Recent progress on the compositeness of near-threshold eigenstates

T. Kinugawa and T. Hyodo, Phys. Rev. C 109 , 045205 (2024);
arXiv:2403.12635 [hep-ph]; arXiv:2604.17813[hep-ph].





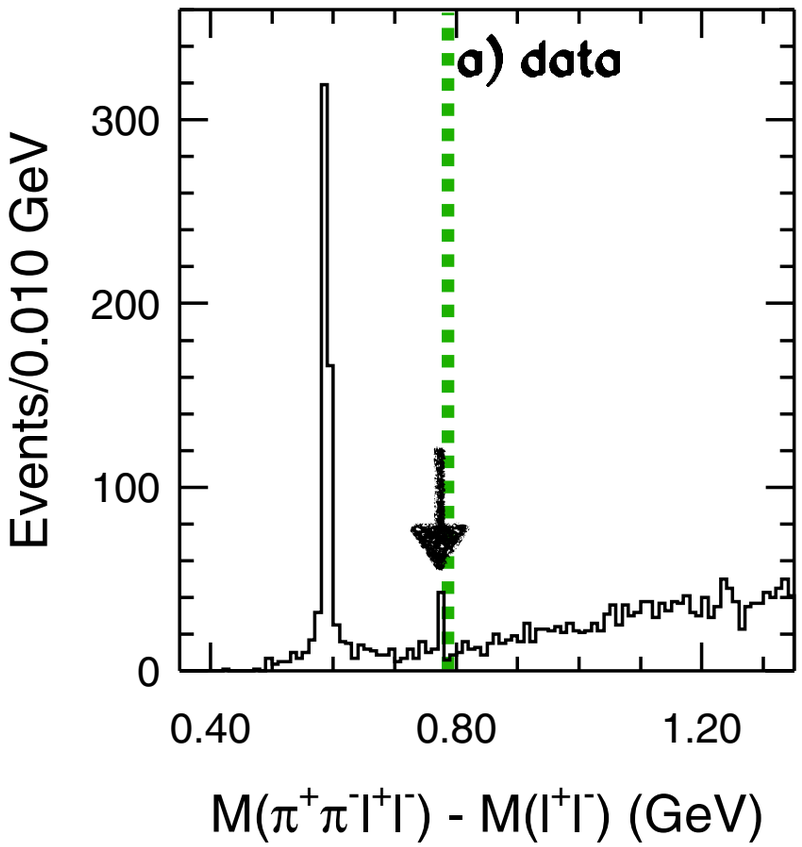
Tomona Kinugawa (衣川 友那)
Nishina Center, RIKEN

September 23rd,
Workshop on Few-Body Problems in Nuclear and Hadron Physics @ITP-CAS

Near-threshold exotic hadrons

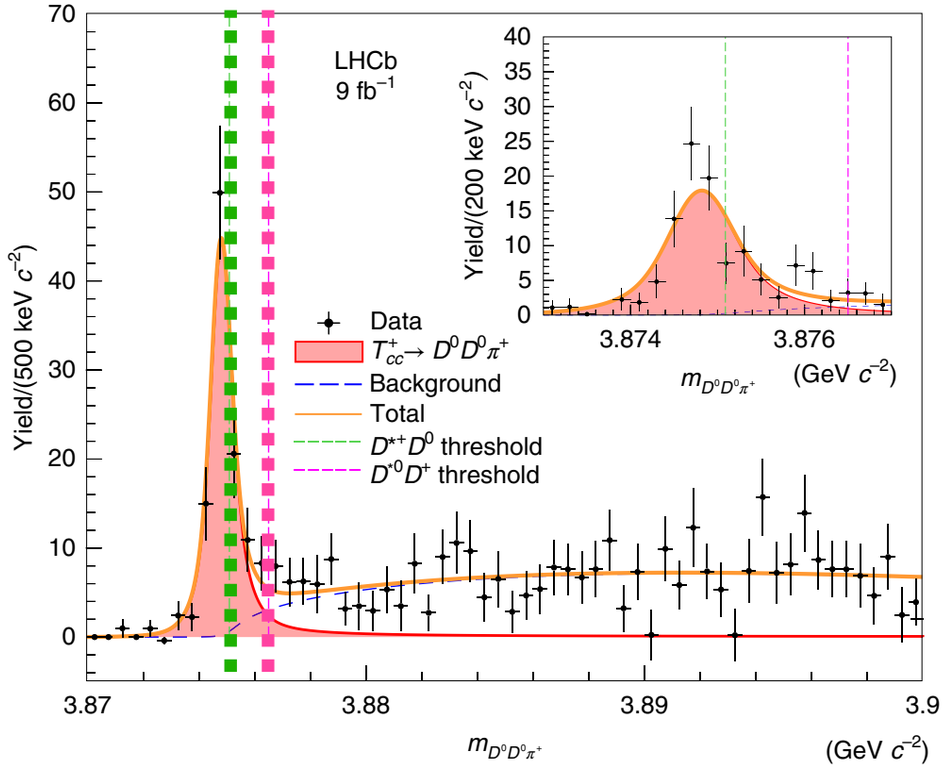
- exotic hadrons tend to be observed near threshold

$X(3872) \to \pi^+\pi^- J/\psi$



S. K. Choi *et al*. (Belle), Phys. Rev. Lett. **91**, 262001 (2003).

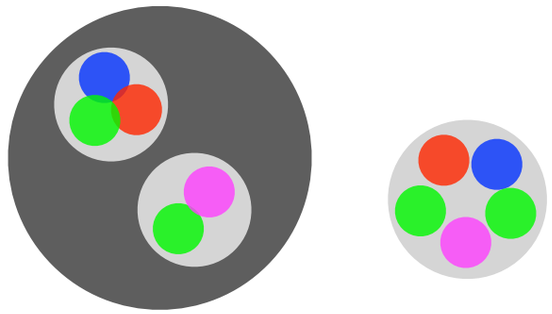
$T_{cc}(3875)^+ \to D^0 D^0 \pi^+$ $(cc\bar{u}\bar{d})$



LHCb Collaboration, Nature Phys. **18** (2022) no.7, 751-754;
LHCb Collaboration, Nat. Commun. **13** 3351 (2022).

→ internal structure of near-threshold states?

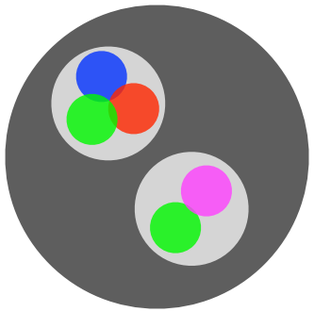
hadronic molecule, multiquark, etc…?



Compositeness

Weinberg, S. Phys. Rev. 137, 672–678 (1965);
T. Hyodo, Int. J. Mod. Phys. A 28, 1330045 (2013);
T. Kinugawa, T. Hyodo, Eur. Phys. J. A 61, 154 (2025).

◉ Definition





$$|\text{state}\rangle = \sqrt{X}\,|\text{composite}\rangle + \sqrt{Z}\,|\text{others}\rangle$$

compositeness **elementarity**

- quantitative measure

$$0 \le X \le 1 \qquad X + Z = 1$$

$$\longrightarrow$$

$X > 0.5 \Leftrightarrow$ **composite dominant**

$X < 0.5 \Leftrightarrow$ **non-composite dominant**

- applicable to various states

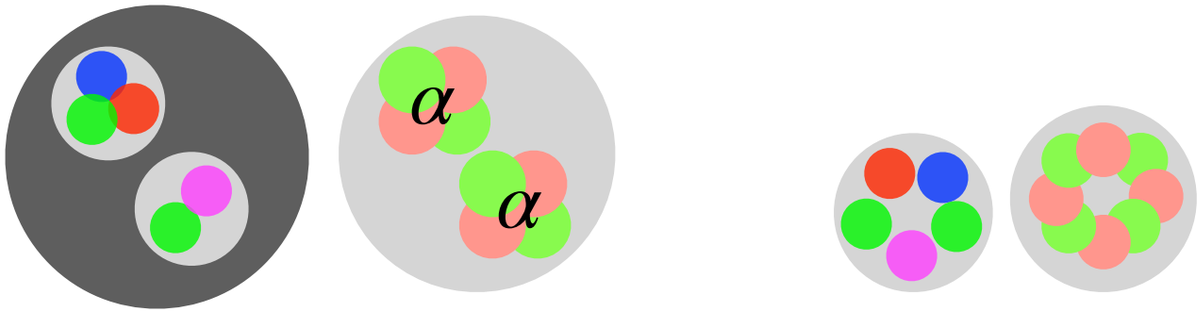
$f_0(980)$, $a_0(980)$

V. Baru, J. Haidenbauer, C. Hanhart, Y. Kalashnikova, A.E. Kudryavtsev, Phys. Lett. B 586, 53–61 (2004);
Y. Kamiya and T. Hyodo, PTEP 2017; Phys. Rev. C 93, 035203 (2016) etc.

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$f_0(980)$, $a_0(980)$ V. Baru, J. Haidenbauer, C. Hanhart, Y. Kalashnikova, A.E. Kudryavtsev, Phys. Lett. B 586, 53–61 (2004);
Y. Kamiya and T. Hyodo, PTEP 2017; Phys. Rev. C 93, 035203 (2016) etc.

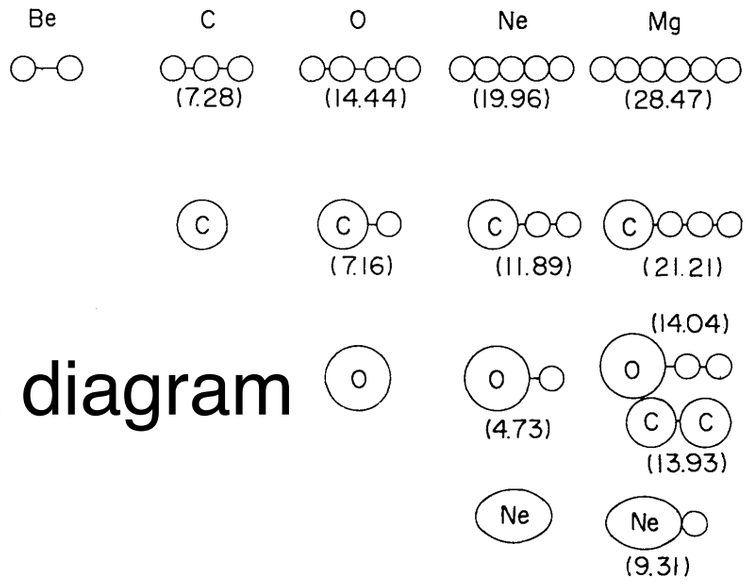
nuclei, atomic systems E. Braaten, H. W. Hammer, and M. Kusunoki, (2003), cond-mat/0301489;
T. Kinugawa, T. Hyodo, Phys. Rev. C 106, 015205 (2022).

Clustering phenomena

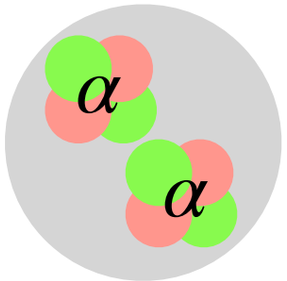
- near-threshold states ⟶ clustering phenomena

- α clustering phenomena

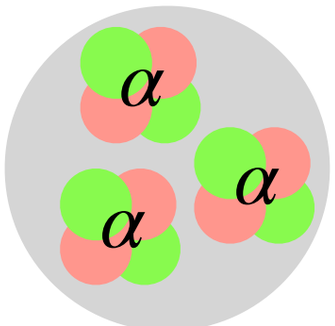
K. Ikeda, and N. Takigawa, and H. Horiuchi, Prog. Theor. Phys. Suppl. E68, 464-475 (1968).



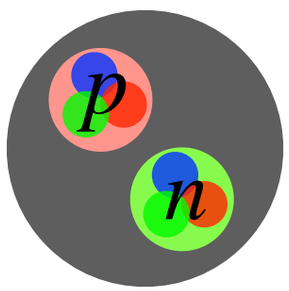
Ikeda diagram



^{8}Be nuclei



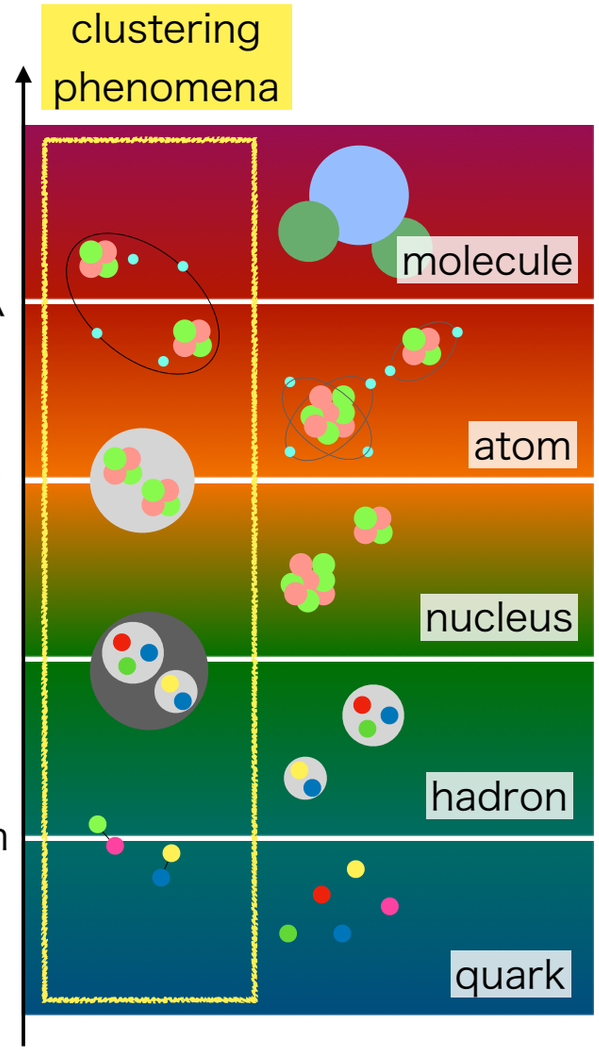
^{12}C Hoyle state



- deuteron $B = 2.22$ MeV

hadronic molecule ⟶ clustering of quarks

- clustering phenomena: **universal**!

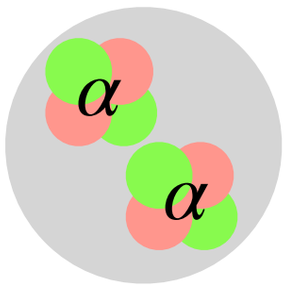


Near-threshold states

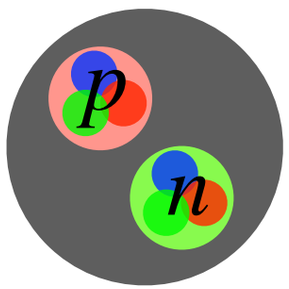
◉ Are near-threshold states composite dominant ($X \sim 1$)?

- **empiricaly** : Yes! “threshold energy rule”

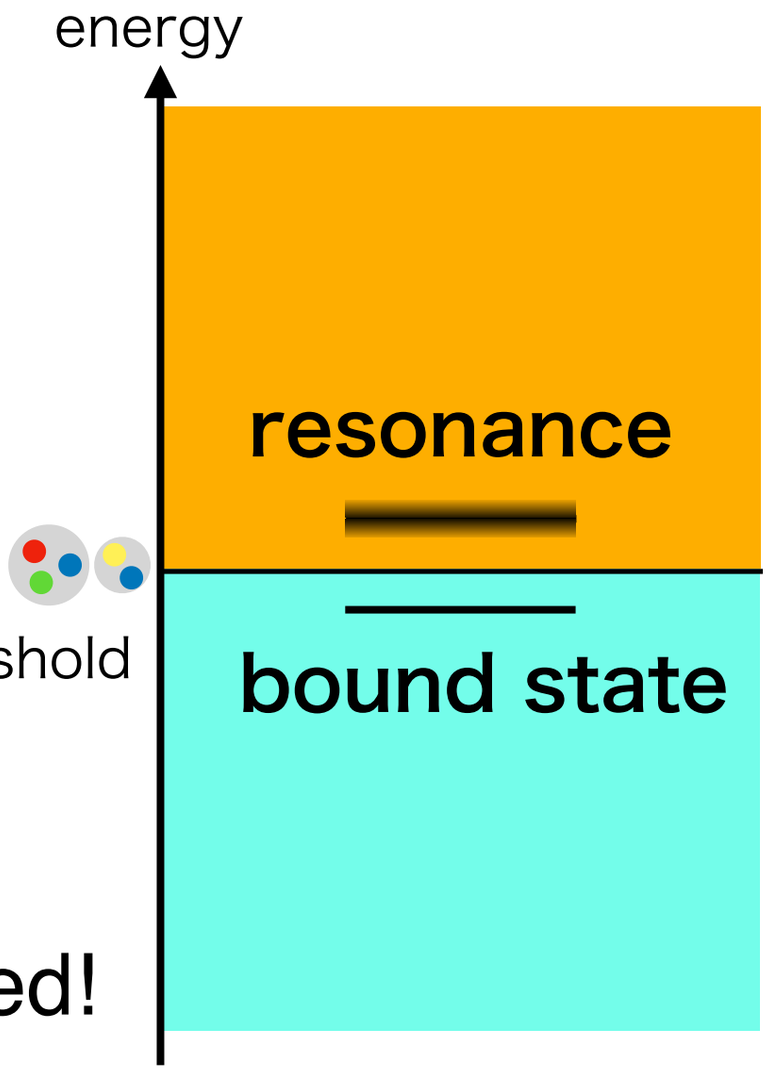
K. Ikeda, and N. Takigawa, and H. Horiuchi, Prog. Theor. Phys. Suppl. E68, 464-475 (1968).



^{8}Be nuclei



deuteron



- **theoretically**

- $X \to 1$ at threshold ($B = 0$)

T. Hyodo, Phys. Rev. C **90**, 055208 (2014)

- for $B \neq 0$, even non-composite dominant shallow bound states ($X \sim 0$) can be realized!

C. Hanhart, J. R. Pelaez, and G. Rios, Phys. Lett. B **739**, 375 (2014).

→ Why do we rarely observe such a state?

- structure of **resonances** slightly above threshold?

→ provide a theoretical foundation of the threshold energy rule

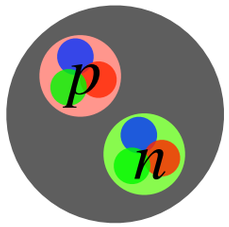
This work

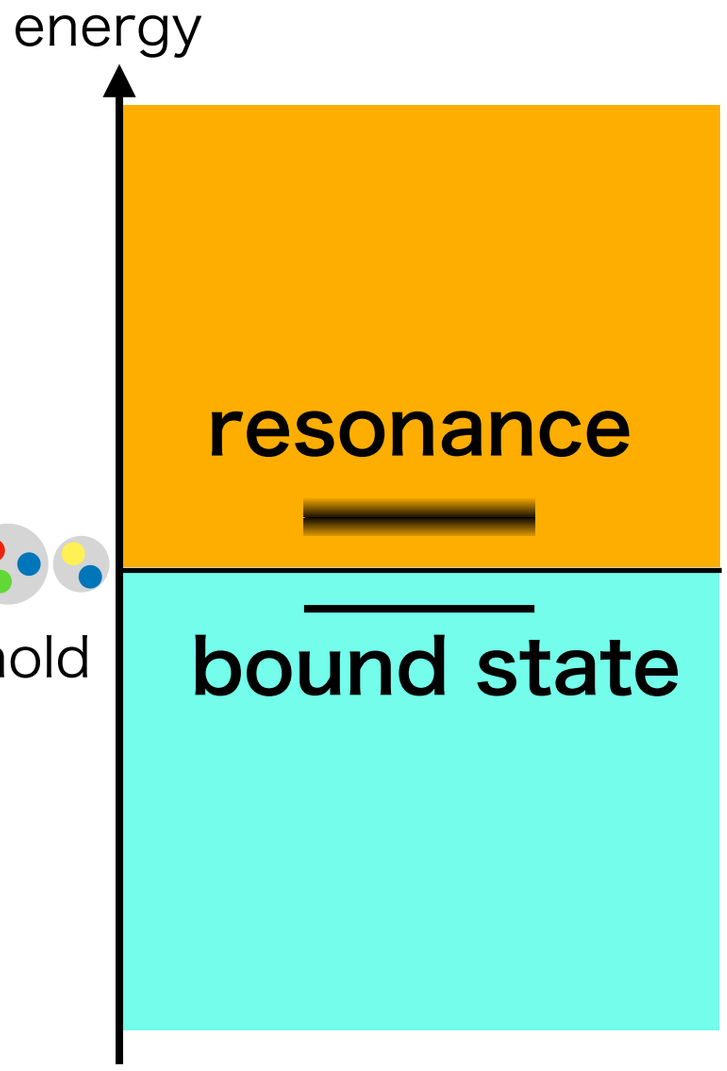
Universal nature of near-threshold s-wave states?

⟶ Can internal structure of near-threshold states be determined indepently of models?

(a) near-threshold bound states

(b) near-threshold resonances





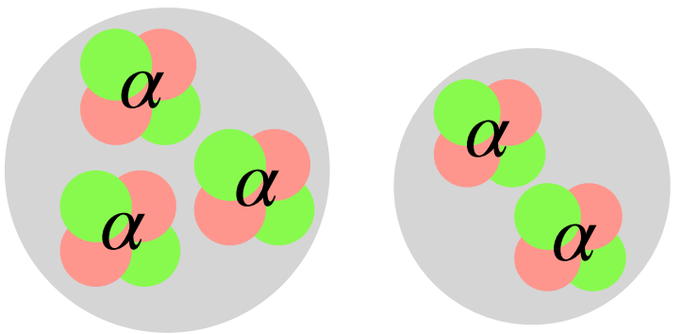
⟶ with **short-range** interaction

T. Kinugawa and T. Hyodo, Phys. Rev. C 109, 045205 (2024);

T. Kinugawa and T. Hyodo, arXiv:2403.12635 [hep-ph], accepted in PTEP.

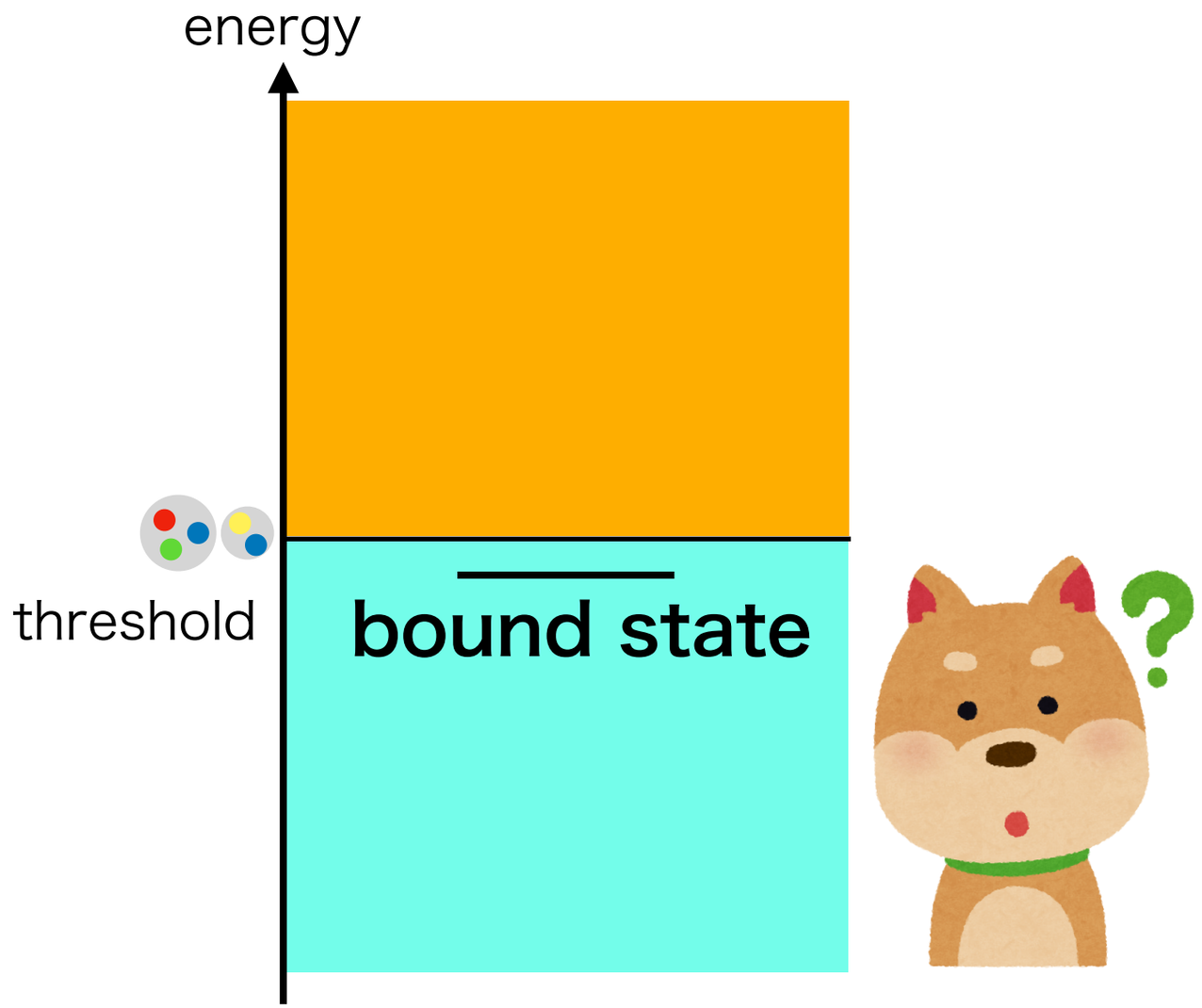
(c) near-threshold states with **Coulonb** + short-range interaction

T. Kinugawa and T. Hyodo, arXiv:2604.17813[hep-ph].



α clustering phenomena

Near-threshold bound states



T. Kinugawa and T. Hyodo, Phys. Rev. C 109 , 045205 (2024).

non-rel. EFT model

◉ single-channel resonance model

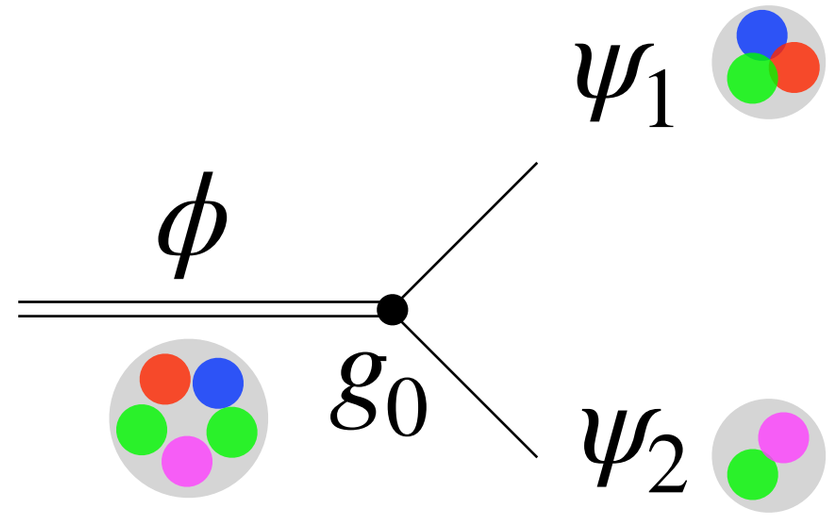
E. Braaten, M. Kusunoki, and D. Zhang, Annals Phys. 323, 1770 (2008).

$$\mathscr{H}_{\text{free}} = \frac{1}{2m_1}\nabla\psi_1^\dagger\cdot\nabla\psi_1 + \frac{1}{2m_2}\nabla\psi_2^\dagger\cdot\nabla\psi_2 + \frac{1}{2m_\phi}\nabla\phi^\dagger\cdot\nabla\phi + \nu_0\phi^\dagger\phi,$$

1.

$$\mathscr{H}_{\text{int}} = g_0(\phi^\dagger\psi_1\psi_2 + \psi_1^\dagger\psi_2^\dagger\phi)\,.$$

2.



1. single-channel scattering of $\psi_{1,2}$
2. couplings of $\psi_{1,2}$ to ϕ

◉ scattering amplitude

$$V = \frac{g_0^2}{E-\nu_0}, \quad G = -\frac{\mu}{\pi^2}\left[\Lambda + ik\arctan\left(\frac{\Lambda}{-ik}\right)\right]. \quad \Lambda : \text{cutoff}$$

$$\xrightarrow[T=\frac{1}{V^{-1}-G}]{} f(k) = -\frac{\mu}{2\pi}\left[\frac{\frac{k^2}{2\mu}-\nu_0}{g_0^2} + \frac{\mu}{\pi^2}\left[\Lambda + ik\arctan\left(\frac{\Lambda}{-ik}\right)\right]\right]^{-1}.$$

Compositeness in model

◉ compositeness X

scattering amplitude : $T = \dfrac{1}{V^{-1} - G}$ Y. Kamiya and T. Hyodo, PTEP 2017, 023D02 (2017).

$$X = \frac{G'(-B)}{G'(-B) - [V^{-1}(-B)]'}, \quad \alpha'(E) = d\alpha/dE$$

$$= \left[1 + \frac{\pi^2 \kappa}{g_0^2 \mu^2}\left(\arctan(\Lambda/\kappa) - \frac{\Lambda/\kappa}{1 + (\Lambda/\kappa)^2}\right)^{-1}\right]^{-1}.$$

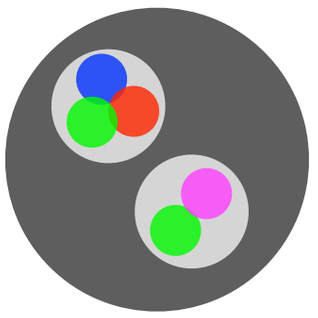
$$\kappa = \sqrt{2\mu B}$$

- compositeness X of bound state with given B?

fix binding energy B

⟶ parameter depenence of X?

$X > 0.5$



or

$X < 0.5$



Model scale and parameters

◉ scale in model

- determined by cutoff Λ: $E_{\mathrm{typ}} = \Lambda^2/(2\mu)$

◉ model parameters g_0, ν_0, Λ

1. fix binding energy B

coupling const. g_0 : $g_0^2(B,\nu_0,\Lambda) = \dfrac{\pi^2}{\mu}(B+\nu_0)\left[\Lambda - \kappa \arctan(\Lambda/\kappa)\right]^{-1}$

$\because$ pole condition $f^{-1}(i\kappa) = 0$ $\qquad \kappa = \sqrt{2\mu B}\,.$

2. use dimentionless quantities by Λ e.g. B/E_{typ}

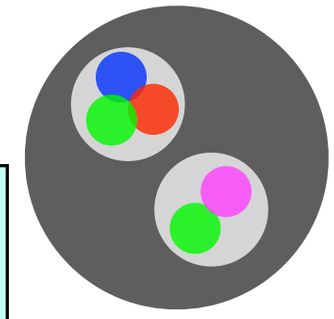
3. free parameter ⟶ bare state energy ν_0

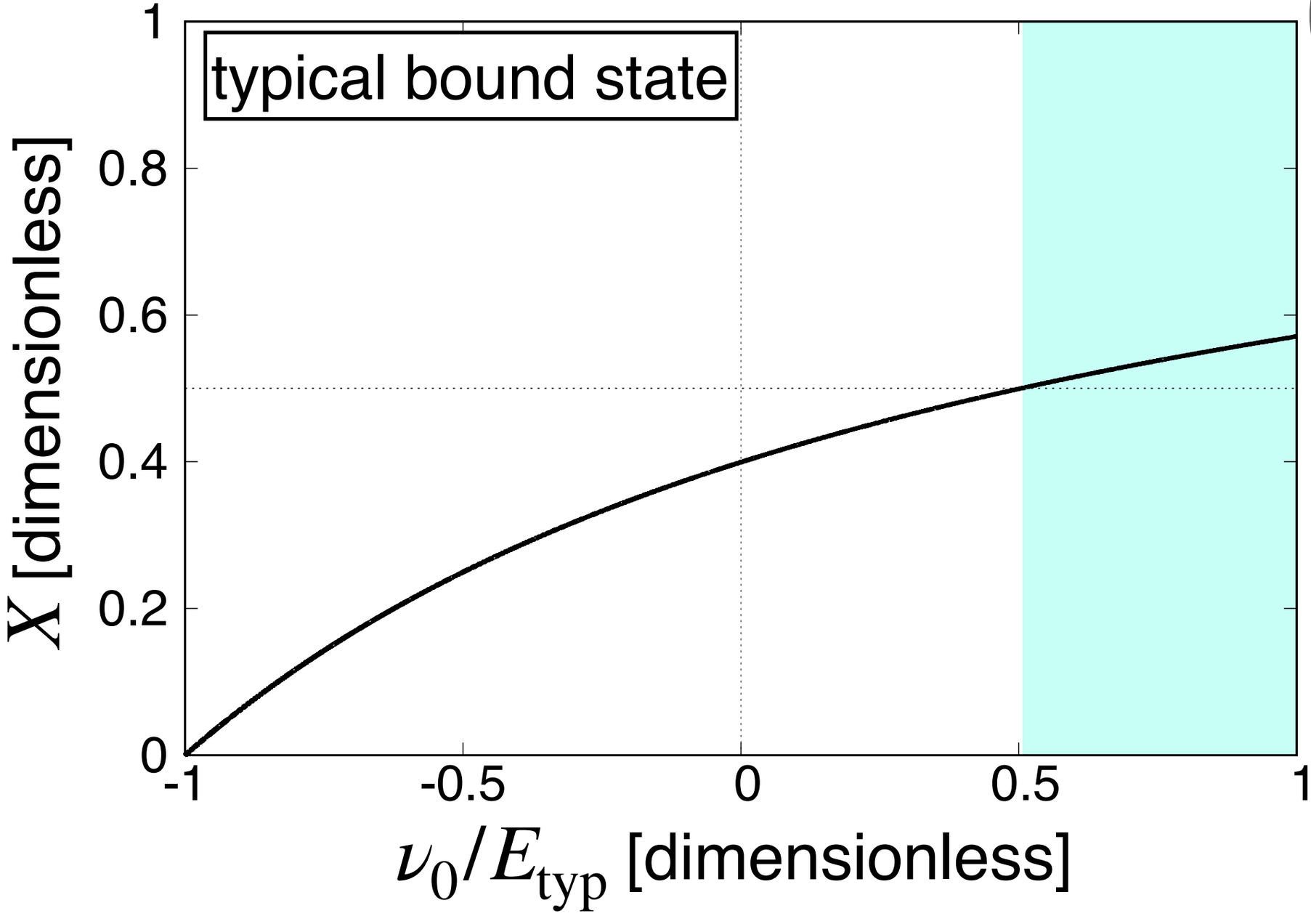
vary in $-B/E_{\mathrm{typ}} \le \nu_0/E_{\mathrm{typ}} \le 1$ region

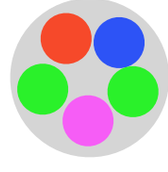
$\because$ $g_0^2 \ge 0$ (lower) & applicable limit of model (upper)

Model-dependence of X

◉ parameter $\nu_0/E_{\rm typ}$ dependence of X ($B = E_{\rm typ}$)

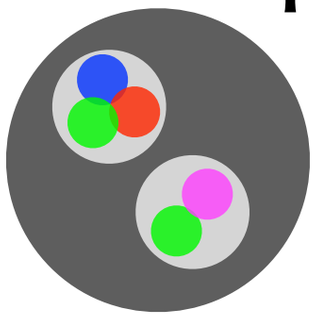


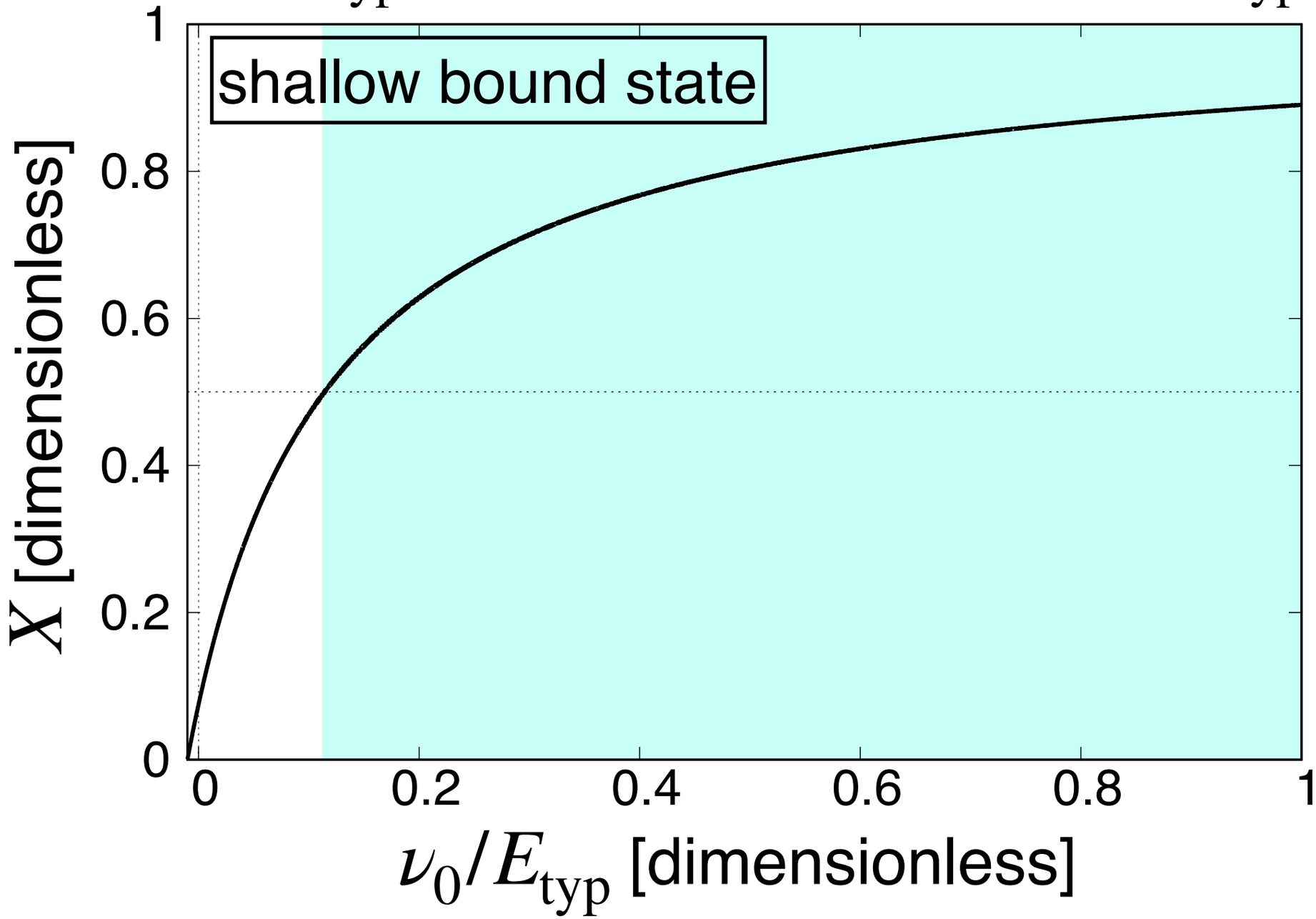




- $X < 0.5$ in $\boxed{75\ \%}$ of ν_0 ∵ **bare state ($X = 0$) origin**

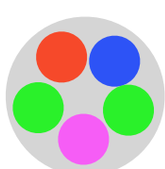
◉ parameter ν_0/E_{typ} dependence of X ($B = 0.01E_{\mathrm{typ}}$)





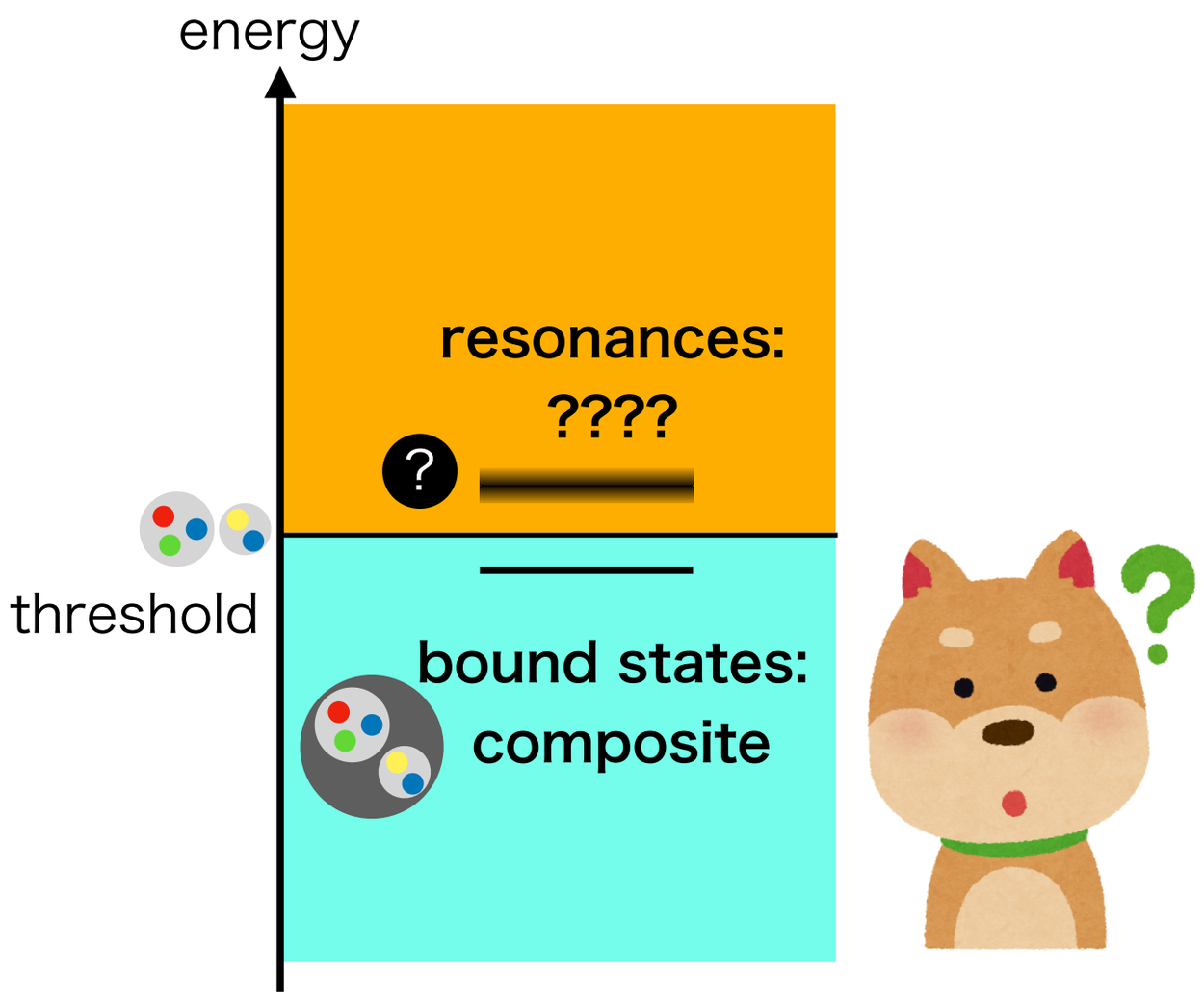


- $X > 0.5$ in 88 % of ν_0 ⟶ emergence of **universality**

- fine tuning of ν_0 is nessecary to obtain  shallow bound state

⟶ shallow bound states tend to be composite dominant

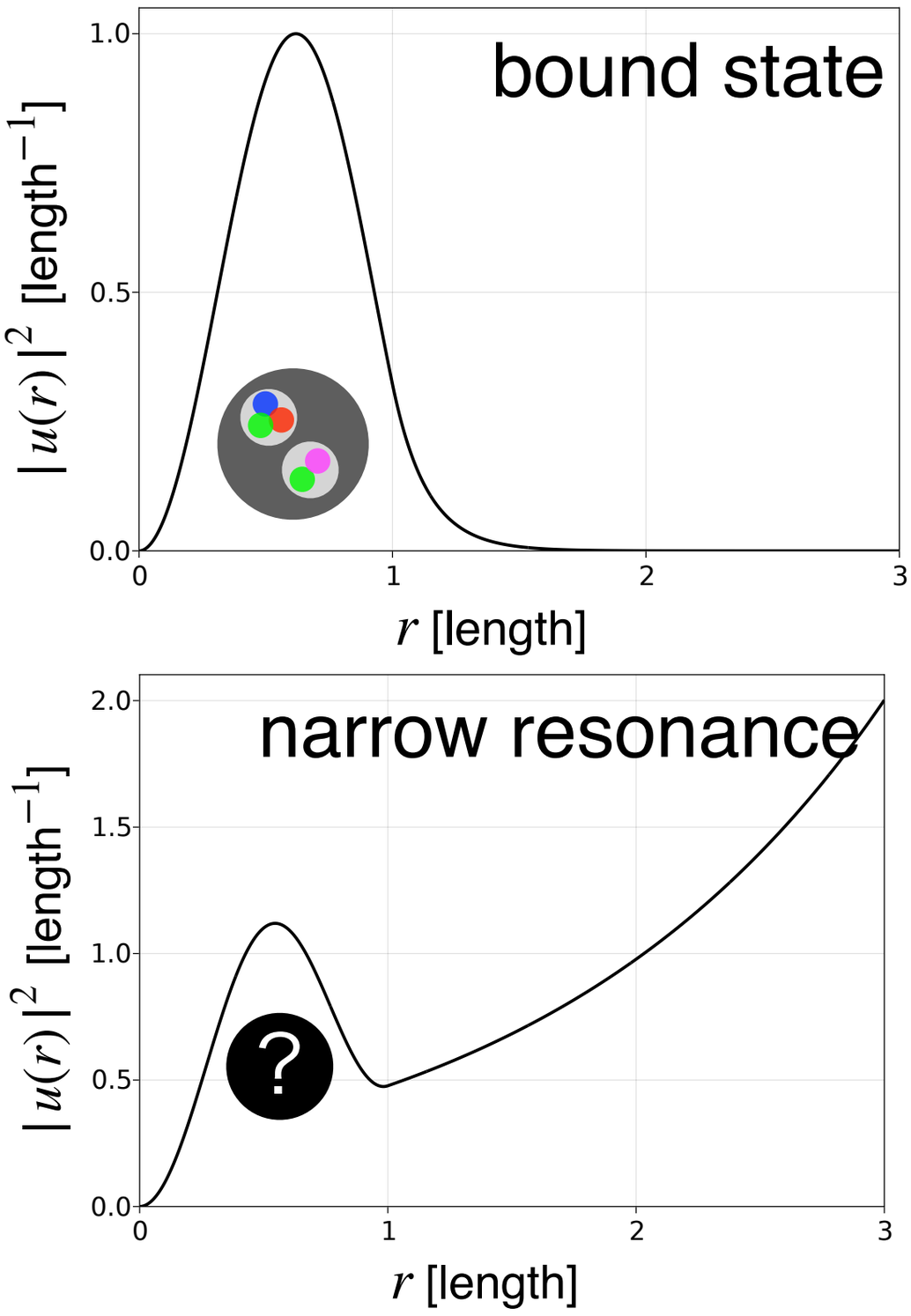
Near-threshold resonances



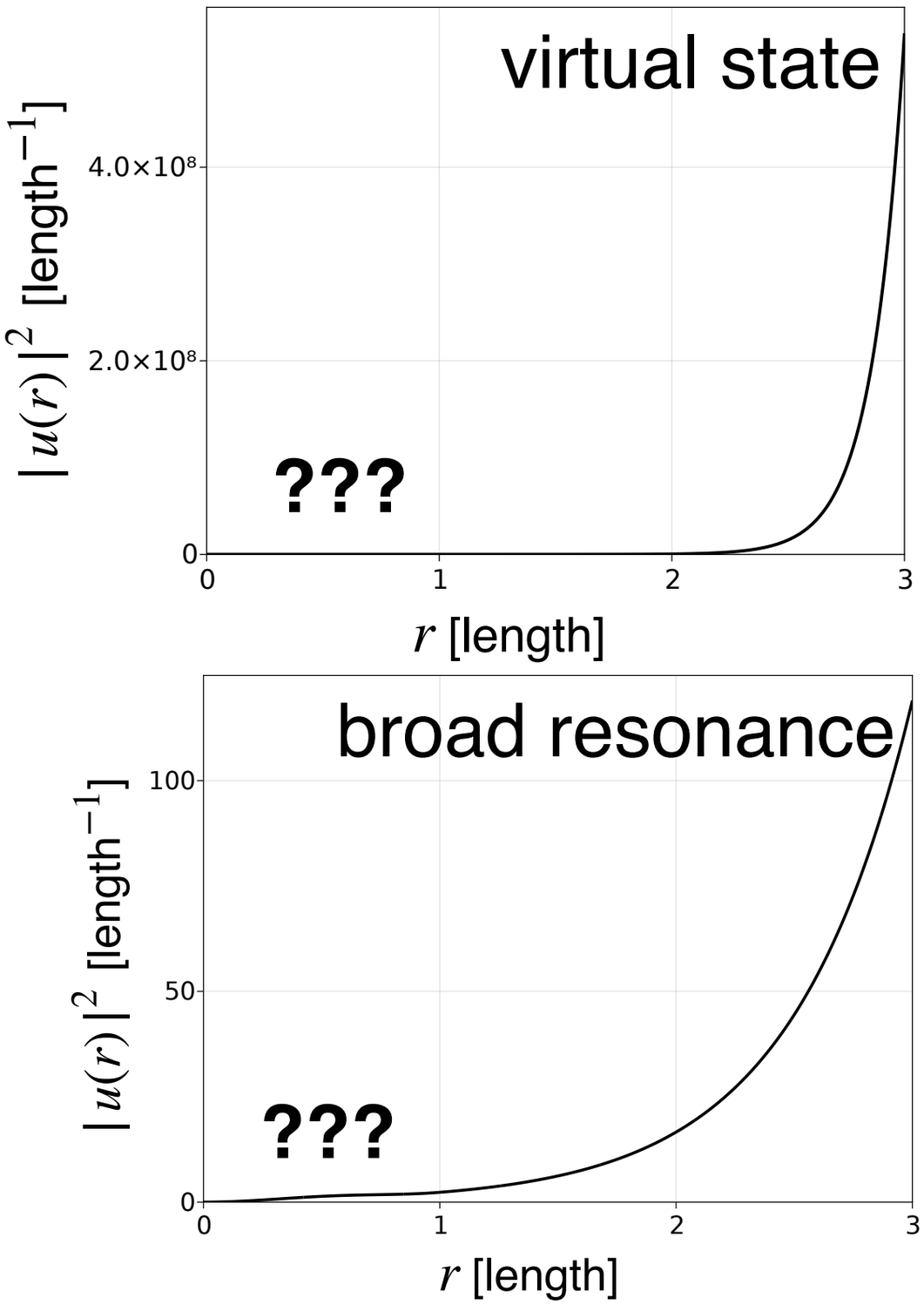
T. Kinugawa and T. Hyodo, arXiv:2403.12635 [hep-ph], accepted in Prog. Theor. Exp. Phys.

Wavefunction of eigenstates

✅ internal “structure”!

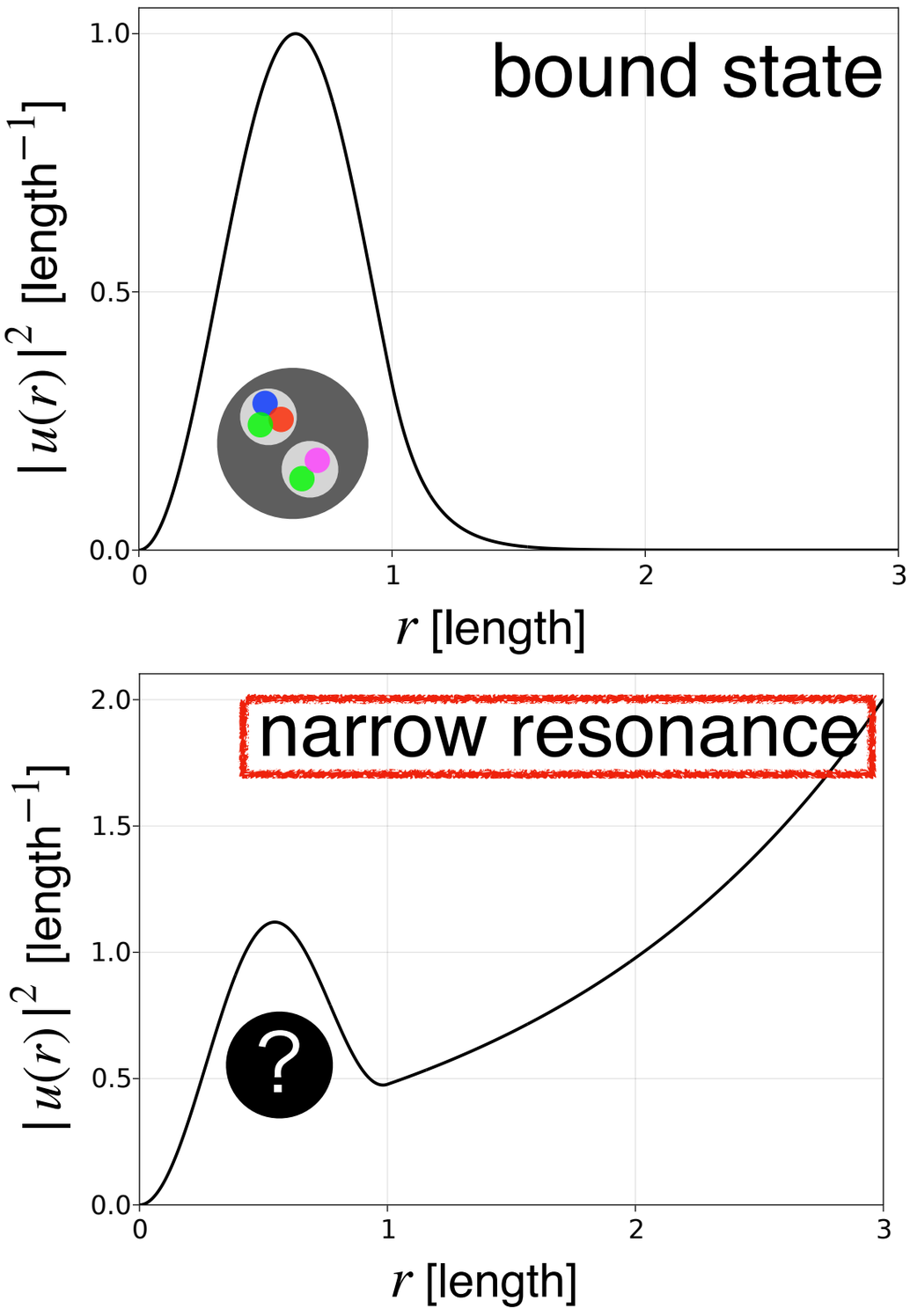


❌ internal “structure”?

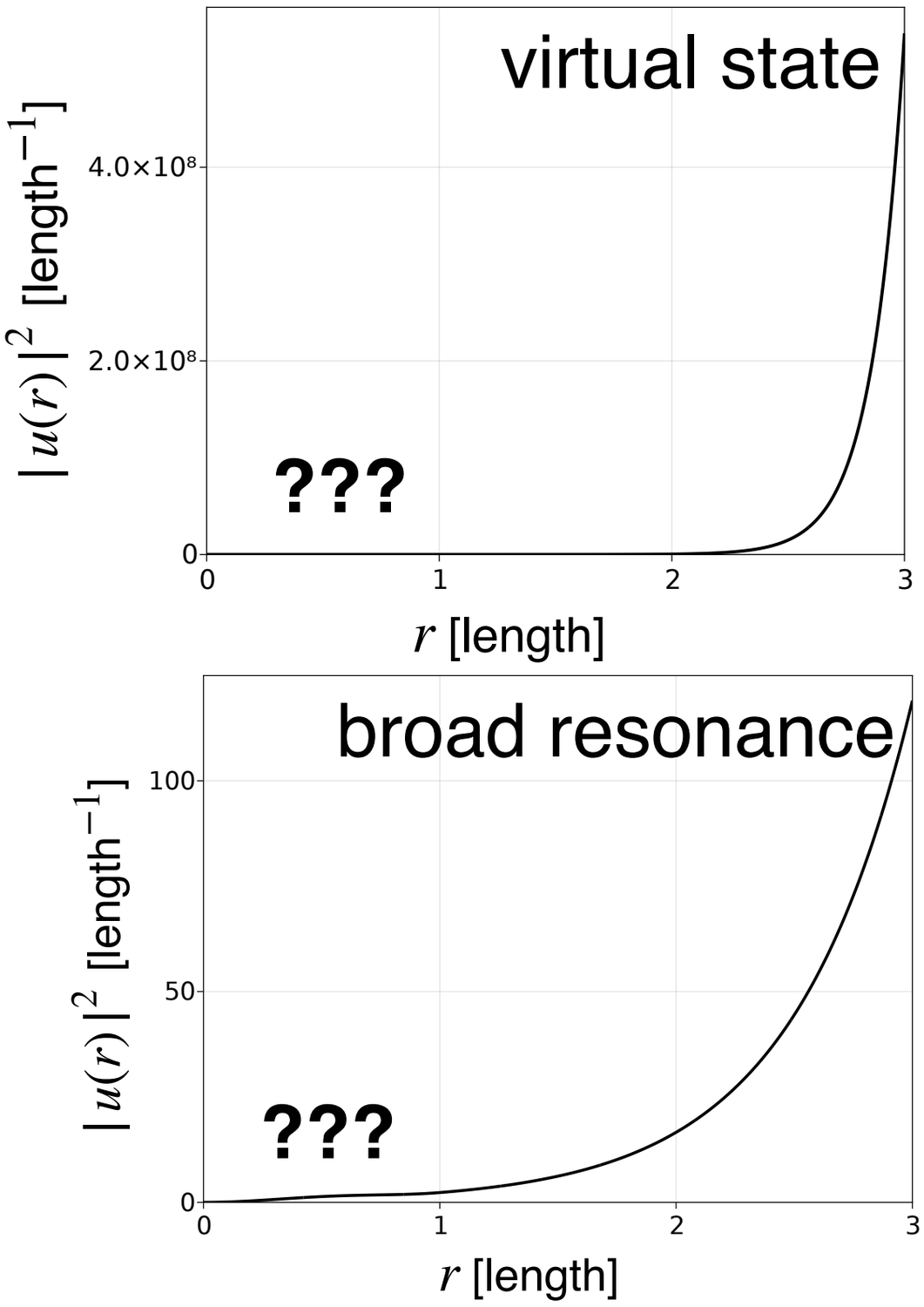


Wavefunction of eigenstates

✅ internal “structure”!



❌ internal “structure”?



Near-th. resonances in ERE

◉ resonance pole written by effective range expansion (ERE)

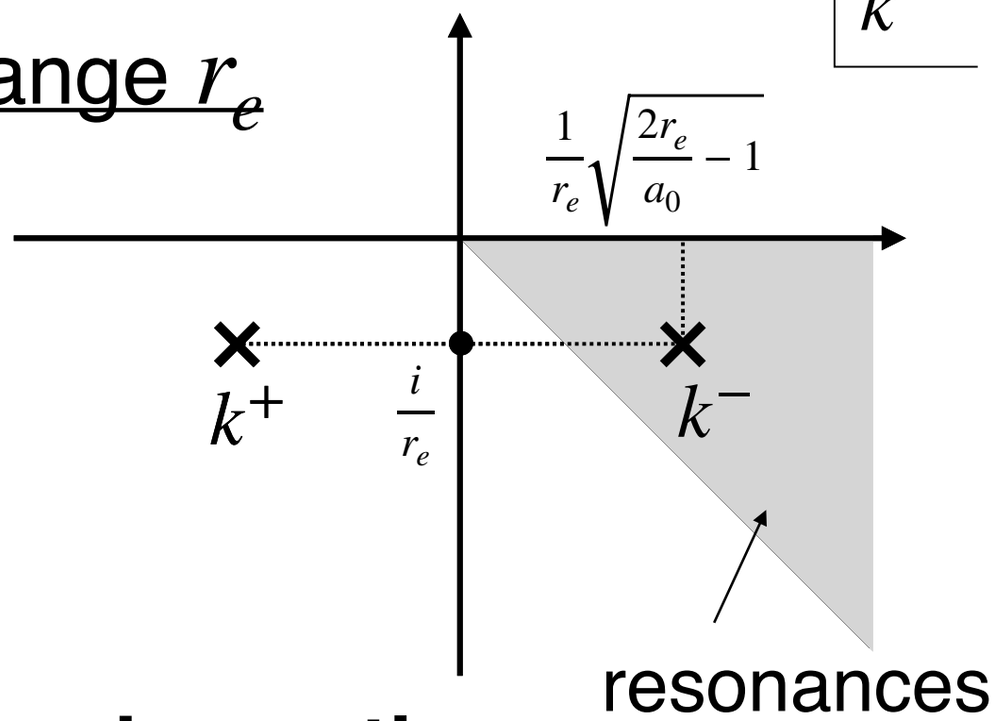
$$f(k)^{-1} = -\frac{1}{a_0} + \frac{r_e}{2}k^2 - ik \longrightarrow k^{\pm} = \frac{i}{r_e} \pm \frac{1}{r_e}\sqrt{\frac{2r_e}{a_0} - 1}$$

T. Hyodo, Phys. Rev. Lett. **111**, 132002 (2013);
I. Matuschek, V. Baru, F.K. Guo, C. Hanhart, Eur. Phys. J. A 57(3), 101 (2021).

- pole position ($k^{\pm}$) ⟷ a_0 and r_e

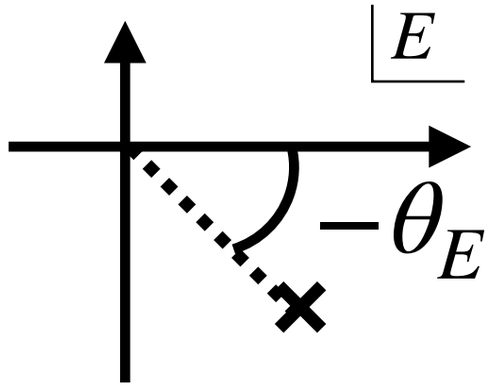
◉ scattering length a_0 and effective range r_e

- r_e should be negative to obtain resonances
- $\mathrm{Im}k = |1/r_e|$ should be small to obtain narrow resonances



→ Effective range should be **large and negative** for near-threshold resonances

Compositeness in ERE



$$X = \sqrt{\frac{1}{1-\frac{2r_e}{a_0}}} = -i\tan\theta_k = -i\tan(\theta_E/2)$$

$$(k = |k|e^{i\theta_k},\ E = |E|e^{i\theta_E})$$

- in general, compositeness X of unstable resonances becomes **complex** by definition

T. Kinugawa, T. Hyodo, Eur. Phys. J. A 61, 154 (2025) etc.



- complex X **cannot** be directly interpreted as a probability

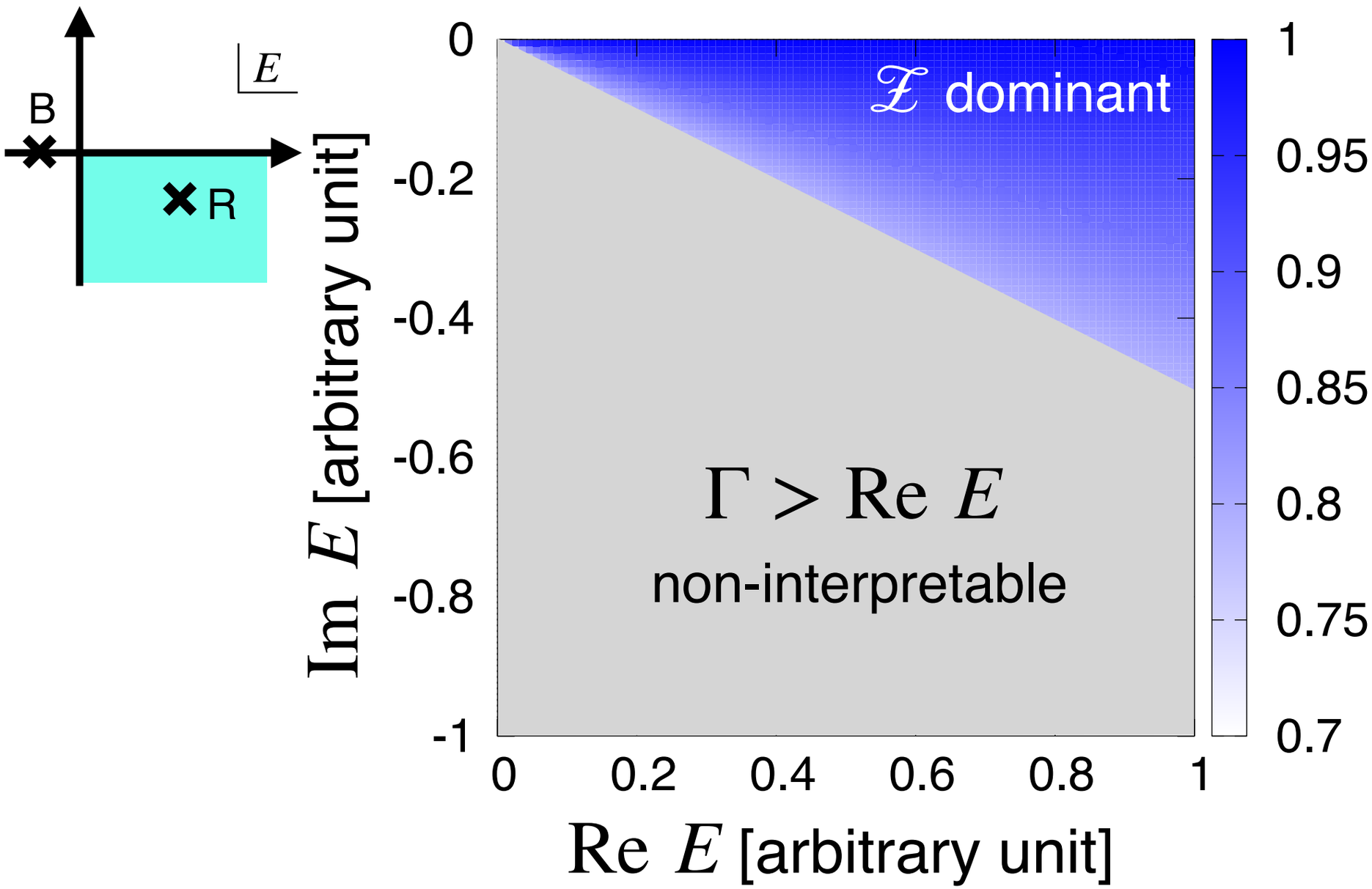
◉ interpretation of complex compositeness

T. Kinugawa and T. Hyodo, arXiv:2403.12635 [hep-ph], accepted in PTEP.

- define new quantities satisfying the following condition…

1. reduce to compositeness of bound states when $\mathrm{Im}\ X \to 0$
2. probabilistic quantities (normalized and between 0 and 1)
3. automatically take into account only **narrow** resonance

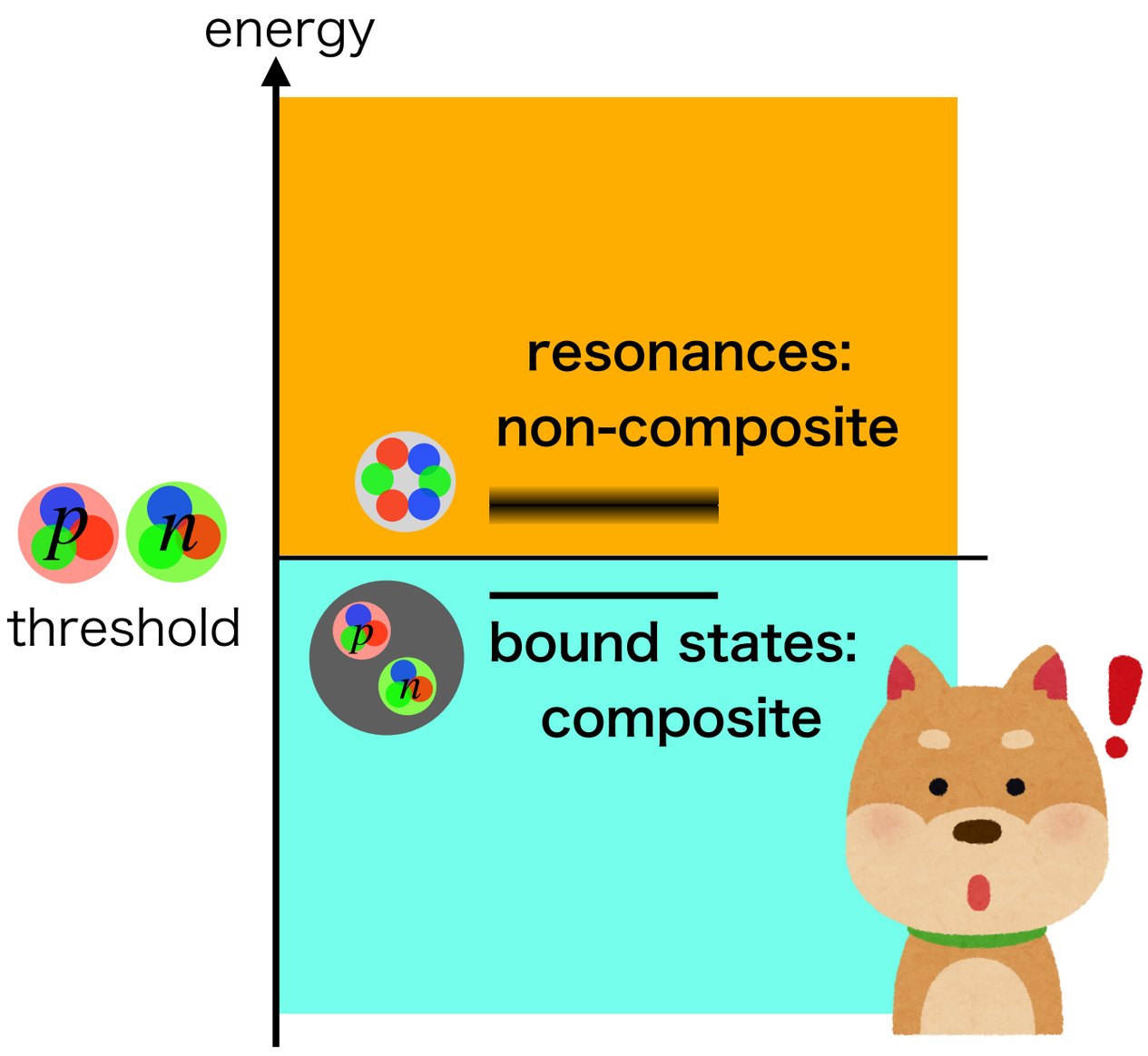
Structure of near-th. resonances



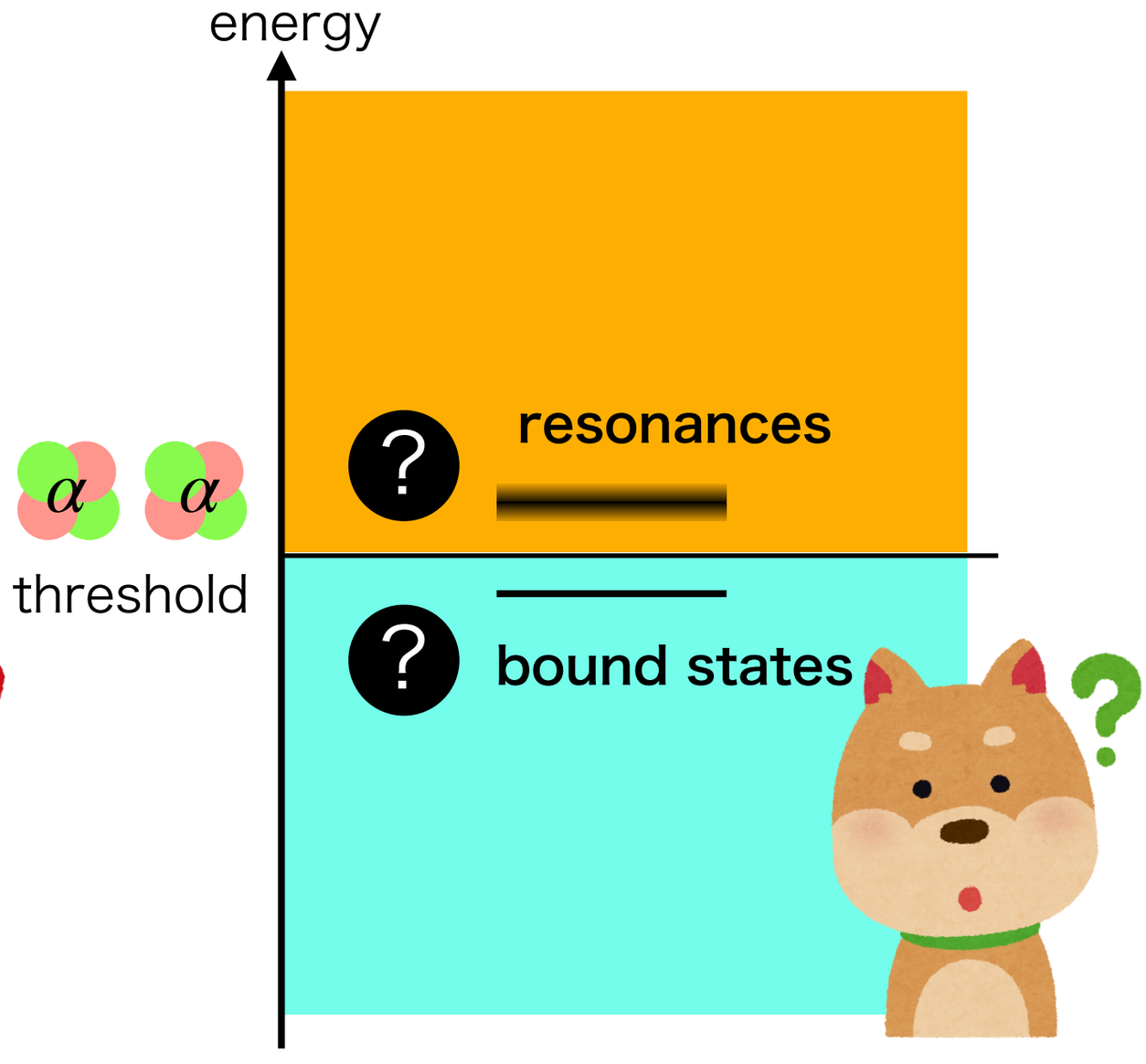
- resonances are **not composite dominant states** ($\mathcal{Z} \gtrsim 0.8$)

- different from near-threshold bound states
  (composite dominant $X \sim 1$ and $Z \sim 0$)

Near-th. states with Coulomb + s.r.



scatterings with **short-range interaction**

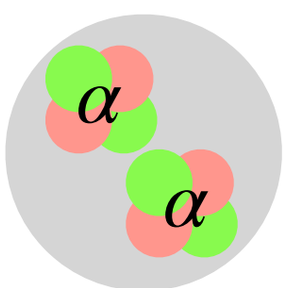


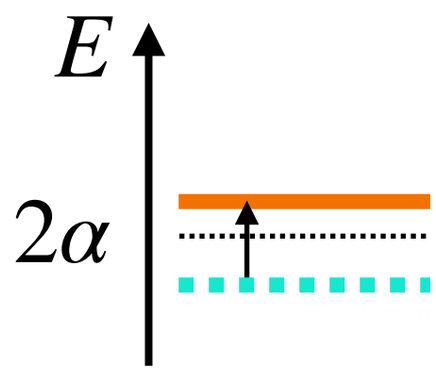
scatterings with **Coulomb + short-range** interaction

T. Kinugawa and T. Hyodo, arXiv:2604.17813[hep-ph].

Coulomb + short-range systems

- ^{8}Be nuclei ($2\alpha^{++}$) J. Hiura, and R. Tamagaki, Prog. Theor. Phys. Suppl. No. 52, 25 (1972);
R. Higa, H.-W. Hammer, and U. van Kolck, Nuclear Physics A 809, 171 (2008).

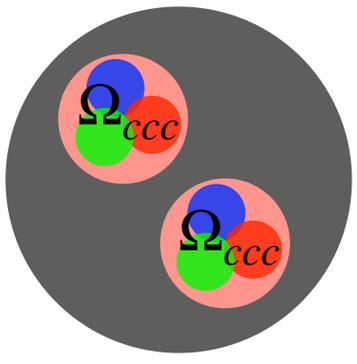


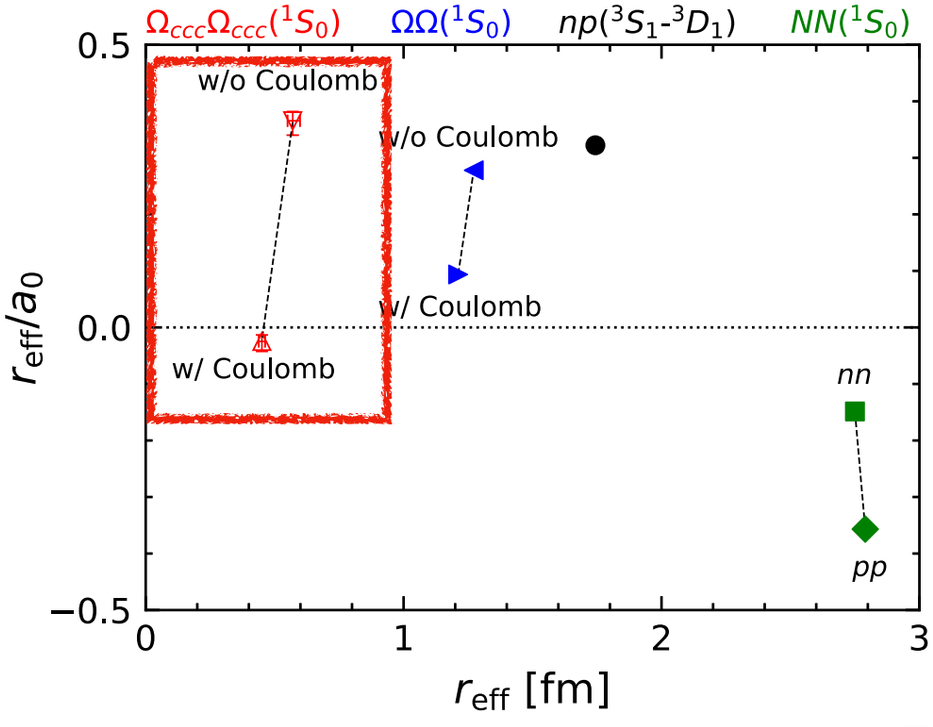


resonance (w/ Coulomb)

bound (w/o Coulomb)

- $\Omega_{ccc}^{++}\,\Omega_{ccc}^{++}$ (HAL QCD) Y. Lyu, H.Tong, *et al*. [HAL QCD Coll.], Phys. Rev. Lett. 127 (2021) 072003.





resonance (w/ Coulomb)

bound (w/o Coulomb)

- $\Xi^-\alpha$: Coulomb assisted bound state E. Hiyama, M. Isaka, T. Doi, and T. Hatsuda, Phys. Rev. C 106, 064318 (2022).

⟶ Coulomb is important for near-threshold states!

Coulomb + short-range systems

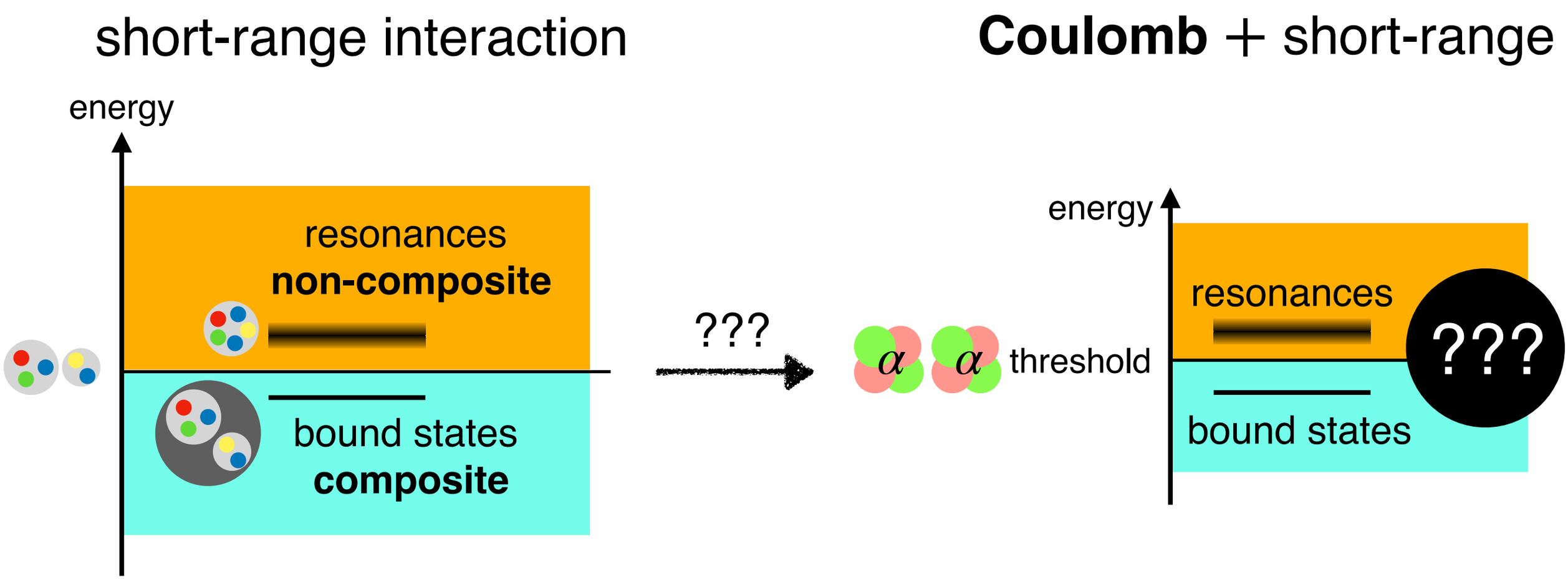
- with/without Coulomb ⟶ different scattering amplitudes

H. A. Bethe, Phys. Rev. 76, 38-50 (1949).

- short-range results cannot be applied to Coulomb+s.r. system !



How are the properties of near-threshold states modified by the Coulomb interaction?

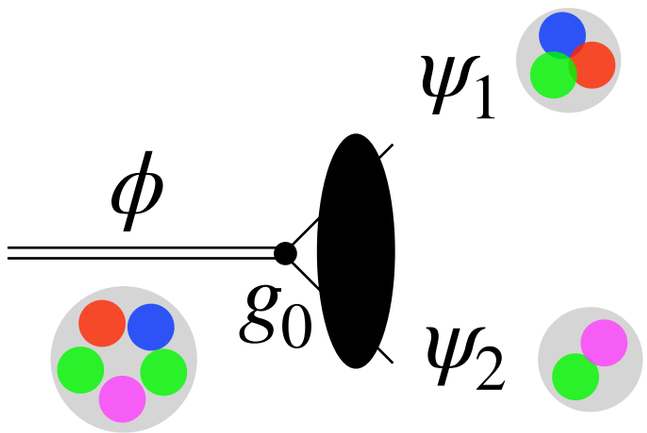


Coulomb + short-range model

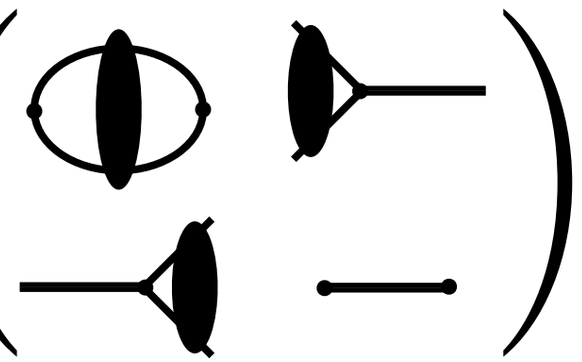
How does compositeness change across the threshold?

◉ model W. Domcke, Atom. Mol. Phys. 16 359 (1983).

R. Higa, H.-W. Hammer, and U. van Kolck, Nuclear Physics A 809 (2008).



$$H = \left(\quad \right)$$



S.-i. Ando, J. W. Shin, C. H. Hyun, and S. W. Hong,Phys. Rev. C 76, 064001 (2007).

see Ando-san's and Bai-san's talks

- pole condition (ERE with Coulomb contribution)

$$-\frac{1}{a_s} + \frac{r_e}{2}k^2 - ik \pm \frac{2}{a_B}\left[\underbrace{\log(-ia_B k)}_{\text{log cut}} + \psi\left(1 \mp \frac{i}{a_B k}\right)\right] = 0$$

Bohr radius

$$a_B = \frac{\hbar c}{\mu c^2 Z_1 Z_2}$$

digamma function

$$\psi(z) = \frac{d}{dz}\log\Gamma(z)$$

Compositeness

S. Weinberg, Phys. Rev. 137, 672–678 (1965);
Y. Kamiya and T. Hyodo, PTEP 2017;
Phys. Rev. C 93, 035203 (2016).

◉ expression of compositeness

$$X = 1 - \frac{1}{1 - \frac{d}{dE}\Sigma(E)} = \left[1 - \frac{r_e}{R^C}\right]^{-1}$$

self energy

c.f. only short-range interaction

$$X = \left[1 - \frac{r_e}{R}\right]^{-1}$$

R^C : radius with Coulomb contribution

$$R^C = = -\frac{1}{k}\left[2i\left(\frac{1}{a_B k}\right)^2 \psi_1\left(\mp i\frac{1}{a_B k}\right) + i - 2\left(\mp\frac{1}{a_B k}\right)\right]$$

◉ interpretation scheme of complex X

X_C: compositeness of resonances in Coulomb + s.r. system

$$X_C = \sqrt{\frac{1}{1 + \left| -\frac{1}{\Sigma'^2} + \frac{2}{\Sigma'}\right|}}$$

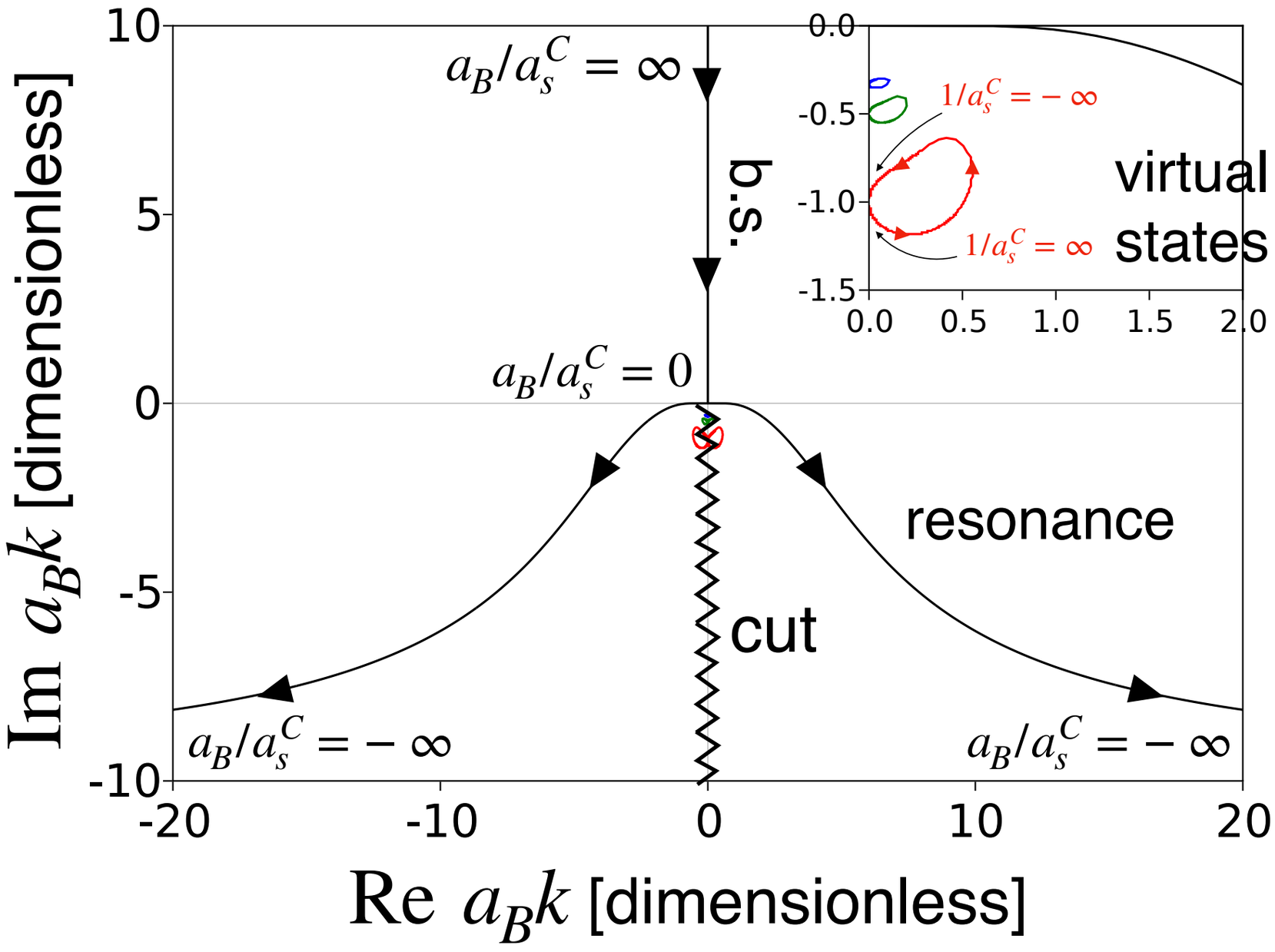
extension of Ref.

I. Matuschek, V. Baru, F.K. Guo, C. Hanhart,
Eur. Phys. J. A 57(3), 101 (2021).

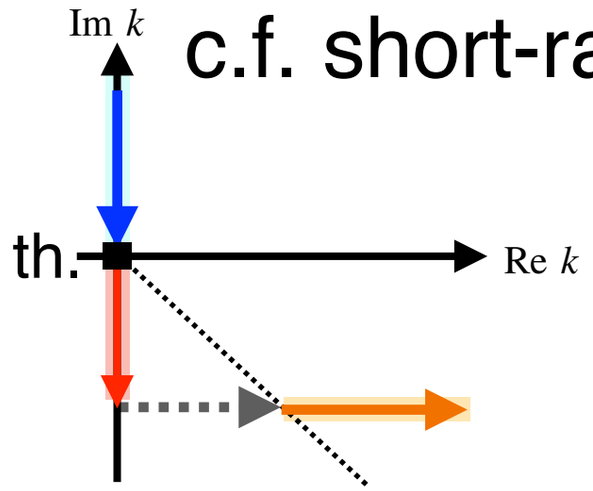
Pole trajectory (repulsive Coulomb)

◉ pole trajectory in complex momentum k plane

- varying Coulomb scattering length a_s with fixed r_e and a_B



- b.s directory goes to resonance



c.f. short-range

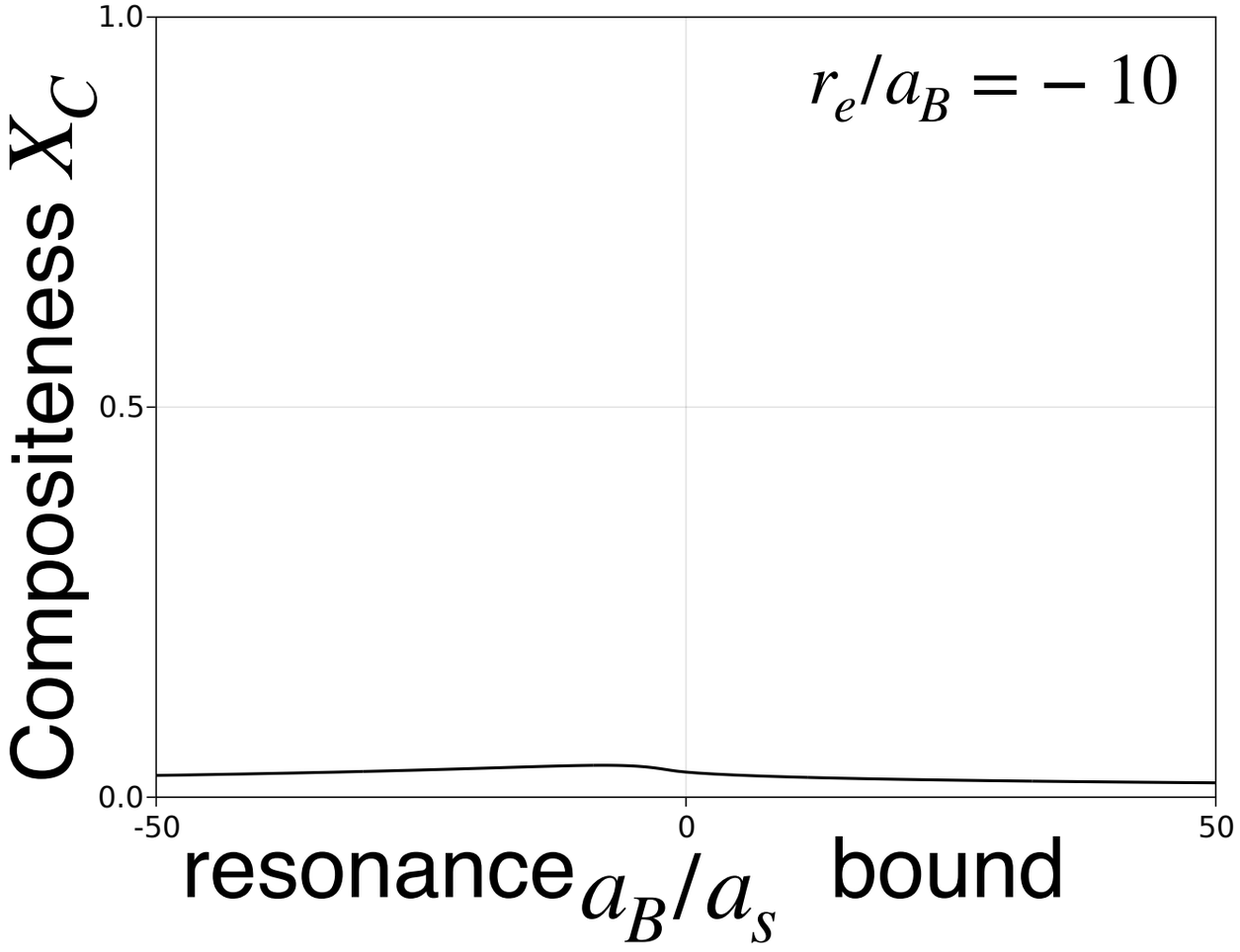
- $a_s \to \infty$ at threshold
- but no universality

$\because$ radius of w.f. $< \infty$

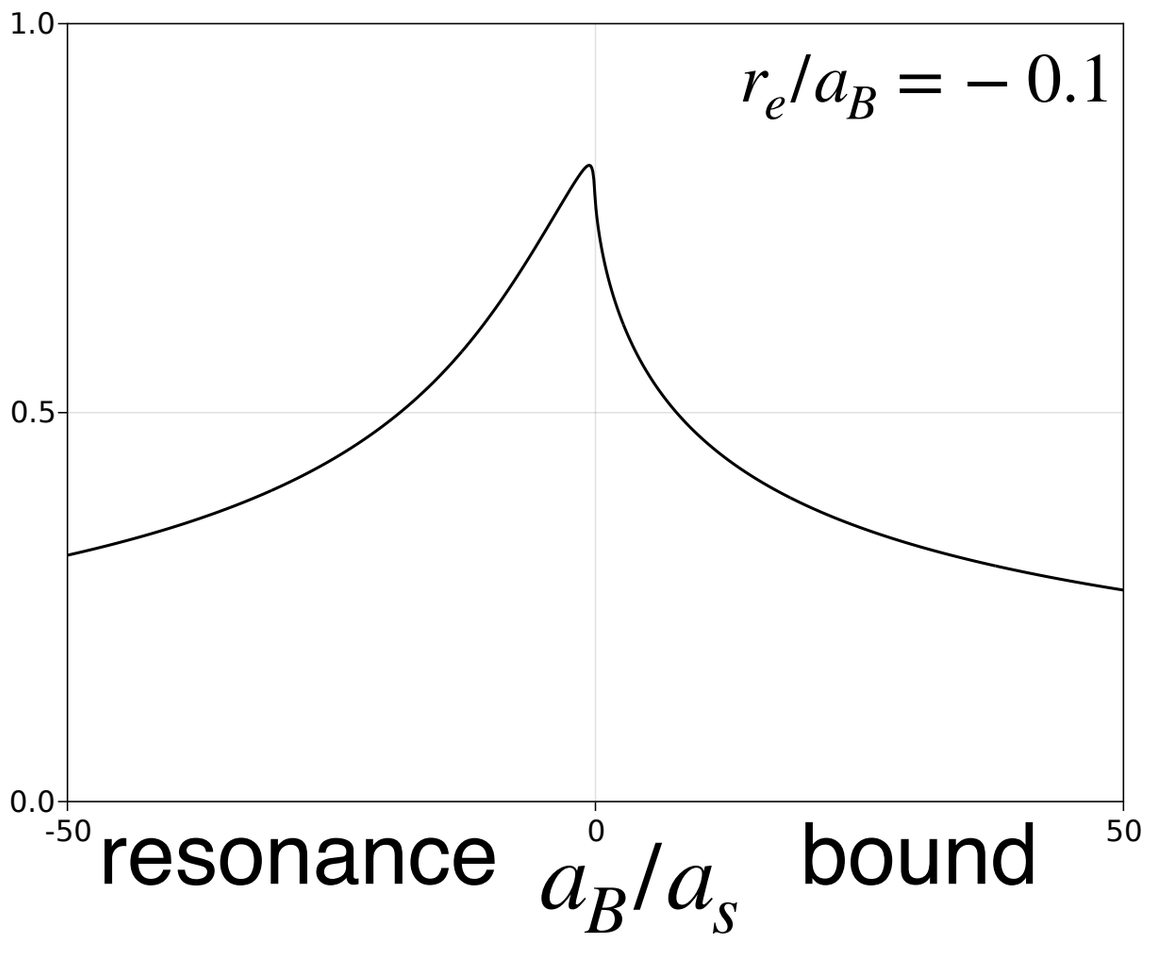
S. Mochizuki, and Y. Nishida,
Phys. Rev. C 110 , 064001 (2024).

Compositeness (repulsive Coulomb)

Coulomb > short-range



Coulomb < short-range

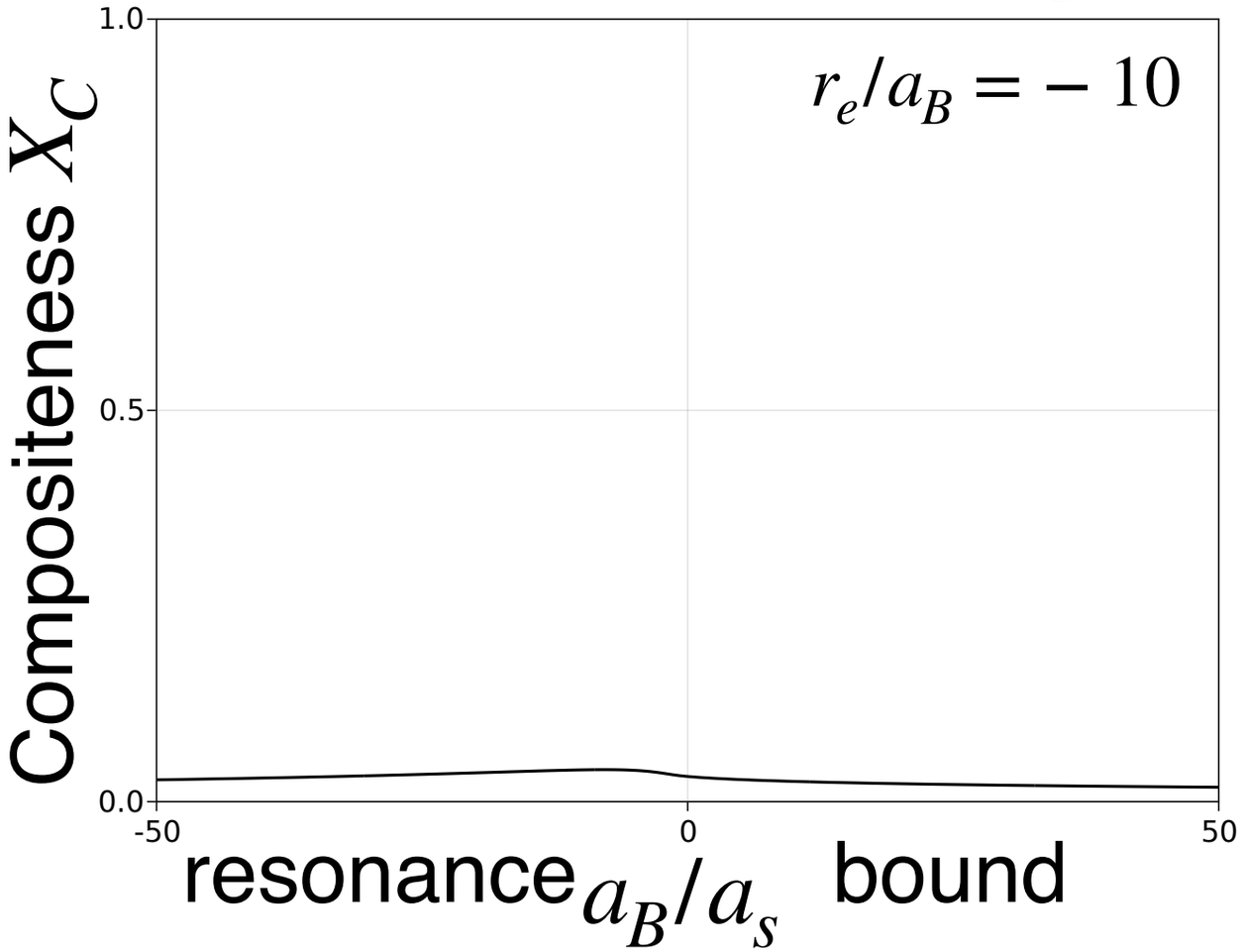


- structure of bound states $\approx$ resonances $\because$ continuous X

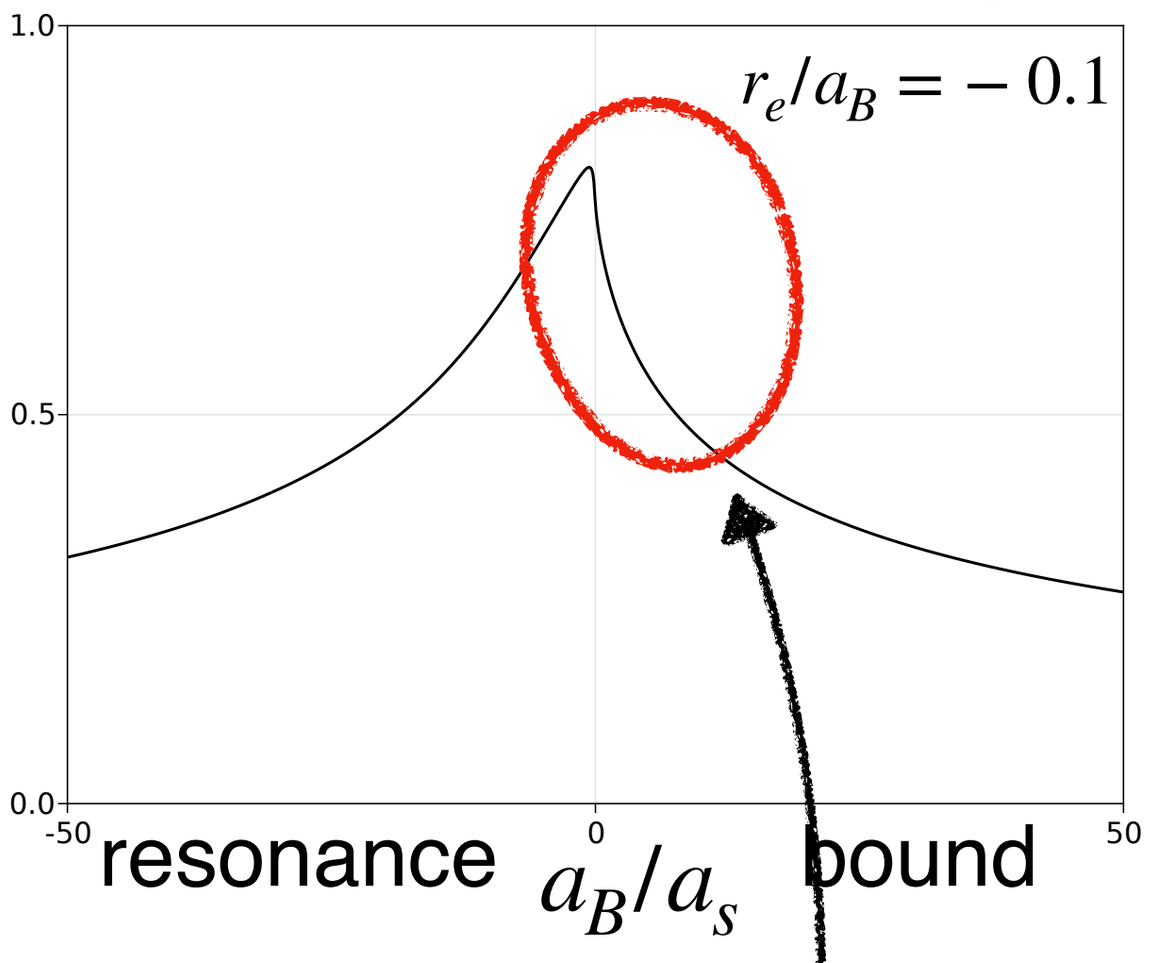
- states with large $|1/a_s|$ are non-composite dominant

Compositeness (repulsive Coulomb)

Coulomb > short-range



Coulomb < short-range



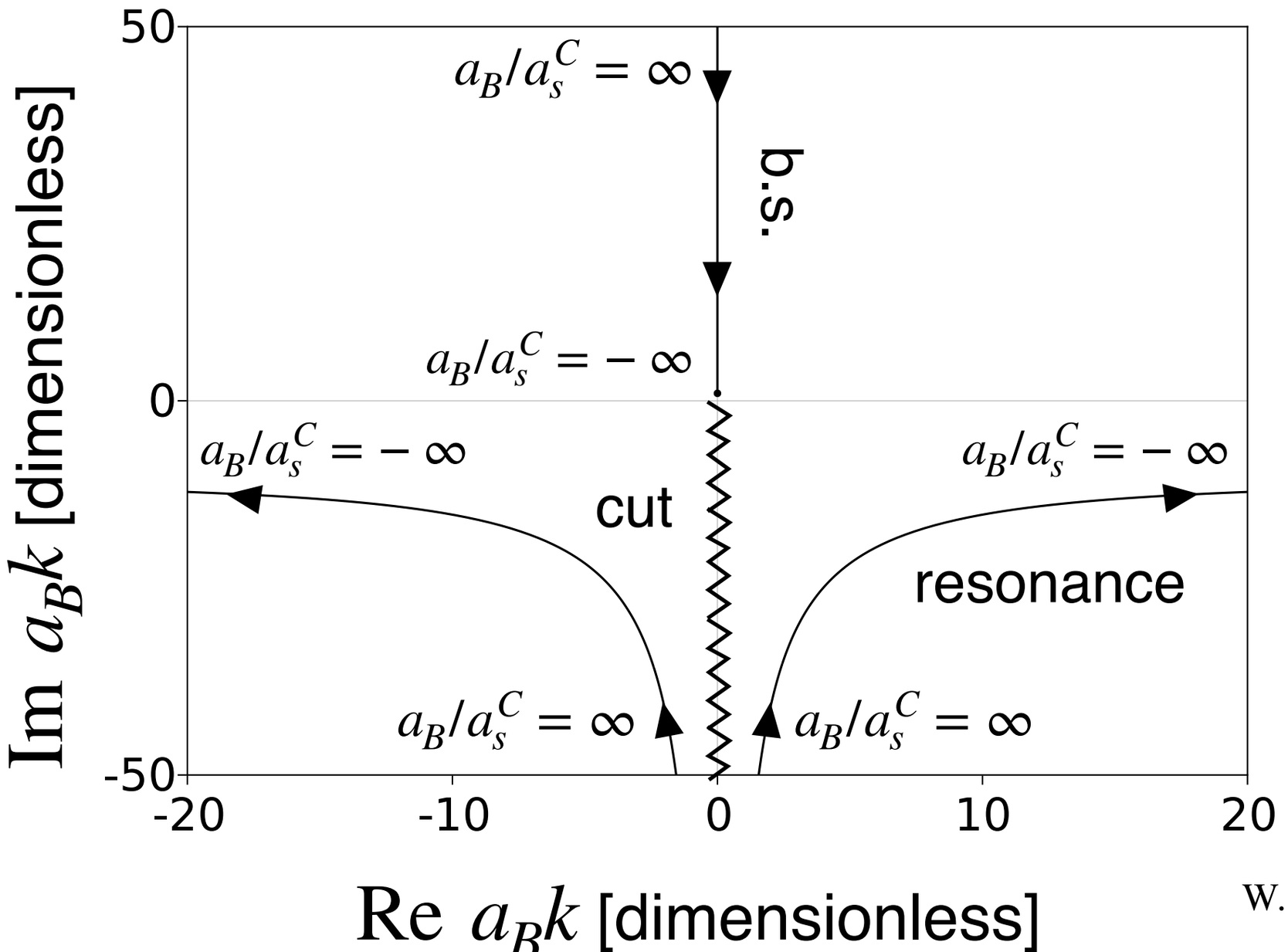
- structure of bound states $\approx$ resonances $\because$ continuous X

- states with large $|1/a_s|$ are non-composite dominant

- remnant of short-range universality in $|r_e| \ll |a_B|$ case

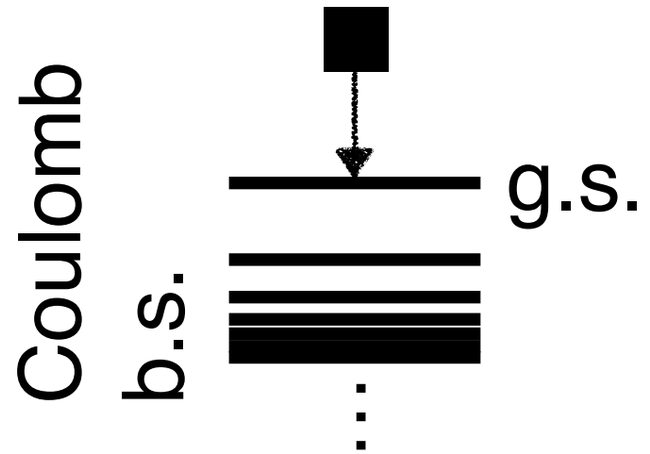
Pole trajectory (attractive Coulomb)

◉ pole trajectory in complex momentum k plane



- bound ≠ resonance

- bound pole
 → Coulomb g.s.

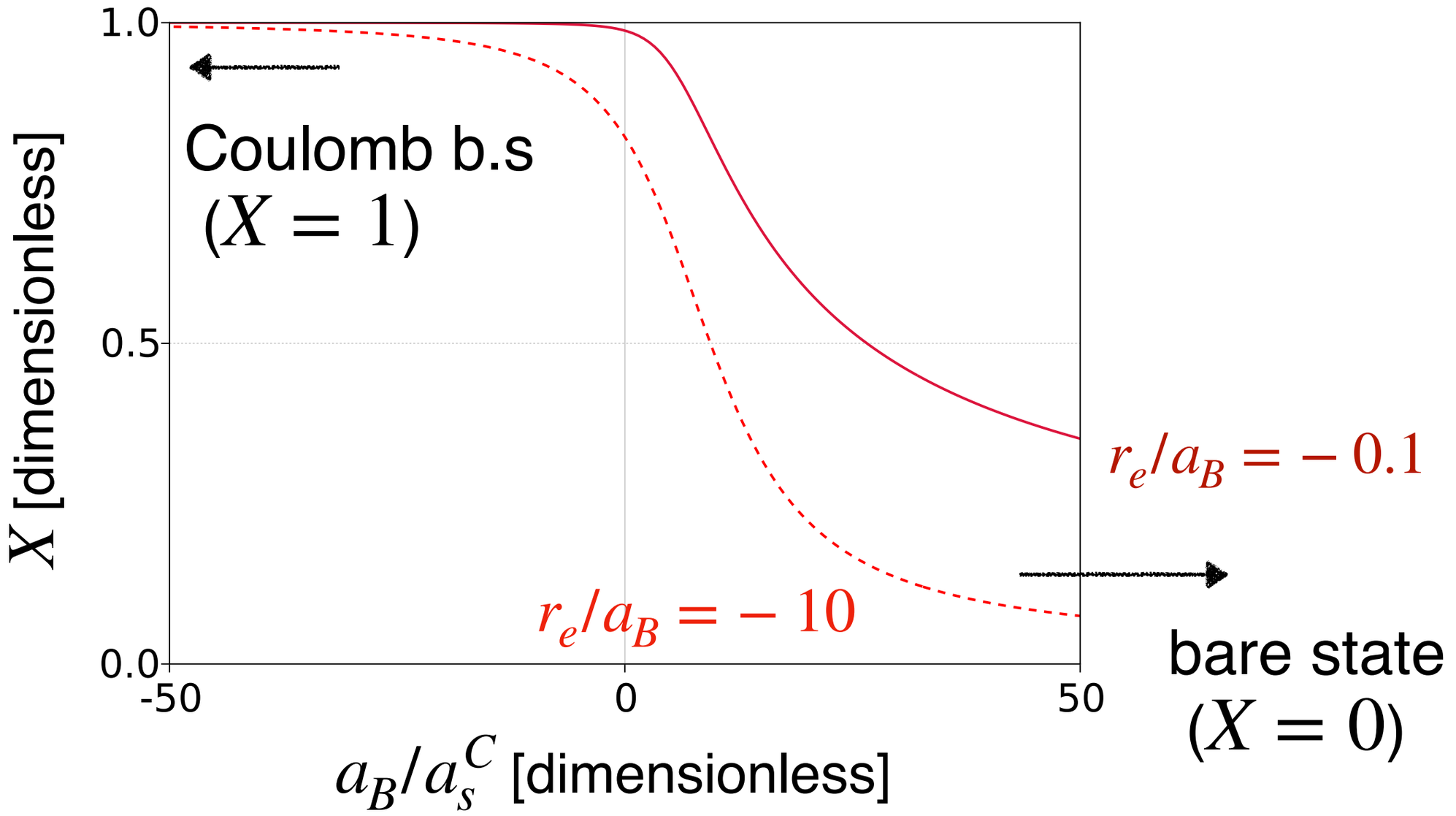


- pole cannot go to $k = 0$

W. Domcke, Atom. Mol. Phys. 16 359 (1983);

S. Mochizuki, and Y. Nishida,
Phys. Rev. C 110 , 064001 (2024).

Compositeness (att. Coulomb b.s.)

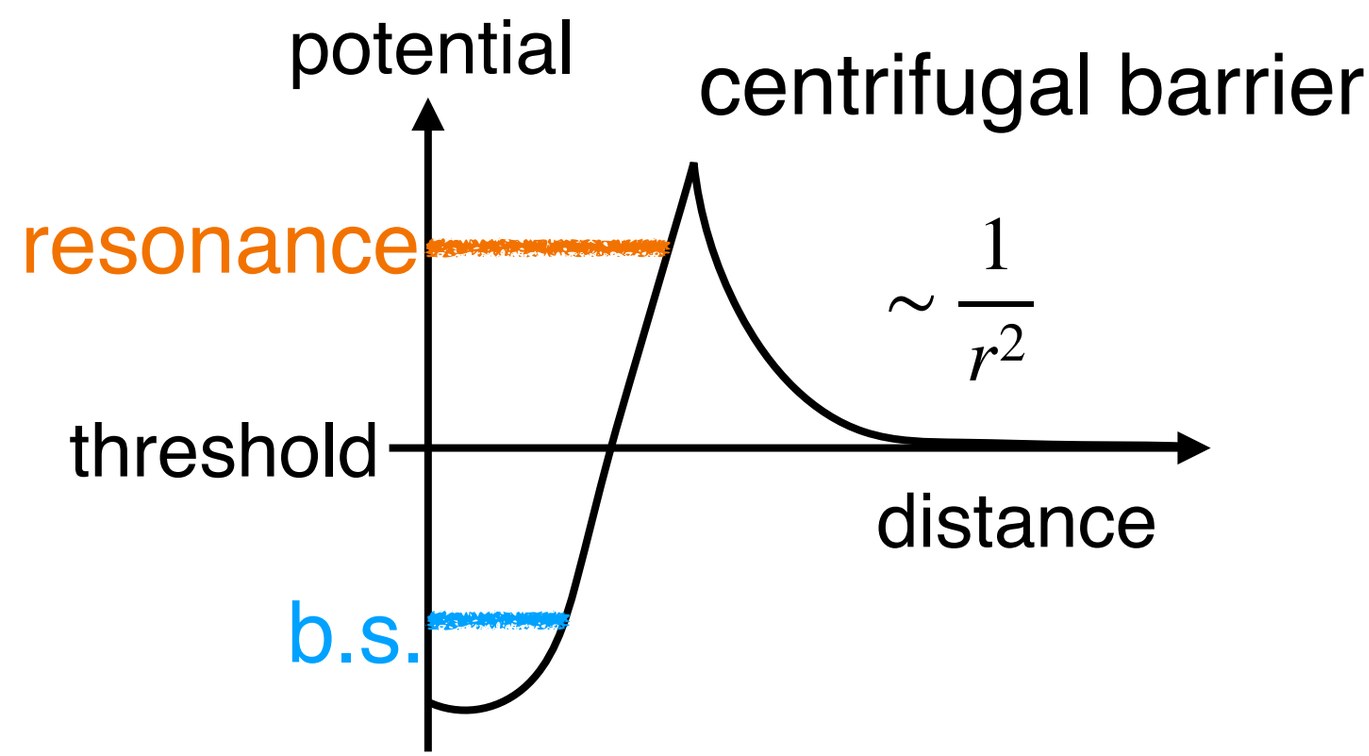


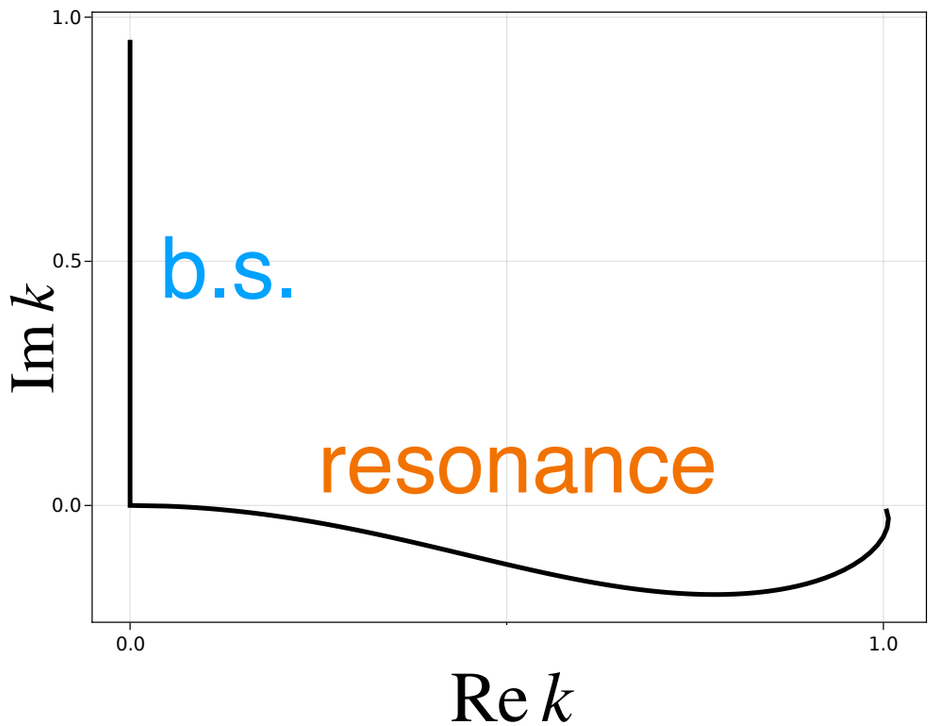
- $1/a_s \to +\infty$: states become elementary dominant ($X \to 0$)
- no short-range universality but $X \to 1$ in $B \to B_{\text{Coulomb g.s.}}$ limit
  ∵ Coulomb g.s. has no bare state contribution (i.e. $X = 1$)
- Coulomb < short-range ($r_e = -0.1$) : remnant of universality

similarity with p-wave s.r.

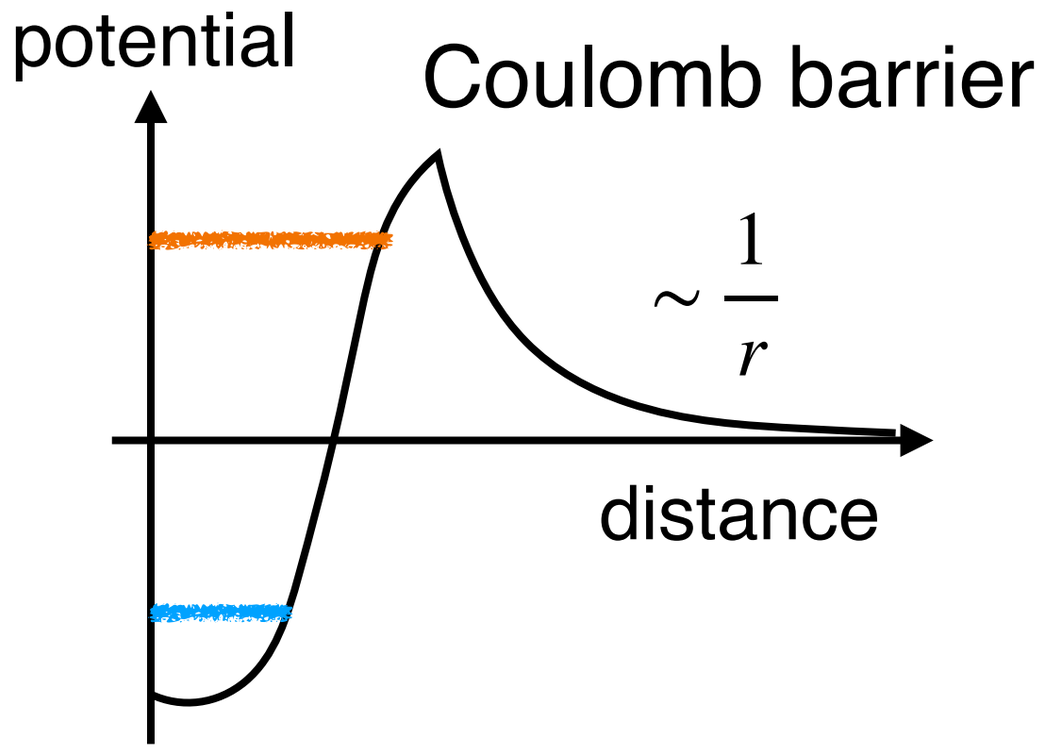
T. Kinugawa, S. Ampuku, K. Miyake, and M. Oka in preparation.

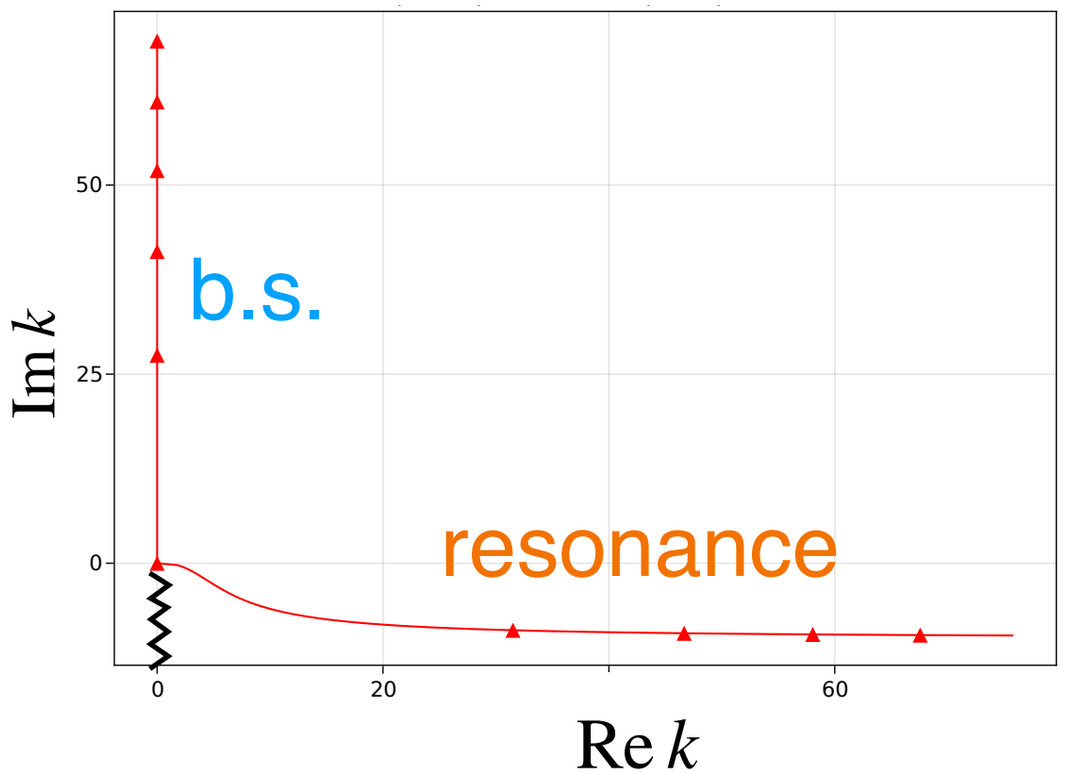
higher-partial wave





repulsive Coulomb + s.r.





similarity with p-wave s.r.

T. Kinugawa, S. Ampuku, K. Miyake, and M. Oka in preparation.

◉ compositeness at threshold

sminar model as on p. 14

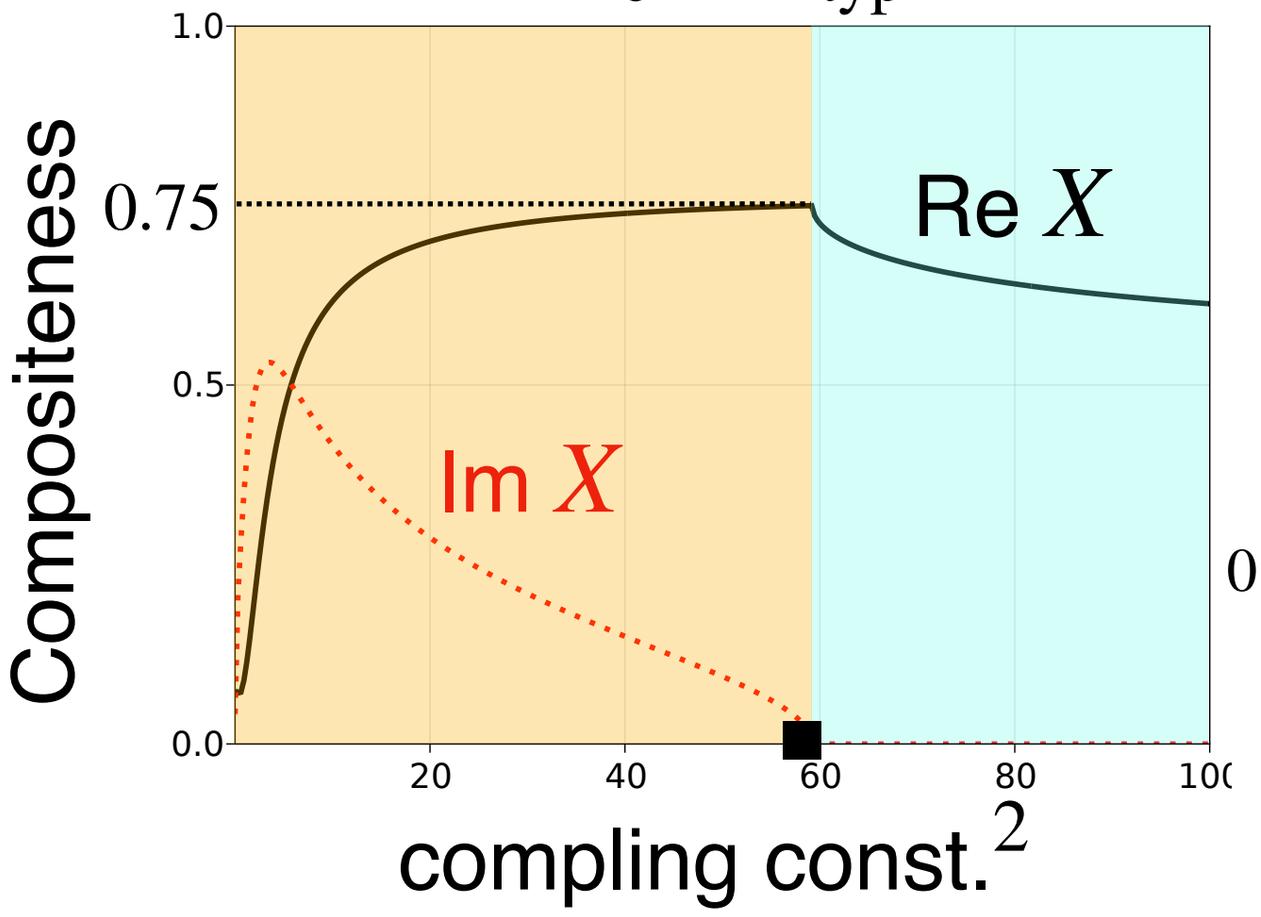
$$X = \left[1 + \frac{\Lambda^2}{6\nu_0\mu}\right]^{-1} = \left[1 + \frac{E_{\rm typ}}{3\nu_0}\right]^{-1}$$

c.f. repulsive Coulomb + s.r.

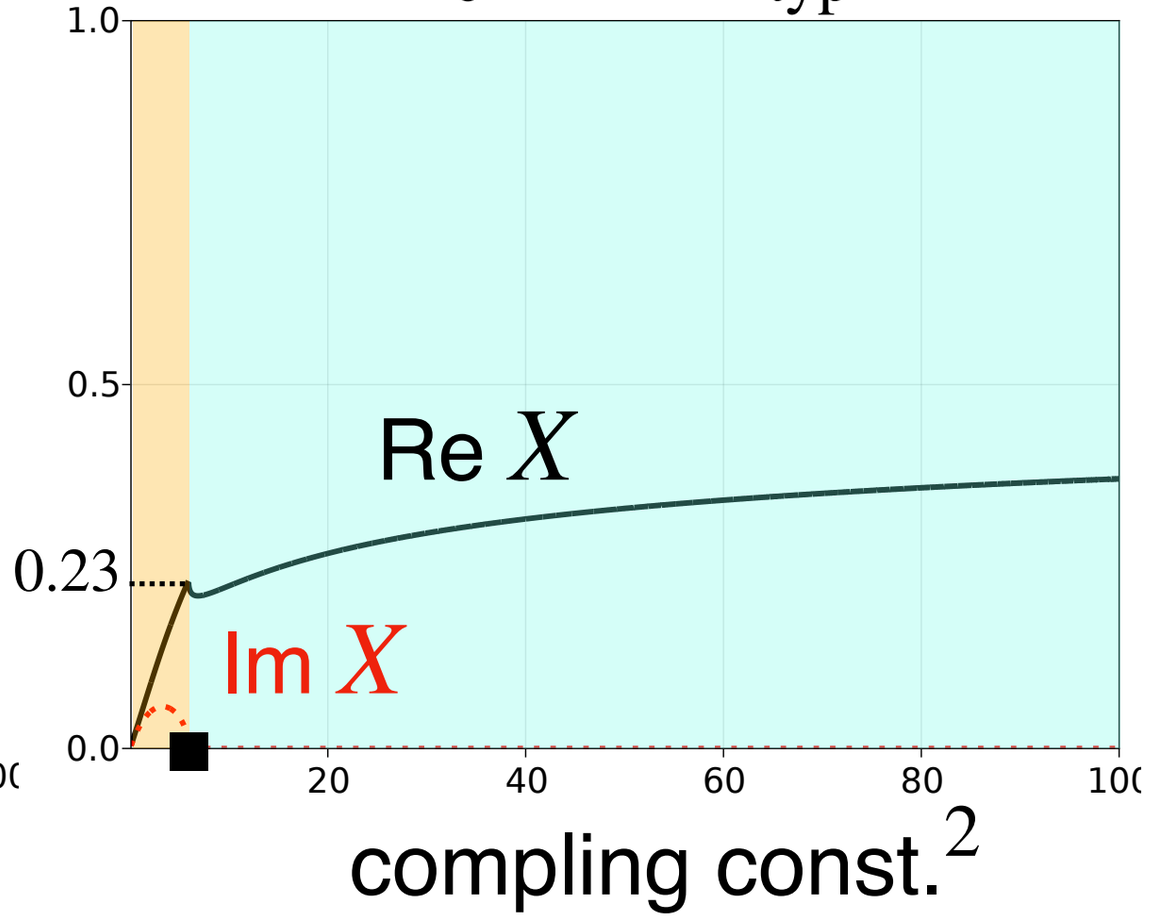
$$X = \left[1 - 3\mathrm{sgn}(Z_1Z_2)\frac{r_e}{a_B}\right]^{-1}$$

parameter dependence → no universality

$\nu_0 = E_{\rm typ}$



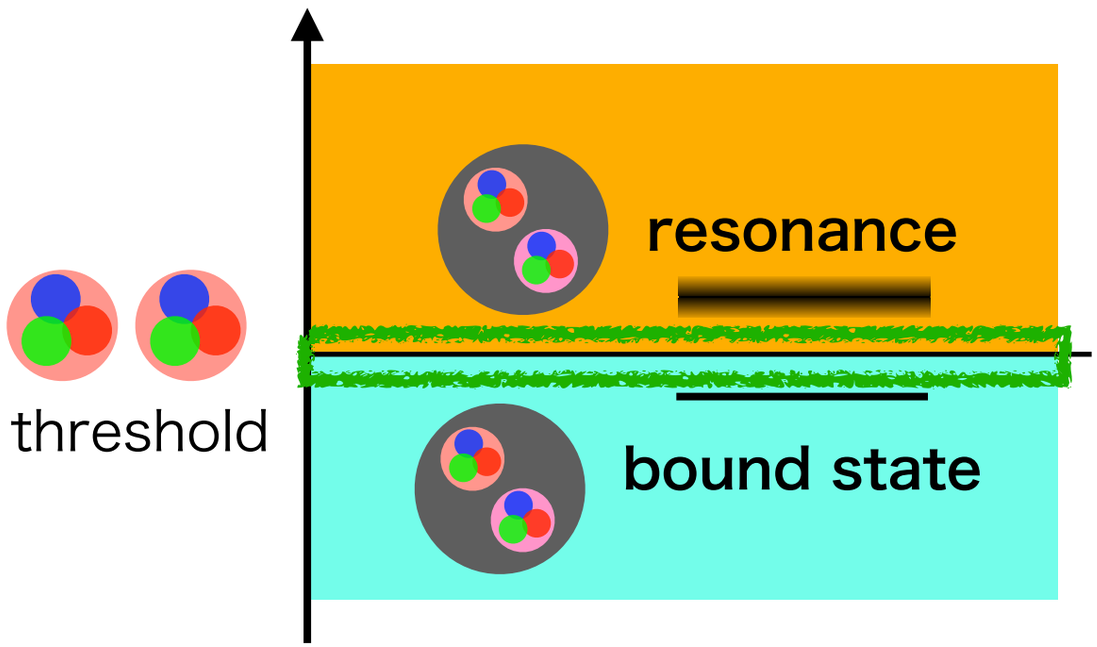
$\nu_0 = 0.1E_{\rm typ}$



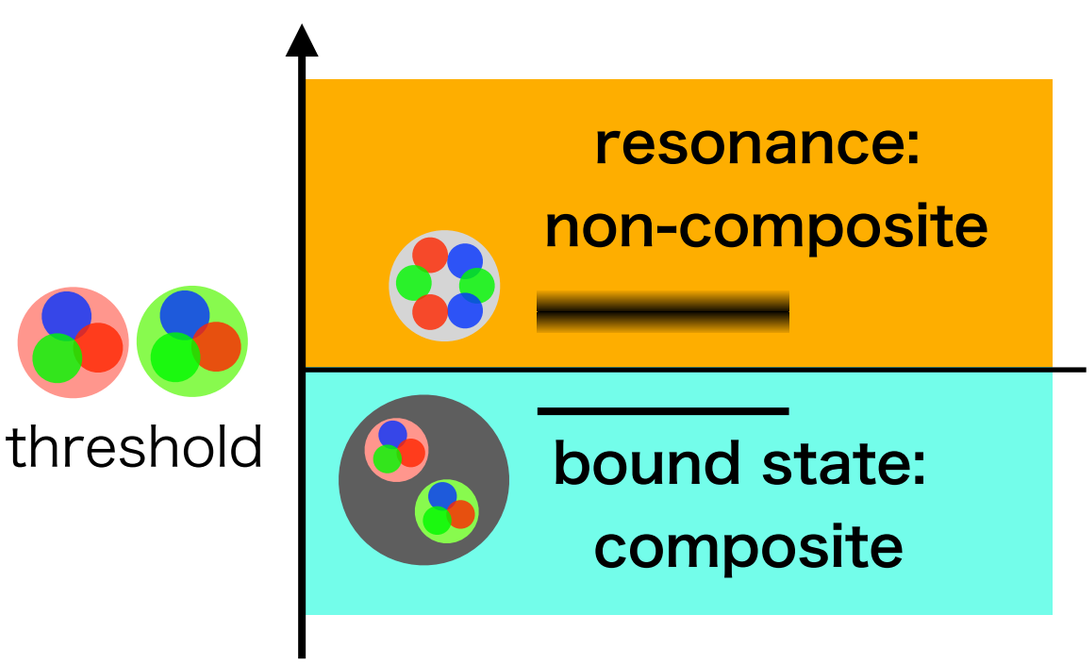
Summary

T. Kinugawa and T. Hyodo, arXiv:2604.17813[hep-ph].

repulsicve Coulomb < short-range



short-range interaction

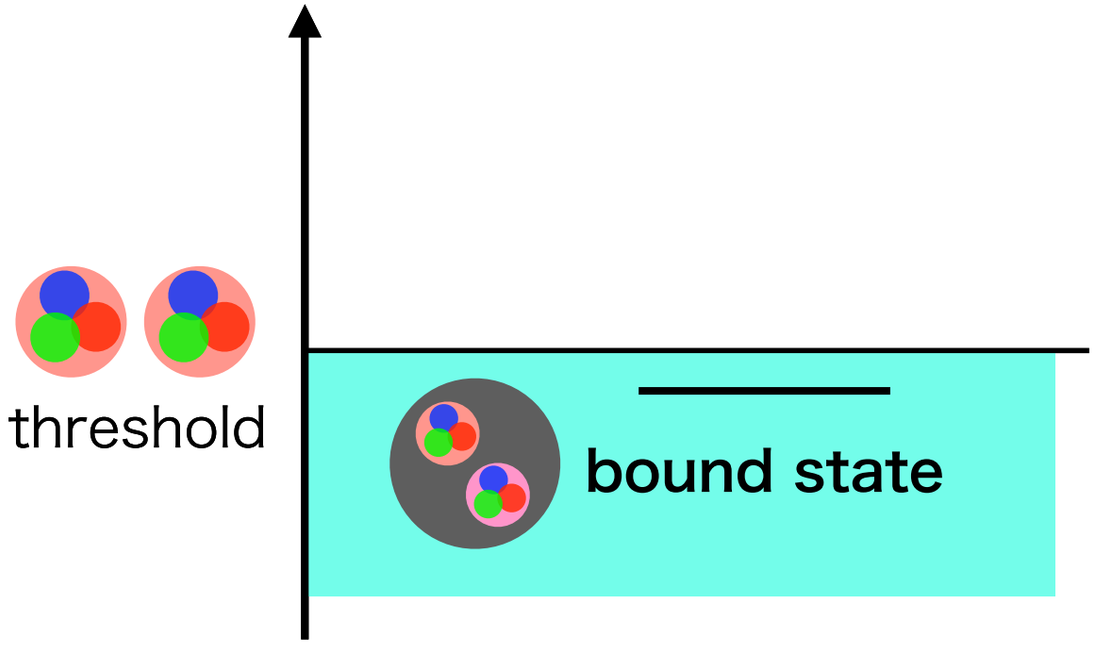


T. Kinugawa and T. Hyodo,

Phys. Rev. C 109 , 045205 (2024);

arXiv:2403.12635 [hep-ph], accepted in PTEP.

attractive Coulomb + short-range





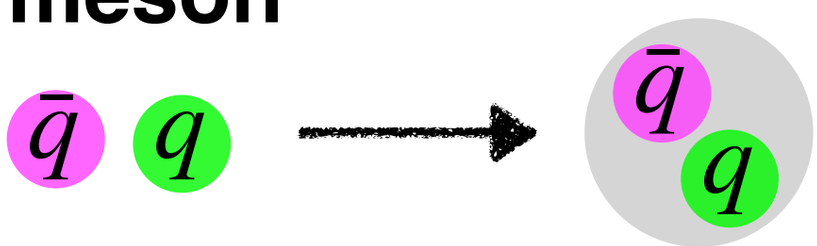
Back up

Exotic hadron

quark q, gluon ⟶ hadron
strong interaction

◉ ordinary hadron

- **meson**



- **baryon**

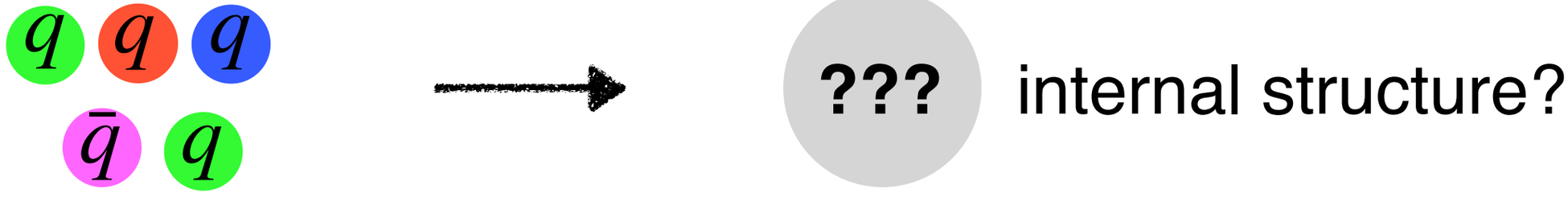


- ∼400 kinds have been observed so far

Particle Data Group, S. Navas et al., Phys. Rev. D 110, 030001 (2024).

◉ exotic hadron

- composed of four or more quarks



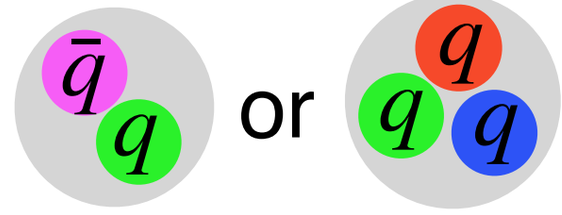
internal structure?

- **rarely** observed ⟵ Why?

- hint for understanding low-energy phenomena in QCD

Exotic hadrons

A. Hosaka, T. Iijima, K. Miyabayashi, Y. Sakai, and S. Yasui, PTEP 2016, 062C01 (2016)

- ordinaly hadron: $q\bar{q}$ (meson) or qqq (baryon) 

- other than $q\bar{q}$ or qqq → **exotic hadron** → structure?





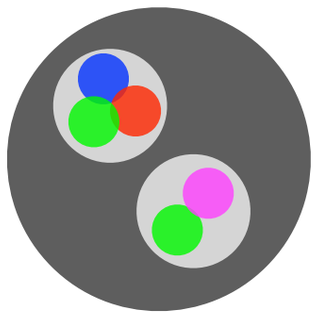
◉ possible structure F-K. Guo, C. Hanhart, Ulf-G. Meißner, Q. Wang, Q. Zhao, and B-S. Zou, RevModPhys.90.015004 (2018).

- **multiquarks**: quarks → hadron



size $\lesssim$ 1 fm (**compact**)

- **hadronic molecule**: quarks → hadrons (subunit) → hadron



size $\gtrsim$ 1 fm (spatially large radious)

e.g. deuteron (pn molecule), $R = 4.32$ fm

→ **superposition** of possible components

internal structure ← investigate weight of each component

History of compositeness

- Weinberg's work (1960s) Weinberg, S. Phys. Rev. 137, 672–678 (1965) etc.
  deuteron is not an elementary particle ← weak-binding relation
  Z : field renormalization factor
- application to exotic hadrons (2000s-)
  X: "compositeness"
  generalization to unstable states
  with spectral function V. Baru, J. Haidenbauer, C. Hanhart, Y. Kalashnikova, A.E. Kudryavtsev, Phys. Lett. B 586, 53–61 (2004) etc.
  with effective range expansion T. Hyodo, Phys. Rev. Lett. **111**, 132002 (2013) etc.
  with effective field theory Y. Kamiya and T. Hyodo, PTEP 2017, 023D02 (2017) etc.

application to …

$f_0(980)$, $a_0(980)$ Y. Kamiya and T. Hyodo, PTEP 2017, Phys. Rev. C 93, 035203 (2016); T. Sekihara, S. Kumano, Phys. Rev. D 92, 034010 (2015) etc.

$\Lambda(1405)$ T. Sekihara, T. Hyodo, Phys. Rev. C 87, 045202 (2013) ; Z.H. Guo, J.A. Oller, Phys. Rev. D 93, 096001 (2016) etc.

nuclei & atomic systems T. Kinugawa, T. Hyodo, Phys. Rev. C 106, 015205 (2022) etc.

Compositeness

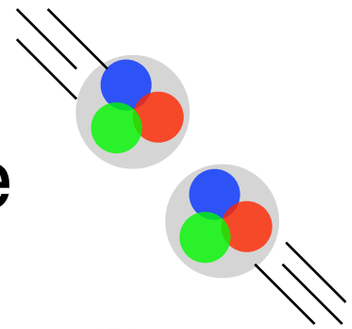
Weinberg, S. Phys. Rev. 137, 672–678 (1965);
T. Hyodo, Int. J. Mod. Phys. A 28, 1330045 (2013);
T. Kinugawa, T. Hyodo, Eur. Phys. J. A 61, 154 (2025).

- decomposing Hamiltonian into free part and remainder

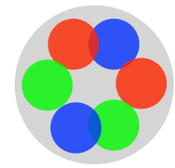
$$\hat{H} = \hat{H}_0 + \hat{V}$$

free Hamiltonian

$$\hat{H}_0 |\boldsymbol{p}\rangle = \frac{\boldsymbol{p}^2}{2\mu} |\boldsymbol{p}\rangle$$ free scattering state



$$\hat{H}_0 |\phi\rangle = \nu_0 |\phi\rangle$$ bare discrete state



- compositeness X, elementarity Z

$$X = \int d\boldsymbol{p} \; |\langle \boldsymbol{p} | B \rangle|^2, \quad Z = |\langle \phi | B \rangle|^2$$

overlaps of bound state $|B\rangle$ and $|\boldsymbol{p}\rangle$ or $|\phi\rangle$

- decomposition is not unique! ⟶ **model dependence** of X

S. Weinberg, Phys. Rev. 130, 776 (1963); D. Luri´e and A. J. Macfarlane, Phys. Rev. 136, B816 (1964).

Expressions of Compositeness

◉ LS equation

T. Hyodo, D. Jido, and A. Hosaka, Phys. Rev. C **85**, 015201 (2012);
F. Aceti and E. Oset, Phys. Rev. D **86**, 014012 (2012).

$$T = \frac{1}{V^{-1} - G}$$

V : effective interaction

G : loop function

residue of scattering amplitude g

$$X = -\,g^2\,G'(E)\Big|_{E=-B} \qquad \alpha'(E) = d\alpha/dE$$

$$= \frac{G'(E)}{G'(E) - [V^{-1}(E)]'}\Bigg|_{E=-B}$$

Y. Kamiya and T. Hyodo, PTEP 2017, 023D02 (2017).

g^2 : model independent ← $T_{\rm on}(-B)$ (observable)

$G(E)$: model dependent ← cutoff dependent

Expressions of Compositeness

◉ self-energy

T. Hyodo, Int. J. Mod. Phys. A 28, 1330045 (2013);
T. Kinugawa, T. Hyodo, Eur. Phys. J. A 61, 154 (2025).

$$Z = \frac{1}{1-\Sigma'} \qquad X = 1 - Z = -\frac{\Sigma'}{1-\Sigma'} \qquad \Sigma' = d\Sigma/dE$$

M. T. Vaughn, R. Aaron, and R. D. Amado, Phys. Rev. 124, 1258 (1961);
A. Salam, Nuovo Cim. 25, 224 (1962);
H. Ezawa, T. Muta, and H. Umezawa, Progress of Theoretical Physics 29, 877 (1963).

⟶ field renormalization constant
or wavefunction renormalization factor

◉ weak-binding relation (for shallow bound states)

$$X = \frac{a_0}{2R - a_0} + \mathcal{O}\left(\frac{R_{\rm typ}}{R}\right)$$

a_0 : scattering length

$R_{\rm typ}$: typical length scale in system

$R = 1/\sqrt{2\mu B}$

- for shallow bound states, $R \gg R_{\rm typ}$

compositeness ⟵ observables a_0, B

Weak-binding relation

S. Weinberg, Phys. Rev. 137, 672–678 (1965).

$$X = \frac{a_0}{2R - a_0} + \mathcal{O}\left(\frac{R_{\rm typ}}{R}\right)$$

a_0 : scattering length

$R_{\rm typ}$: typical length scale in system

$R = 1/\sqrt{2\mu B}$

- for weakly bound states, $R \gg R_{\rm typ}$

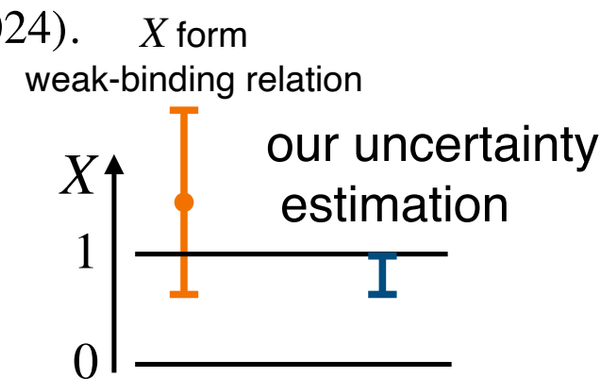
compositeness ⟵ observables (a_0, B)

◉ range correction

Y. Li, F.-K. Guo, J.-Y. Pang, and J.-J. Wu, Phys. Rev. D 105, L071502 (2022);
J. Song, L. R. Dai, and E. Oset, Eur. Phys. J. A 58, 133 (2022);
M. Albaladejo, J. Nieves, Eur. Phys. J. C 82, 724 (2022);
T. Kinugawa, T. Hyodo, Phys. Rev. C 106, 015205 (2022);
Z. Yin and D. Jido, Phys. Rev. C 110, no.5, 055202 (2024).

compositeness of deuteron $X \sim 1.7 > 1$

→ important to consider effective range



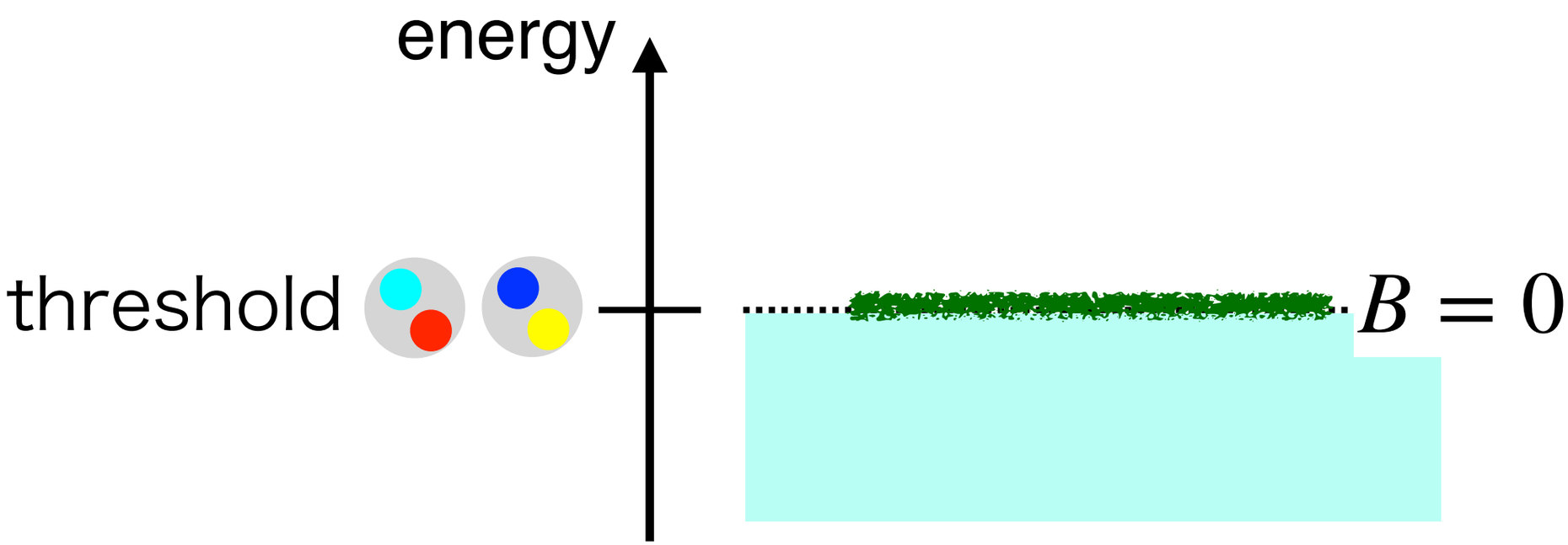
- our work : range correction ⟵ uncertainty estimation

compositeness of deuteron : $0.74 \leq X \leq 1$

Low-energy universality

- near-threshold energy region ← universal phenomena

E. Braaten and H.-W. Hammer, Phys. Rept. **428**, 259 (2006); P. Naidon and S. Endo, Rept. Prog. Phys. **80**, 056001 (2017).



- binding energy $B = 0$ (exactly at threshold)

⟶ that state is **always** completely composite ($X = 1$) 

compositeness is model-**independently** determined

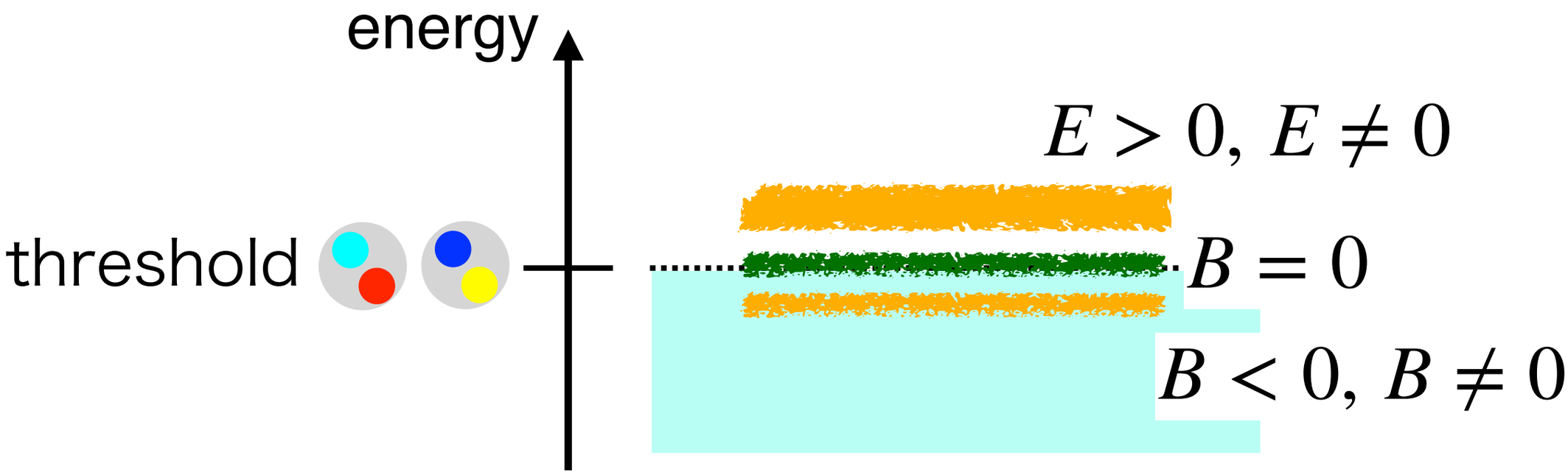
T. Hyodo, Phys. Rev. C **90**, 055208 (2014) .

← this work

Low-energy universality

- near-threshold energy region ← universal phenomena

E. Braaten and H.-W. Hammer, Phys. Rept. **428**, 259 (2006); P. Naidon and S. Endo, Rept. Prog. Phys. **80**, 056001 (2017).



- binding energy $B = 0$ (exactly at threshold)

→ that state is **always** completely composite ($X = 1$)



compositeness is model-**independently** determined

T. Hyodo, Phys. Rev. C **90**, 055208 (2014) .

- structure of **near**-threshold states ($B \neq 0$)? ← this work

similar to $B = 0$ state? $X \sim 1$?

Universality for near-th. resonances

- near-threshold **bound** (and virtual) states

  $a_0 \to \infty$ and universality holds in $B \to 0$ limit

  → $X \to 1$ (completely composite)

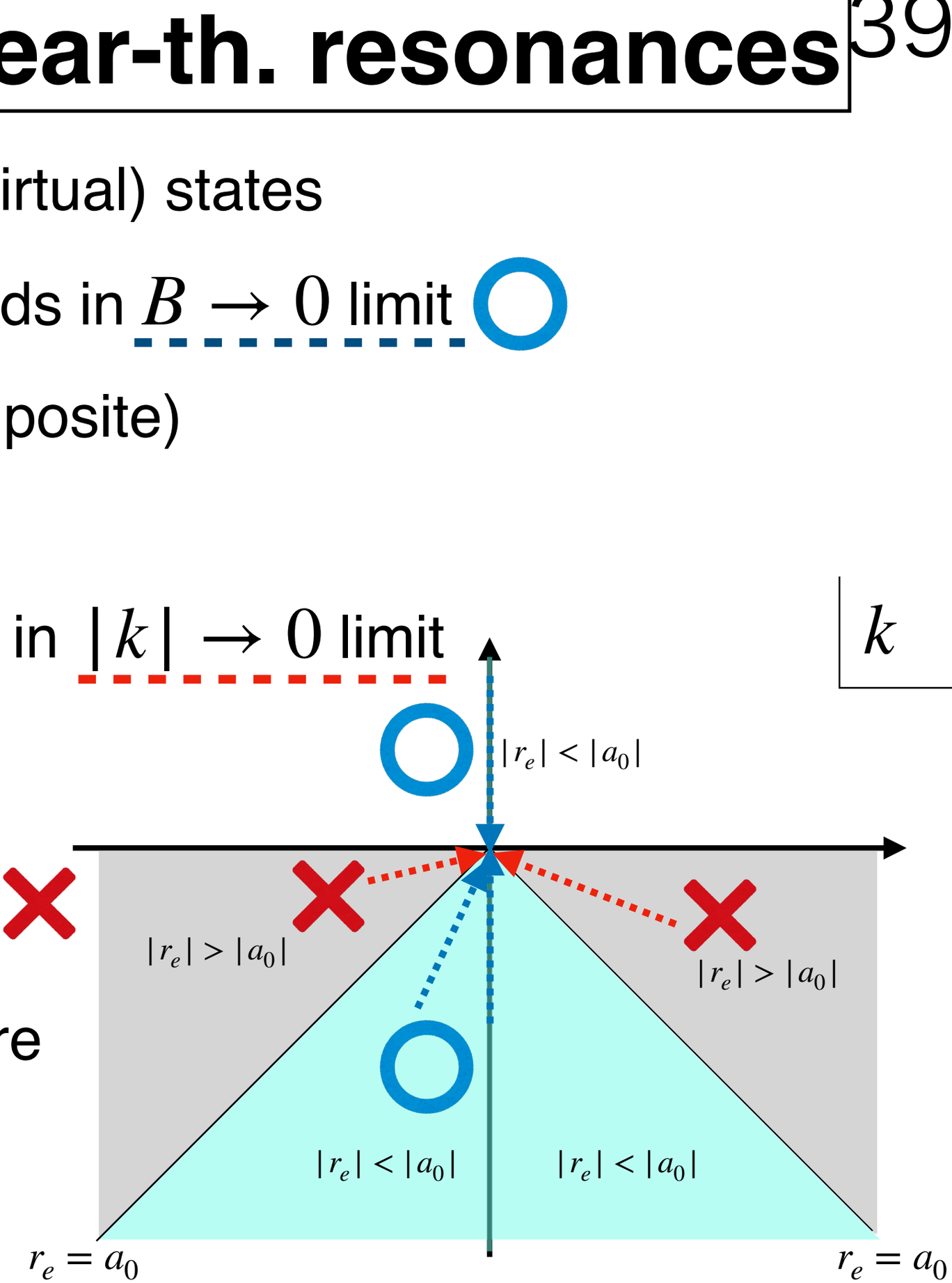
- near-threshold **resonances**

  $a_0 \to \infty$ but also $|r_e| \to \infty$ in $|k| \to 0$ limit

  $\because |a_0| \leq |r_e|$

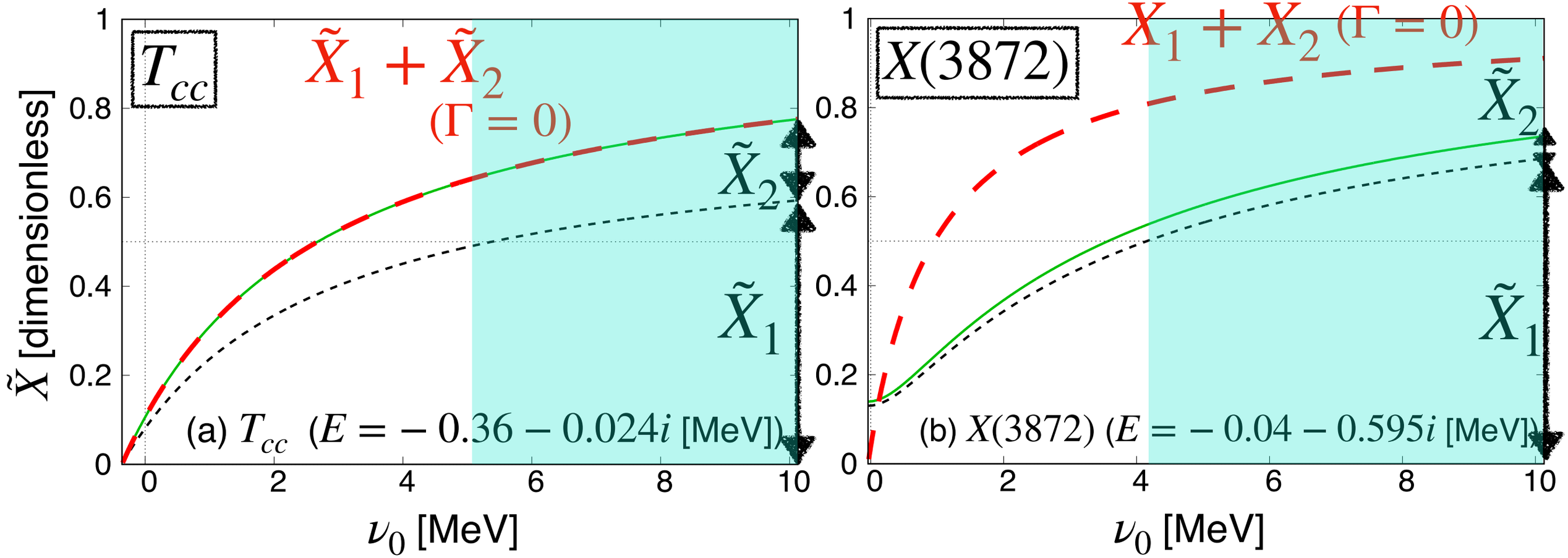
  → universality does not hold

Near-threshold resonances are **not** necessarily composite dominant



Application to T_{cc} and $X(3872)$

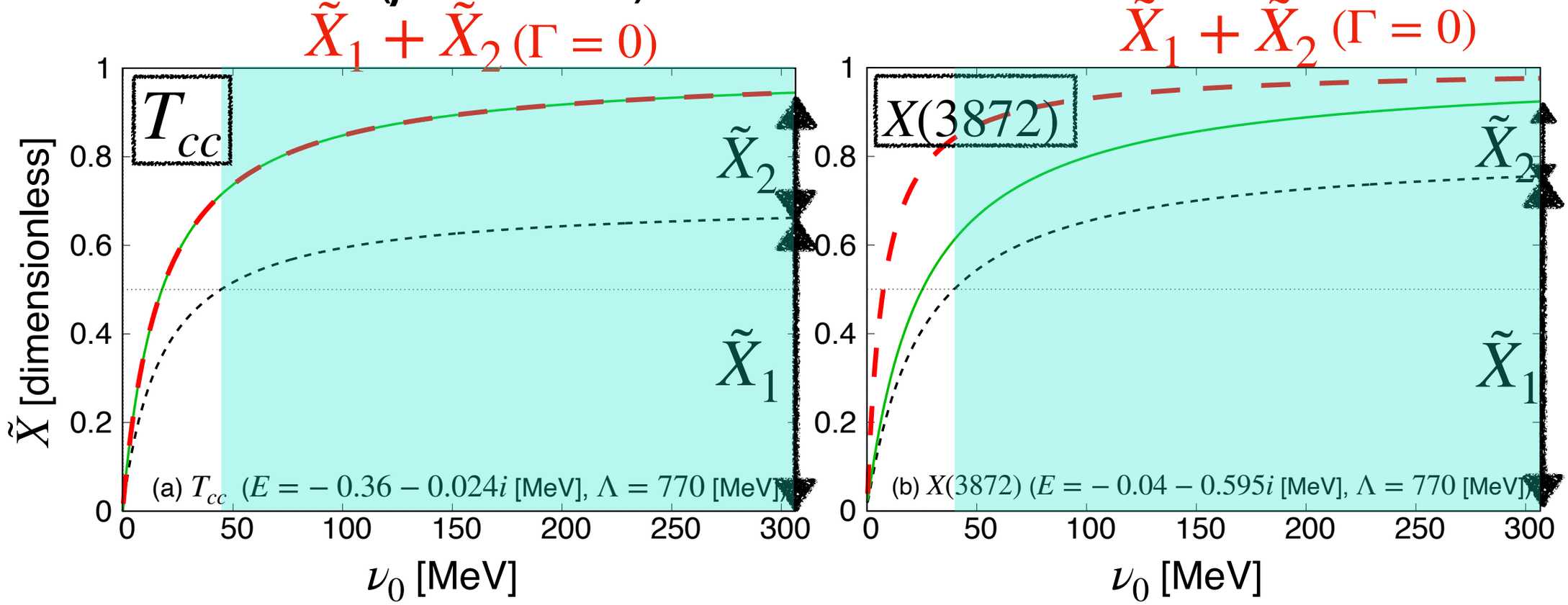
- $\Lambda = 140$ MeV (π meson)



- T_{cc} : $\tilde{X}_1 > 0.5$ for 45 % of ν_0 region
- $X(3872)$: $\tilde{X}_1 > 0.5$ for 59 % of ν_0 region
- coupled ch. effect is more important for T_{cc} than $X(3872)$
- decay effect is more important for $X(3872)$ than T_{cc}

Application to T_{cc} and $X(3872)$

- $\Lambda = 770$ MeV (ρ meson)



- T_{cc} : $\tilde{X}_1 > 0.5$ for 85 % of ν_0 region
- $X(3872)$: $\tilde{X}_1 > 0.5$ for 87 % of ν_0 region
- typical energy scale $E_{\rm typ}$ is larger

  ⟶ states becomes closer to universality limit $X \to 1$

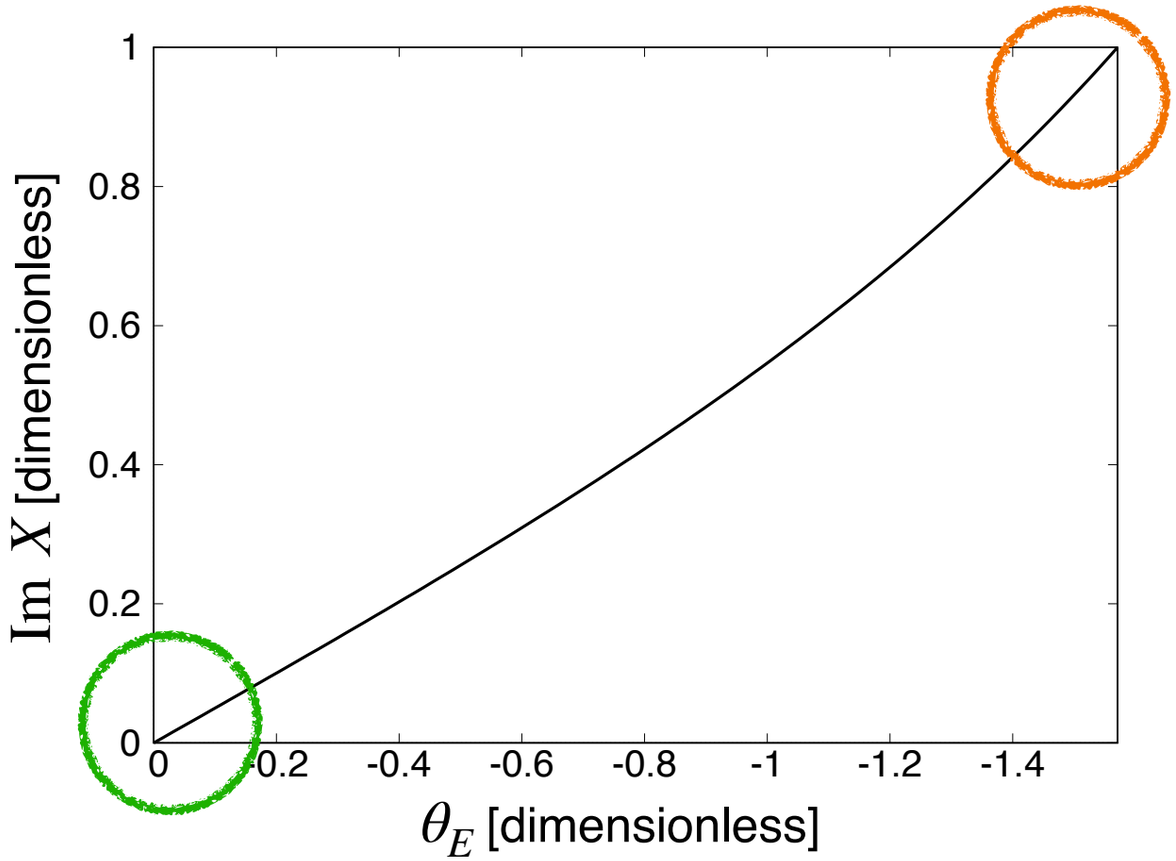
  decay effect : suppressed coupled ch. effect : enhanced

Compositeness in ERE

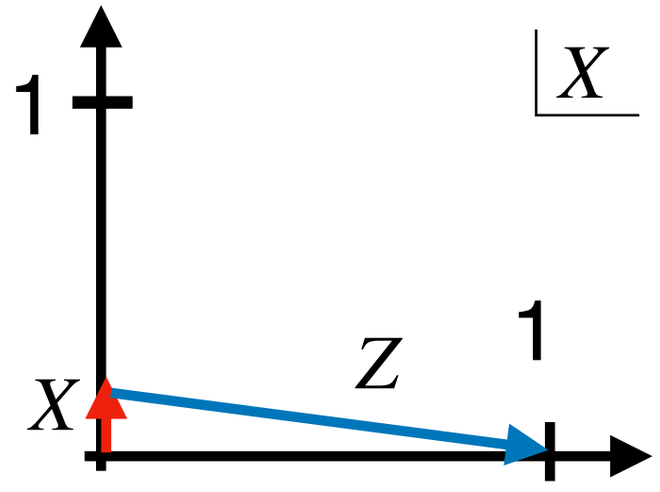
$$X = \sqrt{\frac{1}{1 - \frac{2r_e}{a_0}}} = -i \tan\theta_k \qquad (k = |k| e^{i\theta_k})$$

→ X in ERE is pure imaginary

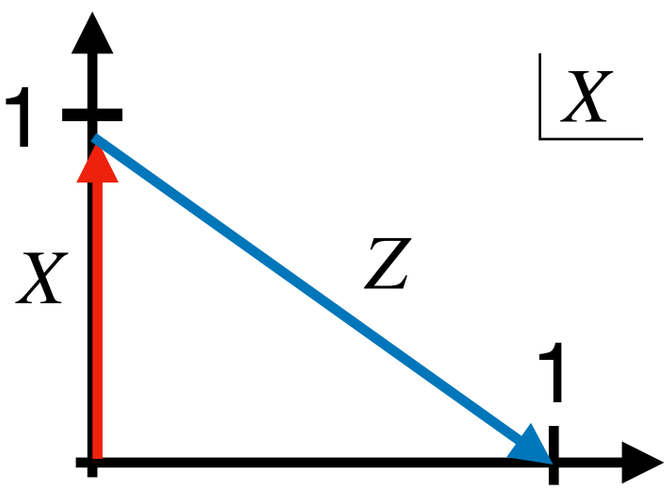
X as a function of θ_E $(E = |E| e^{i\theta_E})$



small width ($\theta_E \sim 0$)



large width ($\theta_E \sim -\pi/2$)



Complex compositeness

- interpretation scheme of complex X is **not** unique

  - $|X|$ F. Aceti, E. Oset, Phys. Rev. D 86, 014012 (2012); T. Hyodo, Int. J. Mod. Phys. A 28, 1330045 (2013); Z.H. Guo, J.A. Oller, Phys. Rev. D 93(9), 096001 (2016); J.A. Oller, Annals Phys. 396, 429 (2018).

  - Re X F. Aceti, L.R. Dai, L.S. Geng, E. Oset, Y. Zhang, Eur. Phys. J. A 50, 57 (2014).

  - introduce new quantities defined by complex X

    Y. Kamiya, T. Hyodo, PTEP 2017(2), 023D02 (2017);
    T. Sekihara, T. Arai, J. Yamagata-Sekihara, S. Yasui, Phys. Rev.C 93(3), 035204 (2016).

  - introduce new quantities based on ERE

    T. Hyodo, Phys. Rev. Lett. 111, 132002 (2013);
    I. Matuschek, V. Baru, F.K. Guo, C. Hanhart, Eur. Phys. J. A 57(3), 101 (2021).

    → How can we propose a “reasonable” scheme?

◉ condition for a reasonable measure

1. reduce to compositeness of bound states when $\mathrm{Im}\ X \to 0$
2. probabilistic quantities (normalized and between 0 and 1)
3. automatically take into account only **narrow** resonance

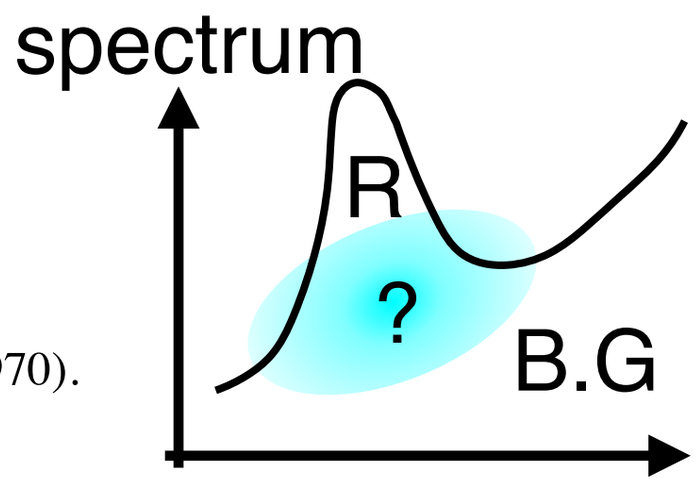
Our proposal

- introduce new quantities defined by complex X

i) $\mathscr{X}$: probability of certainly finding $|\text{composite}\rangle$

ii) $\mathscr{Z}$: probability of certainly finding $|\text{elementary}\rangle$

iii) $\mathscr{Y}$: probability of uncertain identification



uncertainty appears from

- finite lifetime (uncertainty in energy)
- separation from B.G. T. Berggren, Phys. Lett. B 33, 547 (1970).

$$\mathscr{X} = \frac{(\alpha-1)|X| - \alpha|Z| + \alpha}{2\alpha - 1}$$

$$\mathscr{Z} = \frac{(\alpha-1)|Z| - \alpha|X| + \alpha}{2\alpha - 1}$$

$$\mathscr{Y} = \frac{|X| + |Z| - 1}{2\alpha - 1}$$

1. $\mathscr{X} \to X, \mathscr{Z} \to Z$ and $\mathscr{Y} \to 0$ $(\text{Im } X \to 0)$

2. $\mathscr{X} + \mathscr{Y} + \mathscr{Z} = 1$

α reflects uncertain nature of resonances

Definition

- if $\alpha > 1/2$, $\mathscr{Y}$ is always positive but $\mathscr{X}$, $\mathscr{Z}$ can be negative

$\mathscr{X} \geq 0$ and $\mathscr{Z} \geq 0$	$\mathscr{X} > \mathscr{Y}, \mathscr{Z}$	composite dominant
	$\mathscr{Z} > \mathscr{Y}, \mathscr{X}$	elementary dominant
	$\mathscr{Y} > \mathscr{X}, \mathscr{Z}$	uncertain
$\mathscr{X} < 0$ or $\mathscr{Z} < 0$		non-interpretable

3. our criterion for narrow resonance

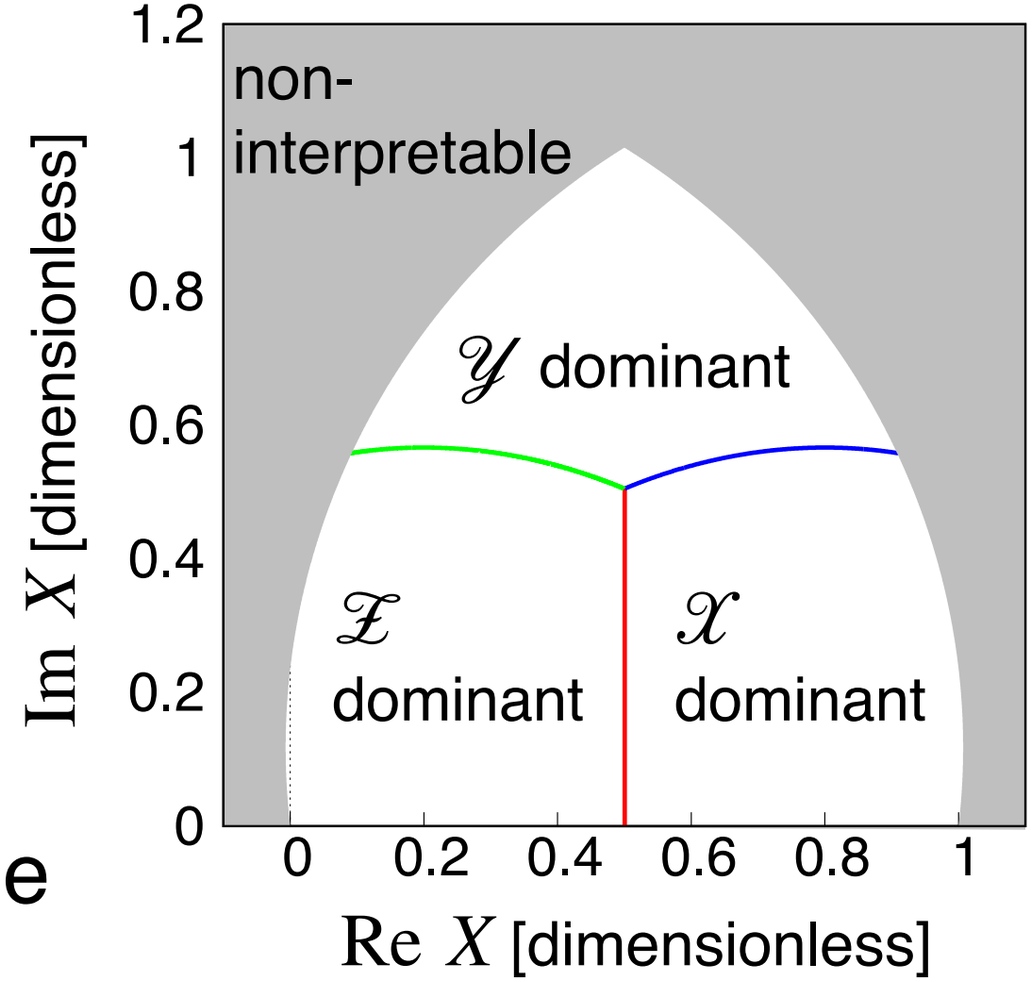
$\longrightarrow \Gamma \leq \mathrm{Re}\ E$

- $\alpha \sim 1.1318$ using ERE

- narrow resonances: interpretable

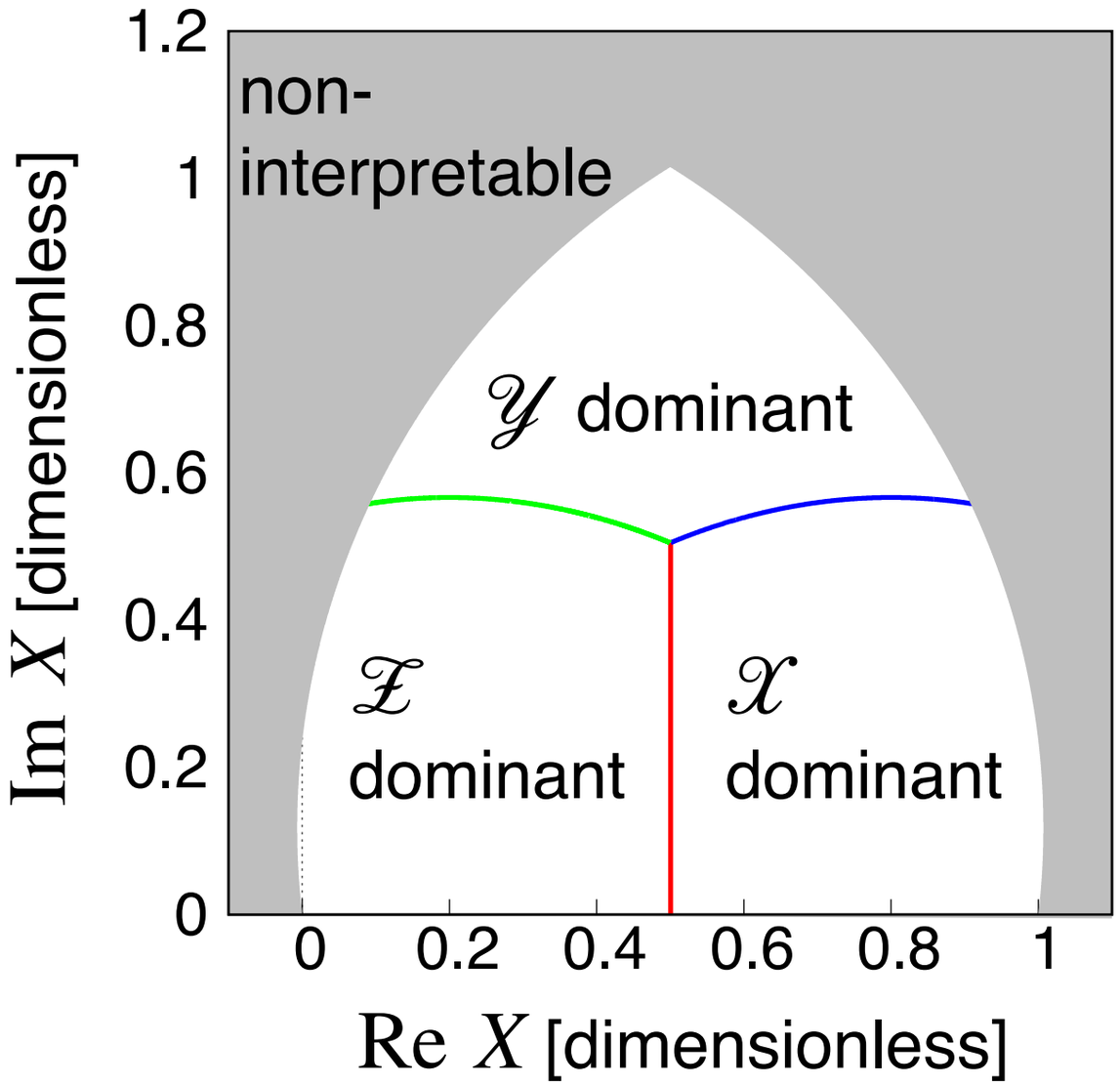
$\longrightarrow \mathscr{X}, \mathscr{Y}$, or $\mathscr{Z}$ dominant

- broad resonances: non-interpretable



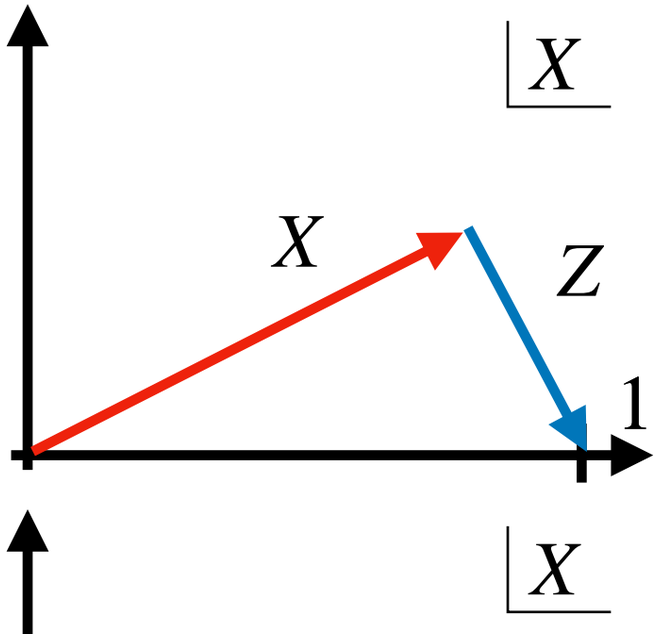
$\mathscr{X}, \mathscr{Y}, \mathscr{Z}$ dominant region

- $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$ dominant region in complex X plane

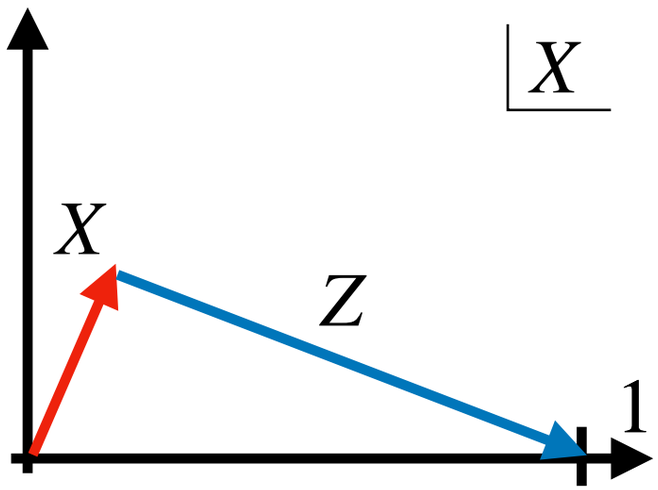


large Im X is assigned to non-interpretable cases

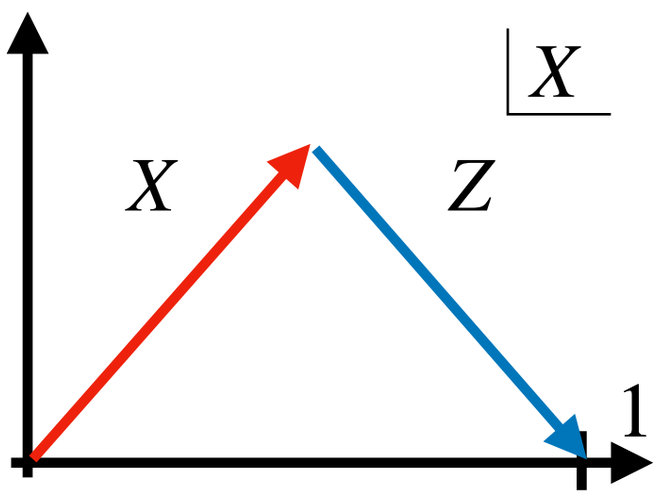
composite dominant



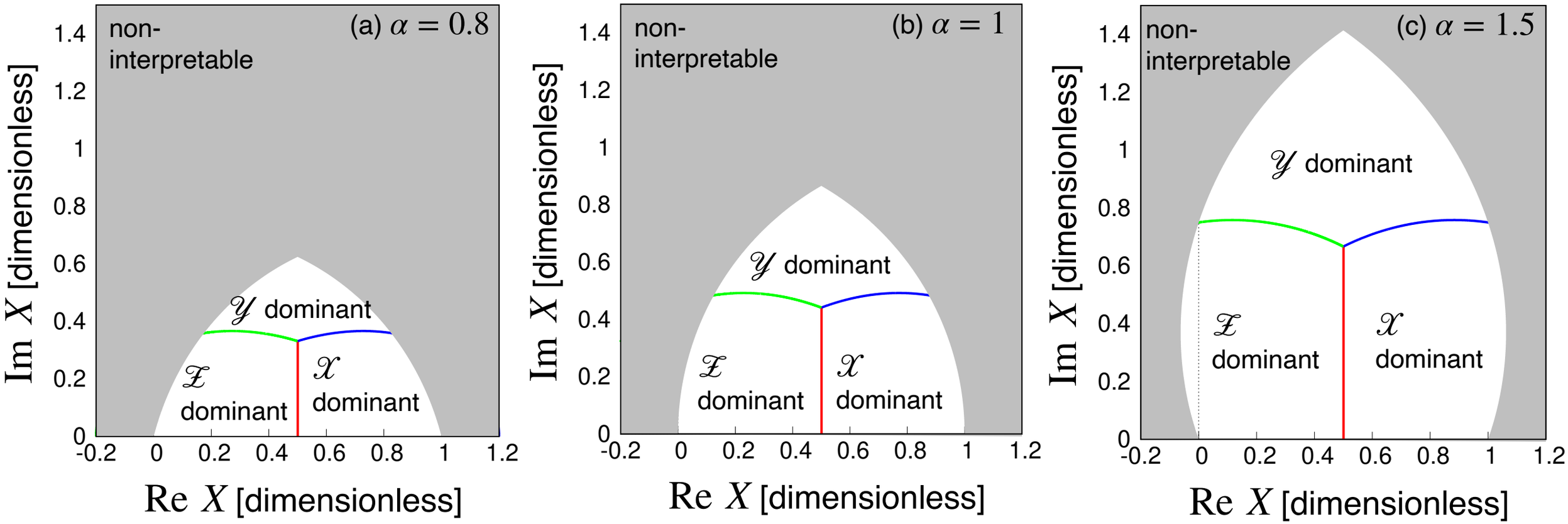
elementary dominant



uncertain



$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ dominant region



- Interpretable regions become larger as α increases

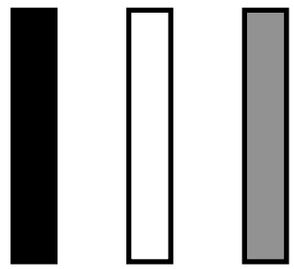
$\alpha \to \infty$ ⟶ $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ reduce to interpretation in previous work

$$\mathcal{X} \to \tilde{X},\ \mathcal{Z} \to \tilde{Z},\ \mathcal{Y} \to 0$$

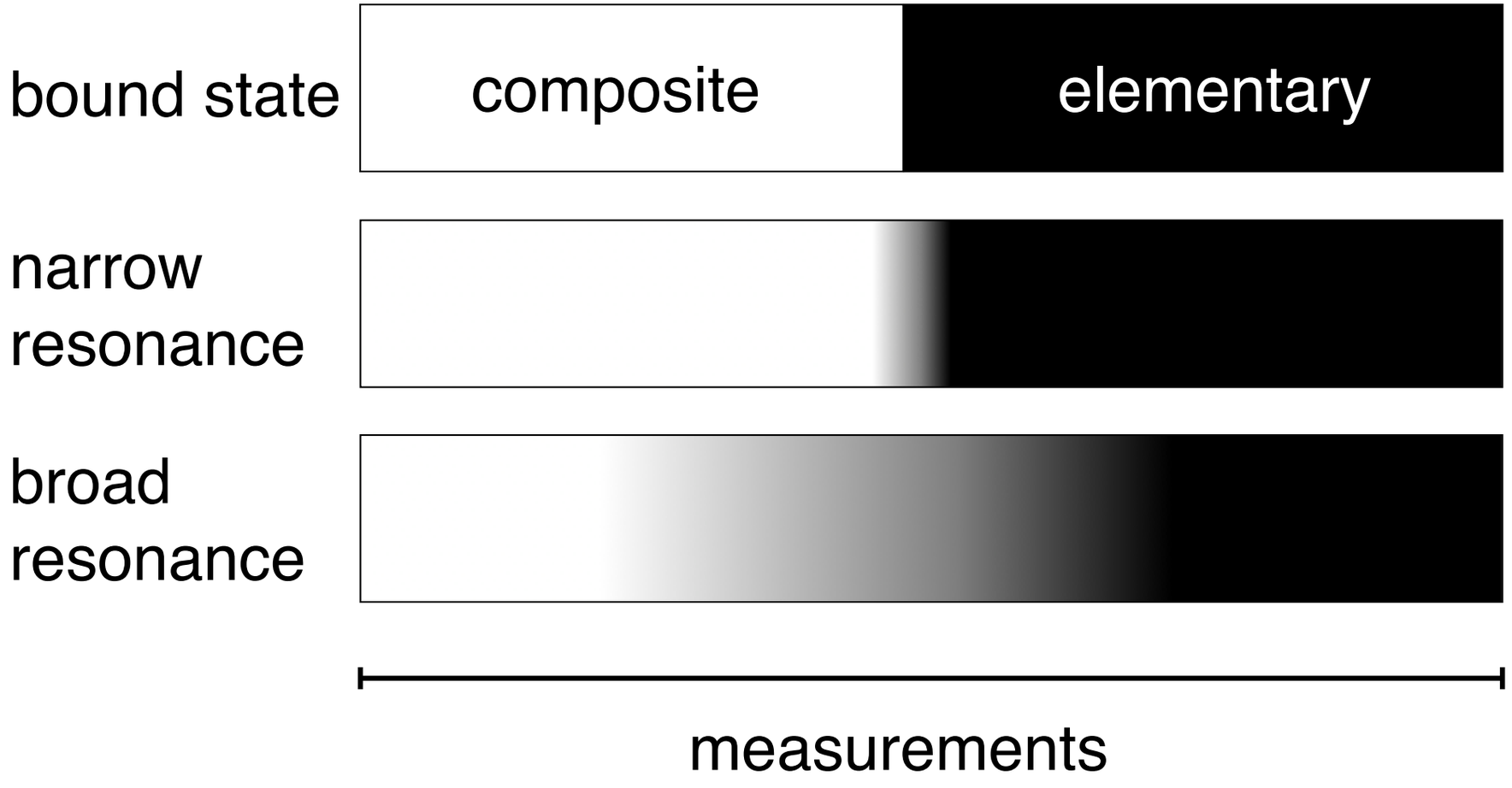
Y. Kamiya and T. Hyodo,
Phys. Rev. C **93**, 035203 (2016).

uncertainty in resonances

a single measurement

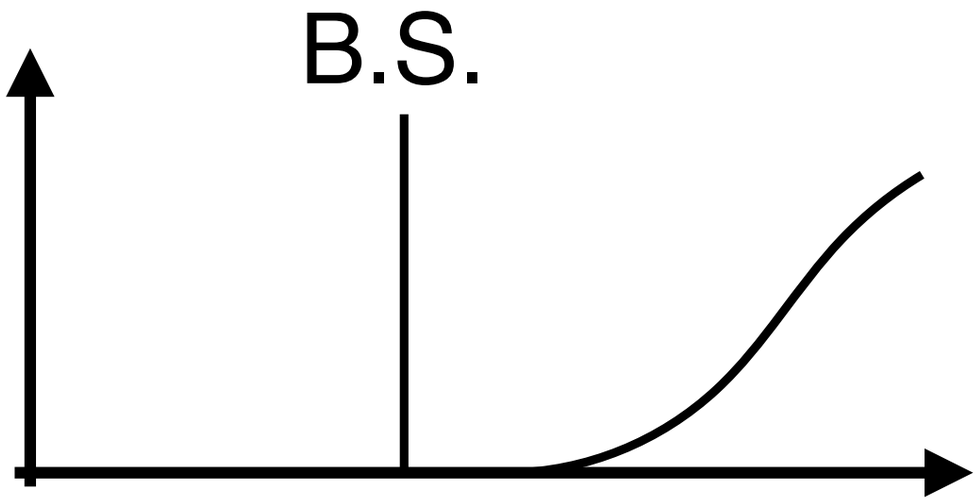


sum of measurements of bound states / resonances

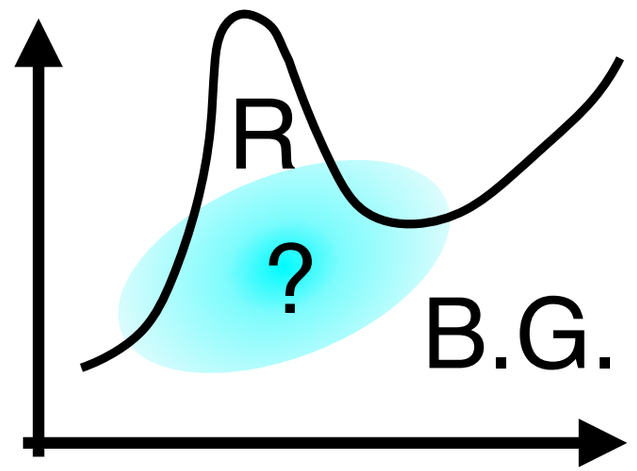


uncertainty in resonances

(a) spectrum



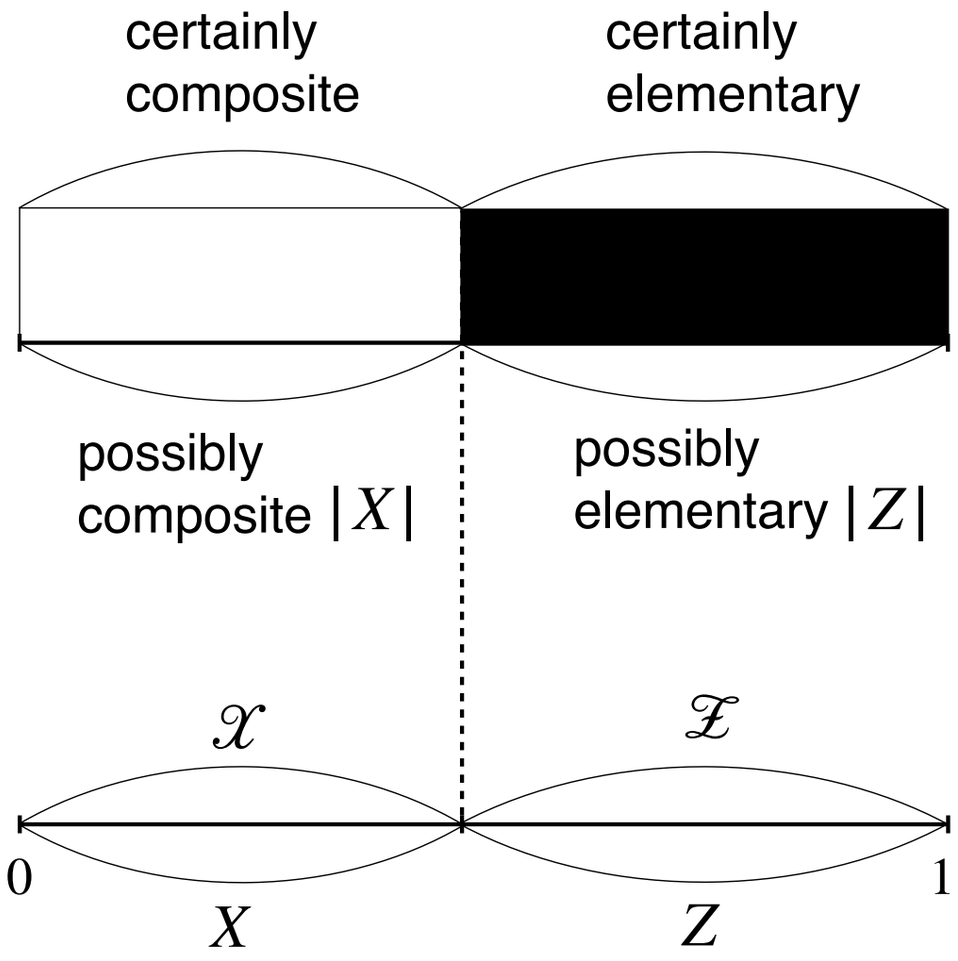
(b) spectrum



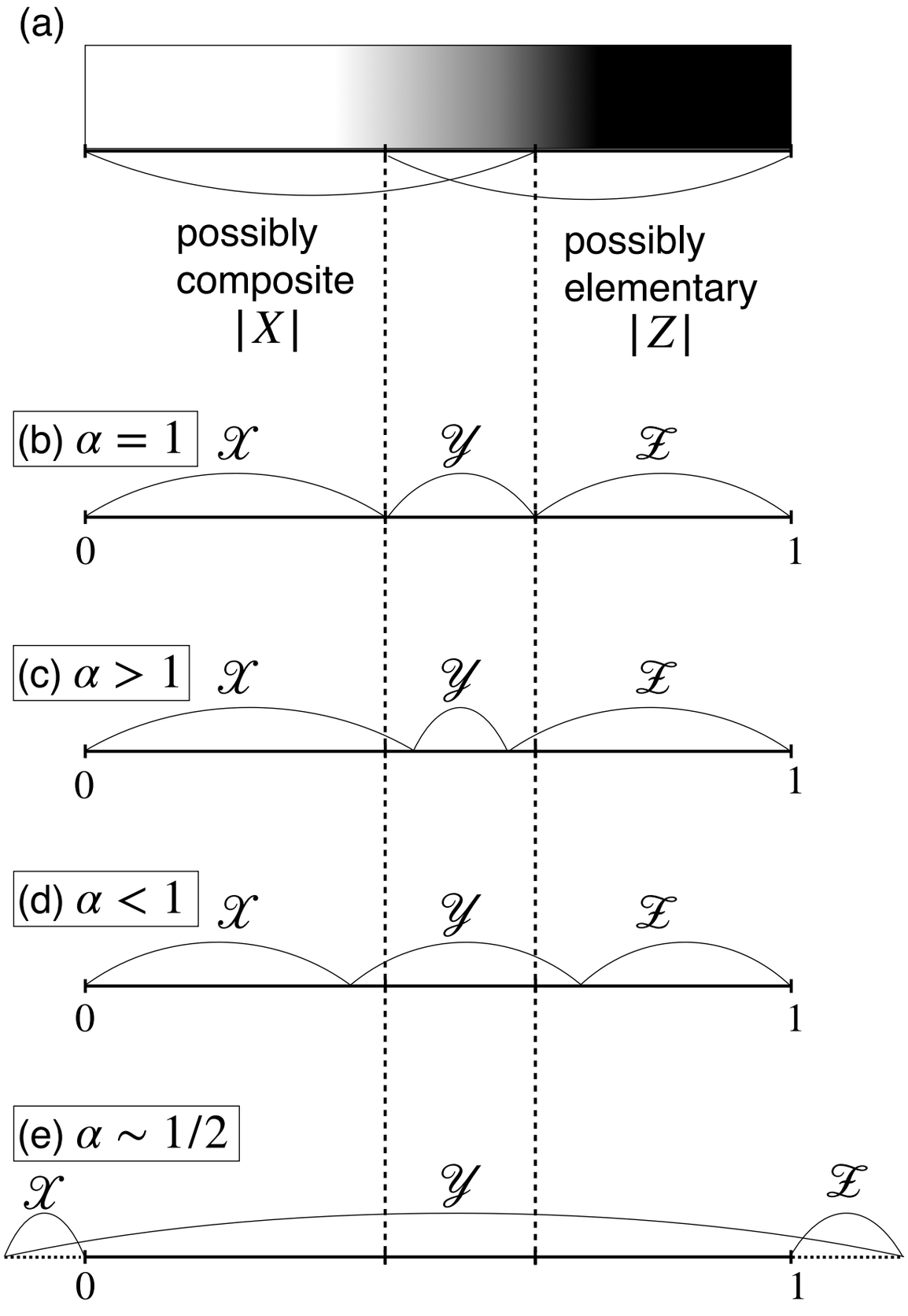
uncertainty appears from

- finite lifetime (uncertainty in energy)
- separation from B.G.

Interpretation

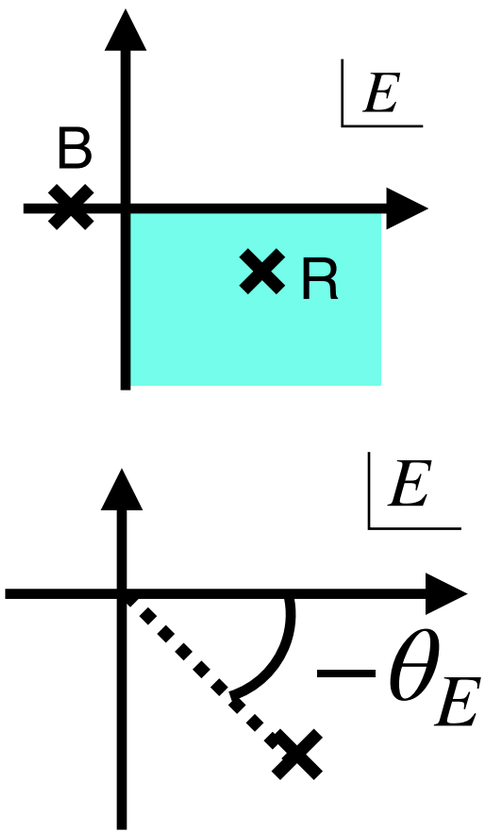


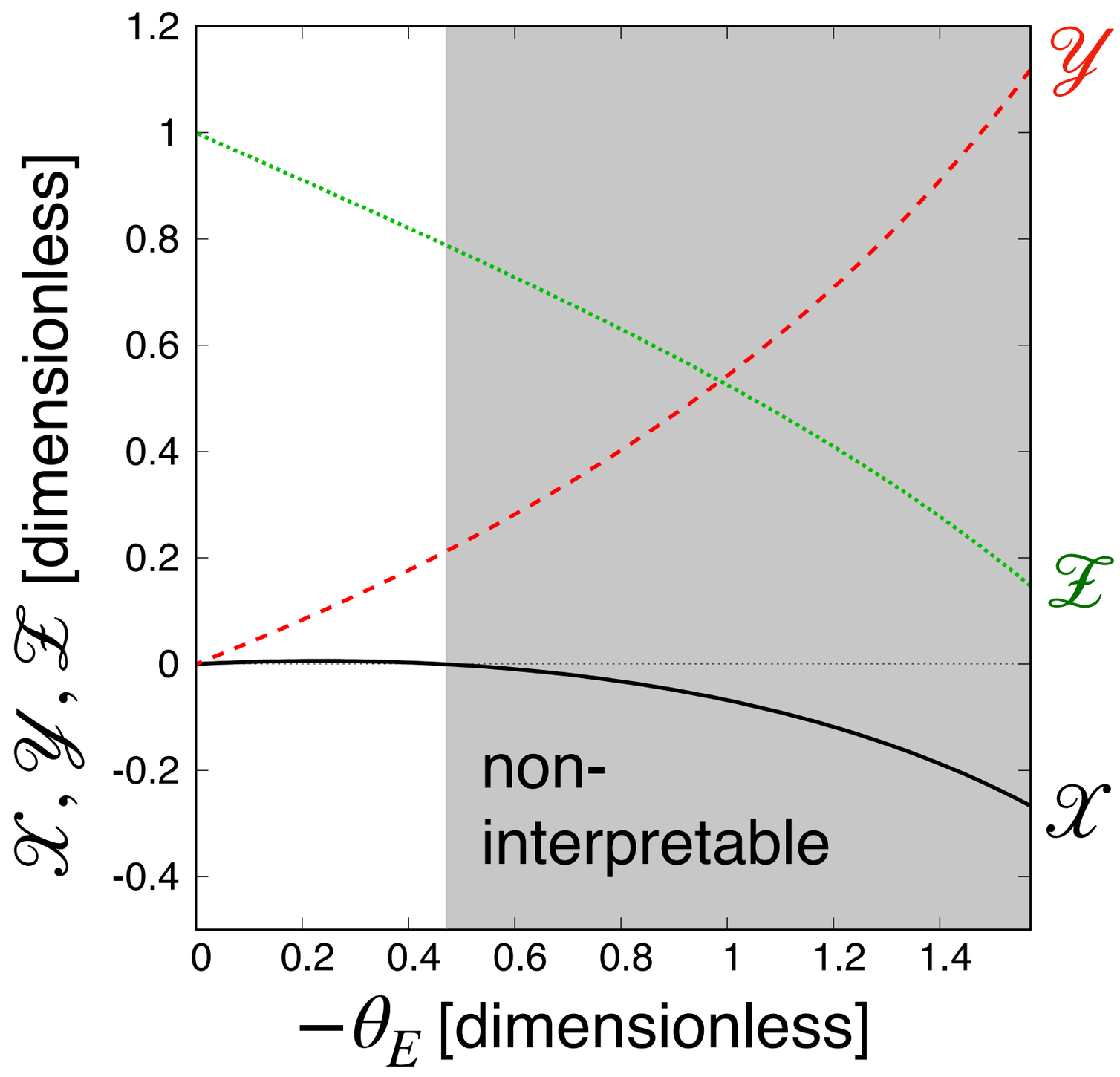
bound state



resonance

Structure of near-th. resonances





Comparison with previous works

$$\bar{Z} = 1 - \sqrt{\left| \frac{1}{1 - 2r_e/a_0} \right|}$$

T. Hyodo, Phys. Rev. Lett. **111**, 132002 (2013).

$$\tilde{Z}_{\rm KH} = \frac{1 + |Z| - |X|}{2}$$

Y. Kamiya and T. Hyodo, Phys. Rev. C **93**, 035203 (2016).

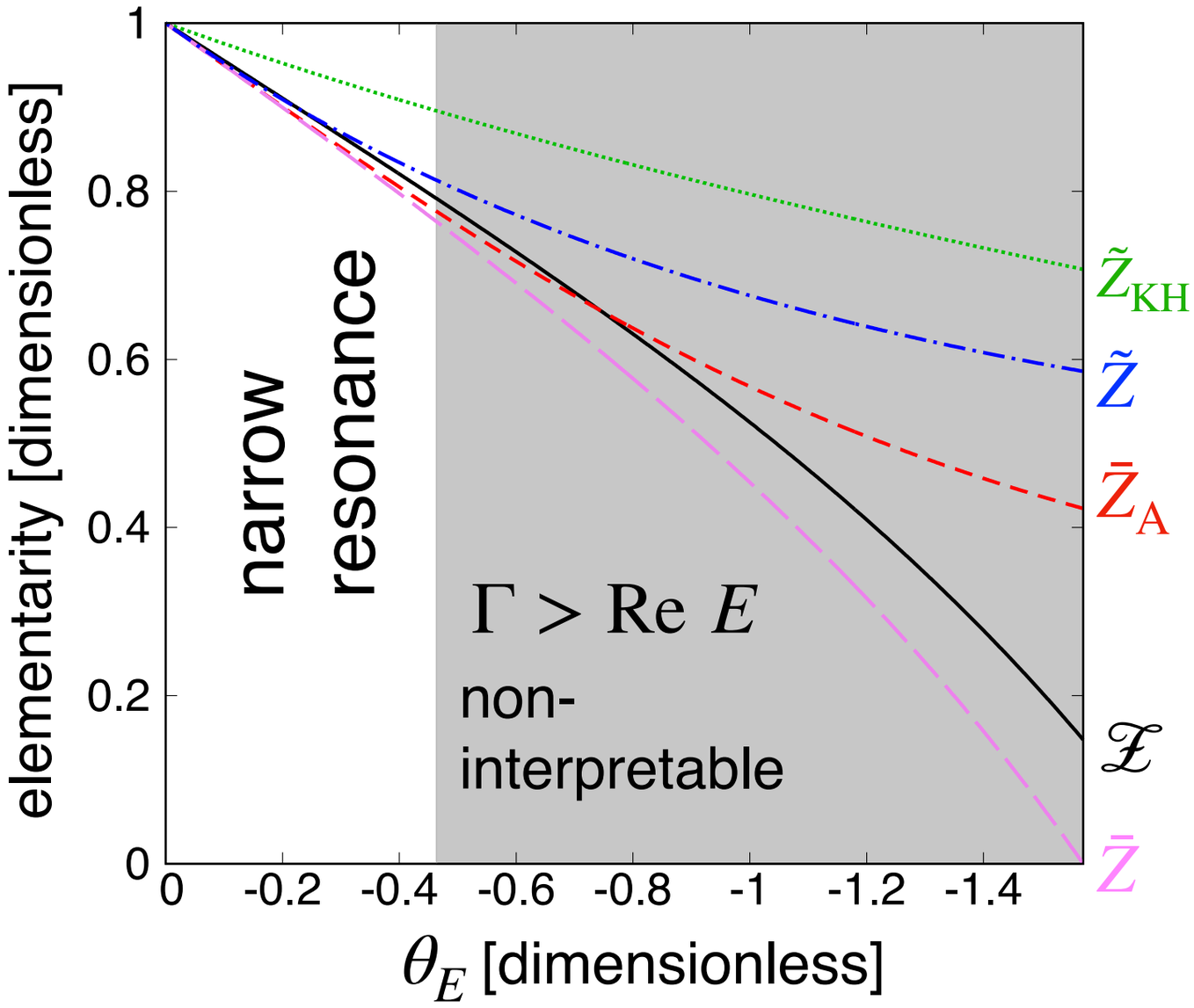
$$\tilde{Z} = \frac{|Z|}{|X| + |Z|}$$

T. Sekihara, T. Arai, J. Yamagata-Sekihara and S. Yasui, PRC **93**, 035204 (2016).

$$\bar{Z}_A = 1 - \sqrt{\frac{1}{1 + |2r_e/a_0|}}$$

I. Matuschek, V. Baru, F.-K. Guo, and C. Hanhart, Eur. Phys. J. A **57**, 101 (2021).

interpretations as a function of θ_E

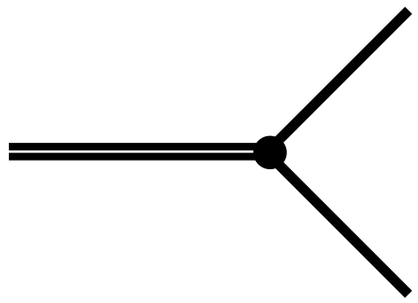


- all schemes show resonances are elementary dominant!

Pole trajectory (only w/ s.r.)

◉ pole trajectory in complex momentum k plane (No Coulomb)

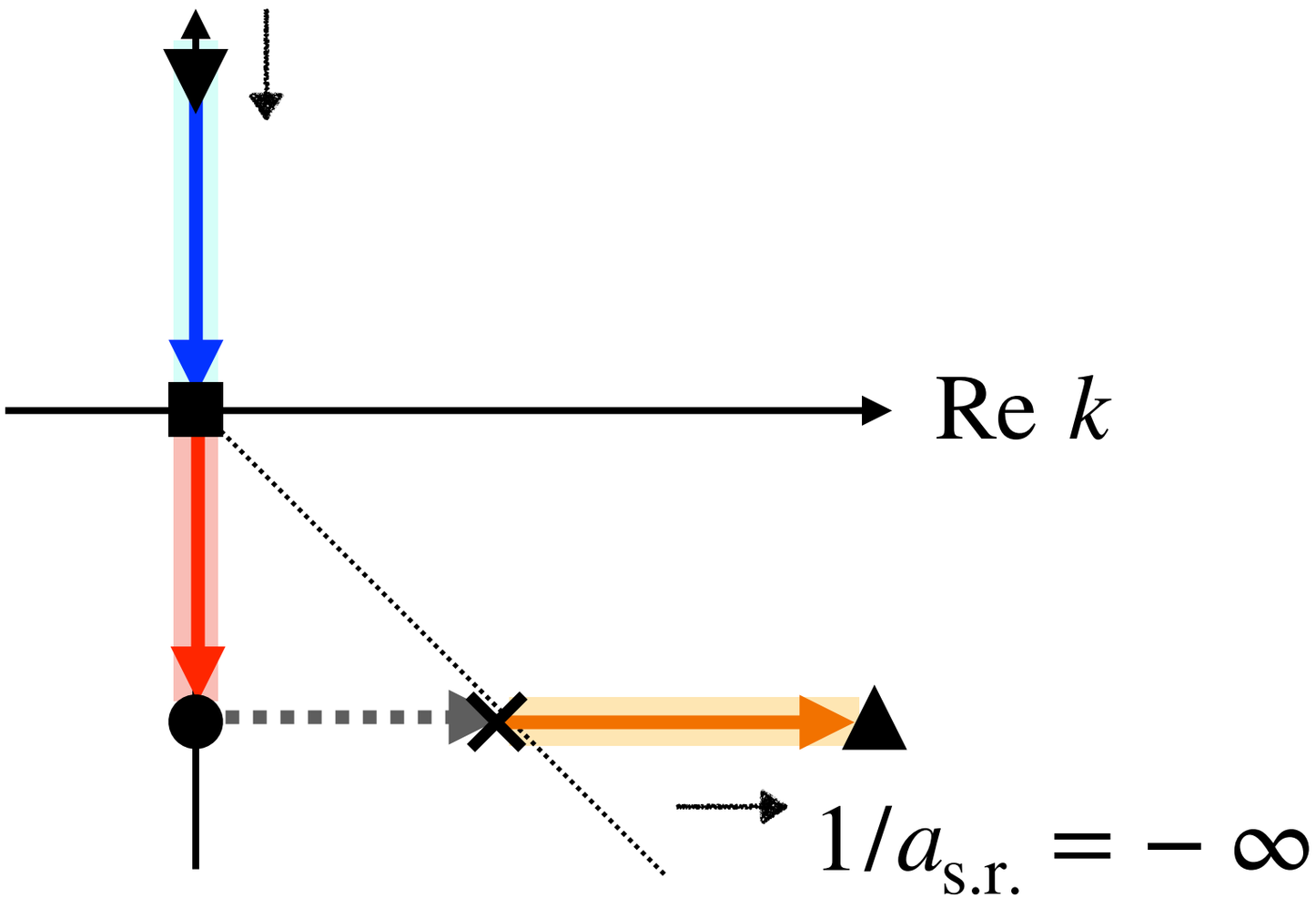
Q channel
⇔ bare state

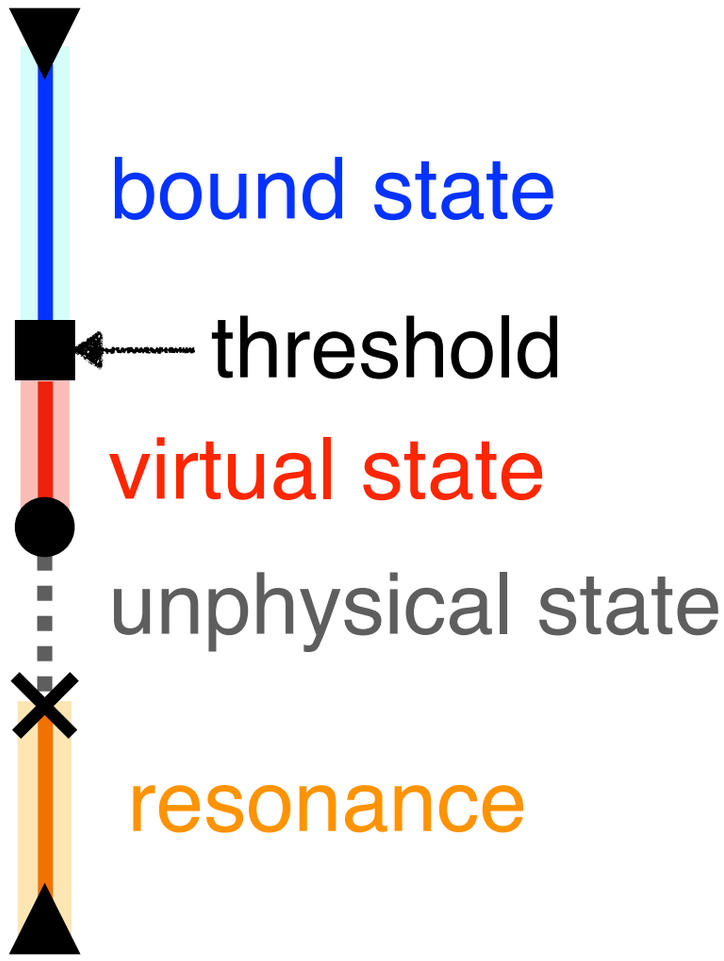


P channel
⇔ free scattering

$\mathrm{Im}\ k \quad 1/a_{\mathrm{s.r.}} = +\infty$

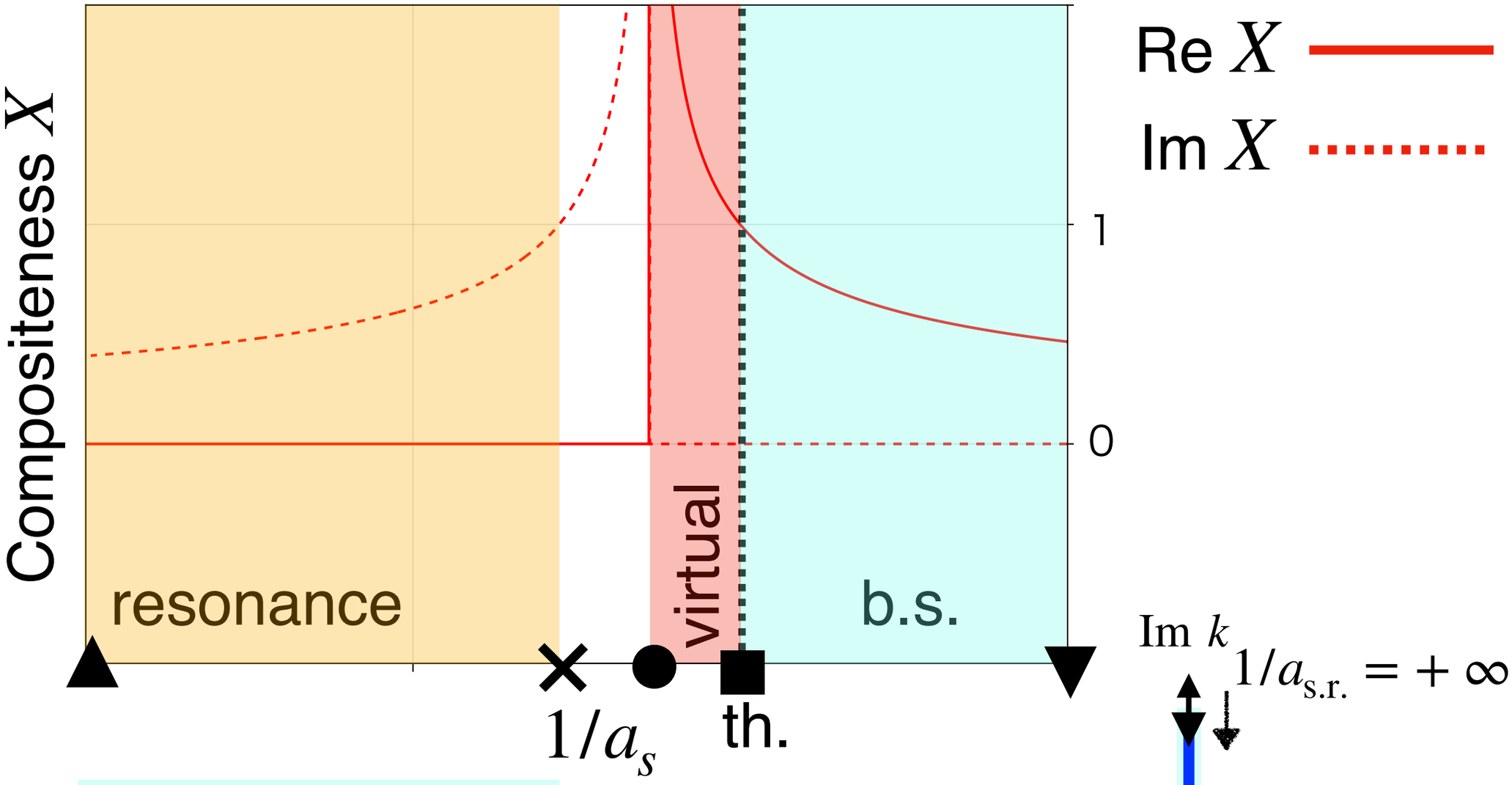
$G_P \to G^0$ free Green's function





T. Hyodo, Phys. Rev. Lett. **111**, 132002 (2013) .

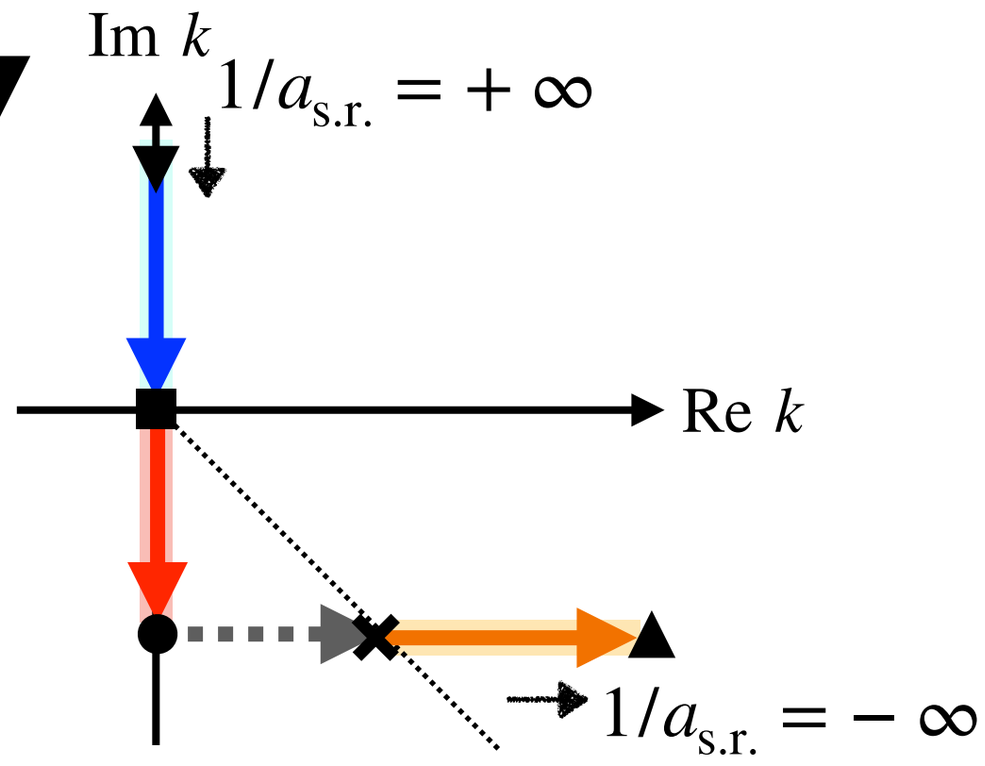
Compositeness (only w/ s.r.)



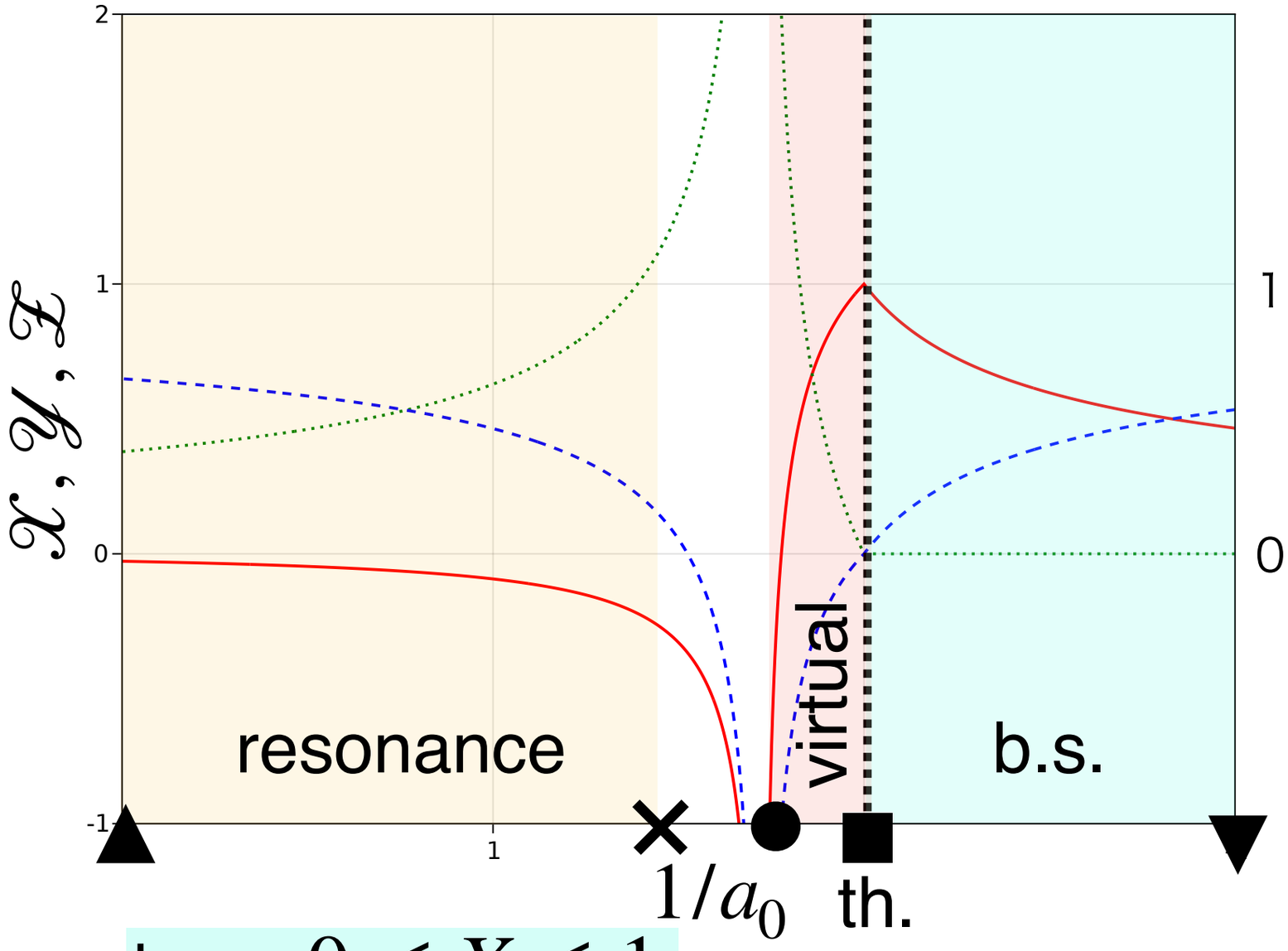
- b.s. : $0 \leq X \leq 1$
- virtual : $1 < X$

● divergence $X \to \infty$

- resonance : $\mathrm{Im}\ X \leq 1$



$\mathscr{X}, \mathscr{Y}, \mathscr{Z}$ from b.s. to resonance

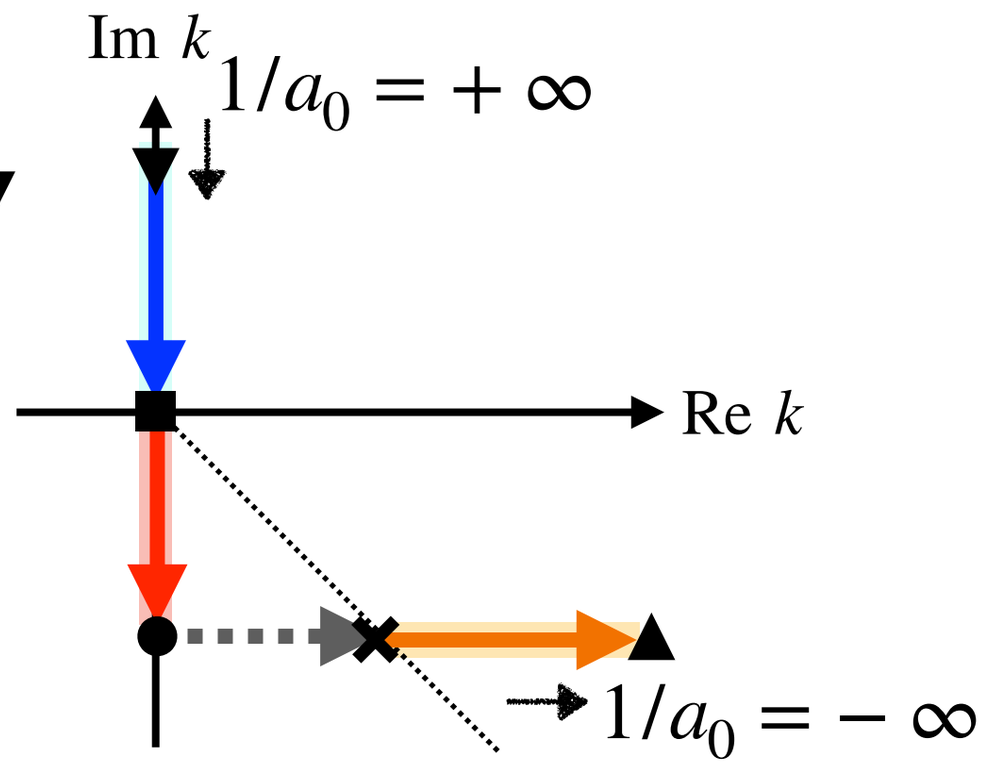


$\mathscr{X}$ ——

$\mathscr{Y}$ ……

$\mathscr{Z}$ ……

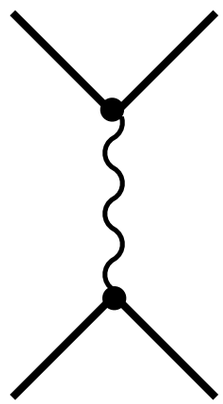
- b.s. : $0 \leq X \leq 1$
- virtual : $\mathscr{Z} < 0$ (non-interpretable)
- ● divergence $\mathscr{X}, \mathscr{Y}, \mathscr{Z} \to \infty$
- resonance : $\mathscr{Z}$ dominant



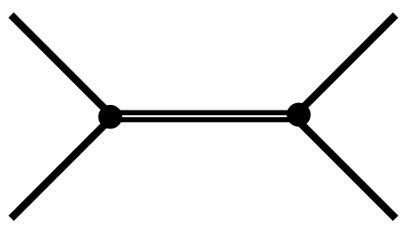
Coulomb+short-range model

◉ model

Coulomb



short-range (s.r.)

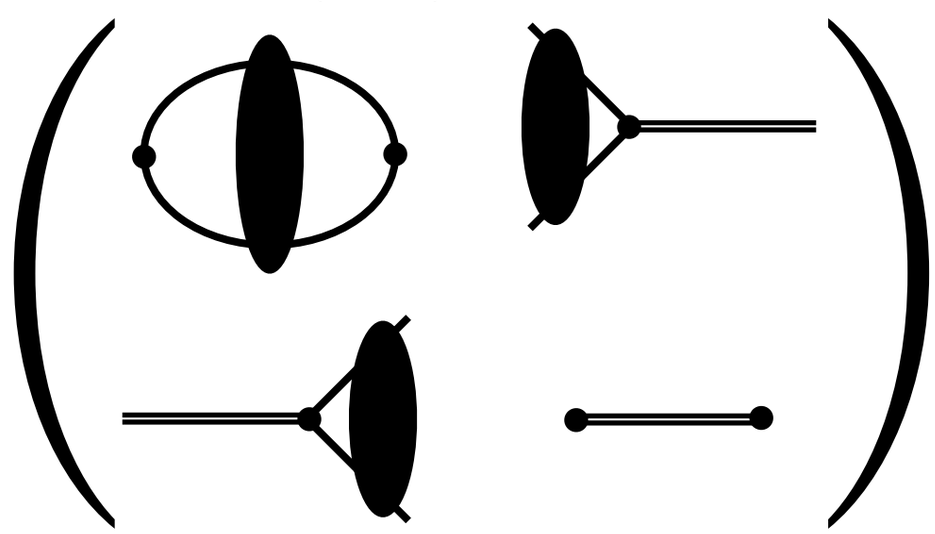


S. Weinberg, Phys. Rev. 137, 672–678 (1965);
V. Baru, J. Haidenbauer, C. Hanhart, Y. Kalashnikova, A.E. Kudryavtsev, Phys. Lett. B 586, 53–61 (2004);
T. Hyodo, Phys. Rev. C **90**, 055208 (2014) .

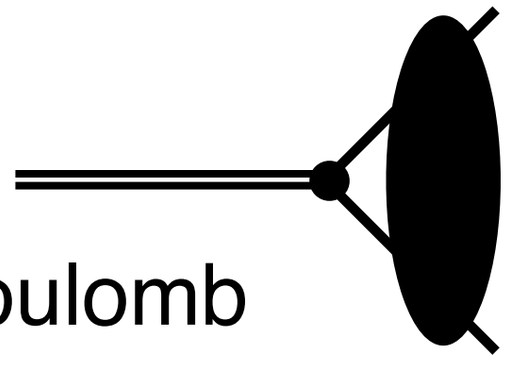
◉ Hamiltonian

W. Domcke, Atom. Mol. Phys. 16 359 (1983);
H. Feshbach, Annals Phys. 19 287-313 (1962).
C. H. Schmickler, H.-W. Hammer, and A.G. Volosniev, Physics Letters B 798, 135016 (2019).

$$\hat{H} = \begin{pmatrix} \hat{H}_{PP} & \hat{H}_{PQ} \\ \hat{H}_{QP} & \hat{H}_{QQ} \end{pmatrix} =$$



Q channel
⇔ b.s. w/o Coulomb



P channel
⇔ scattering w/ Coulomb

Coulomb+short-range model

◉ Schrödinger equation $\hat{H}|\Psi\rangle = E|\Psi\rangle \quad |\Psi\rangle = \begin{pmatrix} |P\rangle \\ |Q\rangle \end{pmatrix}$

$$\hat{H}_{PP}|P\rangle + \hat{H}_{PQ}|Q\rangle = E|P\rangle$$
$$\hat{H}_{QQ}|Q\rangle + \hat{H}_{QP}|P\rangle = E|Q\rangle$$

◉ effective Hamiltonian (channel elimination)

$$\hat{H}_{P\mathrm{ch}}|P\rangle = E|P\rangle \quad \hat{H}_{P\mathrm{ch}} = \hat{H}_{PP} + \hat{H}_{PQ}(E - \hat{H}_{QQ})^{-1}\hat{H}_{QP}$$

Coulomb+short-range model

◉ Schrödinger equation $\hat{H}|\Psi\rangle = E|\Psi\rangle \quad |\Psi\rangle = \begin{pmatrix} |P\rangle \\ |Q\rangle \end{pmatrix}$

$$\hat{H}_{PP}|P\rangle + \hat{H}_{PQ}|Q\rangle = E|P\rangle$$
$$\hat{H}_{QQ}|Q\rangle + \hat{H}_{QP}|P\rangle = E|Q\rangle$$

◉ effective Hamiltonian (channel elimination)

$$\hat{H}_{P\mathrm{ch}}|P\rangle = E|P\rangle \quad \hat{H}_{P\mathrm{ch}} = \hat{H}_{PP} + \hat{H}_{PQ}(E - \hat{H}_{QQ})^{-1}\hat{H}_{QP}$$
$$= \hat{H}^0 + \hat{V}_P \qquad = \hat{V}_Q$$

$$\rightarrow \hat{H}_{P\mathrm{ch}} = \hat{H}^0 + (\hat{V}_P + \hat{V}_Q)$$

$\hat{H}^0$: free Hamiltonian $\hat{V}_P$: pure Coulomb interaction

$\hat{V}_Q$: short-range interaction

Coulomb+short-range model

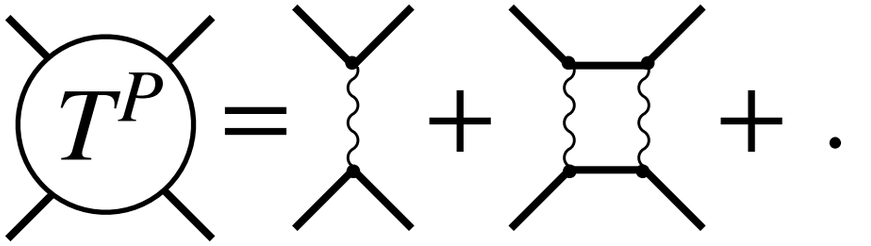
◉ T-matrix

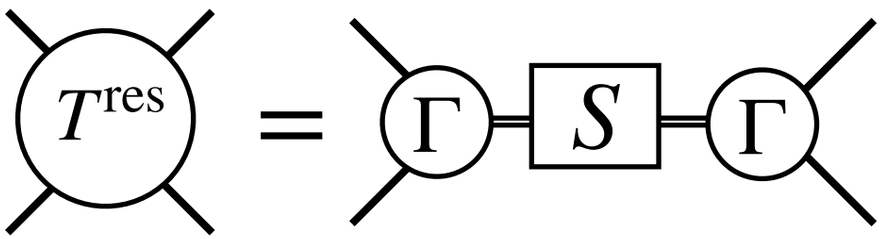
H. Feshbach, Annals Phys. 19 287-313 (1962); R. Higa, H.-W. Hammer, and U. van W. Domcke, Atom. Mol. Phys. 16 359 (1983); Kolck, Nuclear Physics A 809 (2008).

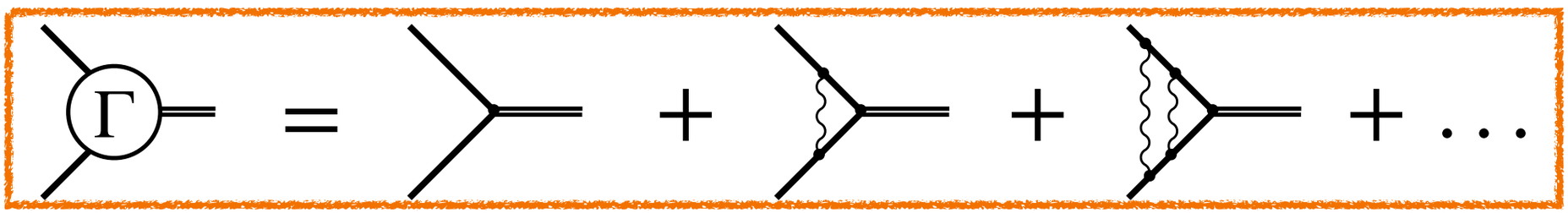
Lippmann-Schwinger eq. : $\hat{T} = [(\hat{V}_P + \hat{V}_Q)^{-1} - \hat{G}^0]^{-1}$

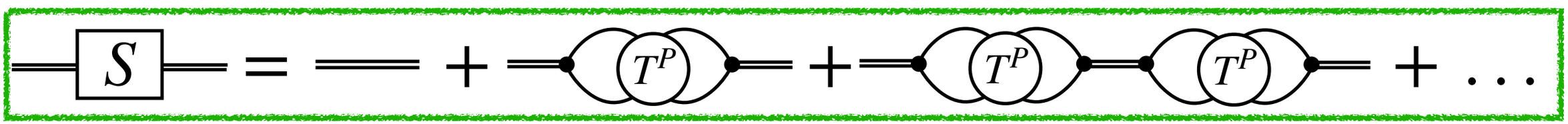
$\Leftrightarrow$ Feshbach method : $\hat{T} = \hat{T}^P + \hat{T}^{\rm res}$

$$\hat{T}^P = [\hat{V}_P^{-1} - \hat{G}^0]^{-1},\ \hat{T}^{\rm res} = \hat{T}^P \hat{V}_P^{-1} [\hat{V}_Q^{-1} - \hat{G}_P]^{-1} \hat{V}_P^{-1} \hat{T}^P$$





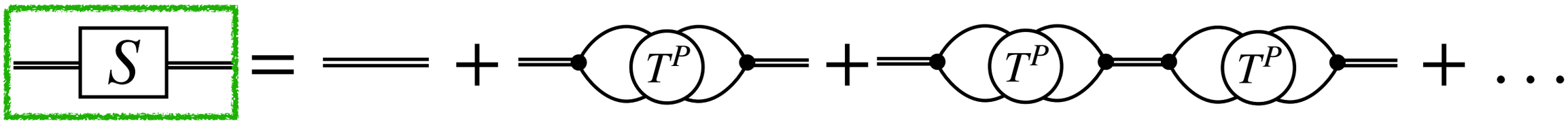


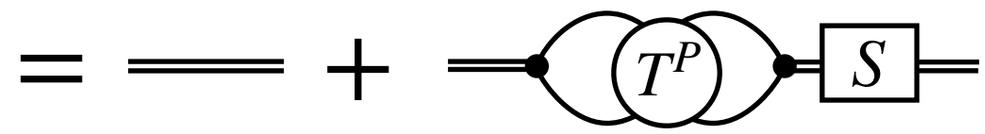


Coulomb+short-range model

◉ pole condition $\hat{T} = \hat{T}^P + \hat{T}^{\rm res}$

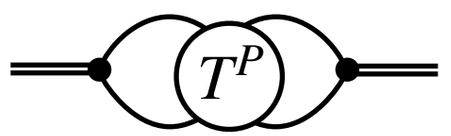
pole of $T(k,k')$ $\Leftrightarrow$ pole of $T^{\rm res}(k,k') \Leftrightarrow \boxed{[V_Q^{-1} - G_P]^{-1}} = \infty$





$H_{QP}G_PH_{PQ}$



$= S(E)$,  $= (E - \varepsilon_d)^{-1}$, 

$= \Sigma(E)$

$\longrightarrow S(E) = (E - \varepsilon_d)^{-1} + (E - \varepsilon_d)^{-1}\Sigma(E)S(E)$

$= [E - \varepsilon_d - \Sigma(E)]^{-1}$

$\longrightarrow$ pole condition : $E - \boxed{\varepsilon_d} - \boxed{\Sigma(E)} = 0$

bare state energy　　self energy

Coulomb+short-range model

◉ self energy $\Sigma(E)$ in low-energy limit

W. Domcke, Atom. Mol. Phys. 16 359 (1983).

- attractive Coulomb

$$\Sigma(k) = \frac{A}{2\pi}\left[c - \frac{1}{2}ia_B k + \log(-ia_B k) + \psi\left(1 - \frac{i}{a_B k}\right)\right]$$

- repulsive Coulomb

$$\Sigma(k) = -\frac{A}{2\pi}\left[c + \frac{1}{2}ia_B k + \log(-ia_B k) + \psi\left(1 + \frac{i}{a_B k}\right)\right]$$

A : constant with dimension of energy

c : dimensionless constant

$\psi(x) = \dfrac{d}{dx}\log(\Gamma(x))$: digamma function

Coulomb+short-range model

◉ short-range limit $a_B \to \infty$

$$a_B = \frac{\hbar c}{\mu c^2 Z_1 Z_2}$$

$$-\frac{1}{a_s} + \frac{r_e}{2}k^2 - ik \pm \frac{2}{a_B}\left[\log(-ia_B k) + \psi\left(1 + \frac{i}{a_B k}\right)\right] = 0$$

$$\to 0$$

$$\longrightarrow -\frac{1}{a_s} + \frac{r_e}{2}k^2 - ik = 0 \quad \text{short-range interaction}$$

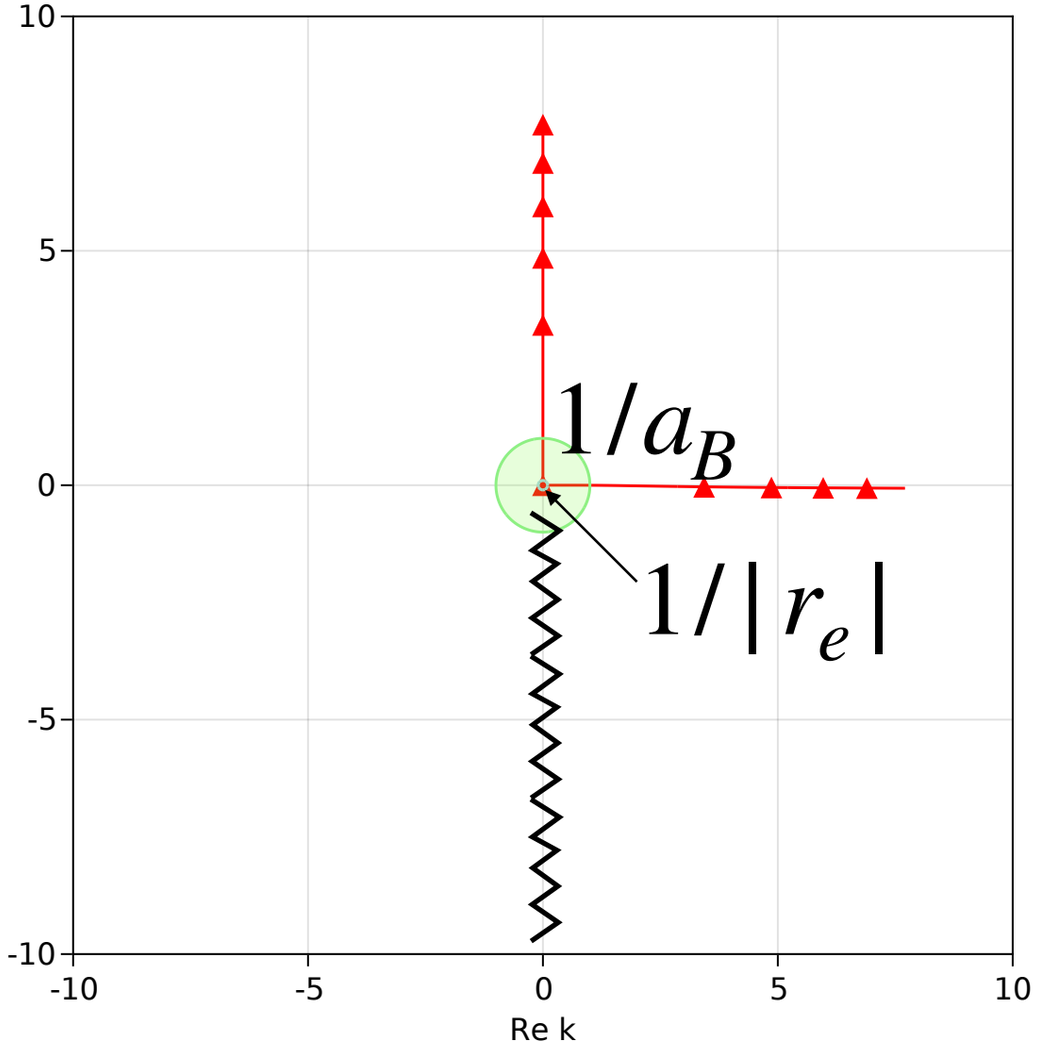
◉ further low-energy limit $r_e \to 0$

- zero-range theory S. Mochizuki, and Y. Nishida, Phys. Rev. C 110 , 064001 (2024).

$$\frac{ia_B k}{2} \mp \log(-ia_B k) + \psi\left(1 + \frac{i}{a_B k}\right) + \frac{a_B}{2a_s} = 0$$

Pole trajectory (repulsive Coulomb)

$a_B = 1, r_e = -10$

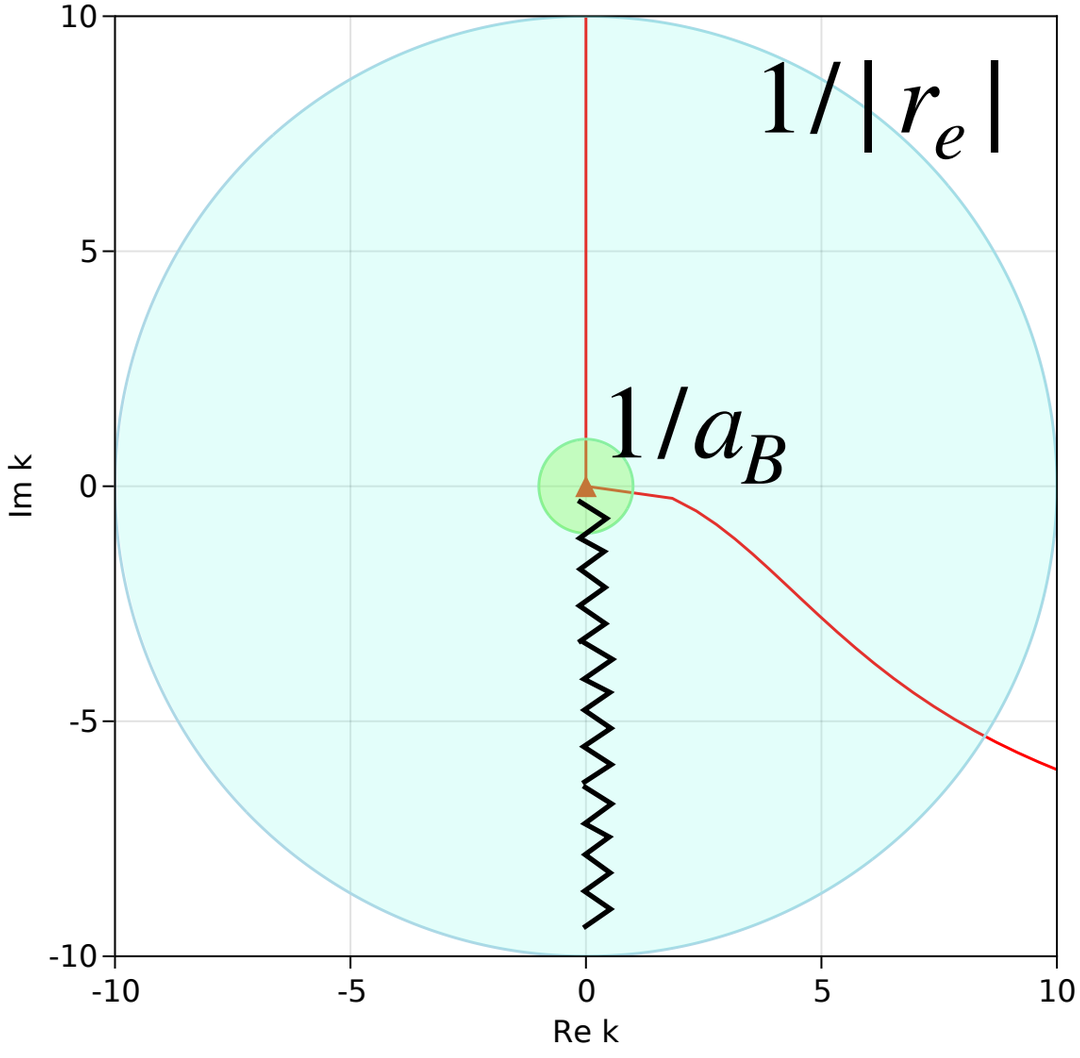


short-range universality

∧

Coulomb dominant region

$a_B = 1, r_e = -0.1$



short-range universality

∨

Coulomb dominant region

Application to near-th. states

	System	a_s^C (fm)	r_e^C (fm)	$\lvert r_e^C/a_B\rvert$
repulsive	pp	-7.82	2.83	0.049
	$^8\mathrm{Be}$ $(\alpha\alpha)$	-1.80×10^3	1.083	0.30
	$\Omega^-\Omega^-$	12.93	1.21	0.038
	$\Omega_{ccc}^{++}\Omega_{ccc}^{++}$	-19	0.45	0.16
attractive	$\Xi^-\alpha$	-12.6	6.66	0.48
	$\Omega^- p$	3.2	1.2	0.029

pp X. Kong and F. Ravndal, Phys. Lett. B 450, 320 (1999)

^{8}Be R. Higa, H. W. Hammer, and U. van Kolck, Nucl. Phys. A 809, 171 (2008)

$\Omega\Omega$ and Ωp Y. Lyu, private communication

$\Omega_{ccc}\Omega_{ccc}$ Y. Lyu, H. Tong, T. Sugiura, S. Aoki, T. Doi, T. Hatsuda, J. Meng, and T. Miyamoto, Phys. Rev. Lett. 127, 072003 (2021)

$\Xi\alpha$ Y. Kamiya, private communication

Application to near-th. states

	System	Eigenenergy (MeV)	Property	X	X_C
	pp	$-0.14-0.47i$	V w/ width	$0.85-0.13i$	0.81
	^{8}Be $(\alpha\alpha)$	$0.083-8.3\times10^{-7}i$	R	$7.9-0.0013i$	0.71
rep	$\Omega^-\Omega^-$	-0.85	B	1.5	0.81
	$\Omega_{ccc}^{++}\Omega_{ccc}^{++}$	$1.3-0.17i$	R	$1.5-0.2i$	0.79
.....					
	$\Xi^-\alpha$	-0.22	B	1.1	1.0
att	Ω^-p	-1.07	B	1.2	0.87

- a_s, r_e and a_B ⟶ eigenenergy using ERE
- pp: virtual with width ⟵ nn virtual state
- ^{8}Be: cluster dominant structure ⟵ experiment & few-body calculations

R. B. Wiringa, S. C. Pieper, J. Carlson, and V. R. Pandharipande, Phys. Rev. C 62, 014001 (2000).
T. Otsuka, T. Abe, T. Yoshida, Y. Tsunoda, N. Shimizu, N. Itagaki, Y. Utsuno, J. Vary, P. Maris, and H. Ueno, Nature Commun. 13, 2234 (2022).

- $\Omega\Omega$ and $\Omega_{ccc}\Omega_{ccc}$: consistent with lattice calculation
- $\Xi\alpha$: purely composite - Ωp: consistent with lattice calc.