigma) = \lambda^{1/2}(\mathrm{s}, \sigma, m_\pi^2)/(8\pi s)$

with the Källén function $\lambda(x,y,z)$

Physical domain: the Kibble function

$\phi(\sigma_2, s, \sigma_3) = \sigma_2\sigma_3(3m_\pi^2 + s - \sigma_2 - \sigma_3) - m_\pi^2(s - m_\pi^2)^2$

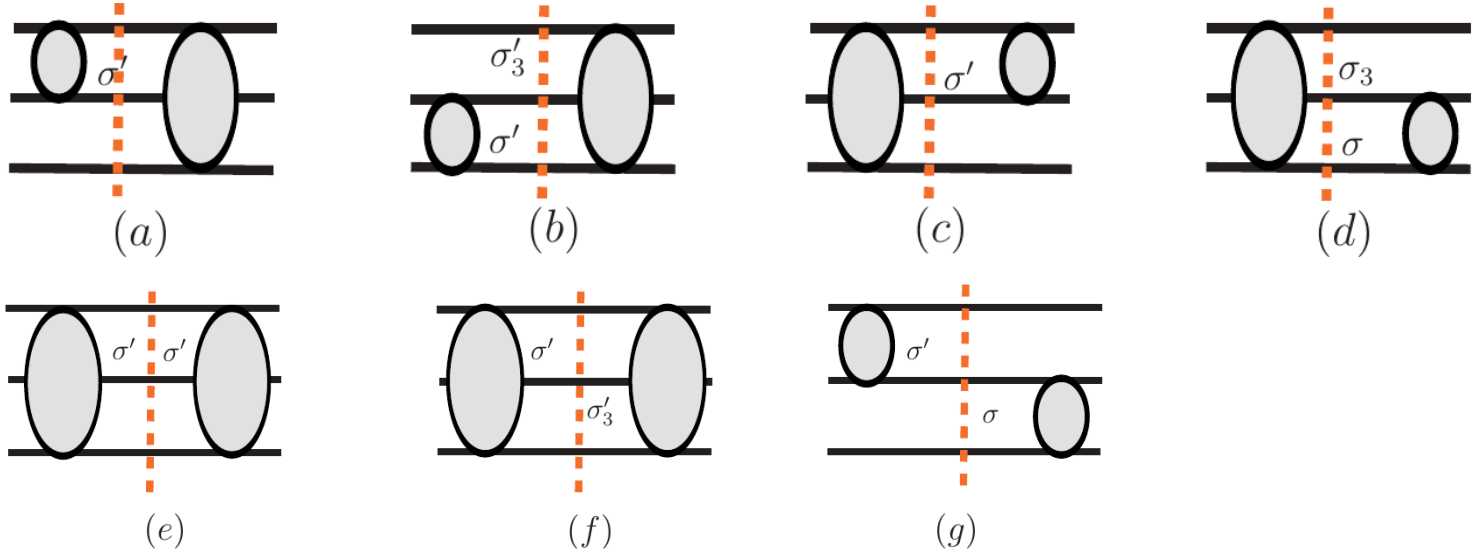
θ^+ enforces the three-body threshold $s > 9m_\pi^2$, the two-body thresholds $\sigma_i > 4m_\pi^2$, and the Dalitz boundary $\phi \geq 0$

Dalitz-plot endpoints $\sigma^{\pm}(\sigma, s) = G(\sigma, s) \pm F(\sigma, s)$, with $G = \frac{1}{2}(s + 3m_\pi^2 - \sigma)$,
$F = \lambda_s^{1/2}(\sigma, m_\pi^2, m_\pi^2)\lambda^{1/2}(\mathrm{s}, \sigma, m_\pi^2)/(2\sigma)$

Two-body and three-body discontinuities

The unitarity relation is decomposed into crossings of the two-particle subenergy cuts and of the total-energy cut

$$[\langle b|T_c - T_c^\dagger|a\rangle]_{dc+cd+dd} = i\int d\Phi_3[\,\langle b|T_d^\dagger|1''\rangle\langle 1''|T_c|a\rangle + 2\langle b|T_d^\dagger|2''\rangle\langle 3''|T_c|a\rangle + \langle b|T_c^\dagger|1''\rangle\langle 1''|T_d|a\rangle + 2\langle b|T_c^\dagger|2''\rangle\langle 3''|T_d|a\rangle + 6\langle b|T_d^\dagger|2''\rangle\langle 3''|T_d|a\rangle\,]$$



Diagrammatic representation of the 3→3 unitarity relation. Dashed lines mark the cuts

disconnected–connected (dc)

connected–disconnected (cd)

connected–connected (cc)

disconnected–disconnected (dd)

Explicit form of the two-body discontinuities

- $x_\pm \equiv x \pm i\epsilon$ are boundary values from the upper/lower edge of the relevant unitarity cut
- σ' is continued first, then σ, at fixed s_+

$$T(\sigma'_+, s_+, \sigma_+) - T(\sigma'_-, s_+, \sigma_-)$$

$$= i\, t^\dagger(\sigma'_+)\rho(\sigma')\, T(\sigma'_+, s_+, \sigma_+) + 2i\, \frac{t^\dagger(\sigma'_+)}{\lambda_s^{1/2}(\sigma')} \frac{1}{8\pi} \int_{\sigma^-(\sigma', s_+)}^{\sigma^+(\sigma', s_+)} d\sigma'_3\; T(\sigma'_3, s_+, \sigma_+)$$

Final-state σ′ cut

$$+ i\, T^\dagger(\sigma'_+, s_+, \sigma_+)\, \rho(\sigma)\, t(\sigma_+) + 2i\, \frac{t(\sigma_+)}{\lambda_s^{1/2}(\sigma)} \frac{1}{8\pi} \int_{\sigma^-(\sigma, s)}^{\sigma^+(\sigma, s)} d\sigma_3\, T^\dagger(\sigma'_+, s_+, \sigma_3)$$

Initial-state σ cut

$$+ 6i\, \frac{2\pi s\, t^\dagger(\sigma'_+)\, t(\sigma_+)}{\lambda_s^{1/2}(\sigma')\, \lambda_s^{1/2}(\sigma)}\, \theta^+(\phi(\sigma', s, \sigma))$$

dd crossed channel

$\rho(\sigma) = \frac{1}{8\pi} \frac{2|\vec{p}^{\,*}|}{\sqrt{\sigma}}$, momentum $|\vec{p}^{\,*}| = \sqrt{\sigma/4 - m_\pi^2}$

This relation is the starting point of the dispersive construction

Dispersive representation

At fixed s_+ and σ_+, choose the final-state subenergy σ' as the dispersion variable. Then one obtains the dispersive representation of $T(\sigma', s, \sigma\ \ )$

$$T(\sigma'_+, s_+, \sigma_+) = \int_{4m_\pi^2}^{\infty} \frac{d\sigma''}{\sigma''-\sigma'-i\epsilon} \frac{6s\, t^\dagger(\sigma''_+)\, t(\sigma_+)}{\lambda_s^{1/2}(\sigma'')\, \lambda_s^{1/2}(\sigma)}\, \theta^+(\phi(\sigma'', s, \sigma)) + \frac{1}{2\pi} \int_{4m_\pi^2}^{\infty} \frac{d\sigma''}{\sigma''-\sigma'-i\epsilon}\, t^\dagger(\sigma''_+)\rho(\sigma'')\, T(\sigma''_+, s_+, \sigma_+)$$
$$+ \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{d\sigma''}{\sigma''-\sigma'-i\epsilon} \frac{t^\dagger(\sigma''_+)}{\lambda_s^{1/2}(\sigma'')} \frac{1}{8\pi} \int_{\sigma^-(\sigma'',s)}^{\sigma^+(\sigma'',s)} d\sigma_3\, T(\sigma_3, s_+, \sigma_+)$$

Term 1 • driving term (one-particle exchange)

- θ^+ constrains the integral to the physical Dalitz region $4m_\pi^2 \le \sigma'' \le (\sqrt{s} - m_\pi)^2$

Terms 2–3 • rescattering kernels

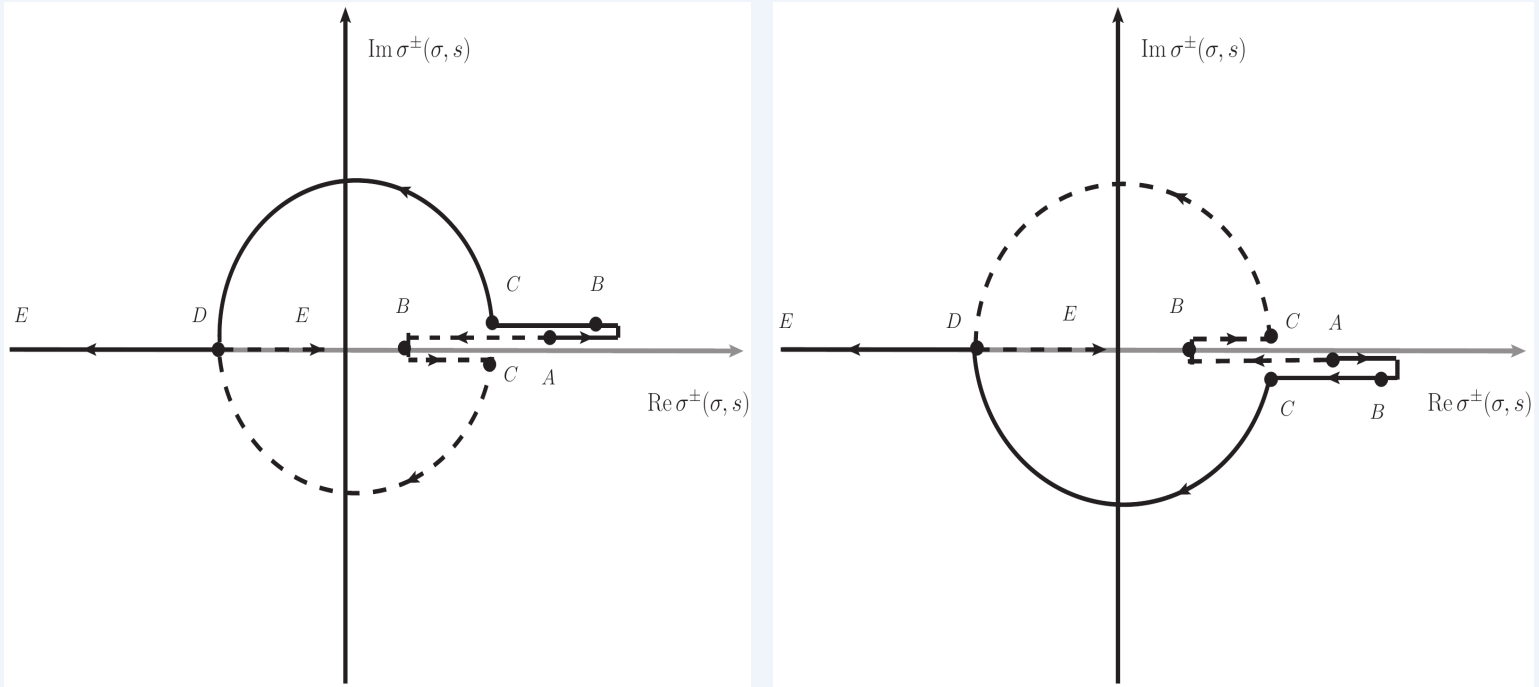
- With the inner integral running between the Dalitz endpoints $\sigma^- \to \sigma^+$
- Both contain the unknown T itself

Endpoint contours

Four intervals of the endpoint contour

As σ increases, the endpoints $\sigma^{\pm}(\sigma, s)$ lie along segments [A,B], [B,C], [C,D], [D,E]

- $[4m_\pi^2, (s-m_\pi^2)/2] \rightarrow$ [A,B]
- $[(s-m_\pi^2)/2, (\sqrt{s}-m_\pi)^2] \rightarrow$ [B,C]
- $[(\sqrt{s}-m_\pi)^2, (\sqrt{s}+m_\pi)^2] \rightarrow$ [C,D]
- $\sigma \geq (\sqrt{s}+m_\pi)^2 \rightarrow$ [D,E]



Integration contours of the endpoints $\sigma^{\pm}$ for $s + i\epsilon$ (left) and $s - i\epsilon$ (right)

Three-body discontinuities

The resulting $\Delta T(\sigma_3, s_+, \sigma_+) \equiv T(\sigma_3, s_+, \sigma_+) - T(\sigma_3, s_-, \sigma_+)$

$$T(\sigma'_+, s_+, \sigma_+) - T(\sigma'_+, s_-, \sigma_+) = \frac{i}{6\pi} \int_{4m_\pi^2}^{(\sqrt{s}-m_\pi)^2} d\sigma''\ T(\sigma'_+, s_+, \sigma'')\, \rho(\sigma'')\rho_s(\sigma'')\, T(\sigma''^*, s_-, \sigma_+)$$

$$+ \frac{i}{3\pi s} \frac{1}{(8\pi)^2} \iint_{\phi>0} d\sigma_3\ d\sigma'_3\, T(\sigma'_+, s_+, \sigma_3)\, T(\sigma'^*_3, s_-, \sigma_+)$$

Same as other works

Mikhail Mikhasenko *et al.*,
JHEP 08, 080 (2019)

First term

- Single integral over the intermediate subenergy σ'', weighted by the two-body phase space ρ and the pair–spectator phase space ρ_s
- One pair rescatters while the spectator is on shell

Second term

- Double integral over the Dalitz region $\phi(\sigma_3, s, \sigma'_3) > 0$
- Both intermediate subenergies vary independently

Dispersive solution automatically satisfies three-body unitarity — generated by two-body unitarity, analyticity and crossing symmetry

Simplification : Separable interactions

The construction simplifies when the two-particle subsystem interactions are separable. The two-body amplitude can factors out of $T(\sigma', s, \sigma)$

$T(\sigma', s, \sigma) = t(\sigma')\, M(\sigma', s, \sigma)\, t(\sigma)$ — the amputated amplitude M satisfies a simpler equation

Dispersive representation for M

$$M(\sigma'_+, s_+, \sigma_+) = \int_{4m_\pi^2}^{\infty} \frac{d\sigma''}{\sigma''-\sigma'-i\epsilon} \frac{6s\,\theta^+(\phi(\sigma'',s,\sigma))}{\lambda_s^{1/2}(\sigma'')\,\lambda_s^{1/2}(\sigma)} + \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{d\sigma''}{\sigma''-\sigma'-i\epsilon} \frac{1}{\lambda_s^{1/2}(\sigma'')} \frac{1}{8\pi}\int_{4m_\pi^2}^{(\sqrt{s}-m_\pi)^2} d\sigma_3'\, \theta^+(\phi)\, t(\sigma_3')\, M(\sigma_3', s, \sigma)$$

Partial-wave one-particle-exchange potential

$$\int_{4m_\pi^2}^{\infty} \frac{d\sigma''}{\sigma''-\sigma'-i\epsilon} \frac{6s\,\theta^+(\phi(\sigma'',s,\sigma))}{\lambda_s^{1/2}(\sigma'')\,\lambda_s^{1/2}(\sigma)} = \frac{6s}{\lambda_s^{1/2}(\sigma')\lambda_s^{1/2}(\sigma)} ln\left(\frac{-s^2+s(\sigma'+\sigma)+(\sigma'-m_\pi^2)(\sigma-m_\pi^2)-\lambda_s^{1/2}(\sigma')\lambda_s^{1/2}(\sigma)}{-s^2+s(\sigma'+\sigma)+(\sigma'-m_\pi^2)(\sigma-m_\pi^2)+\lambda_s^{1/2}(\sigma')\lambda_s^{1/2}(\sigma)}\right)$$

Connection to other parameterizations

Step 1 • Cauchy representation of tM

On the physical sheet, $tM(\sigma', s, \sigma)$ is analytic in σ'. Assuming no bound state, one can write $Mt = \frac{1}{2i\pi}\oint_{\gamma_2} \frac{d\sigma''}{\sigma''-\sigma'-i\epsilon}\, tM$, with γ_2 a clockwise contour encircling the unitarity cut

Step 2 • kernel = triangle loop

Interchanging the order of integration, the kernel

$$K = \frac{1}{2i\pi^2}\frac{1}{8\pi}\oint_{\gamma_2} \frac{d\sigma''}{\sigma''-\sigma'-i\epsilon}\,\frac{1}{\lambda_s^{1/2}(\sigma'')}\int_{\sigma^-}^{\sigma^+}\frac{d\sigma_3'}{\sigma_3-\sigma_3'-i\epsilon}$$

is identical to the triangle-loop diagram

Step 3 • shrinking γ_2 onto contour Γ

Shrinking γ_2 is stopped on the upper side by the endpoint singularities $\sigma^{\pm}(\sigma'_+, s_+)$. The resulting contour Γ runs on the upper side of the real axis from $-\infty$ to $(\sqrt{s}-m_\pi)^2$, only the discontinuity ΔK survives.

Resulting equation

$$M(\sigma'_+, s_+, \sigma_+) = 6s(8\pi^2)\frac{\Delta K(\sigma'_+, s_+, \sigma_+)}{\lambda_s^{1/2}(\sigma)}$$
$$+\ 8\pi s\int_\Gamma d\sigma_3\,\frac{\Delta K(\sigma'_+, s_+, \sigma_{3+})}{\lambda_s^{1/2}(\sigma_3)}\,\tau(\sigma_{3+})\,M(\sigma_{3+}, s_+, \sigma_+)$$

with $\tau(\sigma) = t(\sigma)\rho_s(\sigma)$

Connection to other parameterizations

Several relativistic three-body formalisms have been developed. In the physical region they share the same unitarity discontinuity, but differ in their treatment of the unphysical subenergy region

Formalism	Construction	Representatives
B-matrix	Pairwise parameterization of subsystems, amputated amplitude	Andrew W. Jackura *et al.*, EPJC 79, 56 (2019)
Effective Bethe–Salpeter equation	Isobar propagator dressed by two-body loops, satisfies two- and three-body unitarity	Maxim Mai *et al.*, EPJA 53, 177 (2017)

Dispersive representation has a similar form

Analytic continuation

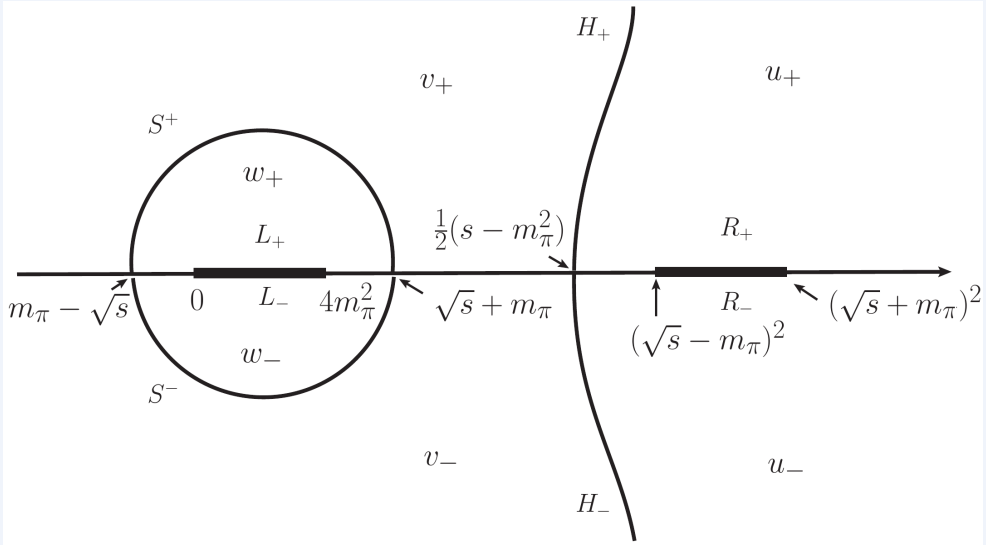
Once the amplitude is fixed on the real axis, its analytic structure is uniquely determined. One can extend $\sigma = x + iy$ to the complex plane. Boundary curves follow from requirement σ^+ or σ^- to be real

$$y^2 = -\frac{\left[x-\frac{1}{2}(s-m_\pi^2)\right]\left[x-(m_\pi^2+m_\pi\sqrt{s})\right]\left[x-(m_\pi^2-m_\pi\sqrt{s})\right]}{x-\frac{1}{2}(s+3m_\pi^2)}$$

The boundary curves form two branches, S and H (each with upper / lower parts $S^\pm$, $H^\pm$), dividing the σ plane into regions $u_\pm$, $v_\pm$, $w_\pm$ that map into one another under σ^+, σ^- (e.g. $\sigma^+: u_\pm \to v_\mp$, $\sigma^-: u_\pm \to w_\pm$)

Purpose

Rotate the integration contour across the three-body unitarity cut to continue the amplitude onto unphysical Riemann sheets of the complex energy plane — poles close to the real axis are resonances.



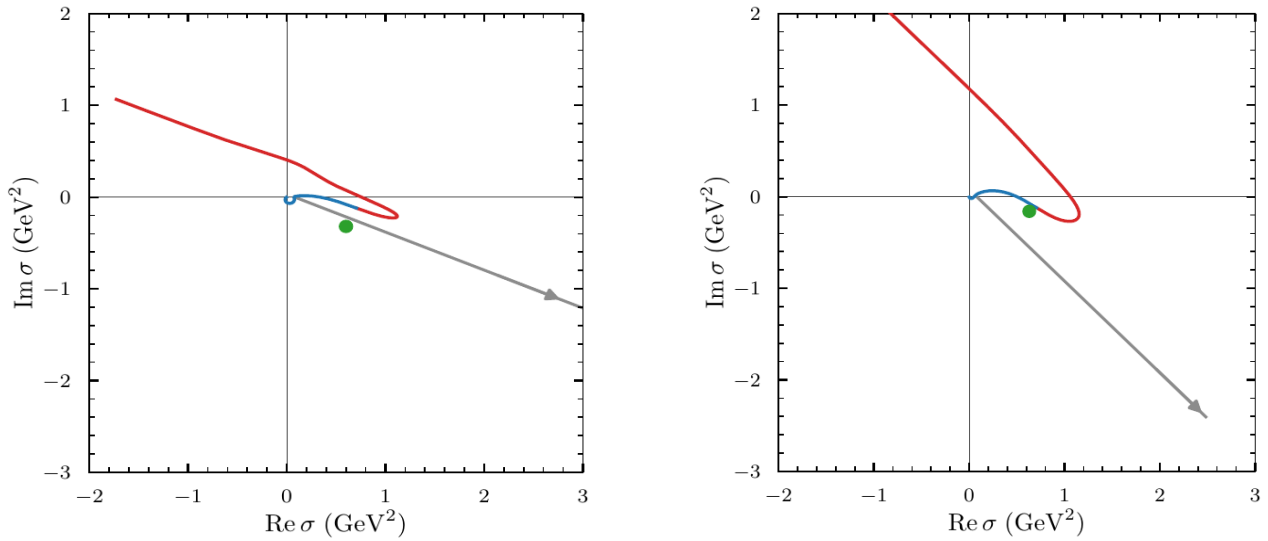
Boundaries of the mapping regions

Riemann-sheet structure

The three-body unitarity cut runs from the threshold $9m_\pi^2$ to infinity. The physical sheet is the one containing the physical values just above the real axis

Analytic continuation

Rotate the dispersive integration contour by an angle ϕ into the lower half-plane: the domain of analyticity of the unphysical sheet is exposed. Beyond this domain, the endpoint contour $\sigma^- \to \sigma^+$ would intersect the dispersive contour σ'' and the continuation fails.



Integration contours in the complex σ plane. Grey: rotated dispersive contour; red / blue: endpoint σ^+ / σ^- contours; green dots: two-body poles on unphysical sheets. Left: $\varphi=\pi/8$, pole $\sqrt{\sigma}=0.8-0.2i$ GeV; right: $\varphi=\pi/4$, pole $\sqrt{\sigma}=0.8-0.1i$ GeV ($\sqrt{s}=1.2-0.1i$ GeV)

How the continuation proceeds

1. As the contour rotates downward, the two-body subsystem moves onto its unphysical Riemann sheet;

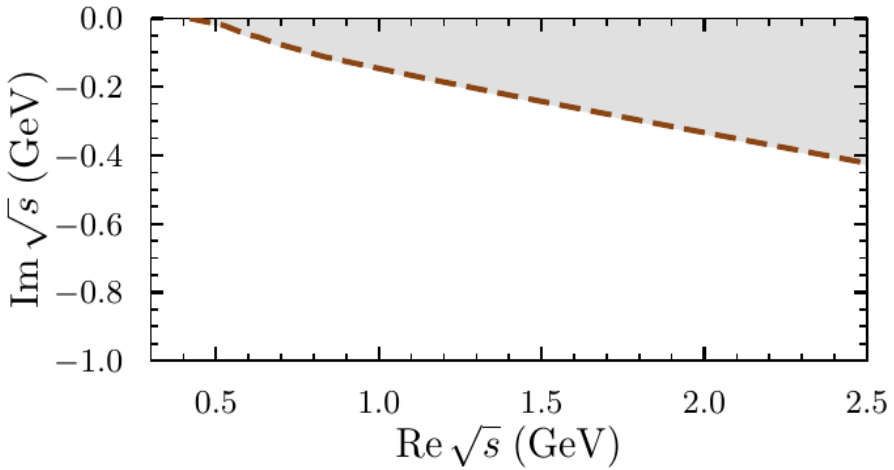
2. When the dispersive contour sweeps past the two-body pole (right panel), the pole residue must be picked up — the pole subtraction of the next page;

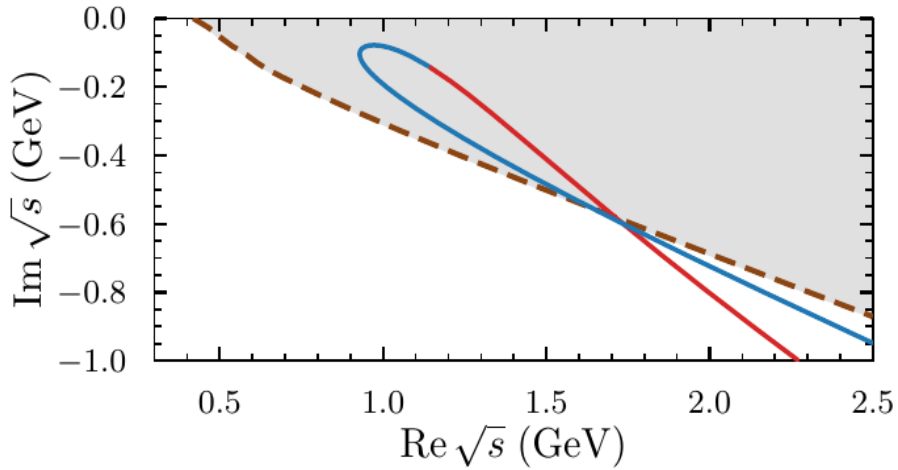
3. A larger rotation angle reaches deeper into the unphysical sheet, limited by the endpoint-contour intersection.

Analytic continuation

By the Schwarz reflection principle, $t^*(\sigma_-) = t^{II}(\sigma_-) = t(\sigma_+)$ along the real axis. When the dispersive contour rotates below the two-body pole, analytic continuation requires subtracting pole terms $r_p = Res_{z=\sigma_{pole}} t^{II}(z)$

$$-i\,\frac{r_p\,\rho(\sigma_{pole})}{\sigma_{pole}-\sigma'}\,T(\sigma_{pole},s_+,\sigma_+) \;-\; 2i\,\frac{r_p}{(\sigma_{pole}-\sigma'-i\epsilon)\,\lambda_s^{1/2}(\sigma_{pole})}\cdot\frac{1}{8\pi}\int_{\sigma^-(\sigma_{pole},s)}^{\sigma^+(\sigma_{pole},s)} d\sigma_3\,T(\sigma_3^*,s_+,\sigma_+)$$





Domain of analyticity of the unphysical sheet in the lower-half √s plane. Brown dashed lines: boundaries of the analyticity region; The red / blue lines correspond to $\sigma_{II}^{\pm}(\sigma,s) = \sigma_{pole}$.

Analytic continuation

Associated with the two-body pole, the three-body amplitude has a complex two-body cut.

Implications

- Each two-body resonance pole generates a complex cut in the $3 \to 3$ amplitude
- For the amputated amplitude M , the dispersive integration contour itself never meets the two-body pole — only $\sigma_{\mathrm{II}}^{\pm}$ does — so the continuation prescription is simpler

Continuation

1. Rotate the dispersive contour by ϕ
2. Subtract two-body pole terms when sweeping past them
3. Track the complex two-body cuts
4. Stay inside the analyticity domain bounded by the endpoint contours.

With these prescriptions, the amplitude can be evaluated on the unphysical sheets adjacent to the physical real axis, on which the resonance pole sits

Summary

1. A relativistic $3 \to 3$ amplitude is constructed with the two-particle subenergy as the dispersion variable. The result is a linear integral equation driven by one-particle exchange.

2. Three-body unitarity is generated by two-body unitarity, analyticity and crossing symmetry. For separable interactions the equation takes the isobar–spectator form, connecting to other parameterizations.

3. Contour deformation continues the amplitude onto adjacent unphysical sheets on which the resonance pole sits.

Thank You!