Recent advances in algorithms and applications of nuclear lattice EFT

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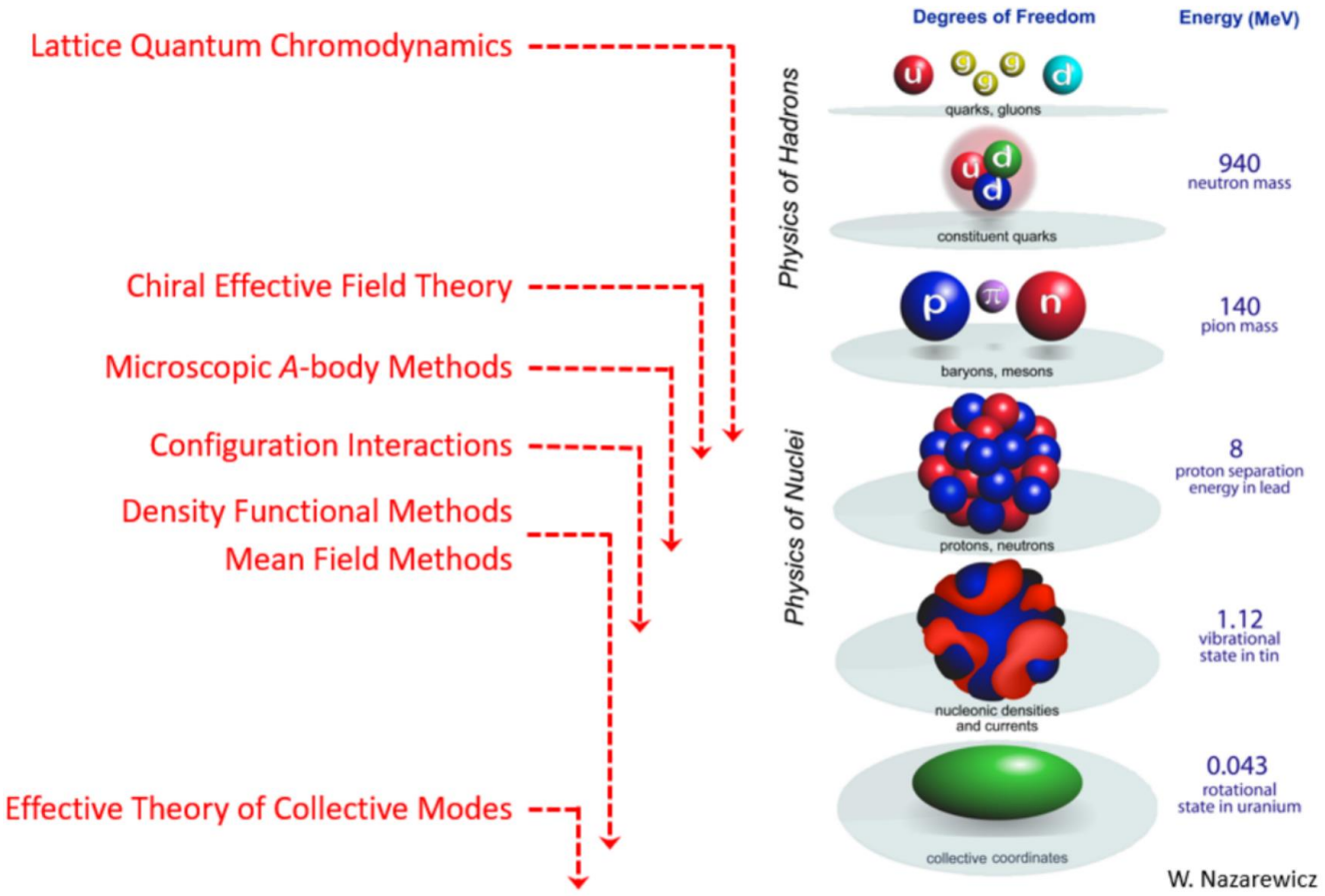
Workshop on Few-Body Problems in Nuclear and Hadron Physics

Sep 21, 2026, ITP-CAS, Beijing

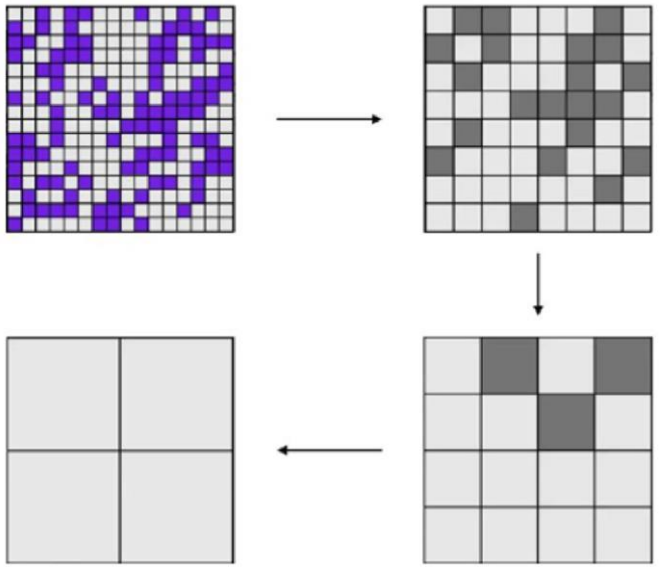
Contents

- **Introduction to Nuclear Lattice Effective Field Theory**
- Non-perturbative QMC and sign problem
- Few-body applications
- Renormalization of NLEFT
- Summary and perspective

What is a nuclear EFT?



- **Modern nuclear force constructions are based on the Effective Field Theory**
- **Theoretical foundation of EFT is the Wilsonian renormalization group**:
  - **High-momentum** details can be integrated out & hidden in LECs
  - **Low-momentum** physics kept invariant under RG transformations

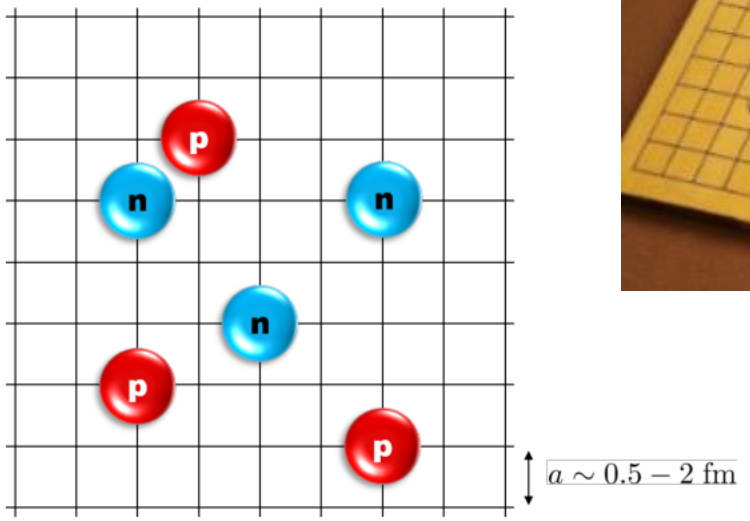


Lattice EFT: A many-body EFT solver

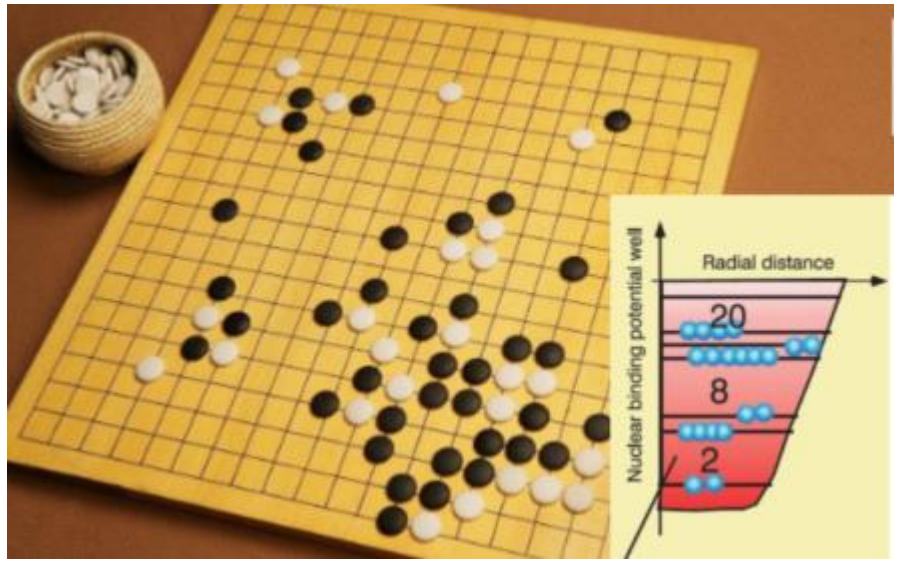
Lattice EFT = Chiral EFT + Lattice + Monte Carlo

Review: Dean Lee, Prog. Part. Nucl. Phys. 63, 117 (2009),
Lähde, Meißner, "Nuclear Lattice Effective Field Theory", Springer (2019)

- Discretized chiral nuclear force
- Lattice spacing $a \approx 1$ fm $= 620$ MeV (∼chiral symmetry breaking scale)
- Protons & neutrons interacting via **short-range, δ-like** and **long-range, pion-exchange** interactions
- Exact method, polynomial scaling ($\sim A^2$)



Lattice adapted for nucleus



- Solve the non-perturbative nuclear many-body problem by sampling all configurations

Many-body correlations emerge naturally without a reference mean-field solution

Lattice EFT: A many-body EFT solver

- Get *interacting g. s.* from imaginary time projection:

$$|\Psi_{g.s.}\rangle \propto \lim_{\tau\to\infty} \exp(-\tau H)|\Psi_A\rangle$$

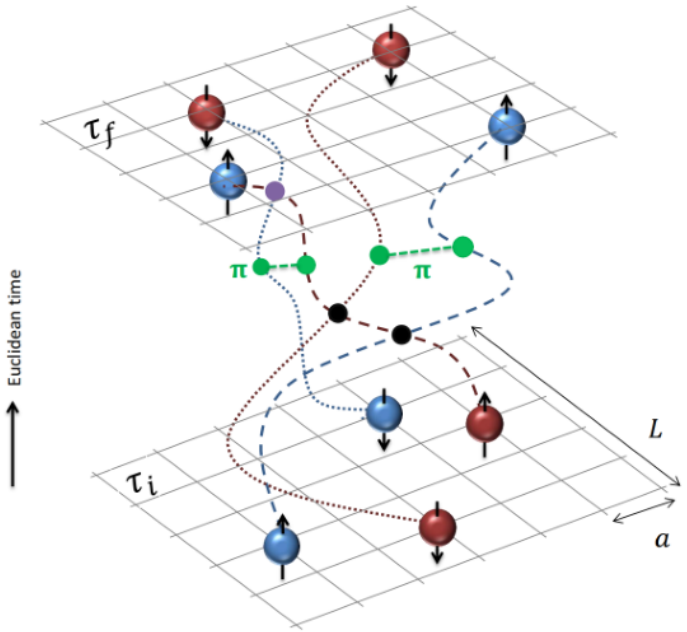
with $|\Psi_A\rangle$ representing A *free* nucleons.

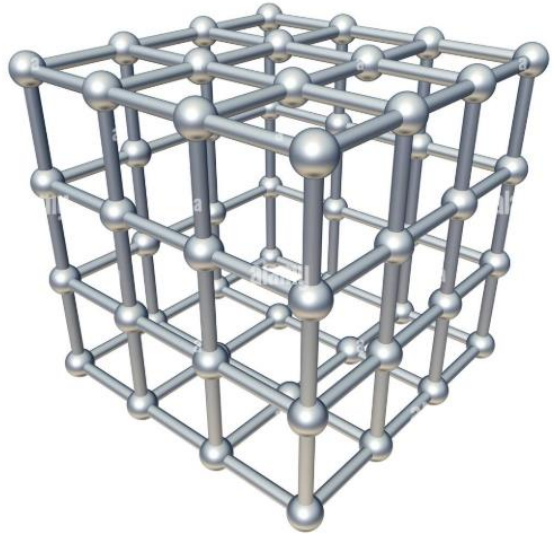
- Expectation value of any operator $\mathscr{O}$:

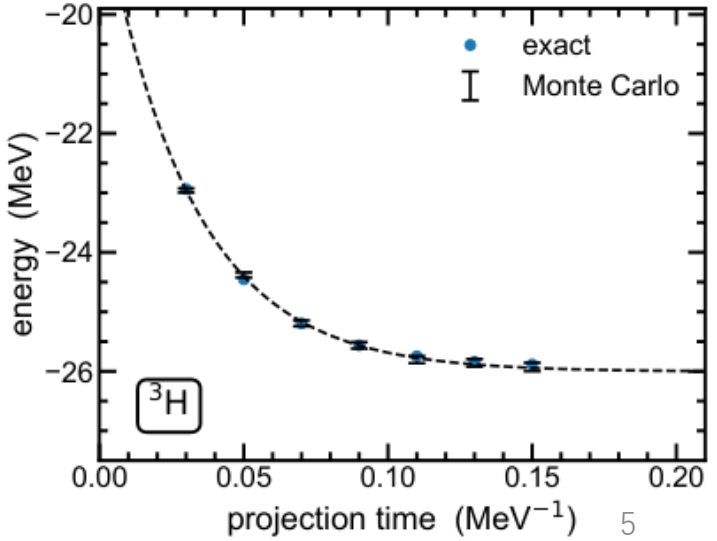
$$\langle O\rangle = \lim_{\tau\to\infty} \frac{\langle\Psi_A|\exp(-\tau H/2)\mathscr{O}\exp(-\tau H/2)|\Psi_A\rangle}{\langle\Psi_A|\exp(-\tau H)|\Psi_A\rangle}$$

- τ is discretized into time slices:

$$\exp(-\tau H) \simeq \left[:\exp(-\frac{\tau}{L_t}H):\right]^{L_t}$$







All possible configurations in $\tau \in [\tau_i, \quad \tau_f]$ are sampled.
Complex structures like nucleon clustering emerges naturally.

Lattice EFT: A many-body EFT solver

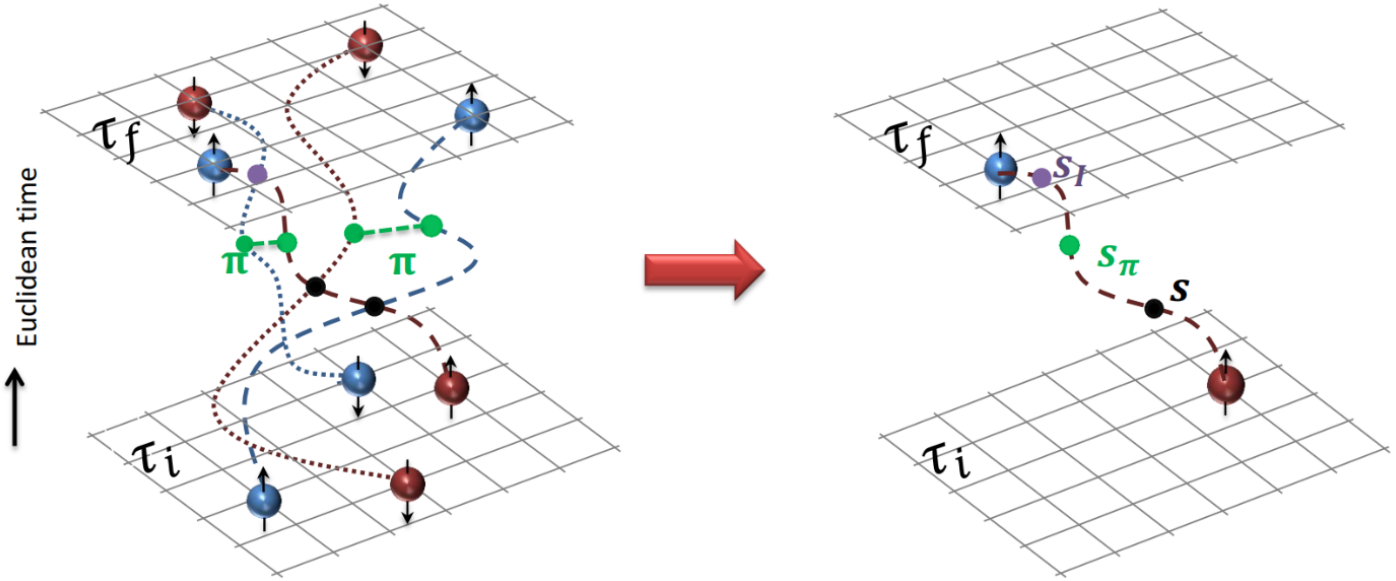
- Quantum correlations between nucleons are represented by fluctuations of the auxiliary fields.

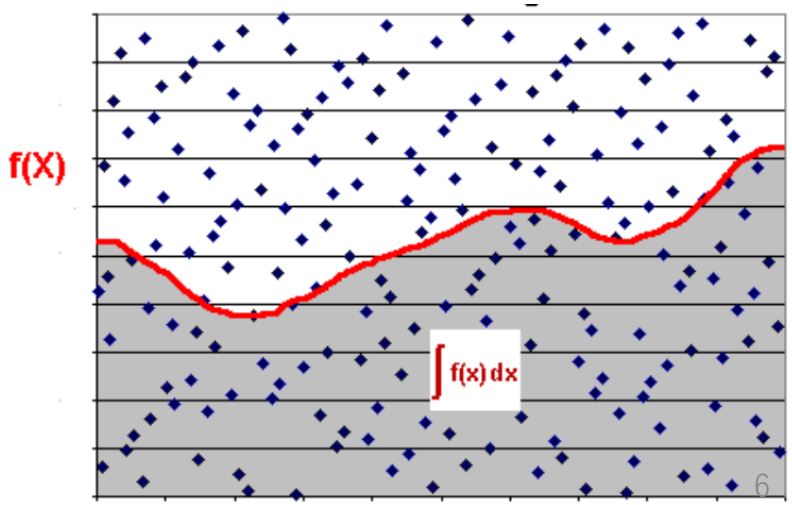
$$:\exp\left[-\frac{a_t C}{2}(\psi^\dagger\psi)^2\right]:=\frac{1}{\sqrt{2\pi}}\int ds:\exp\left[-\frac{s^2}{2}+\sqrt{-a_t C}s(\psi^\dagger\psi)\right]:$$

- Long-range interactions such as OPEP or more complex interactions can be represented similarly.
- For fixed aux. fields, product of s.p. states (e.g., Slater determinant) keep the form of product of s.p. states in propagations. ⟸**No *N*-*N* interaction**

Solve a general Hamiltonian with 5 steps:

1. Rewrite expression into a path integral
2. For each field configuration, calculate the amplitude.
3. Integrate out the field variables using Monte Carlo algorithms.
4. Take the limit $\tau \to \infty$
5. Take the limit $L \to \infty$





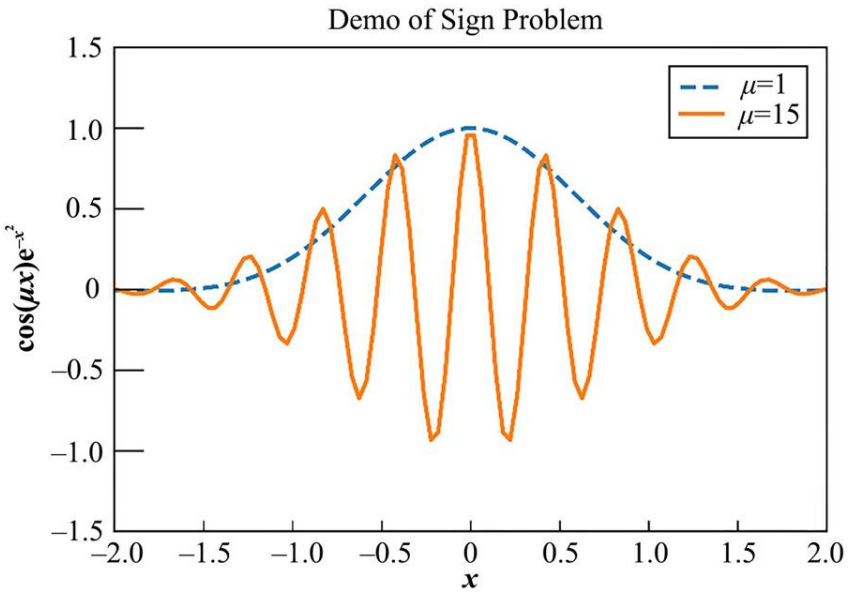
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Sign problem in quantum Monte Carlo

$$\frac{\int f(x)dx}{\int g(x)dx} = \frac{\int |g(x)| \frac{f(x)}{|g(x)|} dx}{\int |g(x)| \frac{g(x)}{|g(x)|} dx} = \frac{\left\langle \frac{f(x)}{|g(x)|} \right\rangle_{p(x)\propto|g(x)|}}{\left\langle \frac{g(x)}{|g(x)|} \right\rangle_{p(x)\propto|g(x)|}}$$

- **Sign** problem arises for **repulsive interactions** or **unbalanced fermions**
  - Coulomb interaction
  - Hard core in NN potentials
  - Tensor forces
  - Three-body forces
  - Unpaired pinholes
  - …
- Very few interactions and systems can be simulated directly
  - Large statistical errors
  - Signal-noise issue
  - Bad scaling → errors increase exponentially with system size
  - No universal solution → NP-hard problem M. Troyer and U.-J. Wiese, PRL94, 170201 (2005)

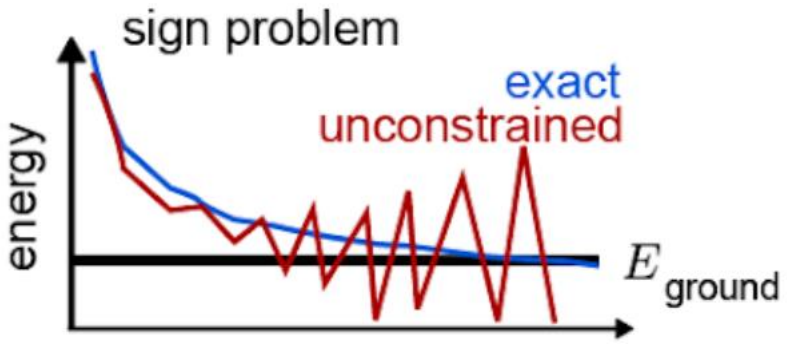


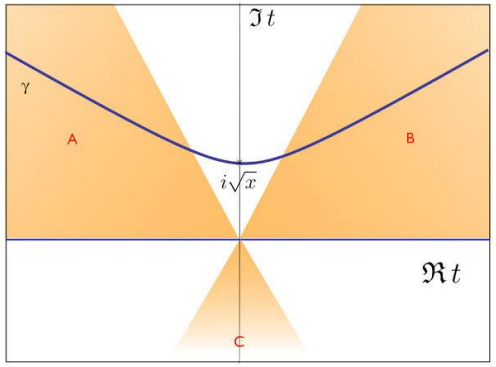


Sign problem in quantum Monte Carlo

- **Quantum Monte Carlo** transform the **quantum many-body problems** into **high-dimensional integrals**
- QMC Statistical error ~$O(N^{-1/2})$ ← **Polynomial scaling!**

- **Sign problem** occurs when the integrand is **NOT** positive definite
- **Sign problem scales exponentially** with particle number
  - Solutions: Constrained-path, fixed-nodes, Lefschetz thimble, etc.
  - Rely on approximations or induces systematic biases
    ← **Not exact even with infinite number of samples!**





No generic solution

Examples of sign-problem-free QMC

Sign-problem-free algorithms exist but mostly confined to specific models

- **Lattice QCD** with two identical quark species

$$Z = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}[U]e^{-S_G[U]}\det[D[U]], \qquad \det[D[U]] = \det[D_u[U]]\det[D_d[U]] = \det[D_u[U]]^2 > 0$$

- **Nuclear Lattice EFT** with Wigner-SU(4) interactions (even-even nuclei)

$$Z = \int \mathcal{D}s e^{-s^2/2}\det[Z(s)], \qquad \det[Z(s)] = \det[Z_\uparrow(s)]\det[Z_\downarrow(s)] = \det[Z_\uparrow(s)]^2 \geq 0$$

- Repulsive Fermi-Hubbard model at half-filling
- Kane-Mele-Hubbard model
- Half-filled Kondo lattice model

Positivity of the fermionic determinant is protected by the time-reversal symmetry

Sufficient condition for absence of the sign problem in the fermionic quantum Monte Carlo algorithm, C.-J. Wu and S.-C. Zhang, PRB71, 155115 (2005)

Sign-problem-free fermionic quantum Monte Carlo: Developments and Applications, Z.-X. Li and H. Yao, Annu. Rev. Condens. Matter Phys. 10, 337 (2019)

“Essential element” Hamiltonian as zeroth order

- **Wigner-SU(4) symmetry** + **paired fermions**
  = Immunity to sign problem
  Wigner, Phys. Rev. 51, 106 (1937)

- Also inspired by the **unexpected effectiveness** of the **locality** in nuclear forces
  Elhatisari et al., PRL 117, 132501 (2016)

$$H_{\mathrm{SU}(4)} = H_{\mathrm{free}} + \frac{1}{2!}C_2 \sum_{\boldsymbol{n}} \tilde{\rho}(\boldsymbol{n})^2 + \frac{1}{3!}C_3 \sum_{\boldsymbol{n}} \tilde{\rho}(\boldsymbol{n})^3$$

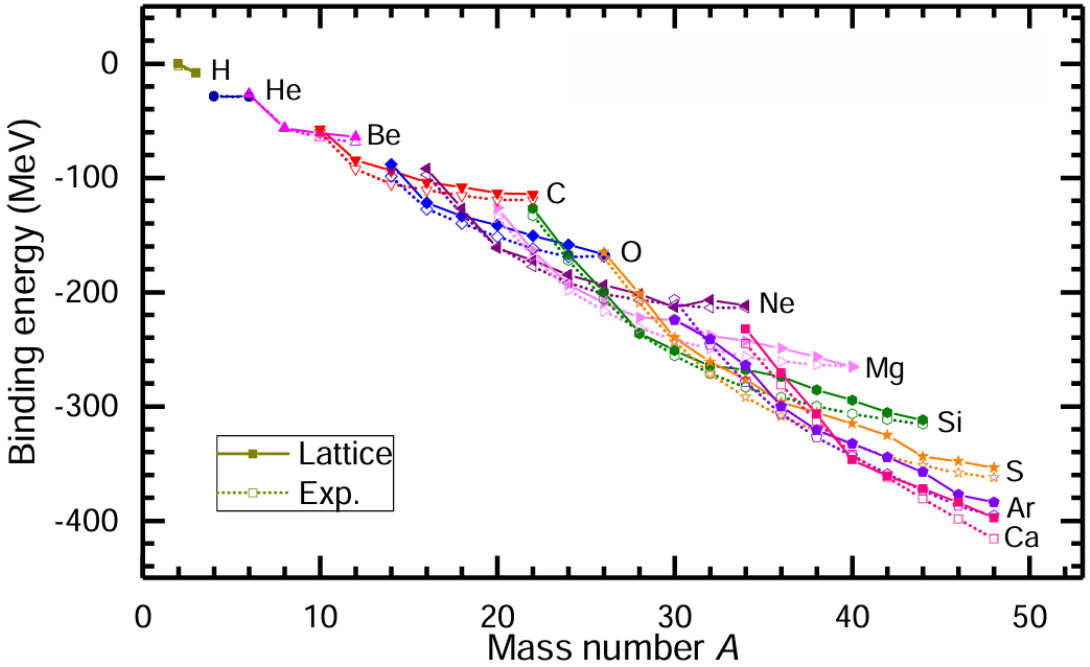
$$\tilde{\rho}(\boldsymbol{n}) = \sum_i \tilde{a}_i^\dagger(\boldsymbol{n})\tilde{a}_i(\boldsymbol{n}) + s_L \sum_{|\boldsymbol{n}'-\boldsymbol{n}|=1} \sum_i \tilde{a}_i^\dagger(\boldsymbol{n}')\tilde{a}_i(\boldsymbol{n}')$$

$$\tilde{a}_i(\boldsymbol{n}) = a_i(\boldsymbol{n}) + s_{NL} \sum_{|\boldsymbol{n}'-\boldsymbol{n}|=1} a_i(\boldsymbol{n}')$$

$C_{\mathrm{SU4}} = -3.41 \times 10^{-7}$ MeV^{-2}, $s_L = 0.061$ and $s_{NL} = 0.5$

BNL et al., PLB 797, 134863 (2019)

$$\det(Z) = \det(Z_\uparrow)^2 \geq 0$$



Applicable in nuclear spectroscopy, clustering, thermodynamics, electromagnetic transition, ⋯

pros: non-perturbative, small statistical errors
cons: low-fidelity, large theory uncertainties

Improved Wigner-SU(4) interaction

PHYSICAL REVIEW LETTERS **135,** 222504 (2025)

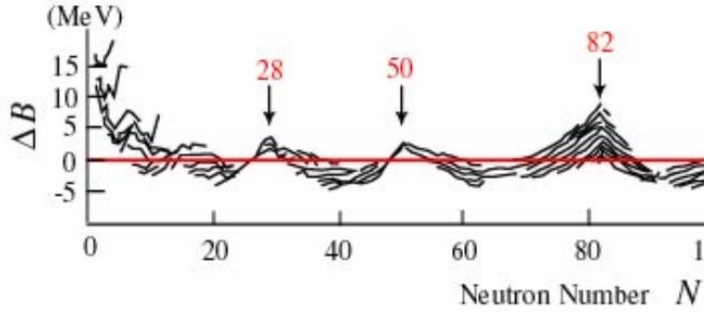
Editors' Suggestion

Sign-Problem-Free Nuclear Quantum Monte Carlo Simulation

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(Received 1 July 2025; revised 28 September 2025; accepted 29 October 2025; published 26 November 2025)



- **Wigner-SU(4) symmetry** is an approximate symmetry
  -Lack correct **shell structure** → Spin-orbit coupling
- We introduce a **spin-orbit coupling**

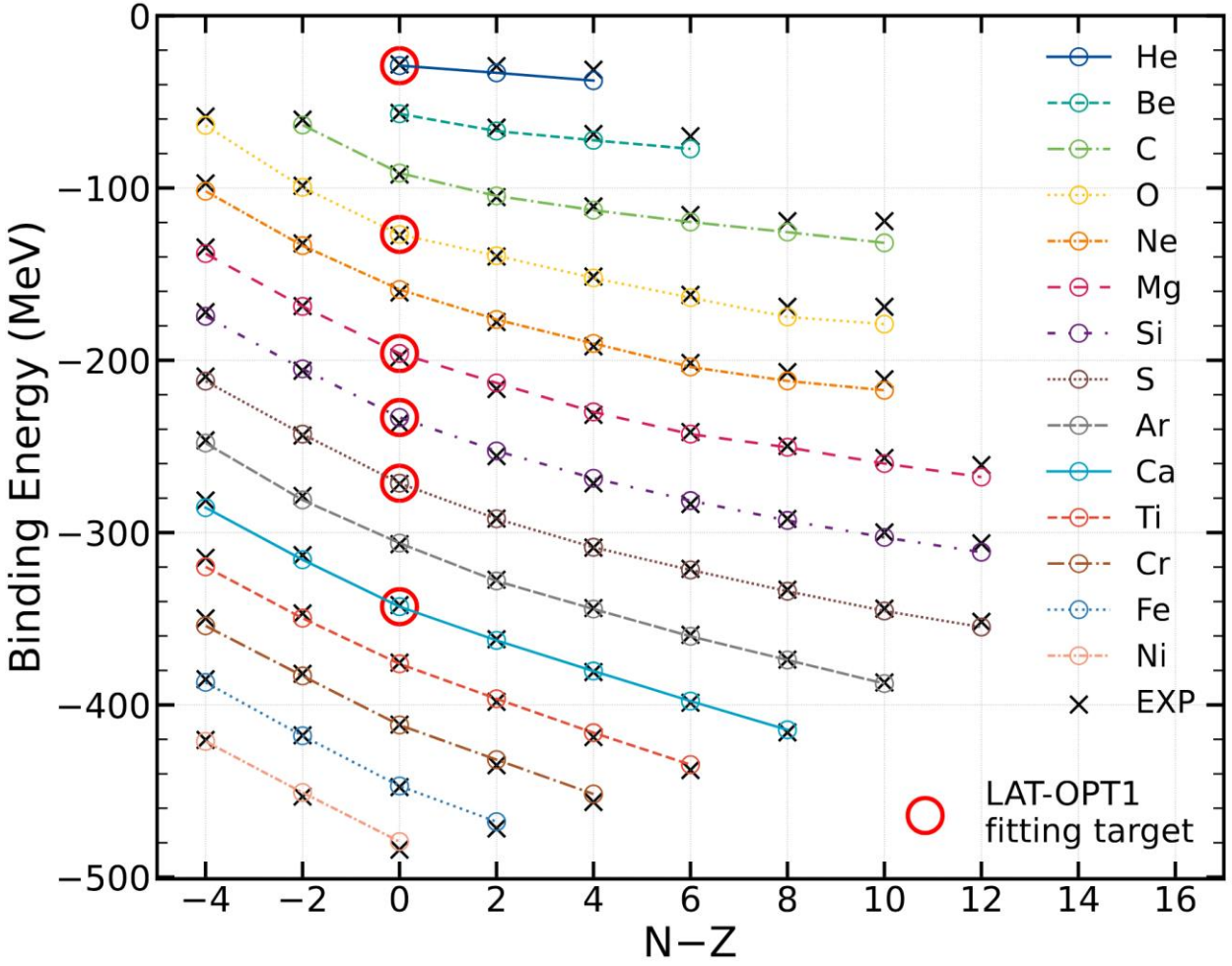
$$H = \sum_{\boldsymbol{n}} \left[-\frac{\Psi^\dagger \nabla^2 \Psi}{2M} + \frac{C_2}{2}\bar{\rho}^2 + \frac{C_3}{6}\bar{\rho}^3 + C_s \bar{\rho}\,\bar{\rho}_s \right]$$

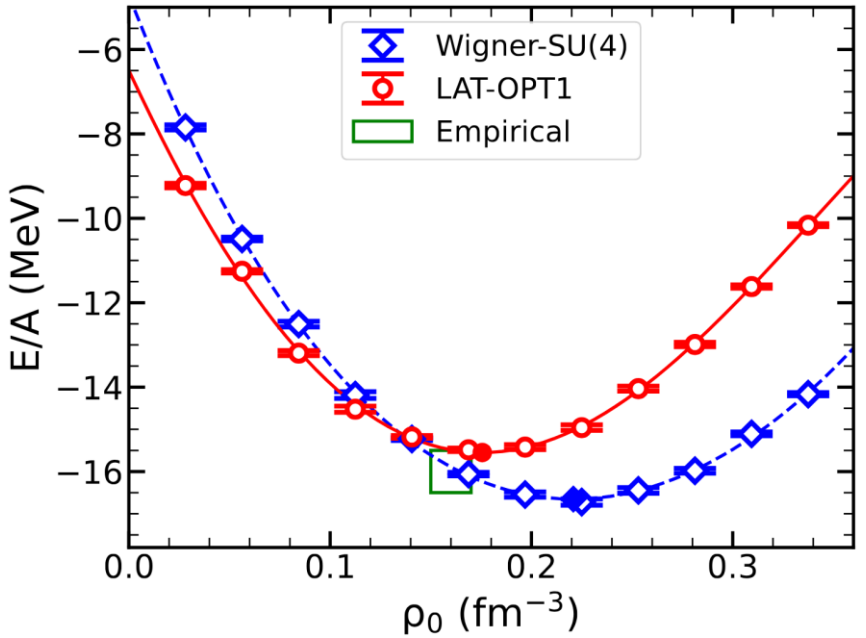
$$\rho = \bar{a}^\dagger \bar{a},$$

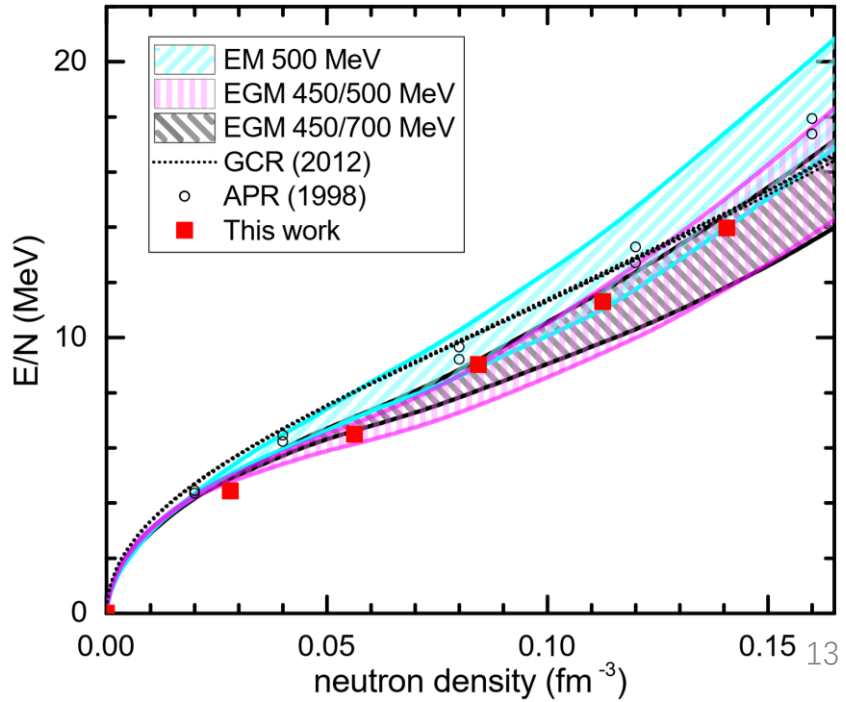
$$\rho_s = \sum_{ijk} \epsilon_{ijk} \nabla_i \left[\bar{a}^\dagger (\overrightarrow{\nabla}_j - \overleftarrow{\nabla}_j) \sigma_k \bar{a} \right]$$

- **Spin-orbit term** act equivalently for **spin-up** and **spin-down** particles
  → **Fermion determinant keeps positive definite**

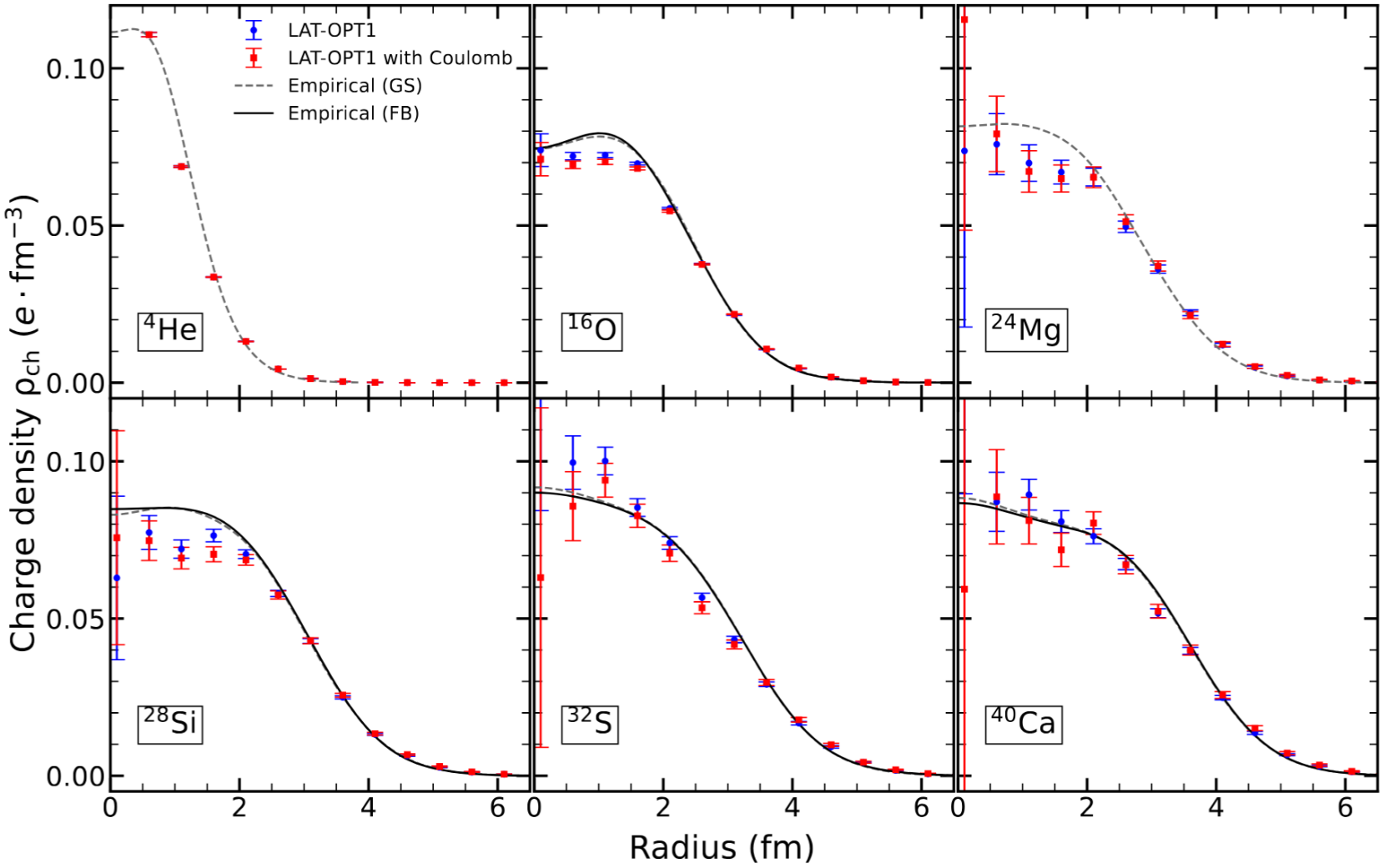
Results from LAT-OPT1





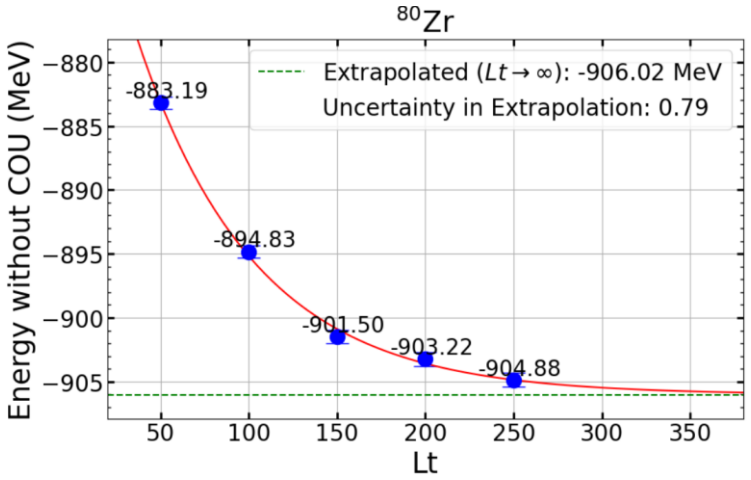


Results from LAT-OPT1: Charge densities



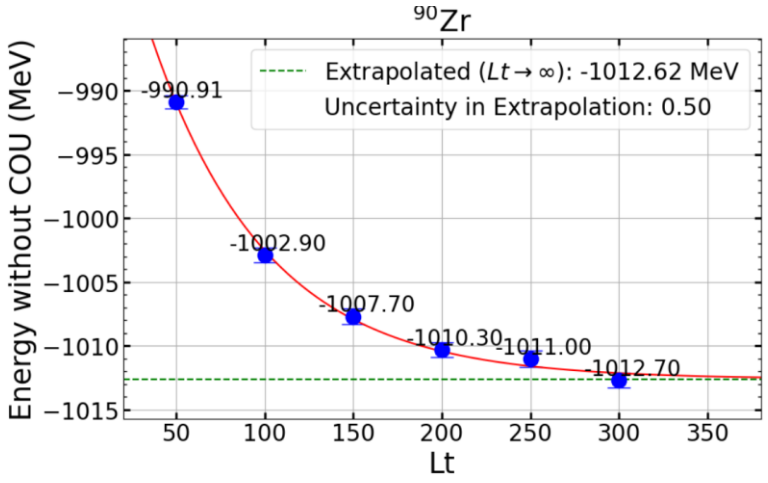
- Charge densities calculated using **pinhole algorithm**
  Elhatisari et al., PRL119, 222505 (2017)
- **Pinhole algorithm** induce **mild sign problem**, not available for A>=40. See **partial pinhole algorithm** for a solution
  Zheng-Xue Ren et al., PRL135-152502 (2025)

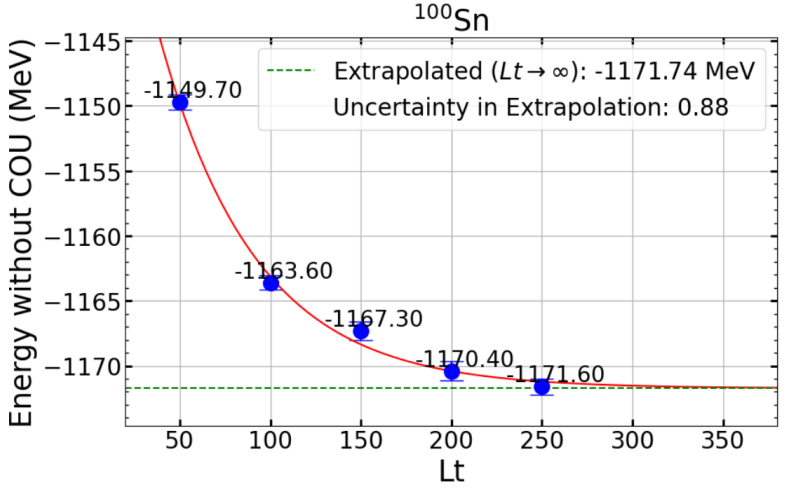
Results from LAT-OPT1: Heavy nuclei

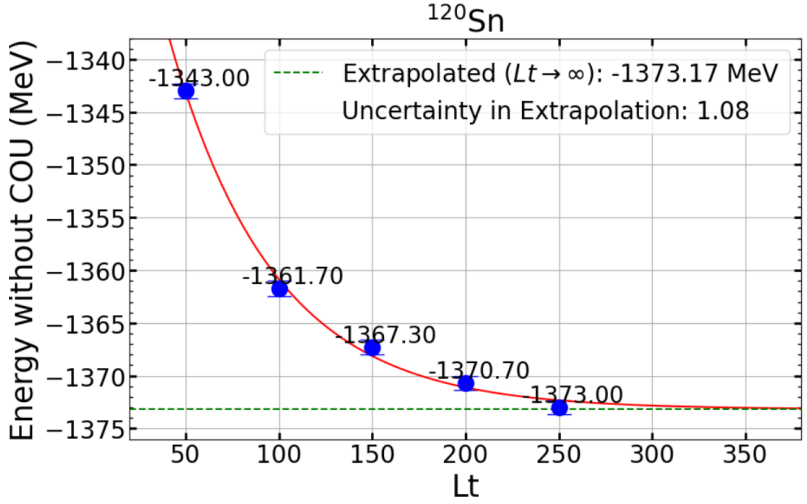


Nucleus	$E_{\rm bind}$ (MeV)	$E_{\rm sl}$ (MeV)	$E_{\rm sl}/E_{\rm bind}$	EXP (MeV)
^{4}He	$-29.0(2)$	-0.3	0.010	-28.3
^{12}C	$-91.3(1)$	-13.3	0.146	-92.2
^{14}C	$-104.6(1)$	-12.7	0.121	-105.3
^{16}O	$-126.9(2)$	-5.6	0.044	-127.6
^{40}Ca	$-343.0(2)$	-13.6	0.040	-342.1
^{48}Ca	$-414.5(3)$	-42.3	0.102	-416.0
^{56}Ni	$-479.3(6)$	-74.6	0.156	-484.0
^{80}Zr	$-672.1(8)$	-23.3	0.035	-669.2
^{90}Zr	$-782.1(5)$	-64.8	0.083	-783.9
^{100}Sn	$-824.7(8)$	-103.0	0.125	-825.2
^{132}Sn	$-1134.2(27)$	-110.9	0.098	-1102.8

- Spin-orbit energy enhanced for **new magic numbers** 28, 50, 82, etc., indicating **shell structure emergence**
- Remarkable **generalization capability** ← all **quantum correlation** included







Exactly solvable phenomenological nuclear force

- ***Ab initio* calculations**
  - Realistic interactions fitted to few-body data
  - Challenging to solve, particularly for heavy nuclei
- **Phenomenological methods**
  - Phenomenological interactions fitted to finite nuclei
  - Easy to solve, however lack correlations
- **Sign-problem-free QMC**
  - Phenomenological interactions fitted to finite nuclei
  - Scalable and unbiased, full quantum correlations ← **spectrum, reaction, clustering, …**

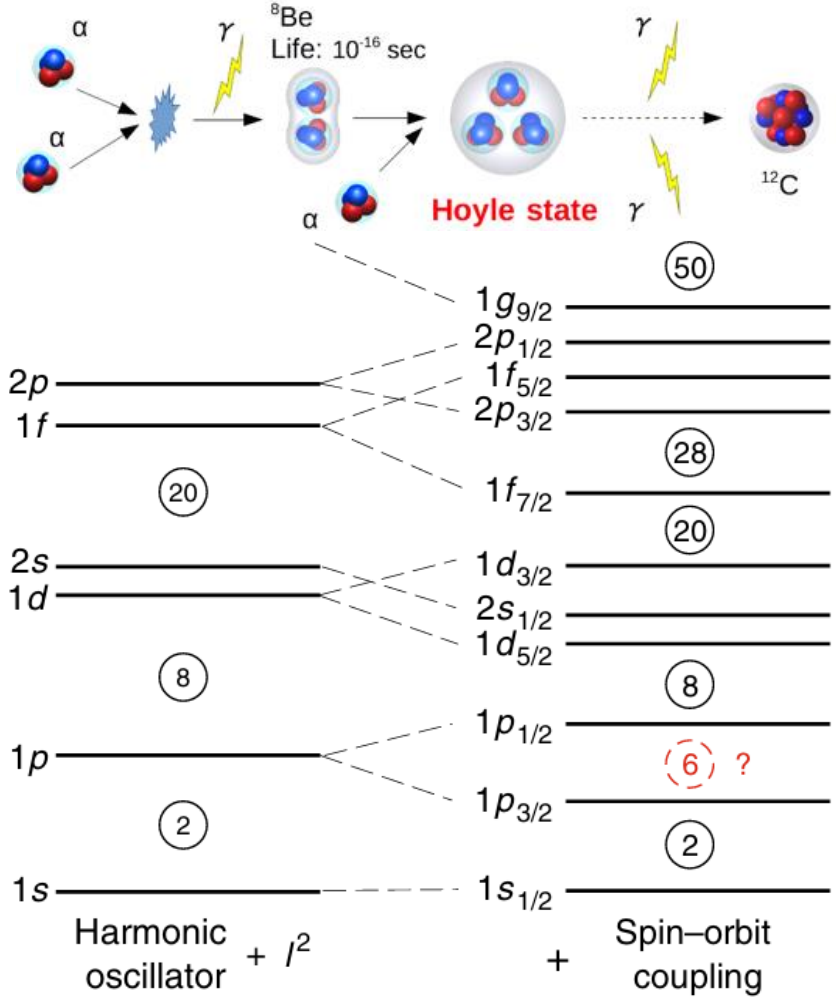
$$V(1,2) = t_0(1 + x_0 P^\sigma)\delta(\mathbf{r}_1 - \mathbf{r}_2) + \tfrac{1}{2}t_1\left[\delta(\mathbf{r}_1 - \mathbf{r}_2)\mathbf{k}^2 + \mathbf{k}^2\delta(\mathbf{r}_1 - \mathbf{r}_2)\right] + t_2\mathbf{k}\delta(\mathbf{r}_1 - \mathbf{r}_2)\mathbf{k} + iW_0(\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)})\mathbf{k} \times \delta(\mathbf{r}_1 - \mathbf{r}_2)\mathbf{k},$$

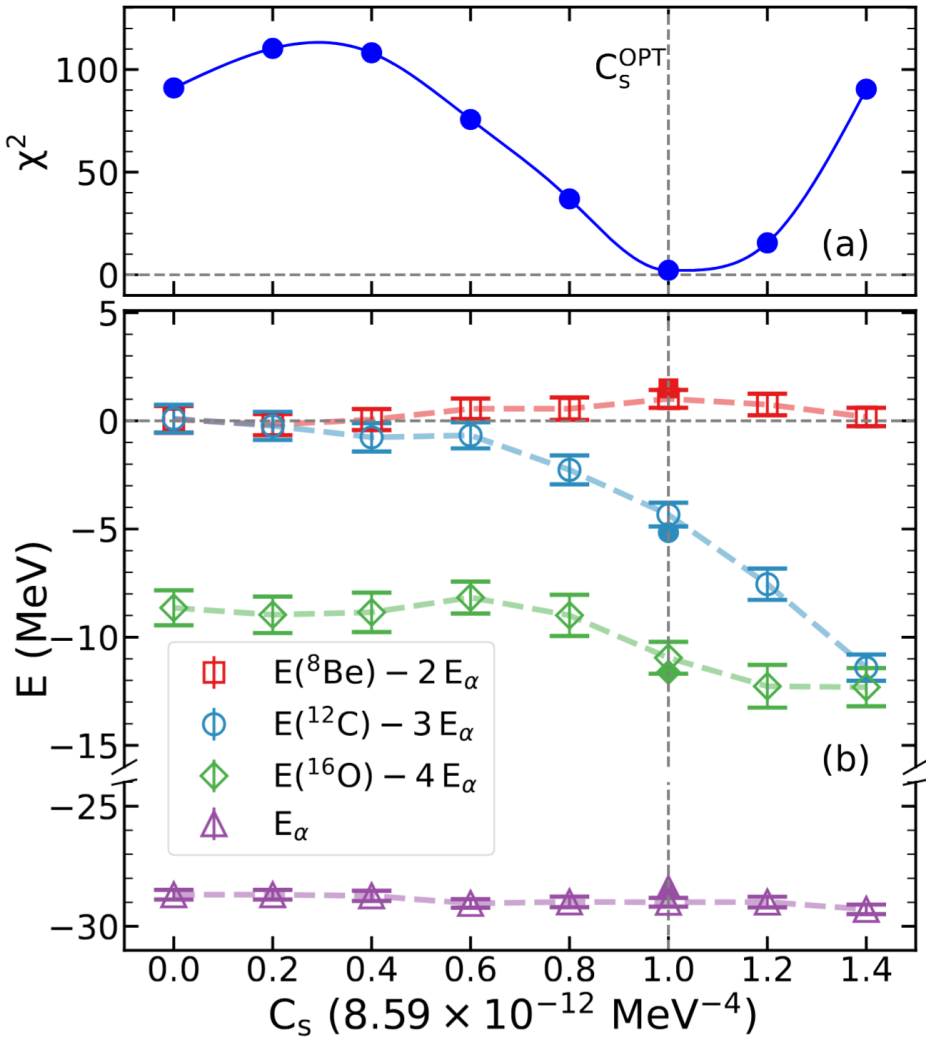
$$V(1,2,3) = t_3\delta(\mathbf{r}_1 - \mathbf{r}_2)\delta(\mathbf{r}_2 - \mathbf{r}_3)$$

LAT-OPT1 has one-to-one correspondence to Skyme force with 6 parameters except for x_0

Model	Parameterization	Parameters	σ (MeV)	Nuclei	σ_{all} (MeV)
Macroscopic-microscopic	FRDM [9]	> 30	1.15	69	0.56
Relativistic mean field	PC-PK1 [10]	11	2.25	60	1.52
Skyrme DFT	UNEDF1 [11]	12	3.43	75	1.91
Lattice EFT	Wigner-SU(4) [1]	4	10.21	55	—
Lattice EFT	LAT-OPT1	5	2.93	76	—

Results from LAT-OPT1: Nuclear clustering





D. T. Tran et al., Nat. Comm. 9, 1594 (2018)

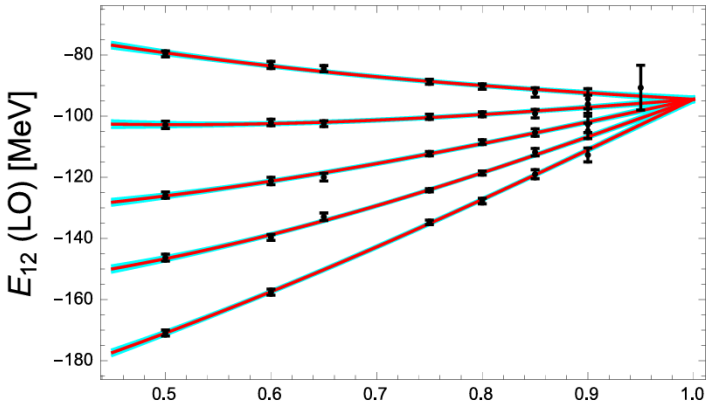
NLEFT calculations with full chiral forces

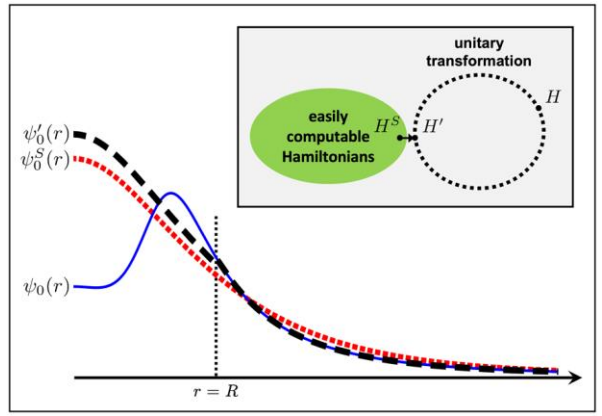
Sign problem is severe for **fermionic systems**

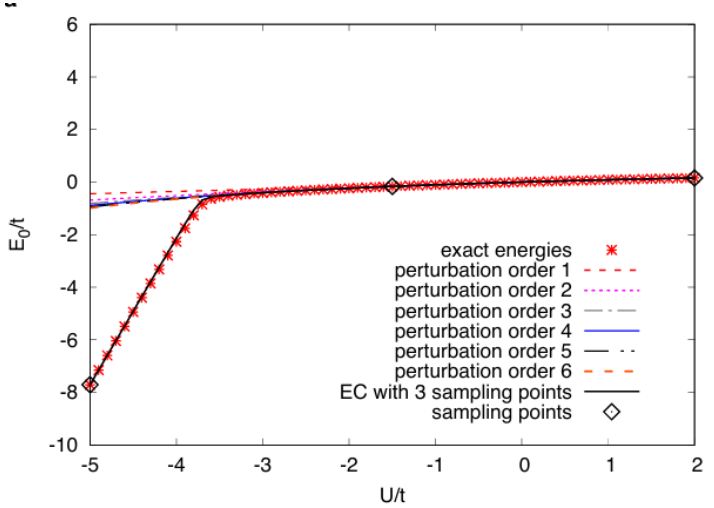
Conventional approach:
Hamiltonian without sign problem (e.g., SU4)
+ first order perturbation theory
Borasoy et al., EPJA 31, 105 (2007)

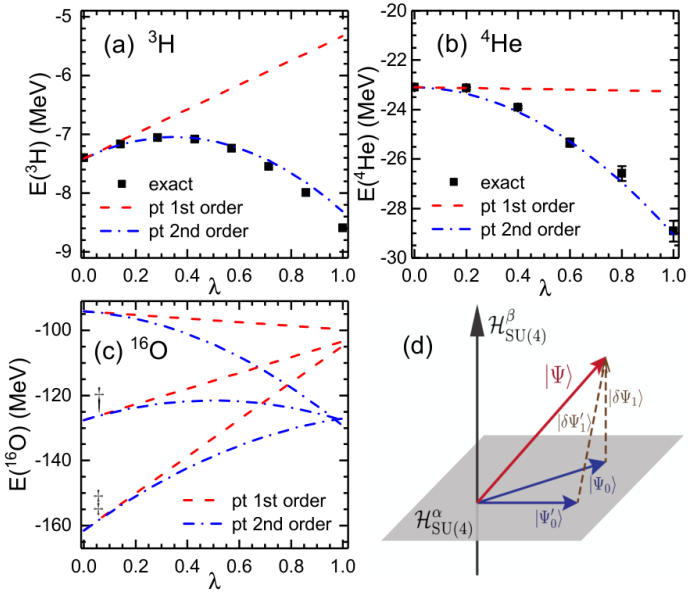
Go beyond the 1st order perturbation theory:

- Symmetry-Sign extrapolation
  Lähde et al., EPJA 51, 92 (2015)
- Eigenvector continuation
  Frame et al., PRL121, 032501 (2018)
- Perturbative quantum Monte Carlo method
  Lu et al., PRL 128, 242501 (2022)
- Wave function matching method
  Elhatisari et al., Nature 630, 59 (2024)
- Rank-one operator method
  Ma et al., PRL 132, 232502 (2024)



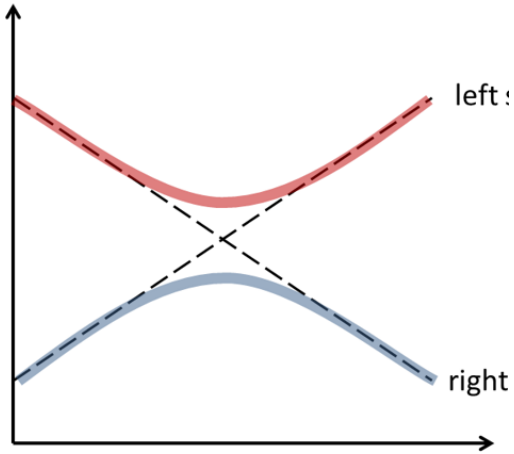






Limitations of first order perturbation theory

- In these cases, we should **not** trust the first order perturbation theory and need to go beyond:
  - First order corrections are too large
  - Hard-core interactions with soft zeroth order
  - States near the bound-continuum threshold
  - Explicit symmetry breaking due to perturbations
  - Near the level crossing

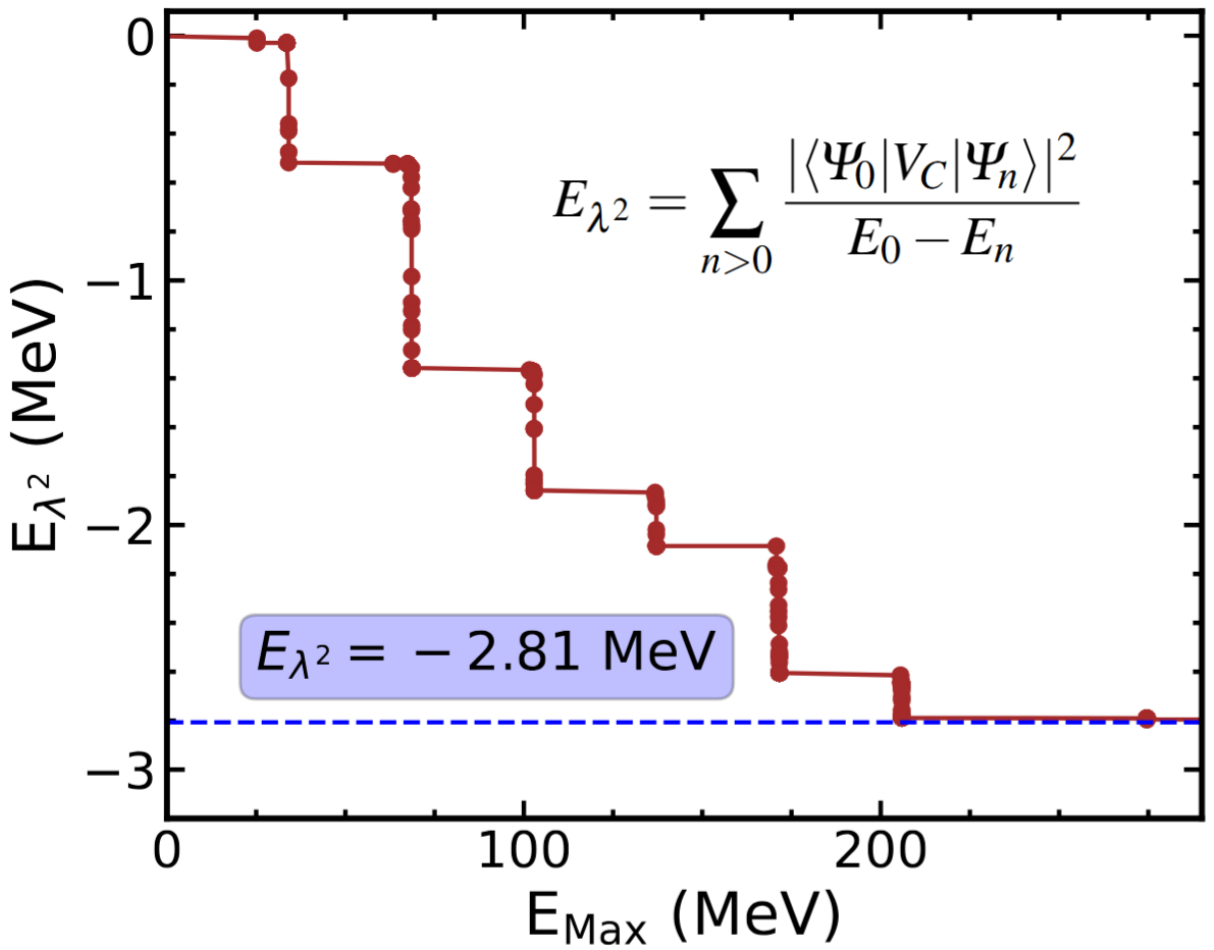


- For improving the precision, we also need higher-order corrections
- Conventional **Rayleigh-Schrödinger perturbation theory** is not applicable for **projection Monte Carlo calculations**

$$E_i = E_i^{(0)} + \lambda \langle \Psi_i^{(0)} | V_C | \Psi_i^{(0)} \rangle + \lambda^2 \sum_{k \neq 0} \frac{\langle \Psi_k^{(0)} | V_C | \Psi_i^{(0)} \rangle}{E_k^{(0)} - E_i^{(0)}} + \mathcal{O}(\lambda^3)$$

Excited states expensive in NLEFT

Comparison with Rayleigh-Schrödinger scheme



- We compare three methods for 2nd correction
  - Perturbative QMC
  - Rayleigh-Schrödinger
  - Cauchy formula
- For ^{2}H in a box all methods agree excellently
- However, R-S method needs hundreds of excited states, converge slowly
- Cauchy formula not applicable for $A > 3$

All state with $E \leq E_{max}$ included
Each point corresponds to a state

Perturbative quantum Monte Carlo

- Expand the projected ground states:

$$|\Psi\rangle = \lim_{L_t\to\infty} M^{L_t/2}|\Psi_T\rangle = |\Psi_0\rangle + |\delta\Psi_1\rangle + \mathcal{O}(V_C^2),$$

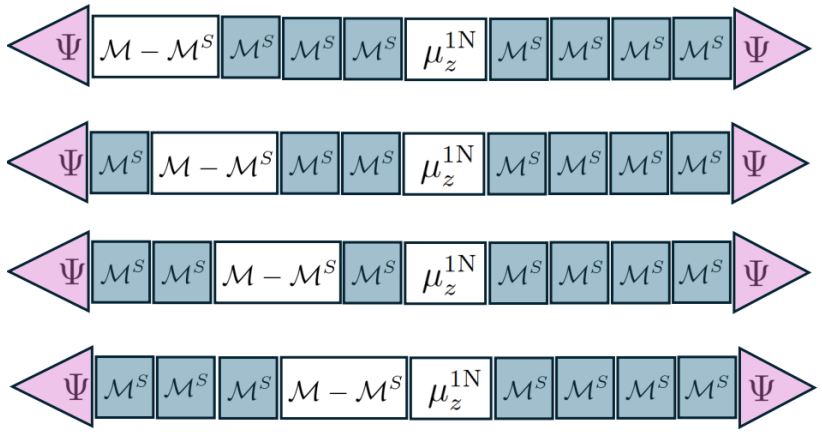
$$|\Psi_0\rangle = \lim_{L_t\to\infty} M_0^{L_t/2}|\Psi_T\rangle,$$

$$|\delta\Psi_1\rangle = \lim_{L_t\to\infty} \sum_{k=1}^{L_t/2} M_0^{L_t/2-k}(M-M_0)M_0^{k-1}|\Psi_T\rangle,$$

- The normalized wave function:

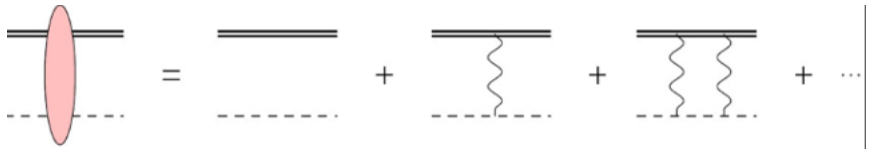
$$|\Psi'\rangle = \frac{|\Psi\rangle}{\sqrt{\langle\Psi|\Psi\rangle}} = \frac{|\Psi_0\rangle}{\sqrt{\langle\Psi_0|\Psi_0\rangle}} + \frac{1}{\sqrt{\langle\Psi_0|\Psi_0\rangle}} \times \left[|\delta\Psi_1\rangle - \frac{\mathrm{Re}\langle\Psi_0|\delta\Psi_1\rangle}{\langle\Psi_0|\Psi_0\rangle}|\Psi_0\rangle\right] + \mathcal{O}(V_C^2)$$

BNL et al., PRL 128, 242501 (2022)



.....

Courtesy of Teng Wang



(Imaginary) time-dependent perturbation theory

Solving realistic chiral interactions

$$V_{2N} = \big[B_1 + B_2(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) + C_1 q^2 + C_2 q^2(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)$$
$$+ C_3 q^2(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) + C_4 q^2(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)$$
$$+ C_5\frac{i}{2}(\boldsymbol{q}\times\boldsymbol{k})\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2) + C_6(\boldsymbol{\sigma}_1\cdot\boldsymbol{q})(\boldsymbol{\sigma}_2\cdot\boldsymbol{q})$$
$$+ C_7(\boldsymbol{\sigma}_1\cdot\boldsymbol{q})(\boldsymbol{\sigma}_2\cdot\boldsymbol{q})(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\big]\, f_{2N}(p_1,p_2,p_1',p_2')$$
$$- \frac{g_A^2 f_\pi(q^2)}{4F_\pi^2}\left[\frac{(\boldsymbol{\sigma}_1\cdot\boldsymbol{q})(\boldsymbol{\sigma}_2\cdot\boldsymbol{q})}{q^2+M_\pi^2} + C_\pi'\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\right](\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)$$

- Target interaction = NLO two-body contact terms
  + One-Pion-Exchange Potential
  + N2LO three-body c_E term
  + Coulomb force

  All determined for $A \leq 3$

- Zeroth order interaction = "essential element"

  $H_{\mathrm{N2LO}} = H_{SU4} + (H_{\mathrm{N2LO}} - \mathrm{H}_{SU4})$

- With different strength of the zeroth order interaction, ^{16}O converge to the same energy

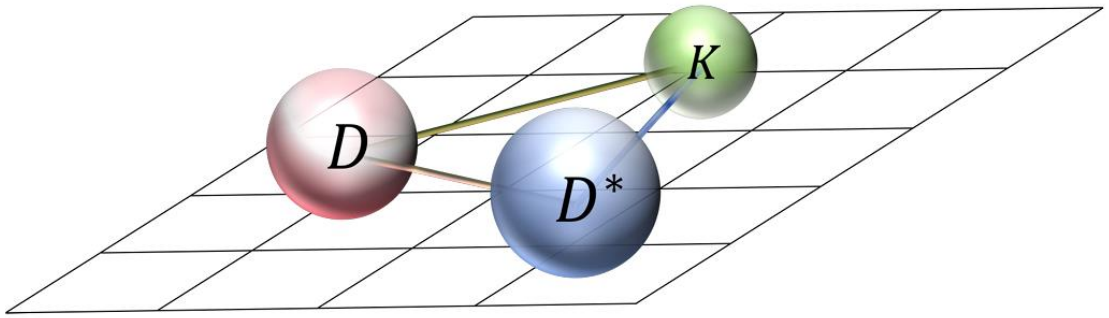
- Large second order corrections, account for 20% of total energies. Unusual. Anything wrong?

	E_0	δE_1	E_1	δE_2	E_2	E_{exp}
^{3}H	$-7.41(3)$	$+2.08$	$-5.33(3)$	-2.99	$-8.32(3)$	-8.48
^{4}He	$-23.1(0)$	-0.2	$-23.3(0)$	-5.8	$-29.1(1)$	-28.3
^{8}Be	$-44.9(4)$	-1.7	$-46.6(4)$	-11.1	$-57.7(4)$	-56.5
^{12}C	$-68.3(4)$	-1.8	$-70.1(4)$	-18.8	$-88.9(3)$	-92.2
^{16}O	$-94.1(2)$	-5.6	$-99.7(2)$	-29.7	$-129.4(2)$	-127.6
^{16}O†	$-127.6(4)$	$+24.2$	$-103.4(4)$	-24.3	$-127.7(2)$	-127.6
^{16}O‡	$-161.5(1)$	$+56.8$	$-104.7(2)$	-22.3	$-127.0(2)$	-127.6

Contents

- Introduction to Nuclear Lattice Effective Field Theory
- Non-perturbative QMC and sign problem
- **Few-body applications**
- Renormalization of NLEFT
- Summary and perspective

Hadronic Lattice EFT

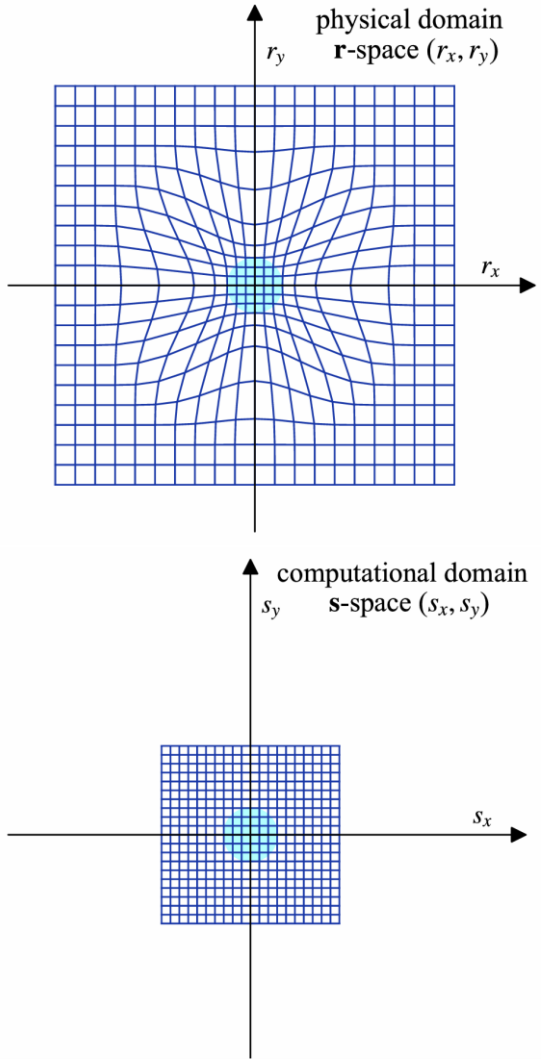


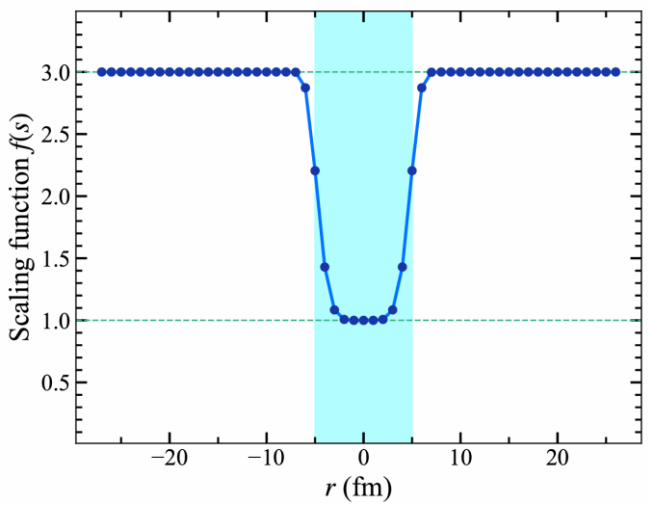
- Nuclear lattice EFT can also solve hypernuclei and more general multi-hadron systems
- Challenge
  - Interactions become more complicated
  - Fewer data for calibrating the EFT

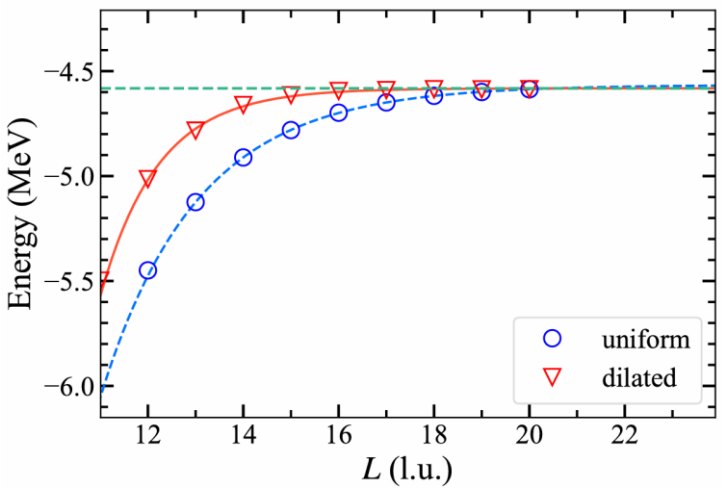
- We employ NLEFT framework to solve asymmetric three-hadron system DD*K
- Two-body interactions are regulated with a soft regulator and calibrated with binding energies of T_{cc}^{+}, $D_{s0}^{*}(2317)$, $D_{s1}(2460)$ molecular states
- Three-body binding energies are calculated for a group of cutoffs and volumes
- Systematic uncertainties due to the finite volume effects, sliding cutoff and leading-order 3NF are examined
- Found an unambiguous three-body bound state even when 3NF is repulsive

Zhang, Hu, He, Liu, Shi, BNL, Wang, PRD 111, 036002 (2025)

Dilated coordinate for lattice calculations



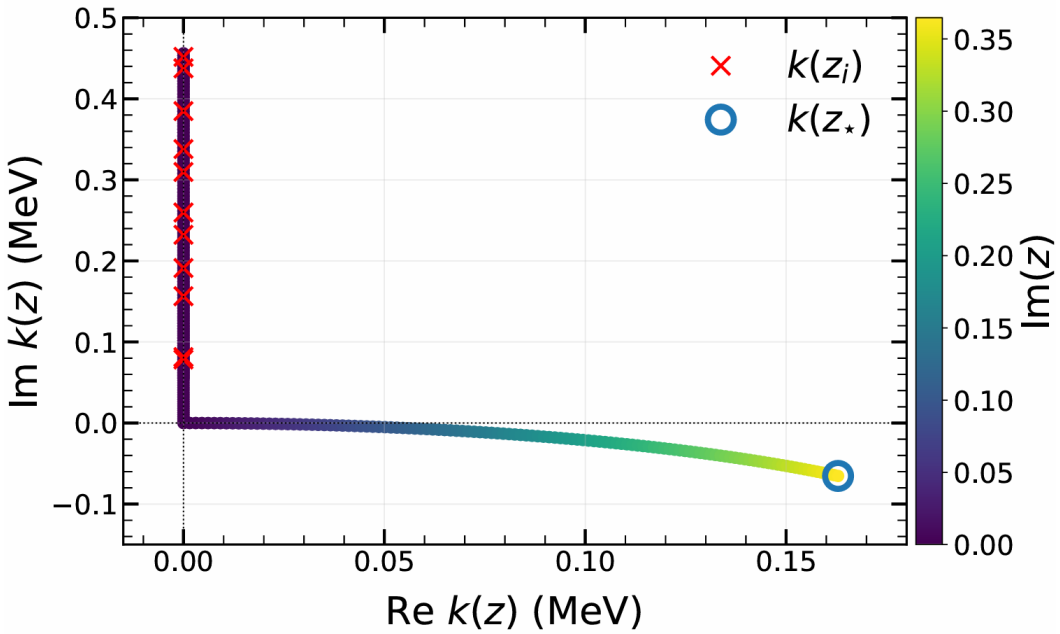




- EFT is associated to a scale separation $Q \ll \Lambda$
- Choose lattice spacing $a \approx \frac{\pi}{\Lambda}$
  → Not efficient for shallow bound states
  E(^{2}H) = -2.224 MeV, E(^{3}H) = -8.482 MeV
  E($^3_\Lambda$H) = -2.326 MeV, Halo nuclei, Astrophysics
- Adaptive mesh refinement
  - Variable lattice spacing & geometry
  - Extensive applications in lattice simulations for GR, Chemistry, Material, fluid, LQCD, ⋯
- Distort lattice in a NLEFT calculation
  - Finer lattice within EFT potential ($a \approx \pi/\Lambda$)
  - Coarser lattice outside potential ($a \approx \pi/Q$)
  - Numerically tested for two- and three-body

  He, Zhang, Wang, Wang, BNL,
  Phys. Rev. C 113, 034001 (2026)

Resonances from analytical continuation



Niu, Zhang, BNL, PLB 880, 140823 (2026)

- NLEFT is good at bound states; however, resonances remain challenging
- Real space finite volume method: spectrum→ scattering
- Complex energy method: bound/resonant states→poles
  - Good for broad resonances or overlapping poles
  - Requires high-precision data to locate poles
- Take ^{5}He ($3/2^-$) as an example, calculate $\mathbf{E}$ and $\mathbf{\Gamma}$ from the Sign-problem-free interaction LAT-OPT1
  - $E = 0.80(10)$ MeV, $\Gamma = 1.05(9)$ MeV
  - $E_{exp} = 0.798$ MeV, $\Gamma_{exp} = 0.648$ MeV

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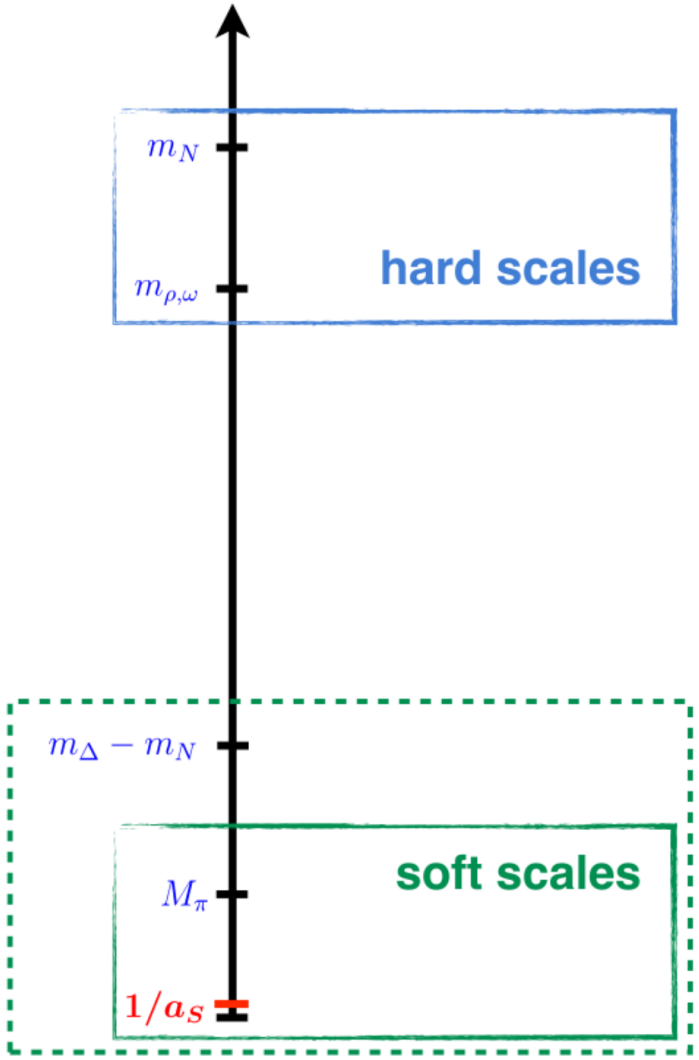
Dual role of the lattice

- Lattice is used to **discretize the action** and **regulate the delta-functions**

- QCD coupling is dimensionless
  → QCD is **renormalizable**
  → Safe to take the continuum limit $a \to 0$

- EFT couplings has negative mass dimensions
  →EFT is **non-renormalizable**
  →Break down at a small lattice spacing

Example: Wigner bound Wigner, PR98, 145 (1955)
For $\Lambda \to \infty$, effective range $r_{\text{eff}} \leq 0$
However, experimental n-p scattering $r_{\text{eff}} \approx 1$ fm
→ implies a break down scale for pionless EFT

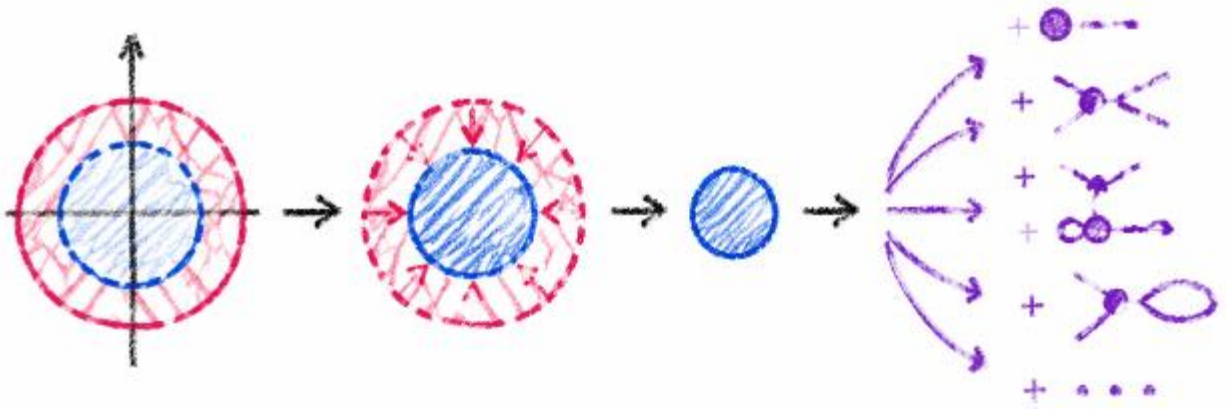


Wilsonian RG: integrate out momentum shells

- Wilson's viewpoint:
  - Variation of Λ induces infinite contact terms
    → Predictions independent of Λ

Including all terms allowed by symmetries eliminate the dependences on Λ

Wilson, PRB 4, 3174 (1971)

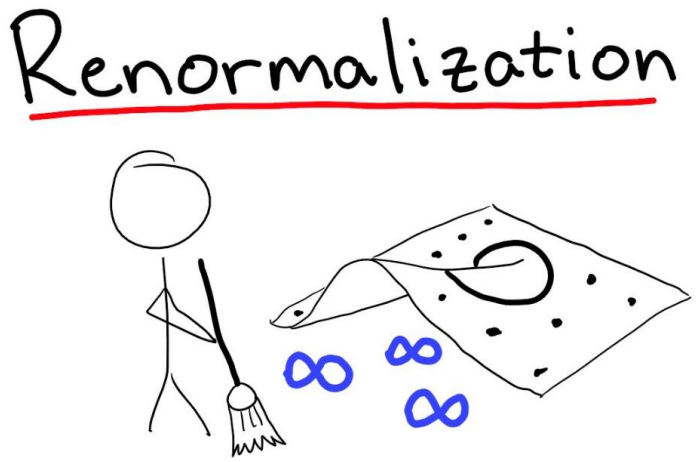




Counter-term renormalization vs. Wilsonian renormalization

Counter-term renormalization	Wilsonian renormalization
Cancel divergences	Integrate out high-energy modes
Order-by-order renormalization	Scale-by-scale evolution
Renormalized LECs	Running couplings
Finite amplitudes	Effective low-energy theory

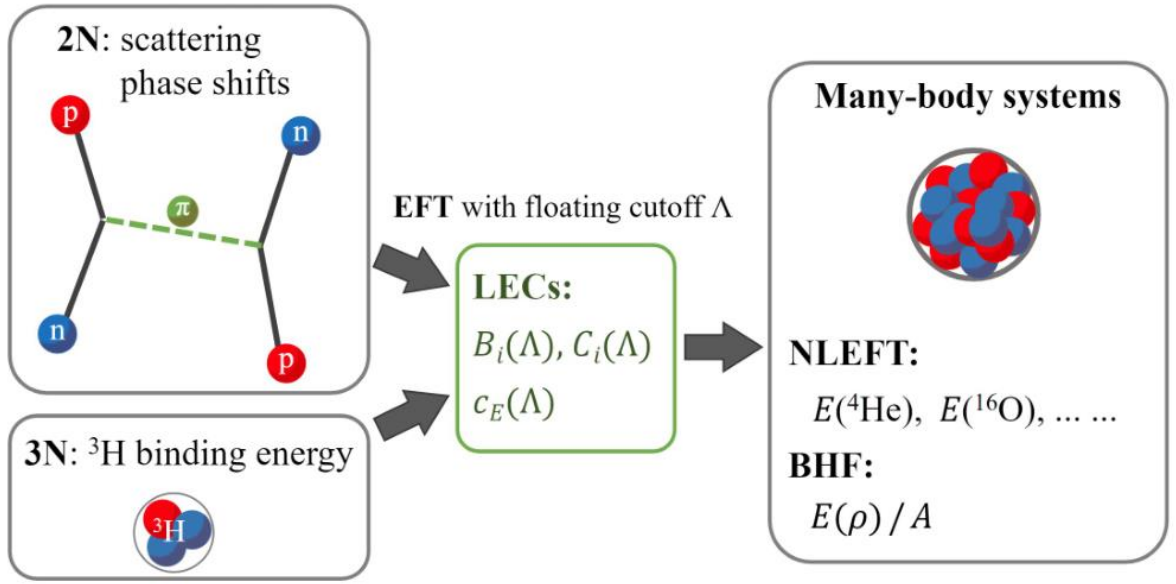
- Literatures often takes **extremely large cutoff** such as Λ~a few GeV, these correspond to **counter-term renormalization**
- Divergences occur for **bare** parameters. After substituting **bare** by **physical** parameters, divergences disappear at $\Lambda \to \infty$
- In **Wilsonian renormalization**, we always use **bare** parameters. (e.g., lattice QCD) Cutoff can not exceed the **breakdown scale**. For chiral EFT, $\Lambda \leq 700$ MeV



Is Wilson's picture holds in nuclear EFT?

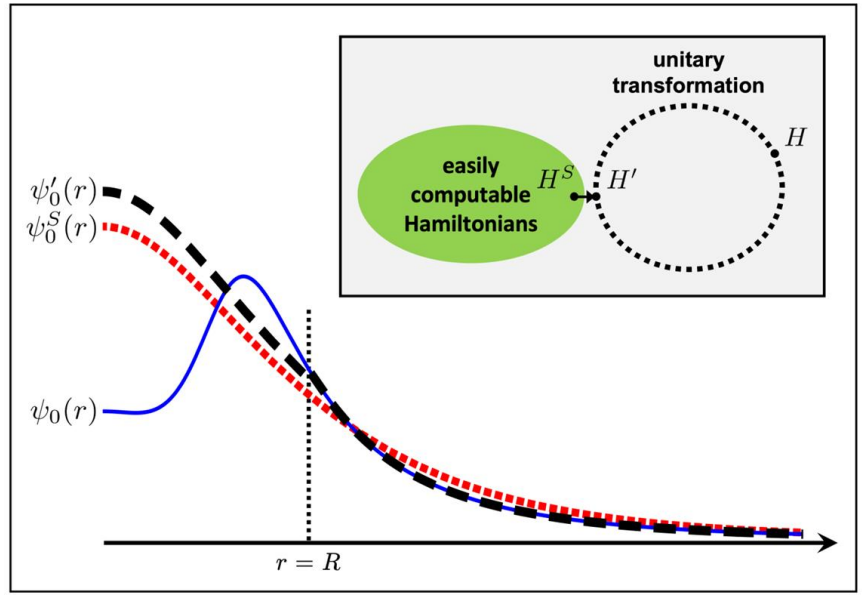
- **According to Wilsonian RG**
  - Always solve the Hamiltonian non-perturbatively
  - Predictions should keep invariant during infinitesimal variation of Λ
  - Improve RG invariance by including higher-orders terms

- **Straightforward to verify**
  - Redo whole calculations using different Λ's (nonperturbatively. Typically below breakdown scale)
  - Examine cutoff dependences of the predictions



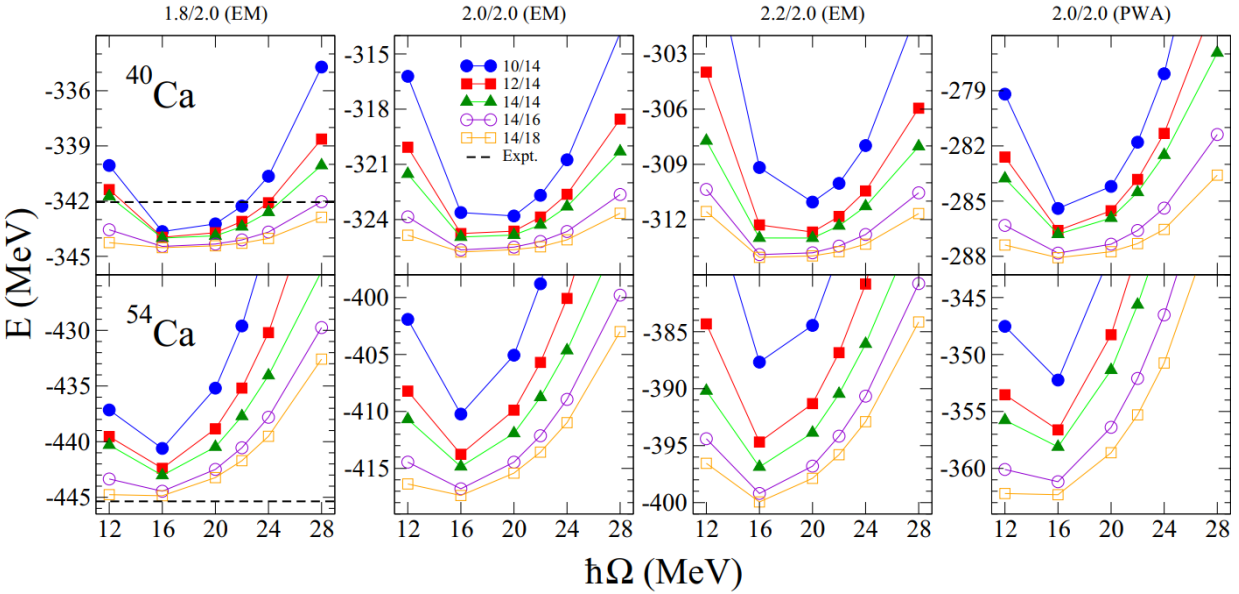
Cutoff effects in many-body EFT calculations

- Verification of Wilsonian RG requires direct solutions of the chiral Hamiltonian, including both 2NF and 3NF, for variable cutoffs
- Modern nuclear ab initio calculations favors transformed chiral EFT interactions: SRG, Lee-Suzuki, WFM, etc.
- Transformed interactions are easier to solve (more soft or perturbative).
  Phaseshift equivalent, phenomenological 3NF adjusted to heavier nuclei are often needed
- Direct calculations with high cutoffs requires large model space or small lattice spacing → often computational inaccessible
- ptQMC provides a framework for solving chiral forces directly, offering a opportunity for examining the RG invariance for medium-mass nuclei

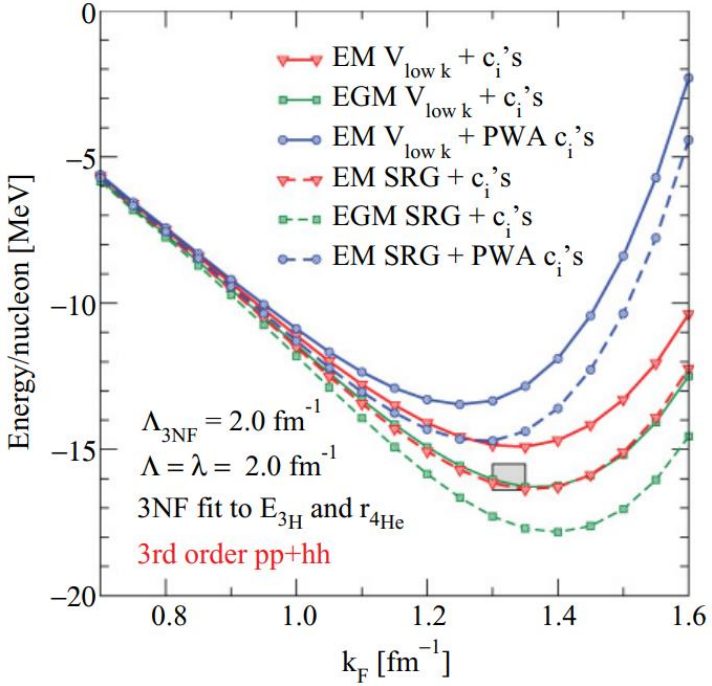


RG-(in)variance of the chiral forces

- Similarity Renormalization Group (SRG) method can be used to generate **phase-shift-equivalent** nuclear forces
- Starting from a bare chiral force, new interactions can be obtained by applying unitary transformations U_λ or U_Λ
- **λ and Λ are effective cutoffs** that can be varied continuously
- While 2N and 3N physics are mostly **independent of λ/Λ,** heavy nuclei & nuclear matter apparently **depend on λ/Λ**



J. Simonis et al., Phys. Rev. C 96, 014303 (2017)



K. Hebeler et al., Phys. Rev. C 83, 031301 (2011)

- Renormalization group invariance of the chiral EFT is a consequence of its construction.
- However, this has **not been proved** and typically **violated** in numerical calculations for heavier systems.

How cutoff affects the predictions

- Calculate difference

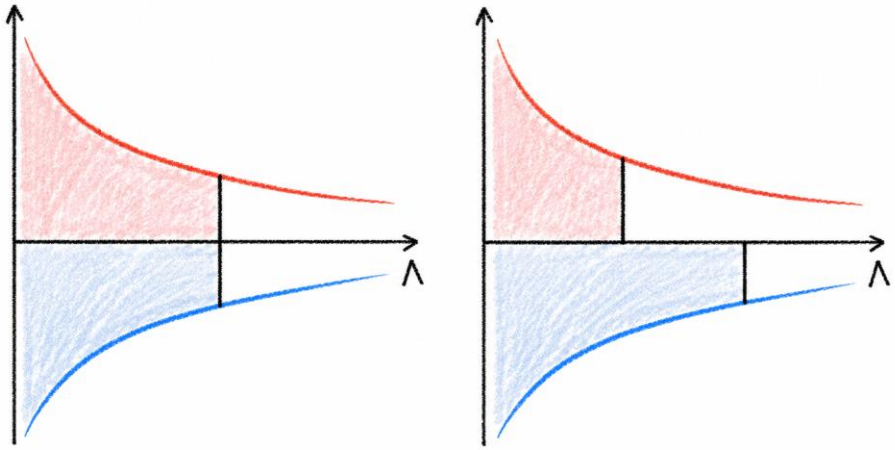
$$I = \int_0^\infty \frac{p^2 dp}{p^2+M^2} - \int_0^\infty \frac{p^2 dp}{p^2+M'^2} = \int_0^\infty \frac{(M'^2-M^2)p^2 dp}{(p^2+M^2)(p^2+M'^2)} = \frac{\pi}{2}(M'-M)$$

- Impose inconsistent cutoff Λ, Λ':

$$I = \lim_{\Lambda,\Lambda'\to\infty} \left[\int_0^\Lambda \frac{p^2 dp}{p^2+M^2} - \int_0^{\Lambda'} \frac{p^2 dp}{p^2+M'^2}\right] = \int_\Lambda^{\Lambda'} \left[\frac{p^2 dp}{p^2+M^2} - \frac{p^2 dp}{p^2+M'^2}\right]$$

$$= \frac{\pi}{2}(M'-M) + \Lambda - \Lambda'$$

- Predictions of field theory relies on **cancelations** between terms
- Inconsistent cutoffs lead to **arbitrary** predictions



Chiral interaction and lattice regulator

$$
\begin{aligned}
V_{Q0} &= \left[B_1 + B_2(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\right] f_{2\mathrm{N}}(\{p_i, p_i'\}), \\
V_{Q2} &= \left[C_1 q^2 + C_2 q^2 (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) + C_3 q^2 (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\right. \\
&\quad + C_4 q^2 (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) \\
&\quad + C_5 \frac{i}{2} (\boldsymbol{q} \times \boldsymbol{k}) \cdot (\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) + C_6 (\boldsymbol{\sigma}_1 \cdot \boldsymbol{q})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{q}) \\
&\quad \left. + C_7 (\boldsymbol{\sigma}_1 \cdot \boldsymbol{q})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{q})(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)\right] f_{2\mathrm{N}}(\{p_i, p_i'\}),
\end{aligned}
$$

$$
V_{3\mathrm{N}} = \frac{c_E}{2 F_\pi^4 \Lambda_\chi} f_{3\mathrm{N}}(\{p_i, p_i'\})
$$

A single 3NF

$$
V_{1\pi} = -\frac{g_A^2 f_\pi(q^2)}{4 F_\pi^2} \left[\frac{(\boldsymbol{\sigma}_1 \cdot \boldsymbol{q})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{q})}{q^2 + M_\pi^2} + C_\pi' \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2\right] (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)
$$

$$
V_{\mathrm{CSB}}^{\mathrm{pp}} = c_{\mathrm{pp}} \left(\frac{1 + \tau_{1z}}{2}\right)\left(\frac{1 + \tau_{2z}}{2}\right) f_{2\mathrm{N}}(\{p_i, p_i'\})
$$

$$
V_{\mathrm{CSB}}^{\mathrm{nn}} = c_{\mathrm{nn}} \left(\frac{1 - \tau_{1z}}{2}\right)\left(\frac{1 - \tau_{2z}}{2}\right) f_{2\mathrm{N}}(\{p_i, p_i'\})
$$

We apply a global lattice-inspired regulator

$$
f_{2\mathrm{N}}(\{p_i, p_i'\}) = \exp\left[-\sum_{i=1}^{2} \left(p_i^6 + p_i'^6\right)/(2\Lambda^6)\right]
$$

Truncation over absolute momentum induces Galilean-invariance breaking terms

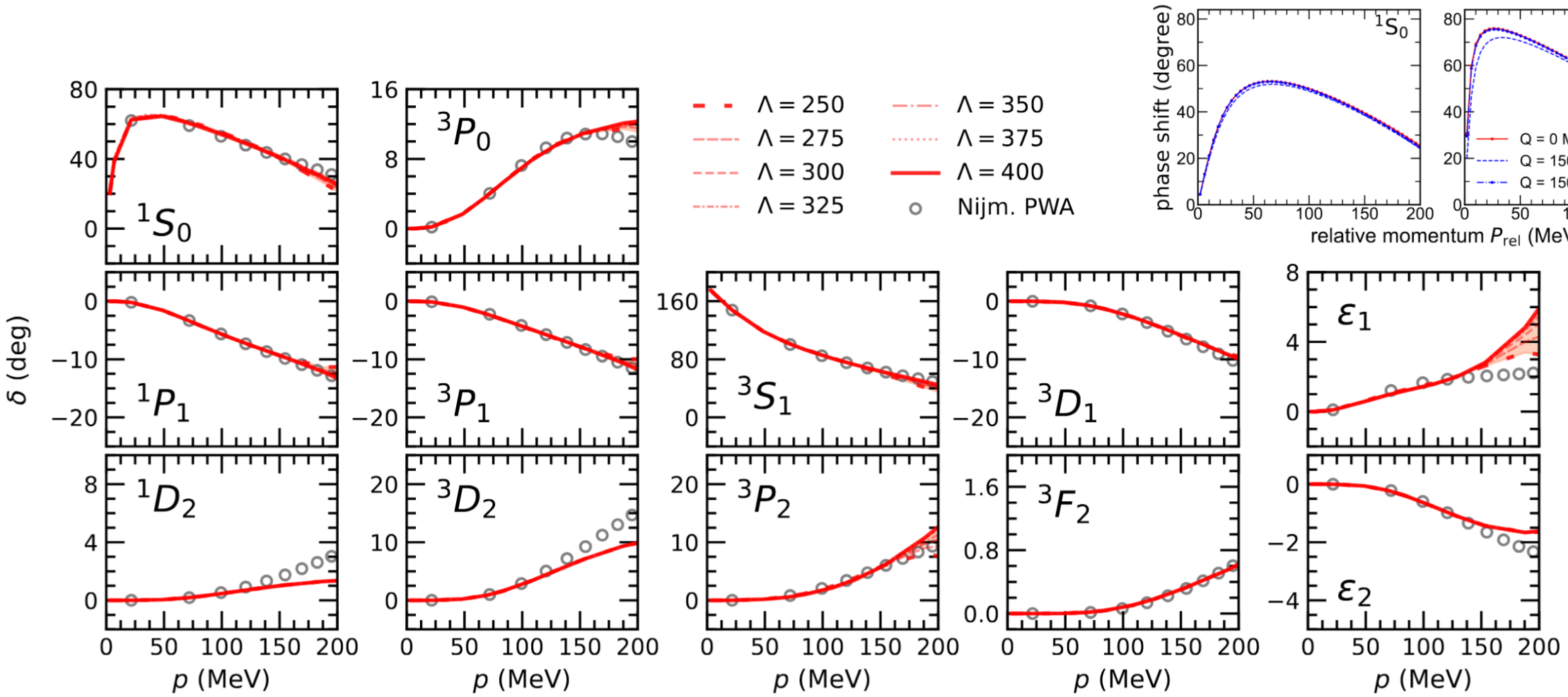
$$
V_{\mathrm{GIR}} = \left[g_1 Q^2 + g_2 Q^2 (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\right] f_{2\mathrm{N}}(\{p_i, p_i'\})
$$

2NF fitted to NN scattering phase shifts
3NF fitted to ^{3}H binding energy
GIR fitted to symmetry constraints

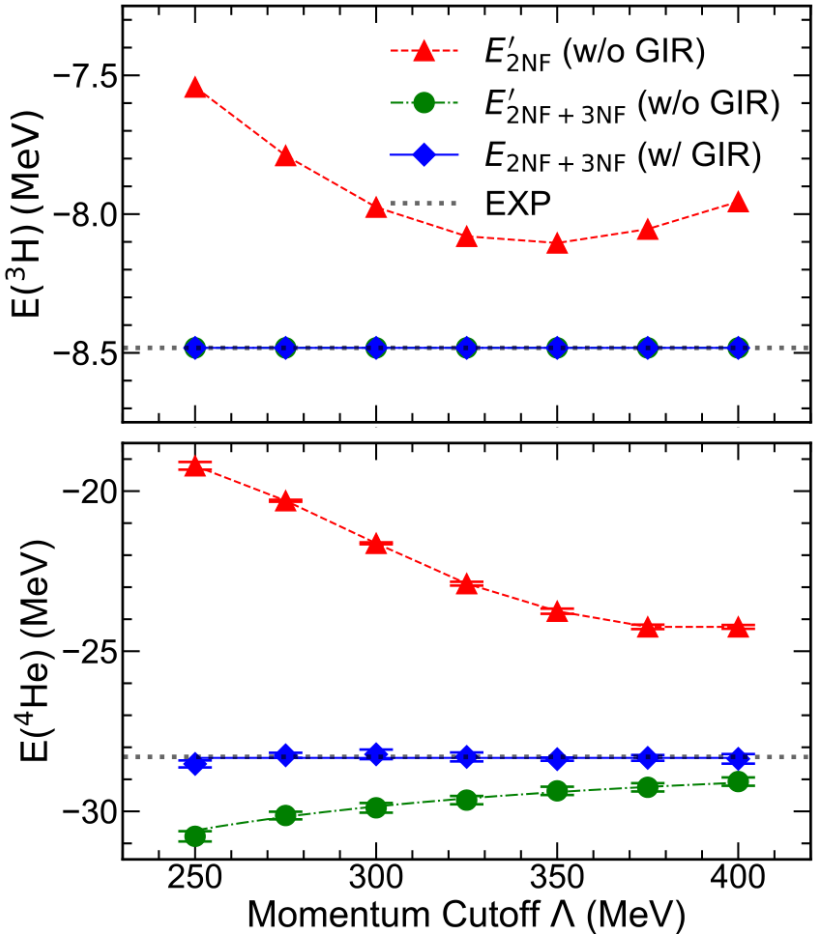
Apply a consistent cutoff to all contact terms

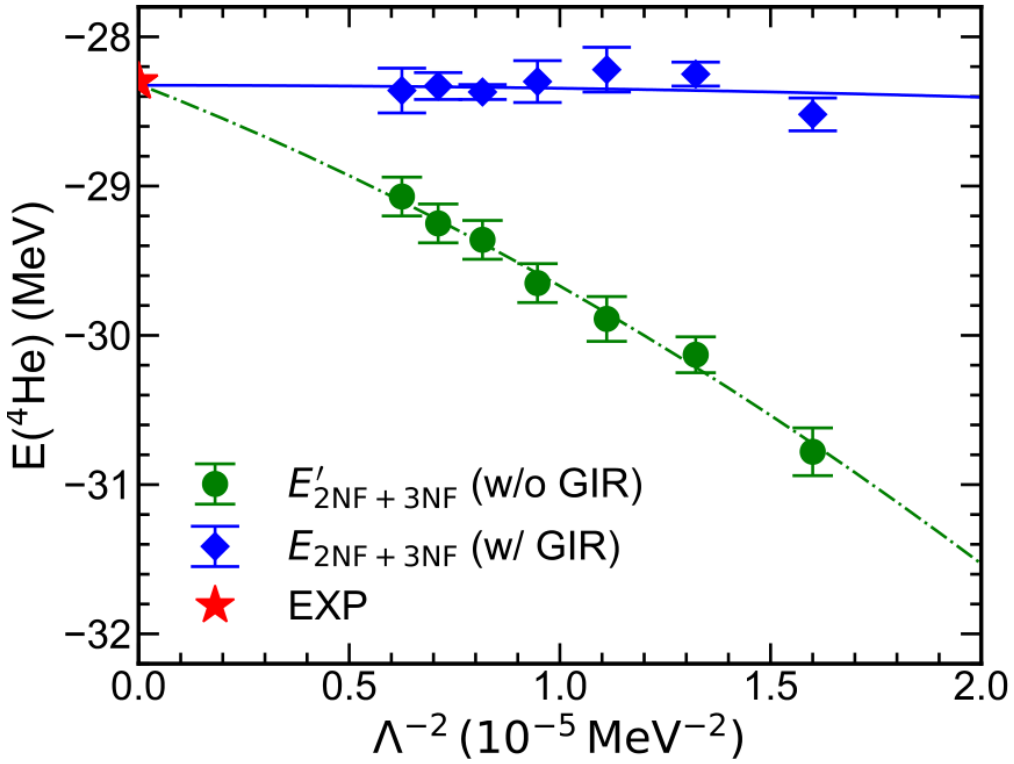
Fitted neutron-proton phase shifts

Determine Galilean invariance restoration terms



Predicted binding energy of ^{4}He





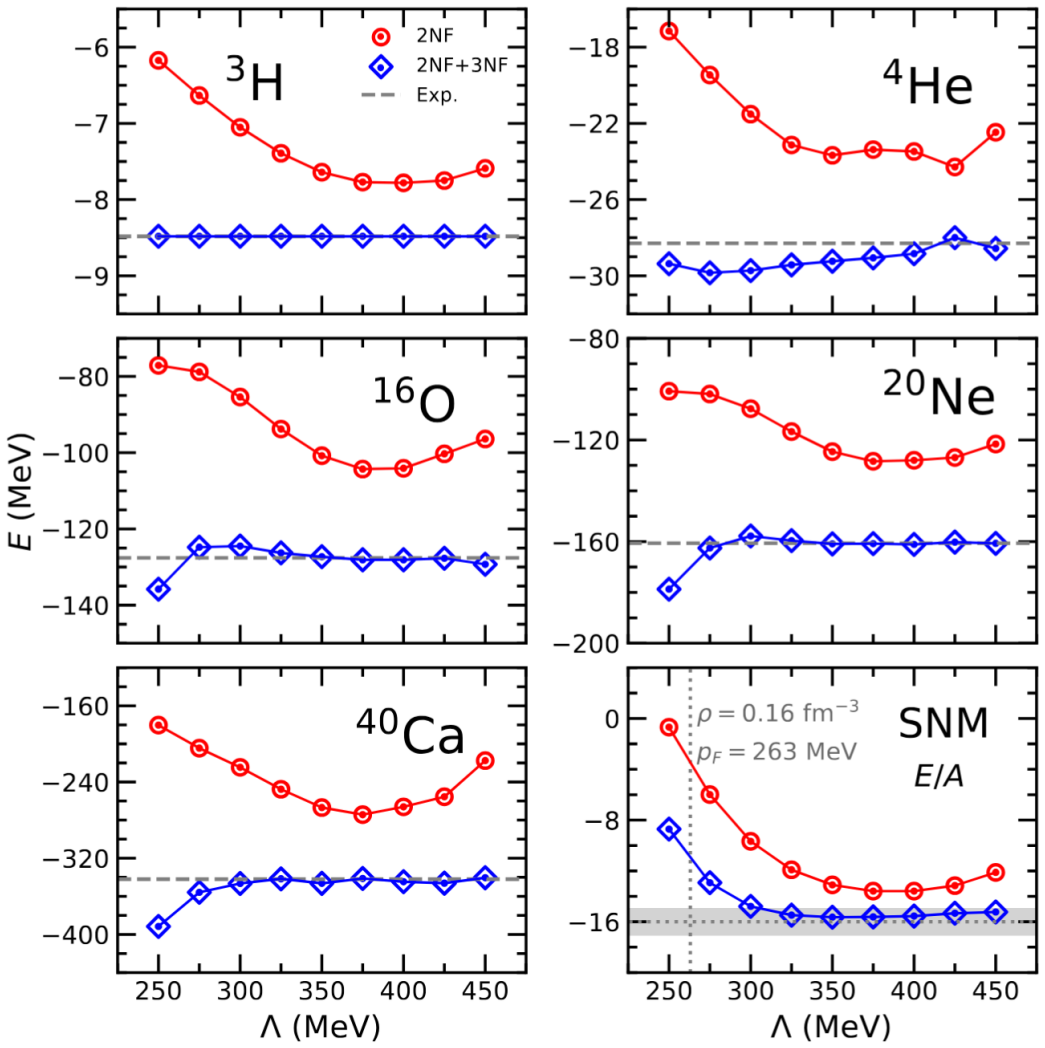
Symmanzik improvement

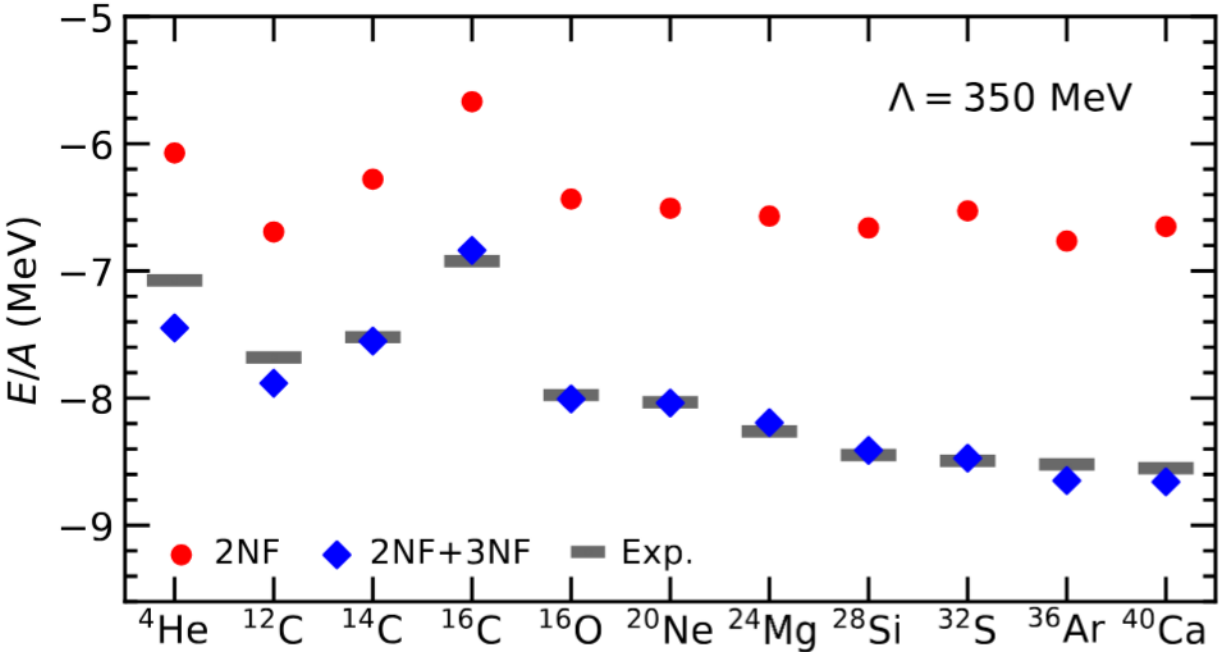
Restore symmetries using counter terms

K. Symanzik, Nucl. Phys. B 226, 187 (1983); Nucl. Phys. B 226, 205 (1983)

Observation of renormalization group invariance in symmetry-restored nuclear lattice effective field theory, J. Shi, C. Wang and BL, Phys. Lett. B 874, 140303 (2026)

Predicted Binding energies of medium-mass nuclei

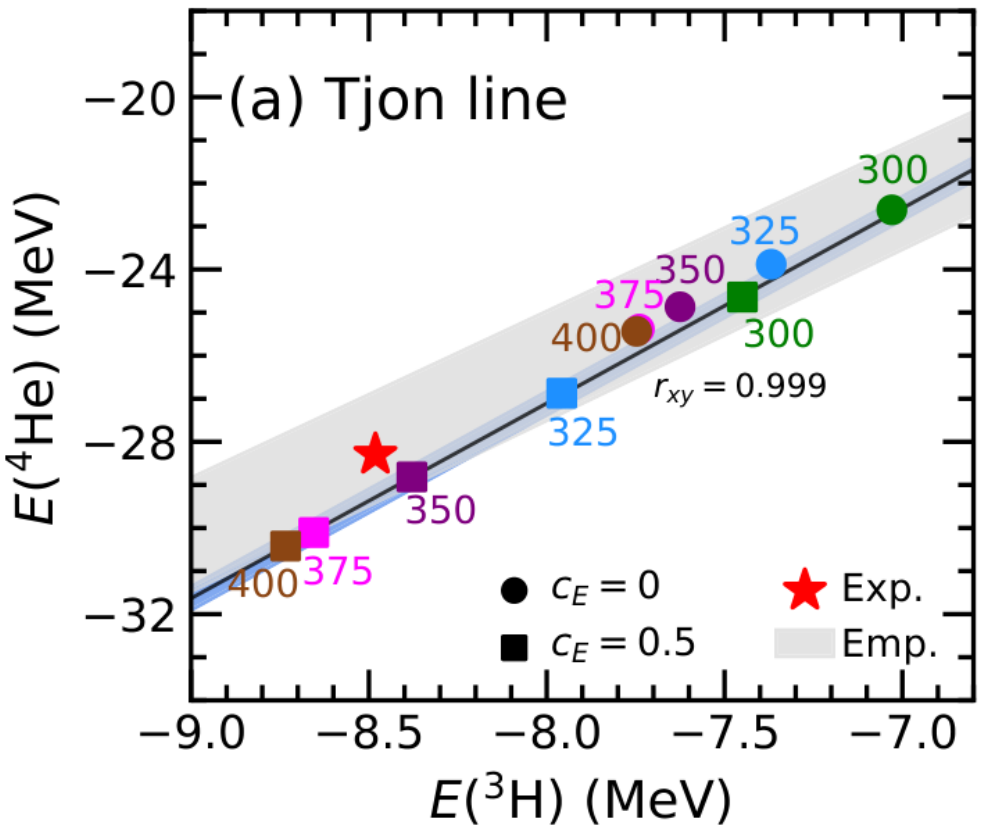




Reproduction of experiments & RG invariance with a single adjustable 3NF

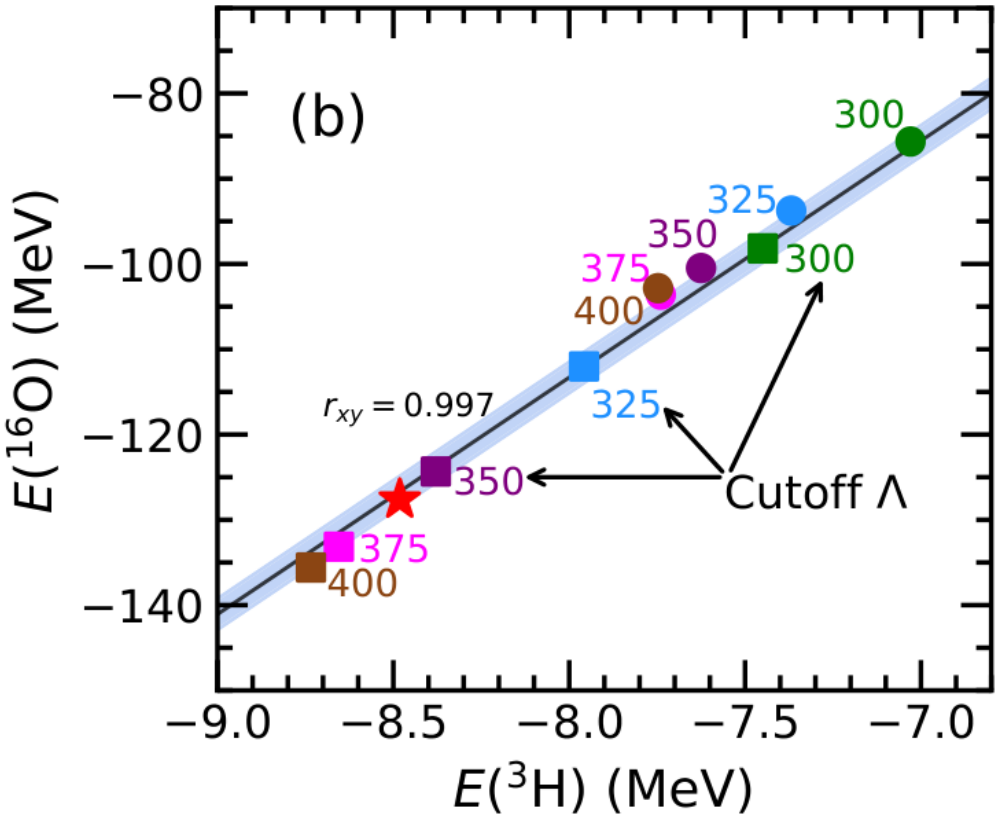
C. Wang, J. Shi, BL, arXiv:2604.20681

Correlations beyond lightest nuclei



Tjon line discovered 50 years ago

J. Tjon, Phys. Lett. B 56, 217 (1975)



A single 3NF is sufficient to cancel cutoff-dependence

Summary and Perspective

- Nuclear Lattice Effective Field Theory is a unified framework for **building NN interactions**, **solving many-body Schrodinger equations** and **extracting observables**.
- Two fundamental challenges in NLEFT calculations
  - Monte Carlo sign problem ← statistical errors
  - Renormalization group invariance (cutoff-independences) ← systematic errors
- Observation of RG-invariance with precision ptQMC calculations
  - Predictions are robust against cutoff variations
  - No need for refitting to medium-mass nuclei or nuclear matter