Workshop on Few-Body Problems in Nuclear and Hadron Physics

21-24 September 2026, Beijing

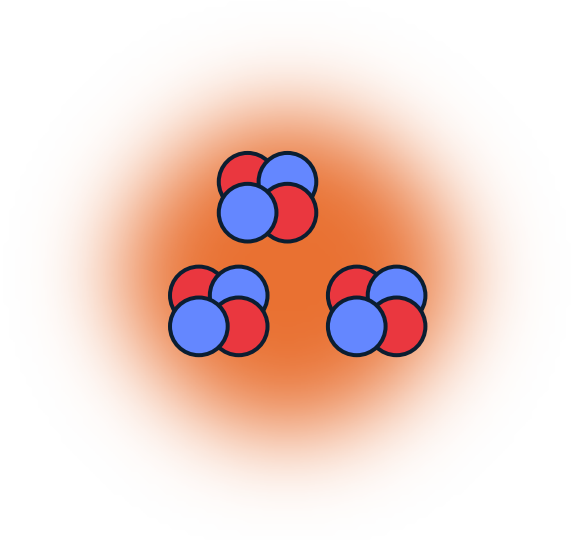
Exploring multi-cluster structures in light nuclei

Bo Zhou (周波)

Fudan University, Shanghai, China

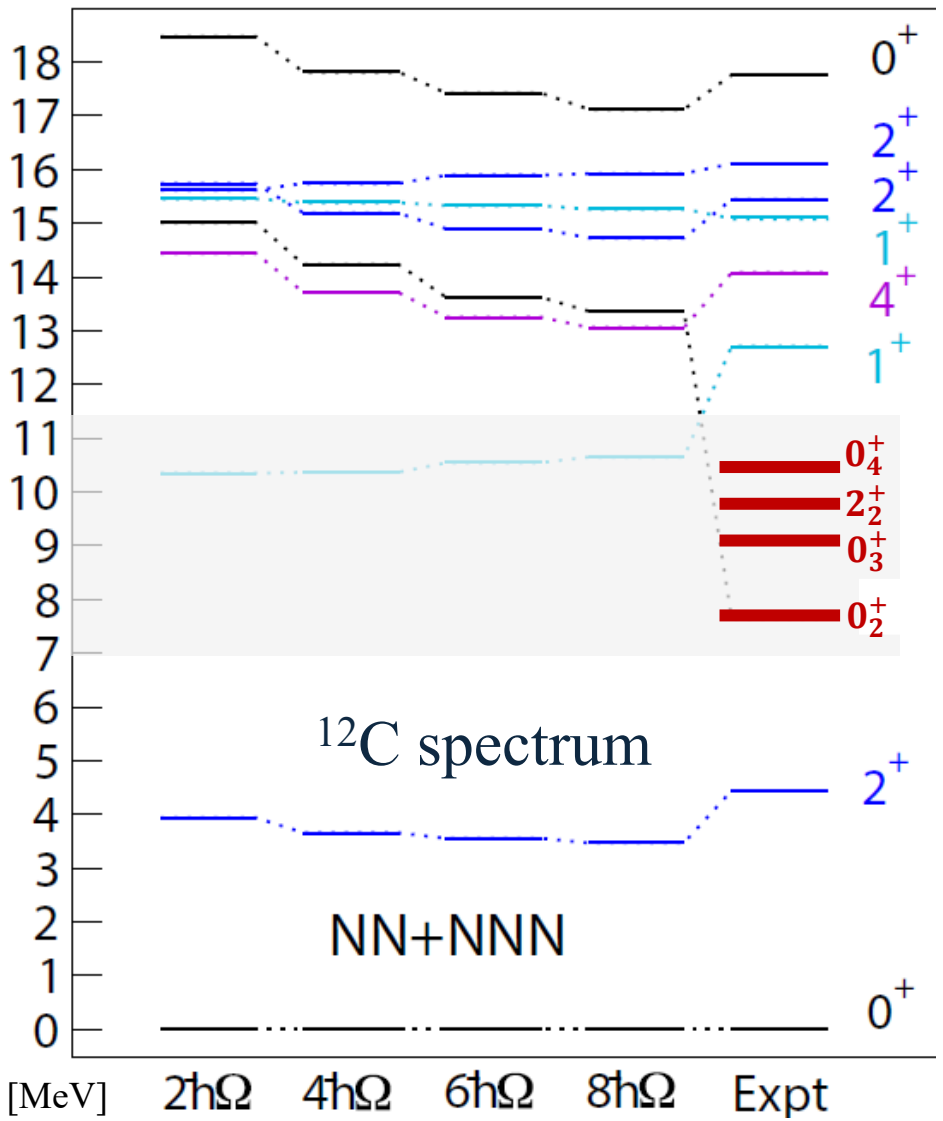
Sep. 22, 2026@ITP

Outline



- Introduction
- The 5α Condensate State
- Hoyle-analog states
- Triple α reaction
- Summary

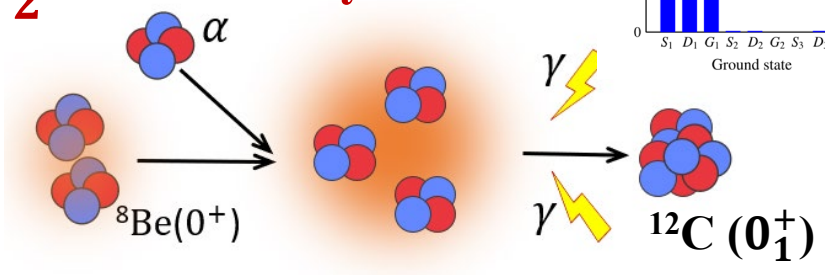
Cluster states of ^{12}C

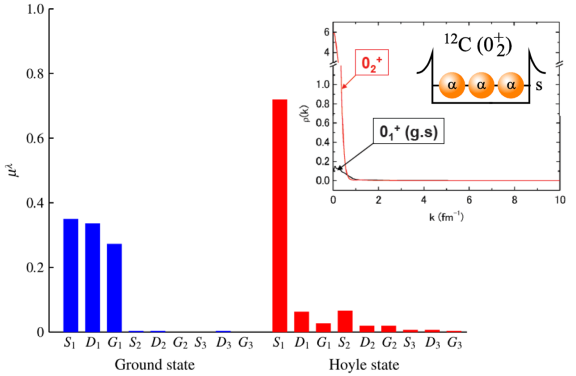


Recent No-Core-Shell-Model calculations

V.Somà,P.Navrátil, *et al.* PRC,101,014318 (2020)

0^+_2 Hoyle State





Hoyle state & Bose-Einstein Condensate.

Rev. Mod. Phys. **89**, 011002 (2017)

$0^+_{3,4}$

Two broad resonance states with large decay width

Phys. Rev. C **84**, 054308 (2011)

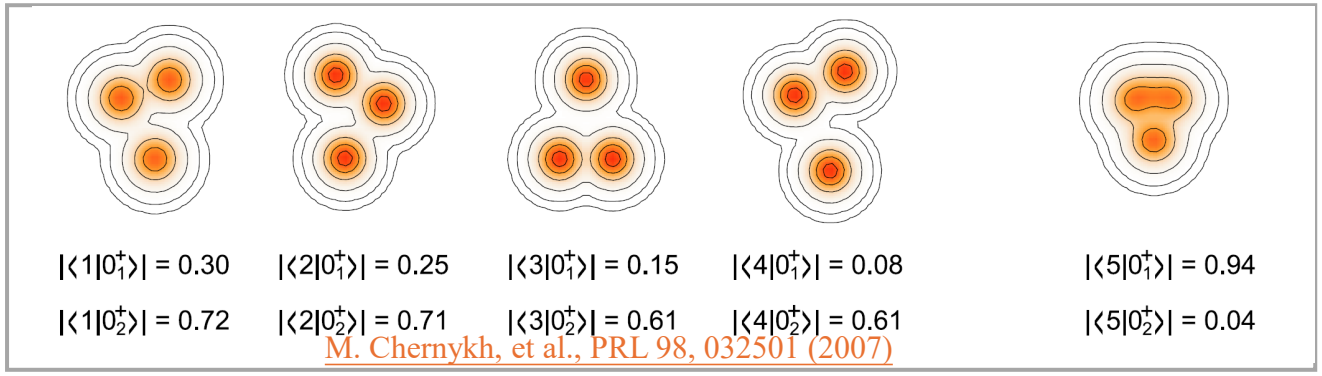
2^+_2

Long puzzle and it now has been confirmed for its existence.

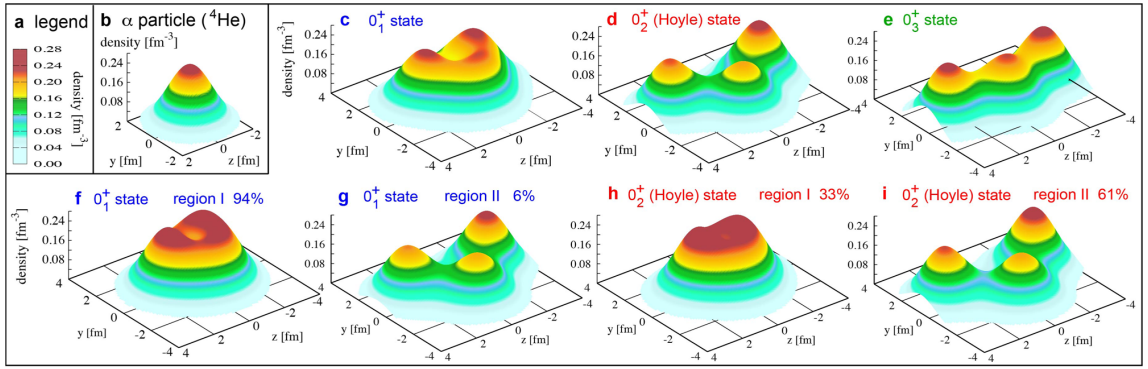
Phys. Rev. Lett. **110**, 152502 (2013)

search for novel clustering states

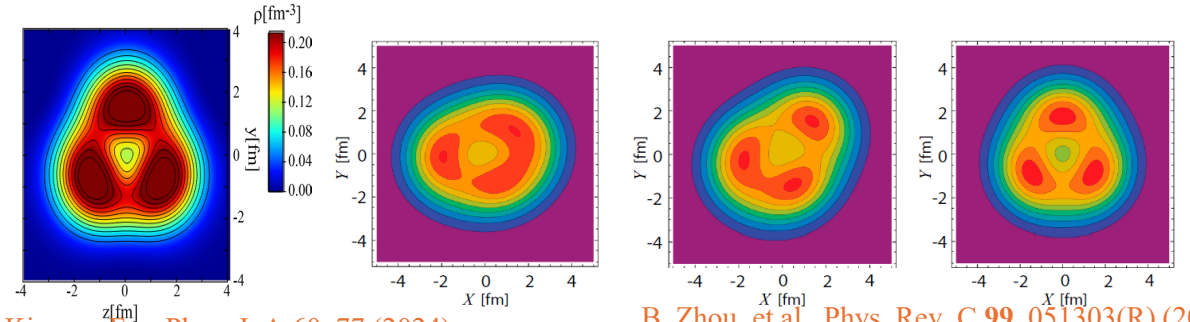
Shape/Structure of the ^{12}C



M. Chernykh, et al., PRL 98, 032501 (2007)

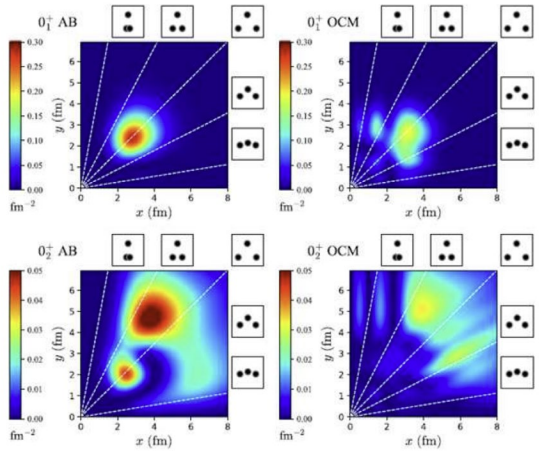


Otsuka, et al., Nat Commun 13, 2234 (2022)

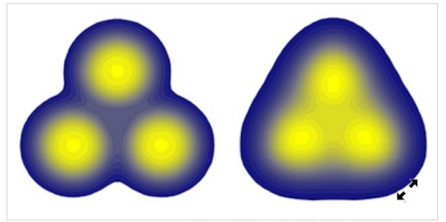


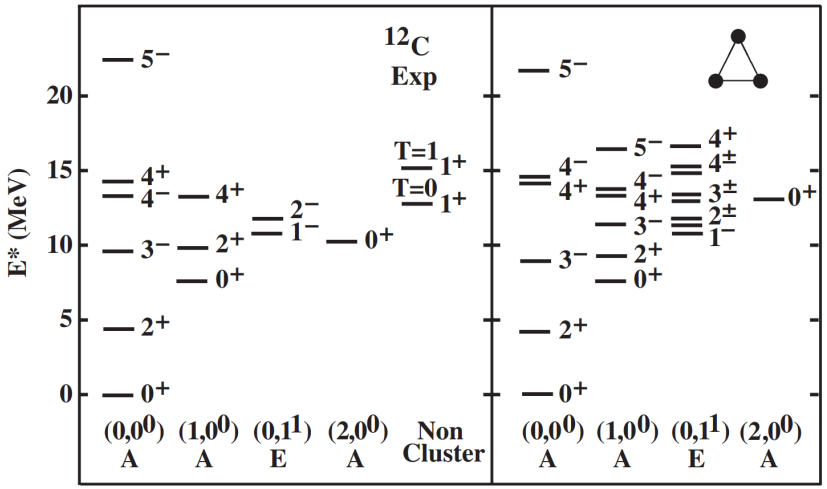
M.Kimura, Eur. Phys. J. A 60, 77 (2024)

B. Zhou, et al., Phys. Rev. C **99**, 051303(R) (2019).

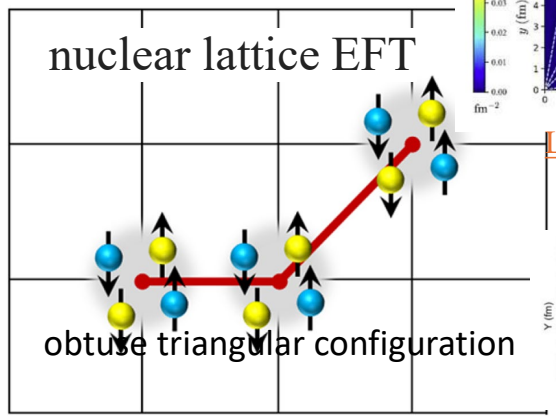


L.Fortunato, *Few-Body Syst* **65**, 1 (2024).

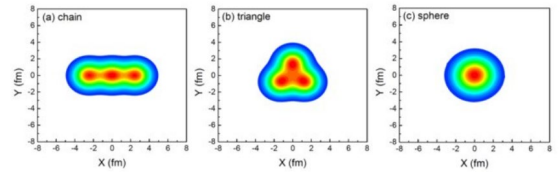




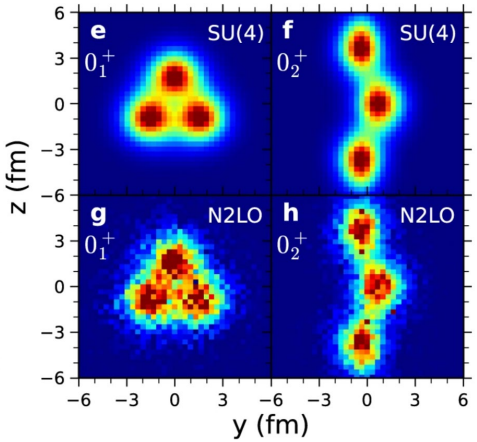
D J Marín-Lámbarri,et al., PRL 113, 012502 (2014)



E.Epelbaum,et al.,PRL 109,252501 (2012)



B. S. Huang, et al., PRC95,034606 (2017)

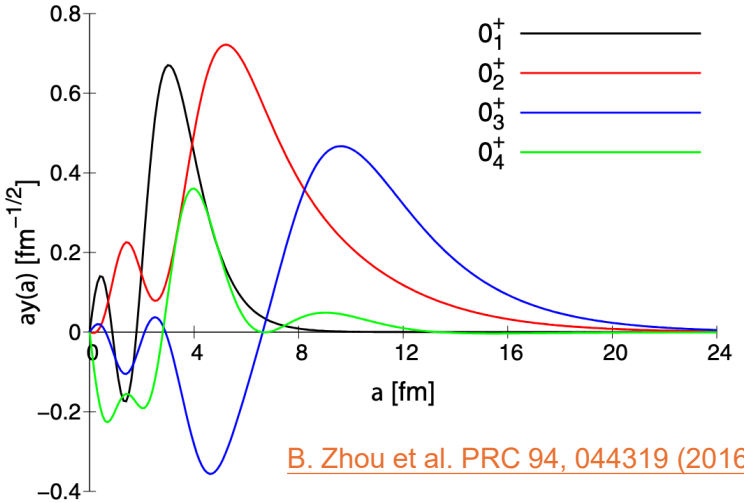


S.Shen, et al., *Nat Commun* **14**, 2777 (2023)

RWA and its three-body extension

Reduced-width amplitude (RWA)

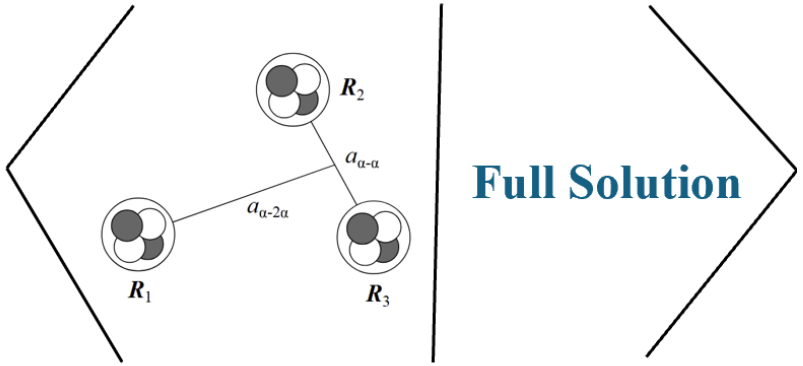
$$y_c^{J\pi}(a) = \sqrt{\frac{A!}{(1+\delta_{C_1C_2})C_1!C_2!}} \times \left\langle \frac{\delta(r-a)}{r^2} \left[Y_l(\hat{\boldsymbol{r}}) \otimes \left[\Phi_{C_1}^{j_1\pi_1} \otimes \Phi_{C_2}^{j_2\pi_2} \right]_{j_{12}} \right]_{JM} \middle| \Psi_M^{J\pi} \right\rangle$$



B. Zhou et al. PRC 94, 044319 (2016).

Two-cluster overlap amplitude (TCOA)

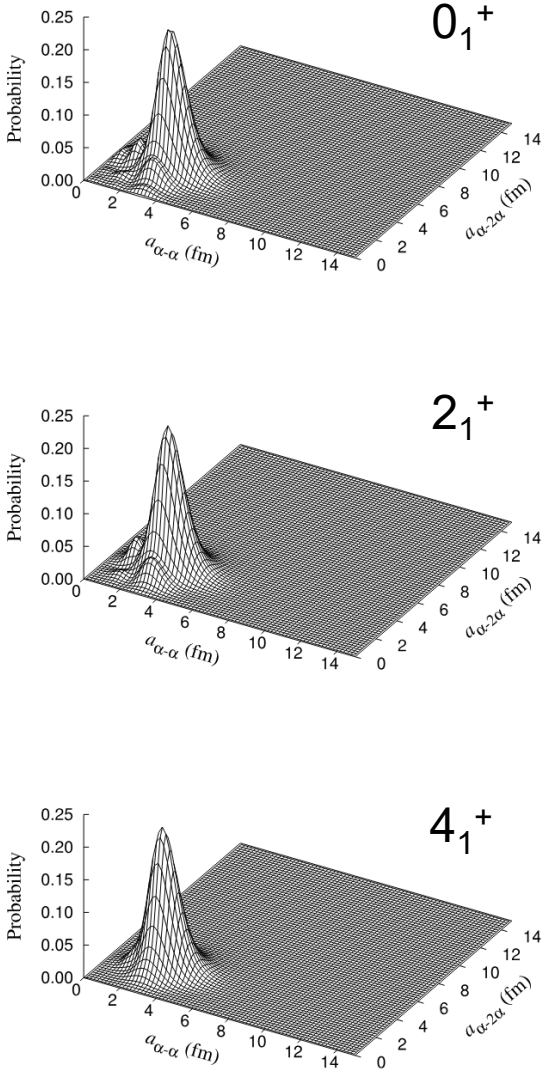
$$\mathcal{Y}_c^{J\pi}(a_1, a_{23}) = \sqrt{\frac{A!}{C_1!C_2!C_3!}} \left\langle \frac{\delta(r_1-a_1)\delta(r_{23}-a_{23})}{r_1^2 r_{23}^2} \times \left[[Y_{l_1}(\hat{r}_1) \otimes Y_{l_{23}}(\hat{r}_{23})]_L \otimes \left[\Phi_{C_1}^{j_1\pi_1} \otimes \left[\Phi_{C_2}^{j_2\pi_2} \otimes \Phi_{C_3}^{j_3\pi_3} \right]_{j_{23}} \right]_{j_{123}} \right]_{JM} \middle| \Psi_M^{J\pi} \right\rangle$$



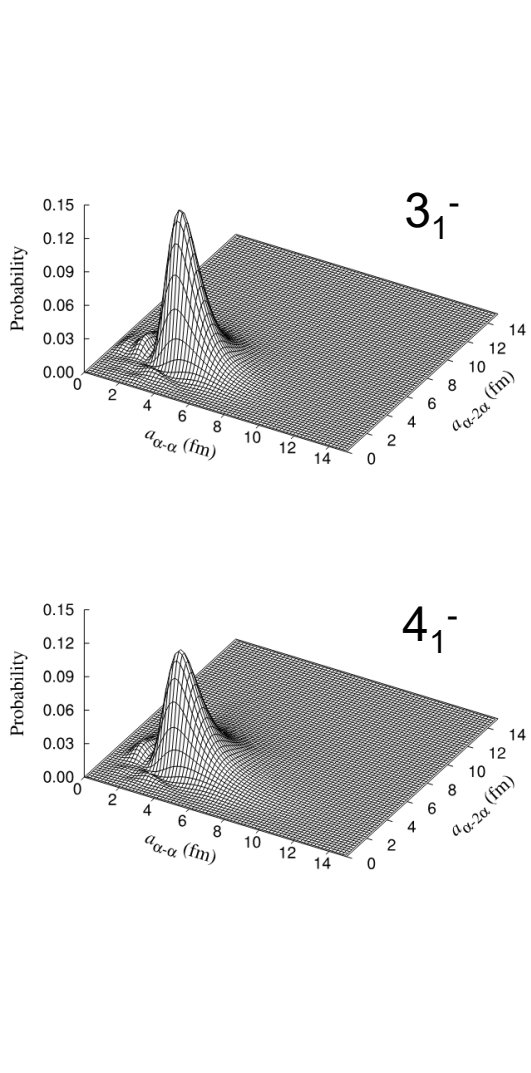
Deye Tao et al., NST 36, 56 (2025).

Visualization of rotational band configurations

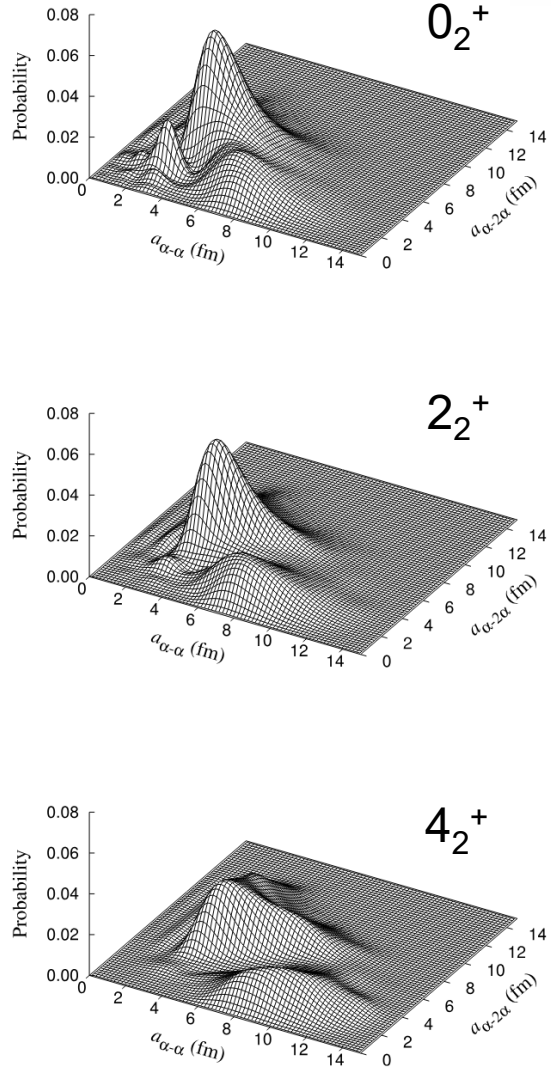
De-Ye Tao, B. Zhou, Y-G. Ma, PLB 874, 140241 (2026)



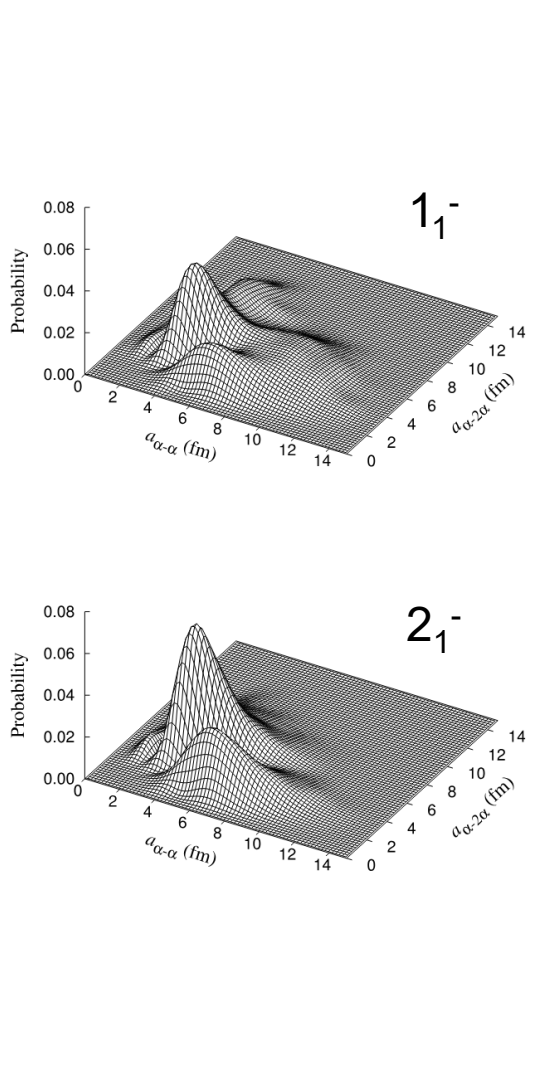
Ground band



$K^\pi=3^-$ band

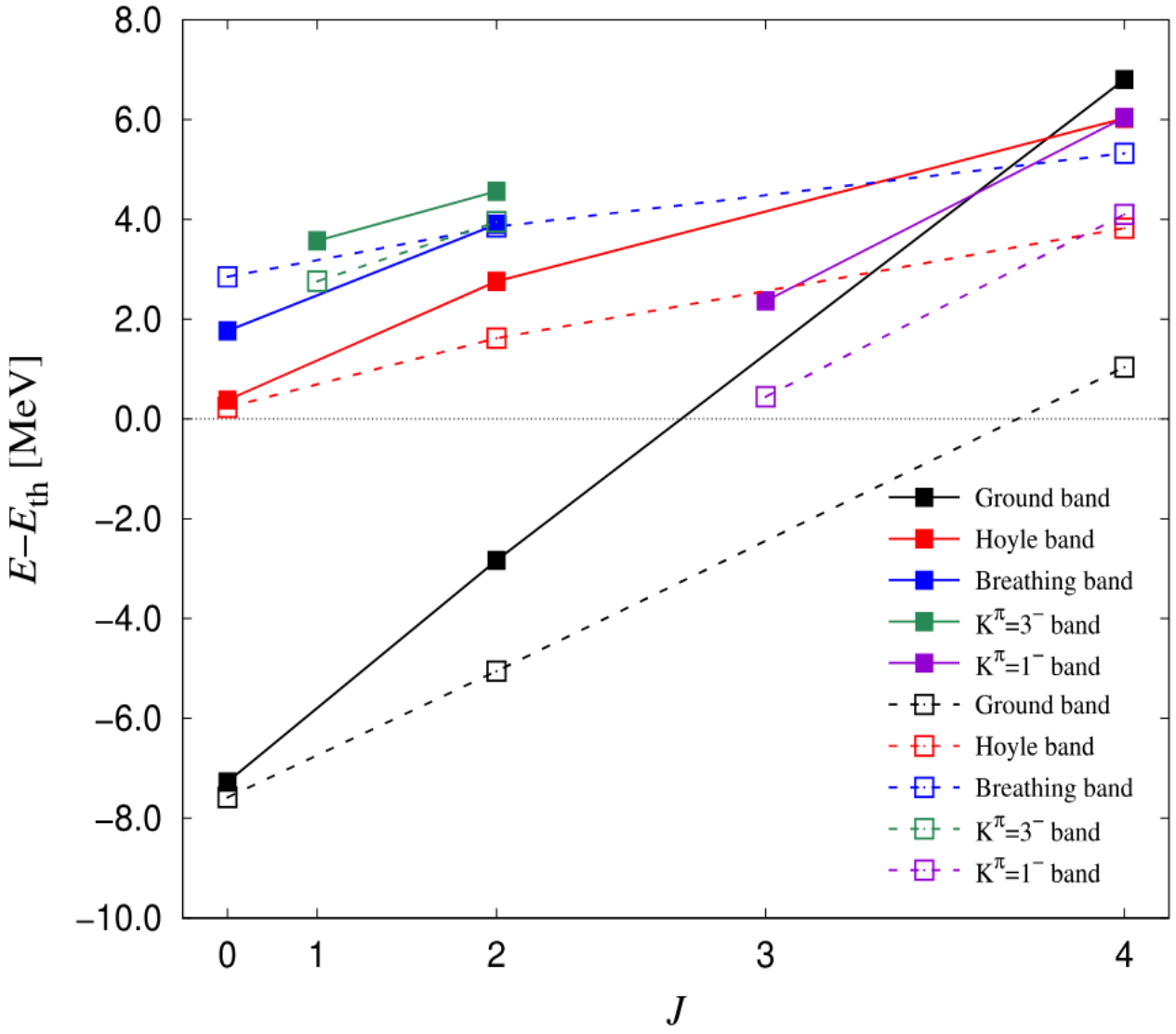


Hoyle band



$K^\pi=1^-$ band

Visualization of rotational band configurations

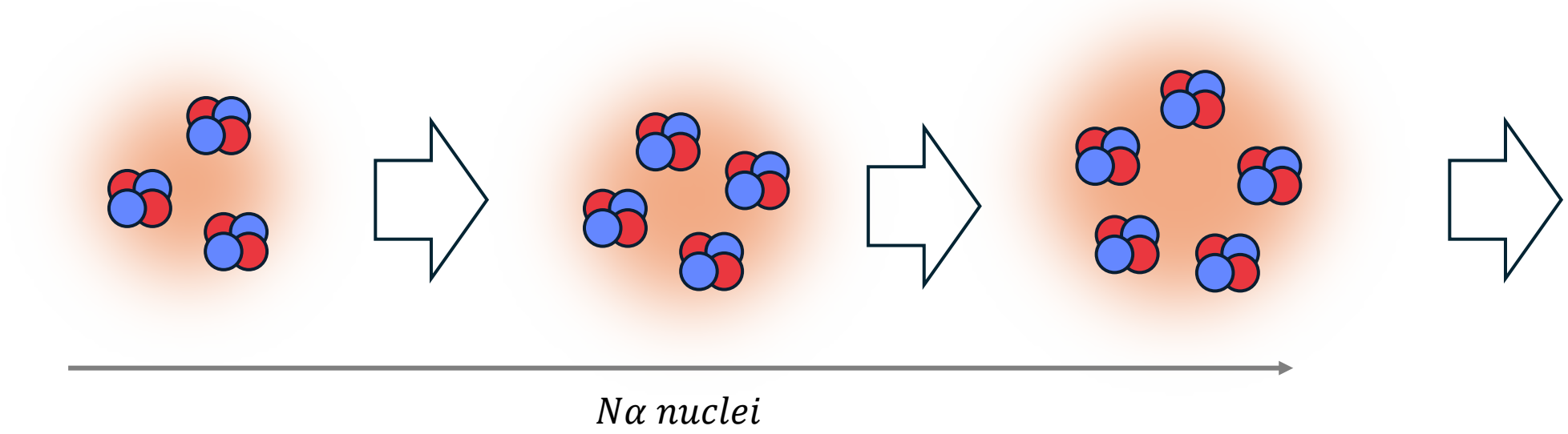


Based directly on the intrinsic clustering structure, we identify the rotation band classification:

- Ground band: 0_1^+, 2_1^+, 4_1^+
- Hoyle band: 0_2^+, 2_2^+, 4_2^+
- Breathing band: 0_3^+, 2_3^+, 4_3^+
- $K^\pi=3^-$ band: 3_1^-, 4_1^-
- $K^\pi=1^-$ band: 1_1^-, 2_1^-

The TCOA analysis of rotational bands can be further extended to other light nuclei.

De-Ye Tao, B. Zhou, Y-G. Ma, PLB 874, 140241 (2026)



$N\alpha$ *nuclei*

Search for the 5α novel cluster states

gas-like cluster state

no-geometry shape

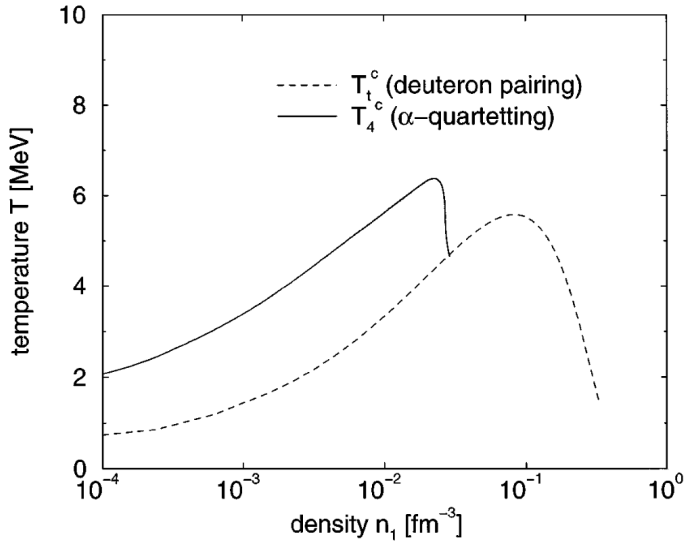
excited states

Four-Particle Condensate in Strongly Coupled Fermion Systems

G. Röpke, A. Schnell, and P. Schuck*
University of Rostock, FB Physik, Universitätsplatz 1, 18051 Rostock, Germany

P. Nozières
Institut Laue-Langevin, B.P. 156, 38042 Grenoble Cedex 9, France
(Received 25 September 1997)

Four-particle correlations in fermion systems at finite temperatures are investigated with special attention to the formation of a condensate. Instead of the instability of the normal state with respect to the onset of pairing described by the Gorkov equation, a new equation is obtained which describes the onset of "quartetting." Applying to symmetric nuclear matter, below a critical density, the four-particle condensation (α-like quartetting) is found to be favored over deuteron condensation (triplet pairing). This pairing-quartetting competition is expected to be a general feature of interacting fermion systems, such as the exciton-biexciton system in excited semiconductors. Possible experimental consequences are pointed out. [S0031-9007(98)05829-3]



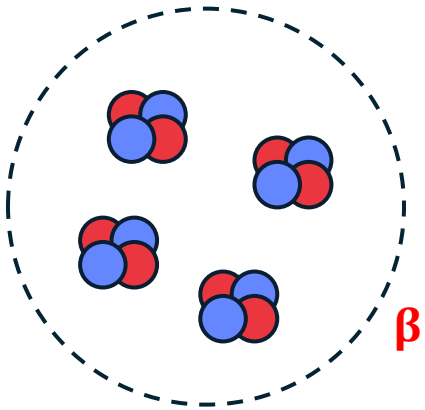
Clustering formation in low density in nuclear matter

THSR wave function

Alpha Cluster Condensation in ^{12}C and ^{16}O

A. Tohsaki,[1] H. Horiuchi,[2] P. Schuck,[3] and G. Röpke[4]
[1]*Department of Fine Materials Engineering, Shinshu University, Ueda 386-8567, Japan*
[2]*Department of Physics, Kyoto University, Kyoto 606-8502, Japan*
[3]*Institut de Physique Nucléaire, F-91406 Orsay Cedex, France*
[4]*FB Physik, Universität Rostock, D-18051 Rostock, Germany*
(Received 29 June 2001; published 17 October 2001)

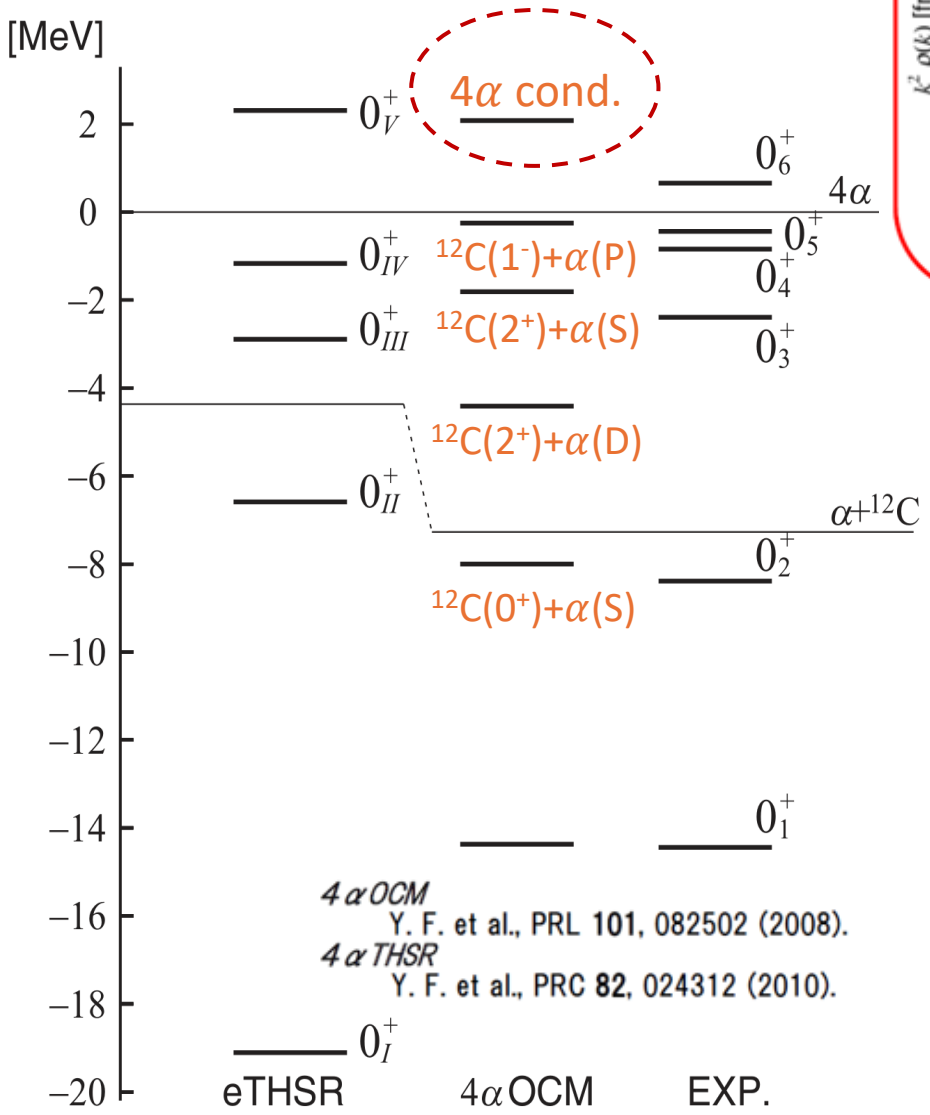
A new α-cluster wave function is proposed which is of the α-particle condensate type. Applications to ^{12}C and ^{16}O show that states of low density close to the 3 and 4 α-particle thresholds in both nuclei are possibly of this kind. It is conjectured that all self-conjugate $4n$ nuclei may show similar features.

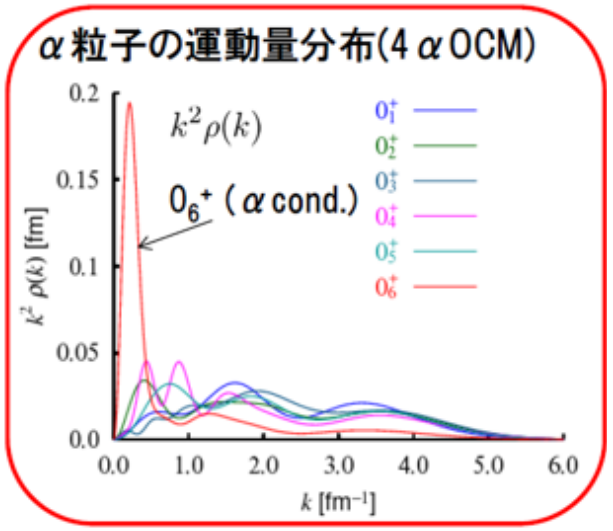


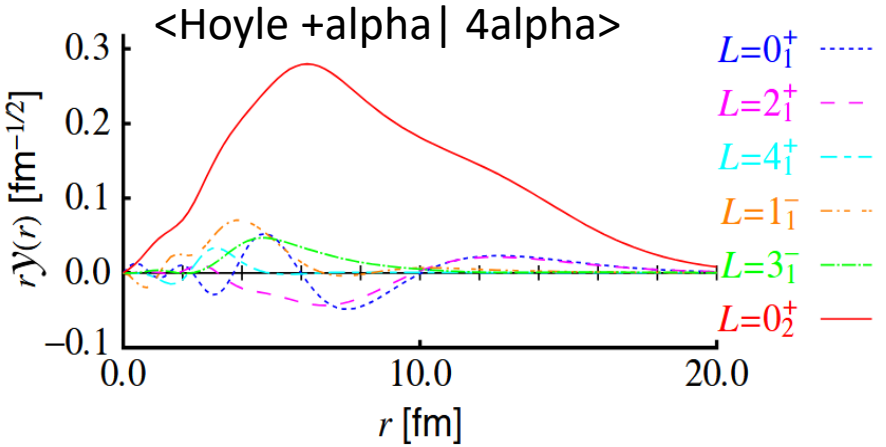
$$\langle \mathbf{r}_1\sigma_1\tau_1,\ldots,\mathbf{r}_{4n}\sigma_{4n}\tau_{4n}|\Phi_{n\alpha}\rangle \propto \mathcal{A}\{e^{-\frac{2}{B^2}(\mathbf{X}_1^2+\cdots+\mathbf{X}_n^2)}\phi(\alpha_1)\cdots\phi(\alpha_n)\},$$

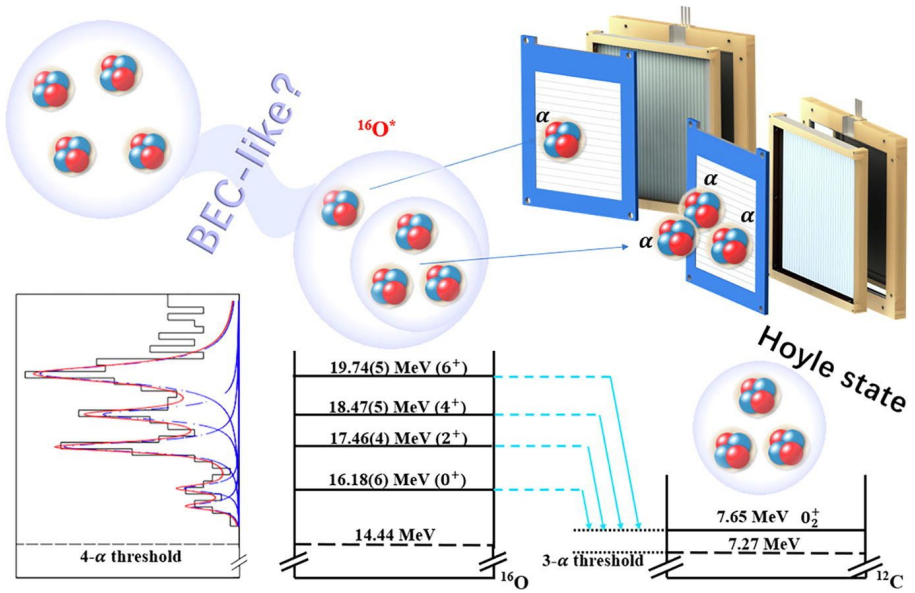
Alpha condensate in finite nuclei, e.g., ^{12}C, ^{16}O.

4α condensate state

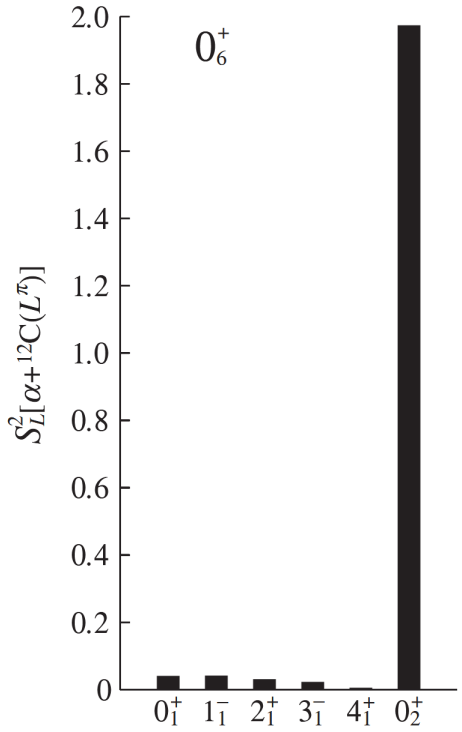








Jiahao Chen et al., Science Bulletin, 68 (2023)



*Yamada et al.PRC **85**, 034315*

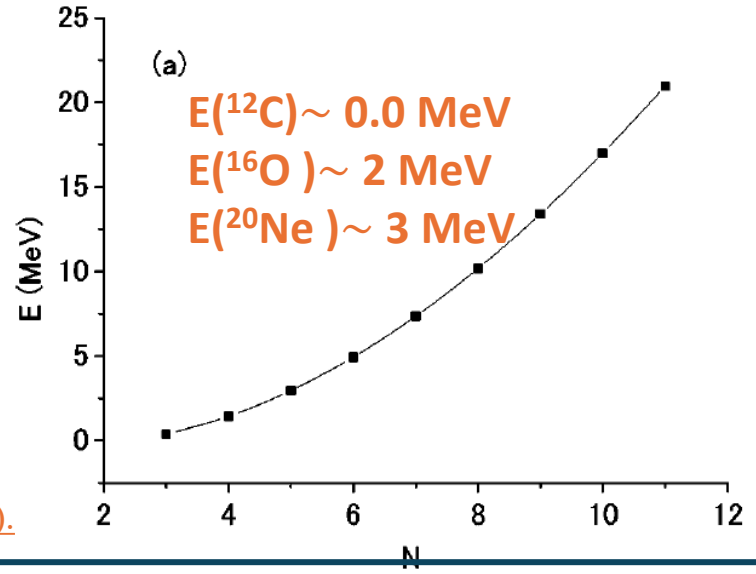
Multi-alpha condensation

Dilute multi-α cluster condensed states with spherical and axially deformed shapes are studied with the Gross-Pitaevskii equation and Hill-Wheeler equation where the α cluster is treated as a structureless boson,

it is predicted to exist in heavier self-conjugate 4*N* nuclei up to *N*=10.

$$-\frac{\hbar^2}{2m}\left(1-\frac{1}{N}\right)\nabla^2\varphi(\boldsymbol{r})+U(\boldsymbol{r})\varphi(\boldsymbol{r})=\varepsilon\varphi(\boldsymbol{r})$$

T. Yamada and P. Schuck, Phys. Rev. C 69, 024309 (2004).



Some candidates for α condensate were found from experiments for ^{12}C and ^{16}O.

Rev. Mod. Phys. **89**, 011002 (2017).

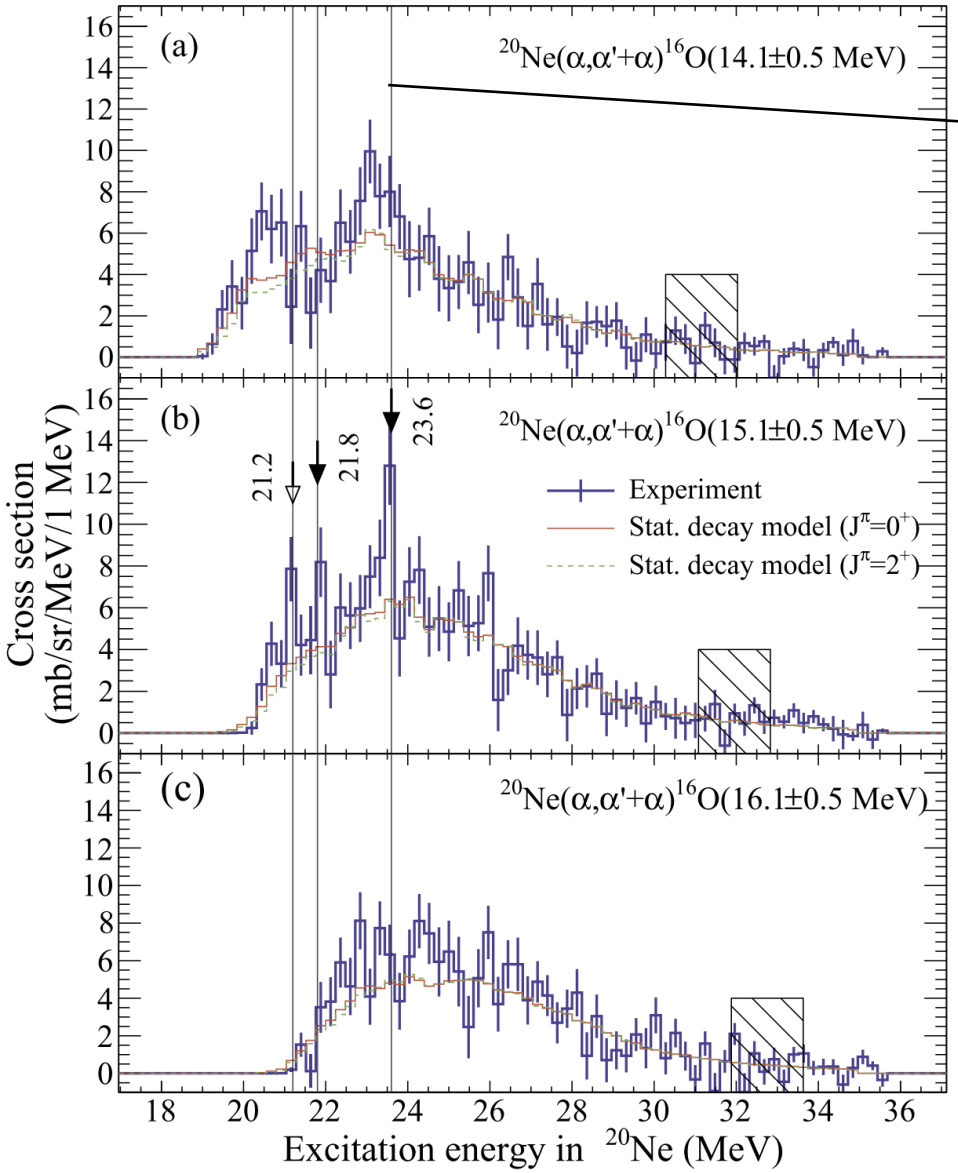
No experimental signatures for α condensation were observed

Phys. Rev. C **100**, 034320 (2019)

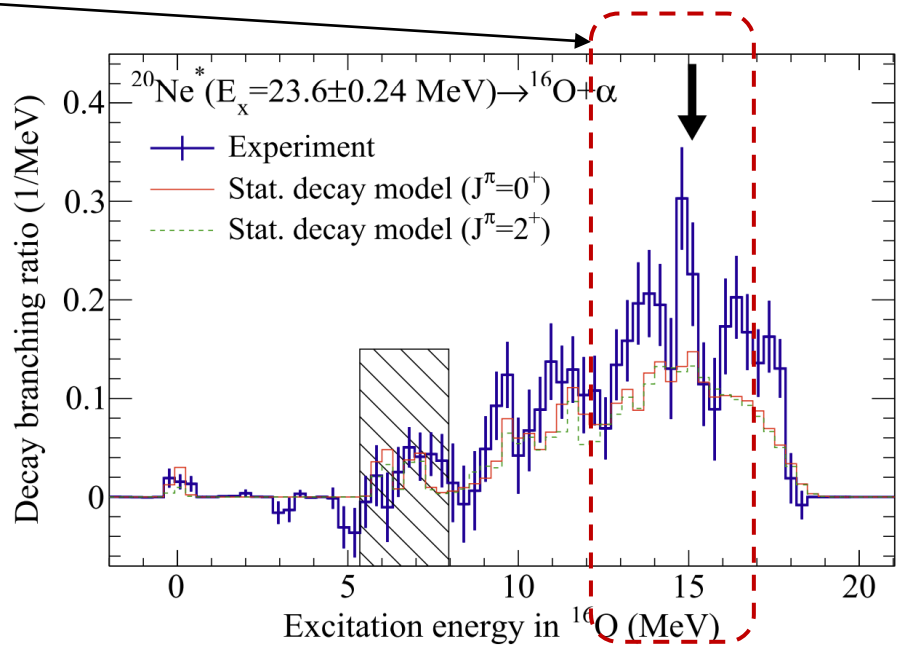
An experimental way of testing Bose-Einstein condensation of clusters in the atomic nucleus is reported. The enhancement of cluster emission and the multiplicity partition of possible emitted clusters could be direct signatures for the condensed states.

PRL **96,** 192502 (2006)

Experiment for 5α condensation



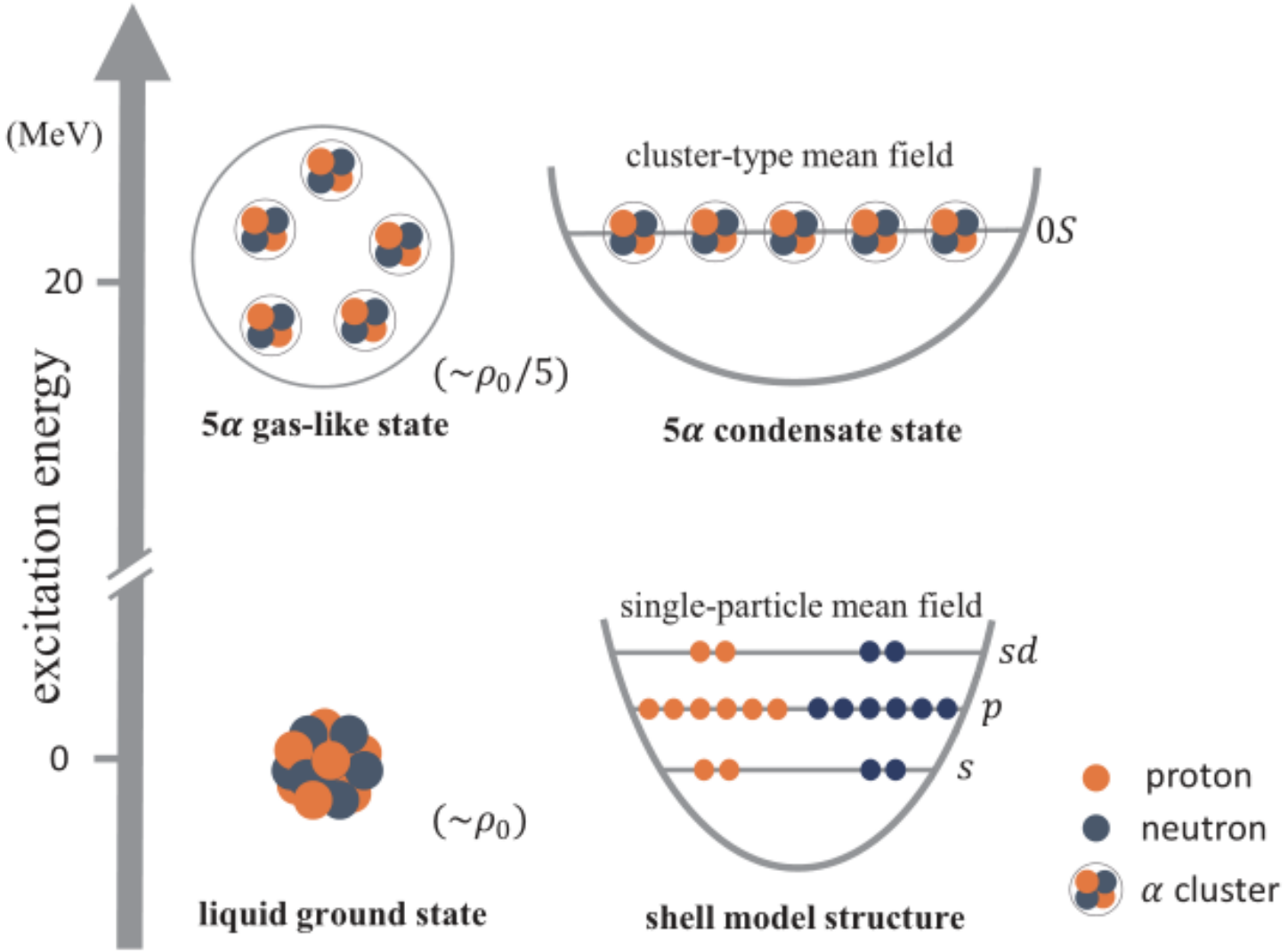
23.6-MeV state enhances in the decay to the ^{16}O (0^{+}_{6}) +α channel

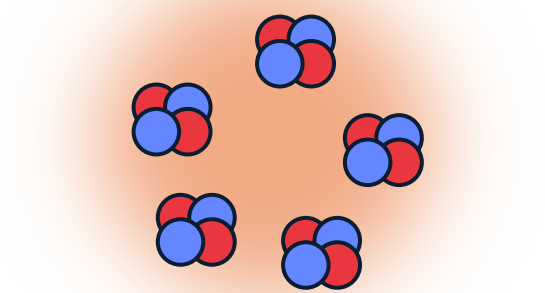


- 4.4 MeV above the 5α threshold
- strongly coupled to ^{16}O(0^{+}_{6}) state

S. Adachi et al., *Candidates for the 5α condensed state in ^{20}Ne*, PLB **819**,136411 (2021).

Search for the 5alpha condensate state



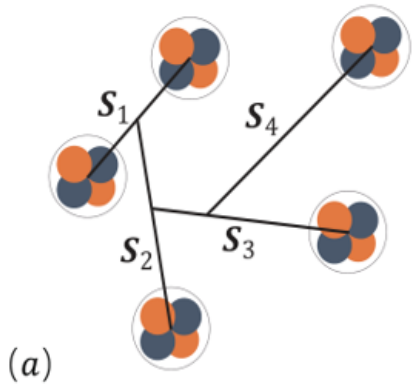


5α BEC state ?

Perform the 5α calculations and study the characters of obtained excited 0^+ states

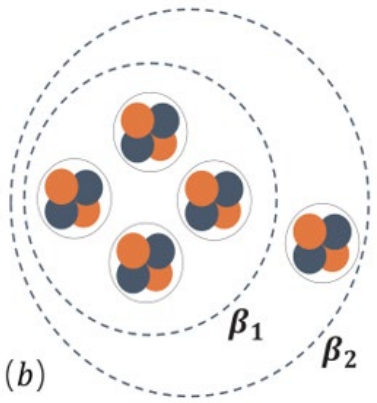
The 5alpha microscopic wave function

configurations problem:



(a)

Conventional cluster model



(b)

Container model

$$\Psi(\beta_1,\beta_2)=\int d^3R_1d^3R_2d^3R_3d^3R_4d^3R_5\ \mathrm{Exp}\left[-\frac{1/2S_1^2+2/3S_2^2+3/4S_3^2}{\beta_1^2}-\frac{4/5S_4^2}{\beta_2^2}\right]\Phi^{\mathrm{B}}(R_1,R_2,R_3,R_4,R_5)$$

$$=n_0\,\mathscr{A}\left\{\mathrm{Exp}\left[-\frac{2\xi_1^2+8/3\xi_2^2+3\xi_3^2}{2(b^2+2\beta_1^2)}\right]\mathrm{Exp}\left[-\frac{16/5\xi_4^2}{2(b^2+2\beta_2^2)}\right]\prod_{i=1}^{5}\varphi_i^{\mathrm{int}}(\alpha)\right\},$$

where the conventional Brink cluster wave function Φ^{B},

$$\Phi^{\mathrm{B}}(R_1,R_2,R_3,R_4,R_5)=\frac{1}{\sqrt{20!}}\mathscr{A}[\phi_1(R_1)...\phi_5(R_2)...\phi_{20}(R_5)],$$

$$\propto\phi_g\,\mathscr{A}\left\{\mathrm{Exp}\left[-\frac{2(\xi_1-S_1)^2+8/3(\xi_2-S_2)^2+3(\xi_3-S_3)^2}{2b^2}\right]\mathrm{Exp}\left[-\frac{16/5(\xi_4-S_4)^2}{2b^2}\right]\prod_{i=1}^{5}\varphi_i^{\mathrm{int}}(\alpha)\right\},$$

with the single-nucleon wave function,

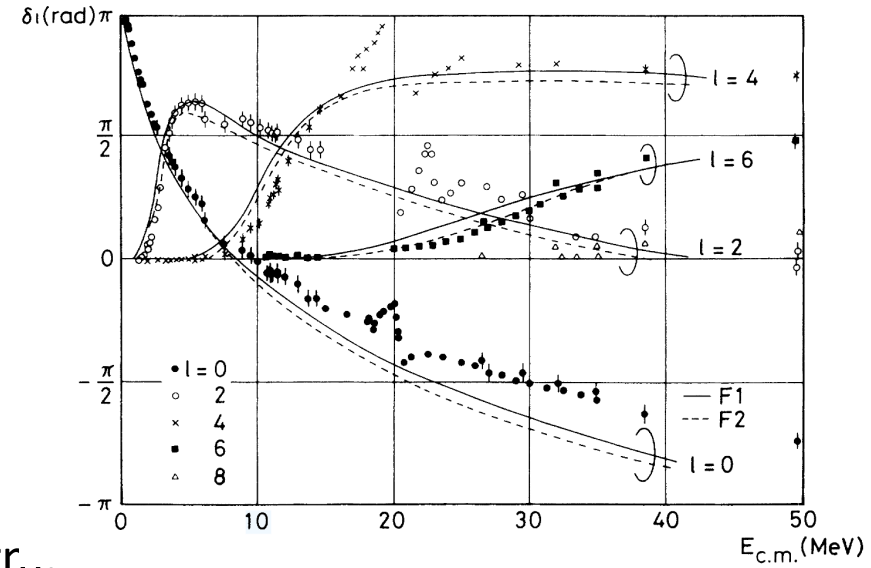
$$\phi_i(R_k)=\left(\frac{1}{\pi b^2}\right)^{\frac{3}{4}}e^{-\frac{1}{2b^2}(r_i-R_k)^2}\chi_i\tau_i.$$

Three-body effective interaction

interaction problem:

The Hamiltonian for ^{20}Ne in this work can be written as:

$$\mathcal{H} = -\frac{\hbar^2}{2M}\sum_i \nabla_i^2 - T_G + \sum_{i<j} V_{ij}^C + \sum_{i<j} V_{ij}^{(2)} + \sum_{i<j<k} V_{ijk}^{(3)},$$



The effective nucleon-nucleon potential part is taken a Gaussian forr.., which is expressed as:

$$V_{ij}^{(2)} = \sum_n v_n^{(2)} \exp\left\{-\left(\frac{r_{ij}}{r_n^{(2)}}\right)^2\right\}(W_n^{(2)} + M_n^{(2)} P_{ij})$$

and

$$V_{ijk}^{(3)} = \sum_n v_n^{(3)} \exp\left\{-\left(\frac{r_{ij}}{r_n^{(3)}}\right)^2 - \left(\frac{r_{jk}}{r_n^{(3)}}\right)^2\right\} \times (W_n^{(3)} + M_n^{(3)} P_{ij})(W_n^{(3)} + M_n^{(3)} P_{jk}),$$

TABLE III. Physical quantities for double closed shell nuclei together with nuclear matter.

		F1	F2	BB	SII	Expt.
α	$E_{\rm bin}$ (MeV)	27.5	27.0	27.4	26.0	28.3
	a (fm^{-2})	0.50	0.50	0.50	0.41	0.51
^{16}O	$E_{\rm bin}$ (MeV)	123.0	124.0	93.0	121.4	127.6
	a (fm^{-2})	0.35	0.35	0.31	0.32	0.34
	$E_{\rm mp}$ (MeV)	23.3	27.7	23.2	32.7	26.4
^{40}Ca	$E_{\rm bin}$ (MeV)	334.0	340.2	250.8	325.5	342.1
	a (fm^{-2})	0.26	0.27	0.25	0.26	0.27
	$E_{\rm mp}$ (MeV)	20.8	24.1	19.0	27.0	23.5
NM	E/A (MeV)	17.0	16.7	15.7	16.0	16.0
	k_f (fm^{-1})	1.27	1.29	1.45	1.30	1.30
	K (MeV)	309	332	193	342	300

Tohsaki F1 three-body interaction was used.

A. Tohsaki,Phys. Rev. C **49**, 1814 (1994).

Radius-Constraint Method + Stabilization Method

resonance problem:

Radius-Constraint Method,

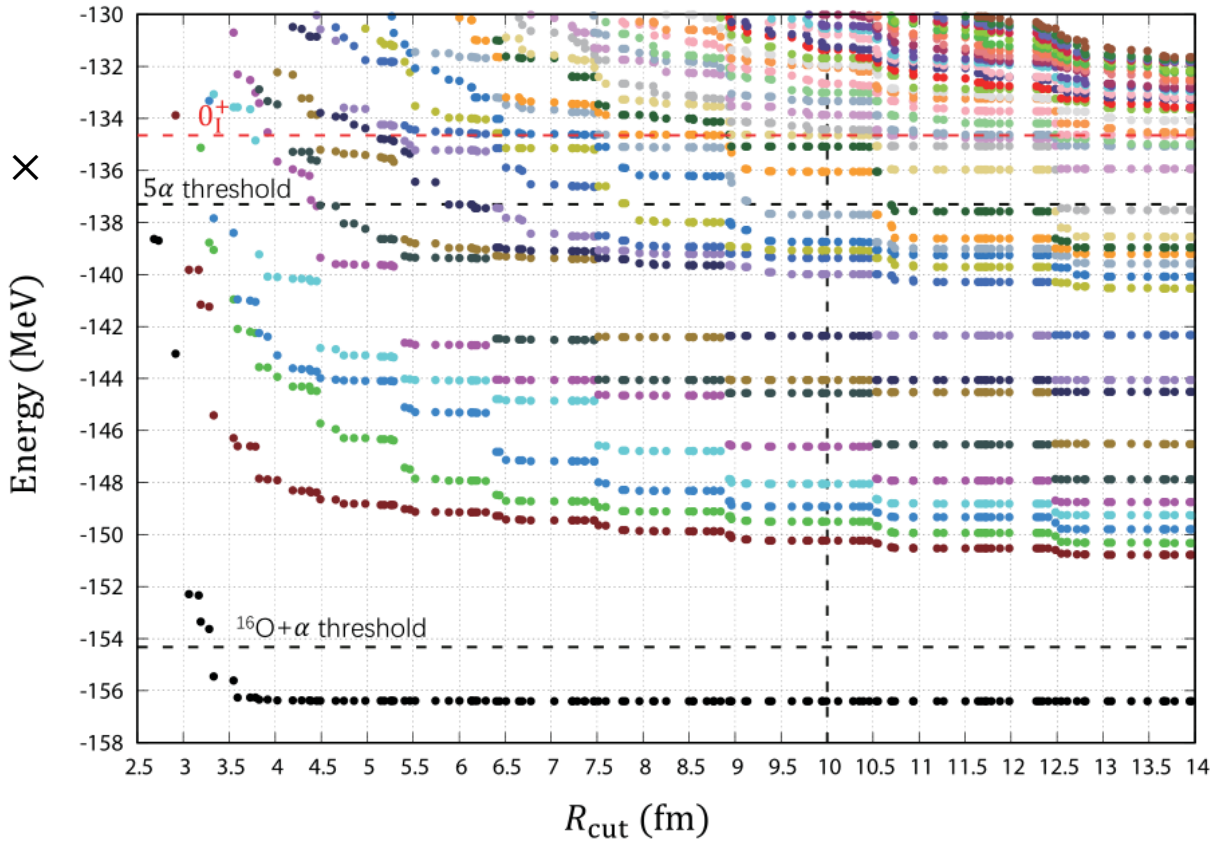
$$\sum_{\beta_1',\beta_2'} \left\langle \hat{\Phi}^{0^+}_{4\alpha+\alpha}(\beta_1,\beta_2) \middle| \sum_{i=1} \frac{1}{20}(r_i - X_G)^2 \middle| \hat{\Phi}^{0^+}_{4\alpha+\alpha}(\beta_1',\beta_2') \right\rangle \times g^{(\gamma)}(\beta_1',\beta_2')$$

$$= \{R^{(\gamma)}\} g^{(\gamma)}(\beta_1,\beta_2) \left\langle \hat{\Phi}^{0^+}_{4\alpha+\alpha}(\beta_1,\beta_2) \middle| \hat{\Phi}^{0^+}_{4\alpha+\alpha}(\beta_1',\beta_2') \right\rangle$$

$$\Psi^{0^+}_{GCM} = \sum_{\beta_1,\beta_2} g^{(\gamma)}(\beta_1,\beta_2) \hat{\Phi}^{0^+}_{4\alpha+\alpha}(\beta_1',\beta_2')$$

Here ,$R^{(\gamma)} \leq R_{cut}$ and R_{cut} is the radius cut-off parameter.

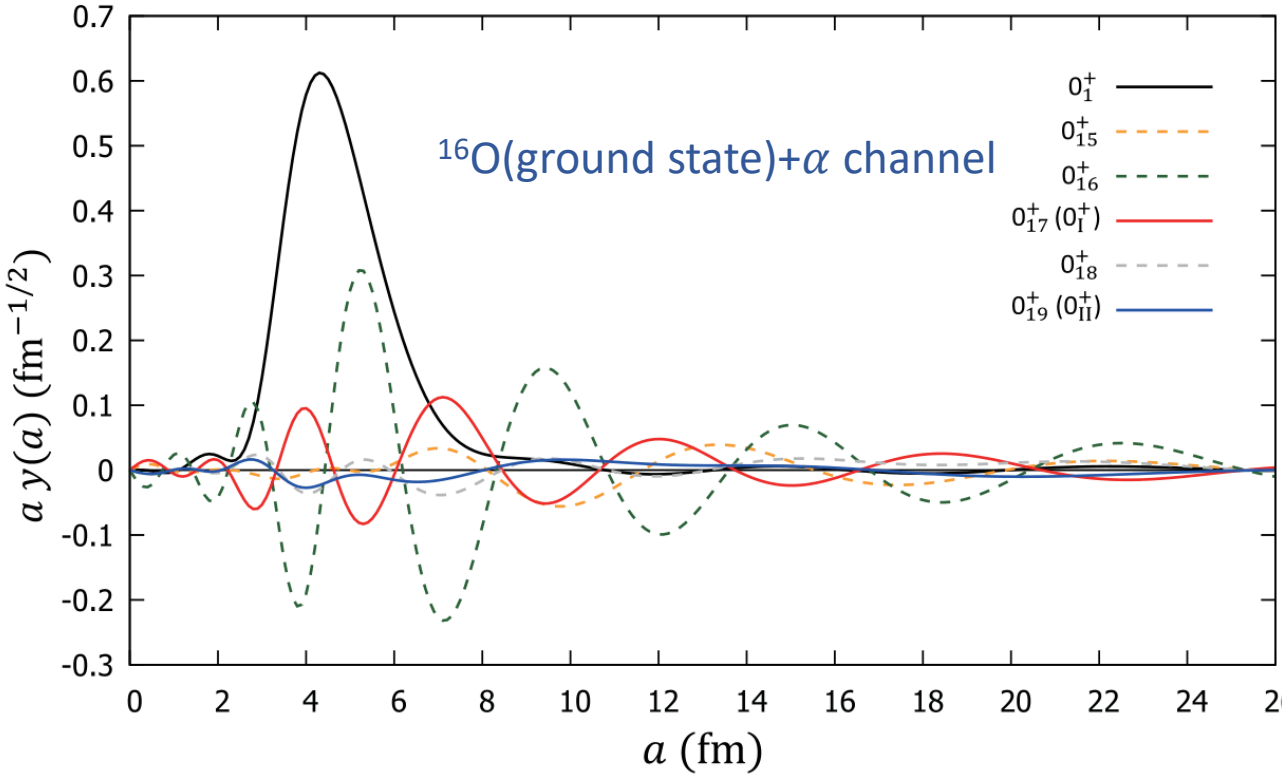
Stabilization Method

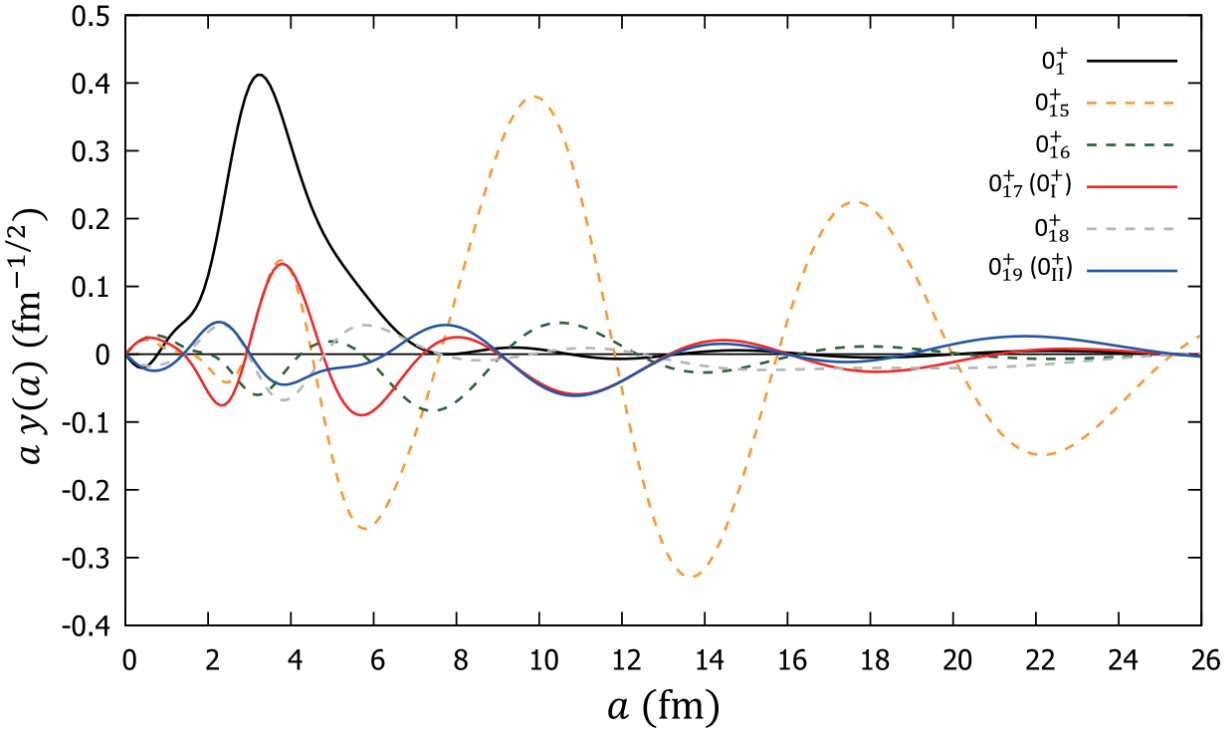


Above the 5alpha threshold: $0^+_{15} \sim 0^+_{19}$

Five 0^+ states above the threshold

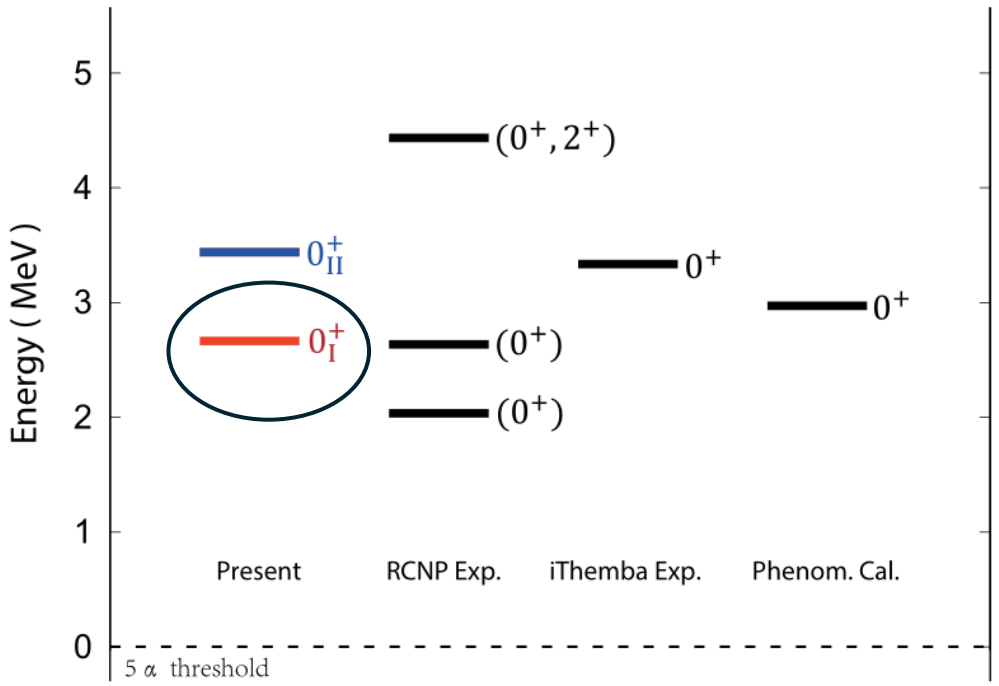
resonance problem



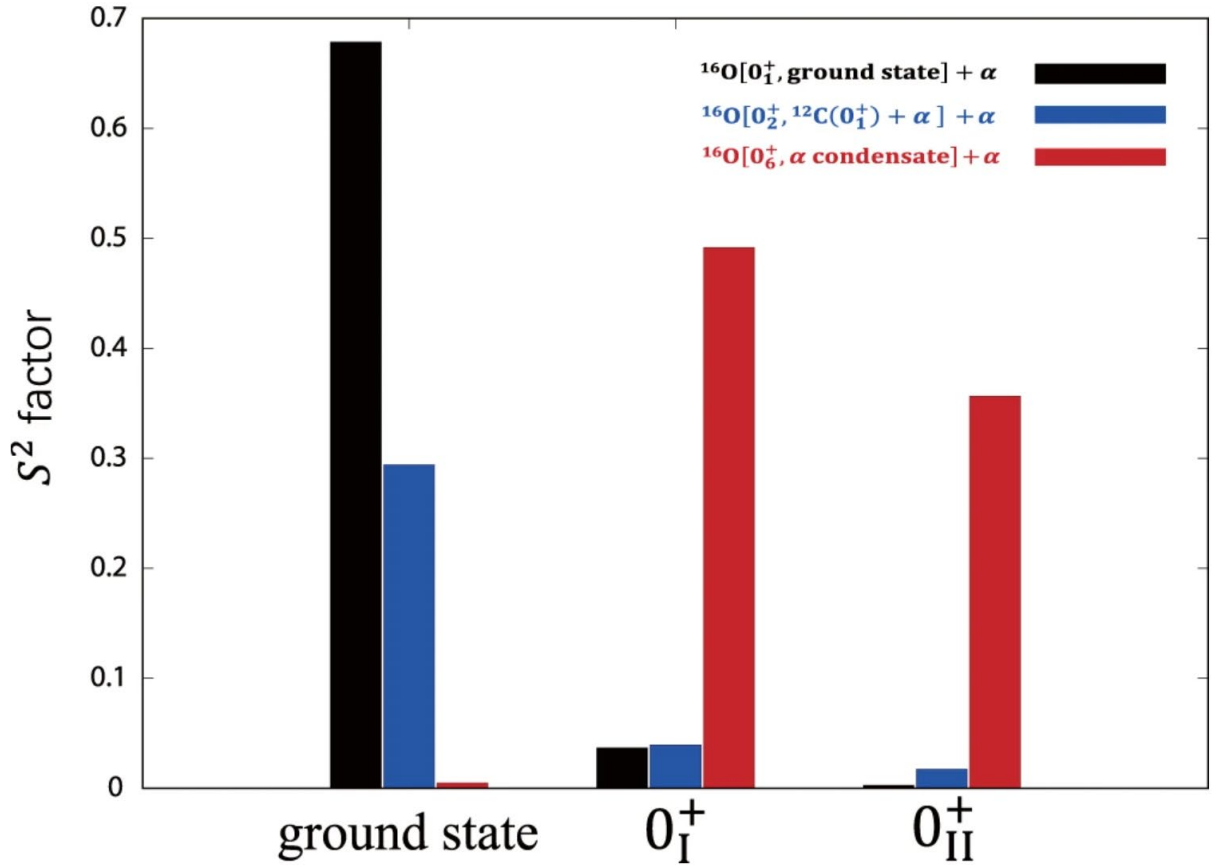


$$y(a) = \sqrt{\frac{20!}{4!16!}} \left\langle \left[\left[\Psi^{0^+_s}_{\rm gcm}(^{16}{\rm O})\varphi_5(\alpha) \right]_{0^+} Y_{00}(\hat{\xi}_4) \right]_{0^+} \frac{\delta(\xi_4 - a)}{\xi_4^2} \middle| \Psi^{0^+_\lambda}_{\rm gcm}(^{20}{\rm Ne}) \right\rangle$$

Two of Five 0^+ states above the threshold



Two 0^+ states around 3 MeV are found.

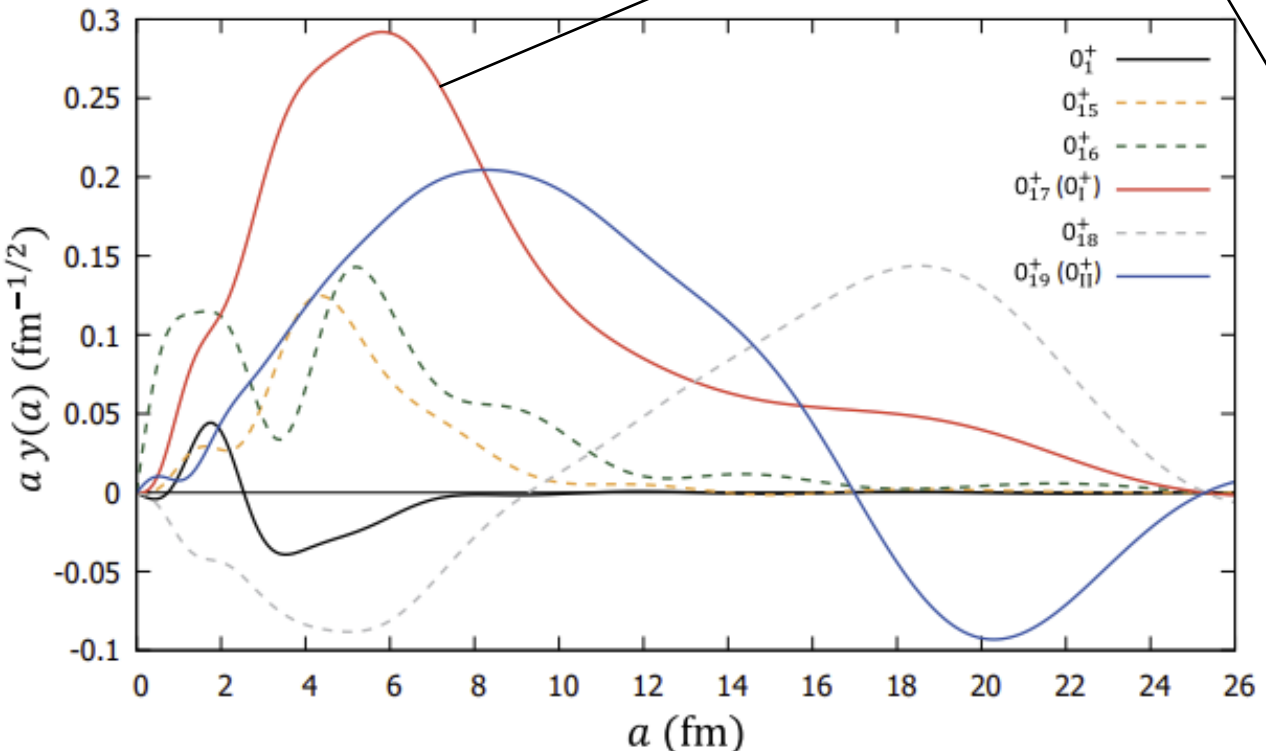


B. Zhou, et al., *Nat. Commun.* **14**, 8206 (2023).

The 5alpha condensate state

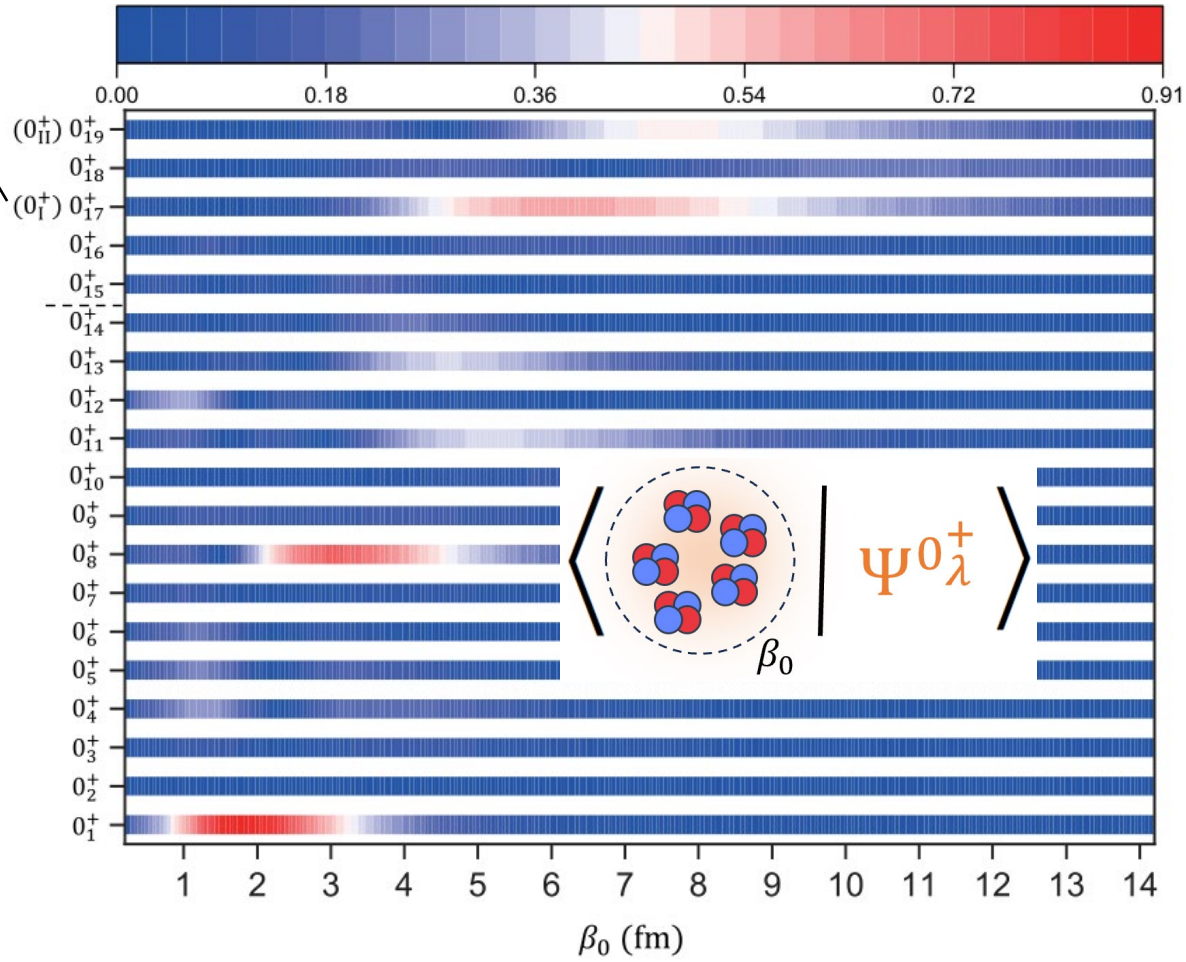
Reduced width amplitude

larger amplitude



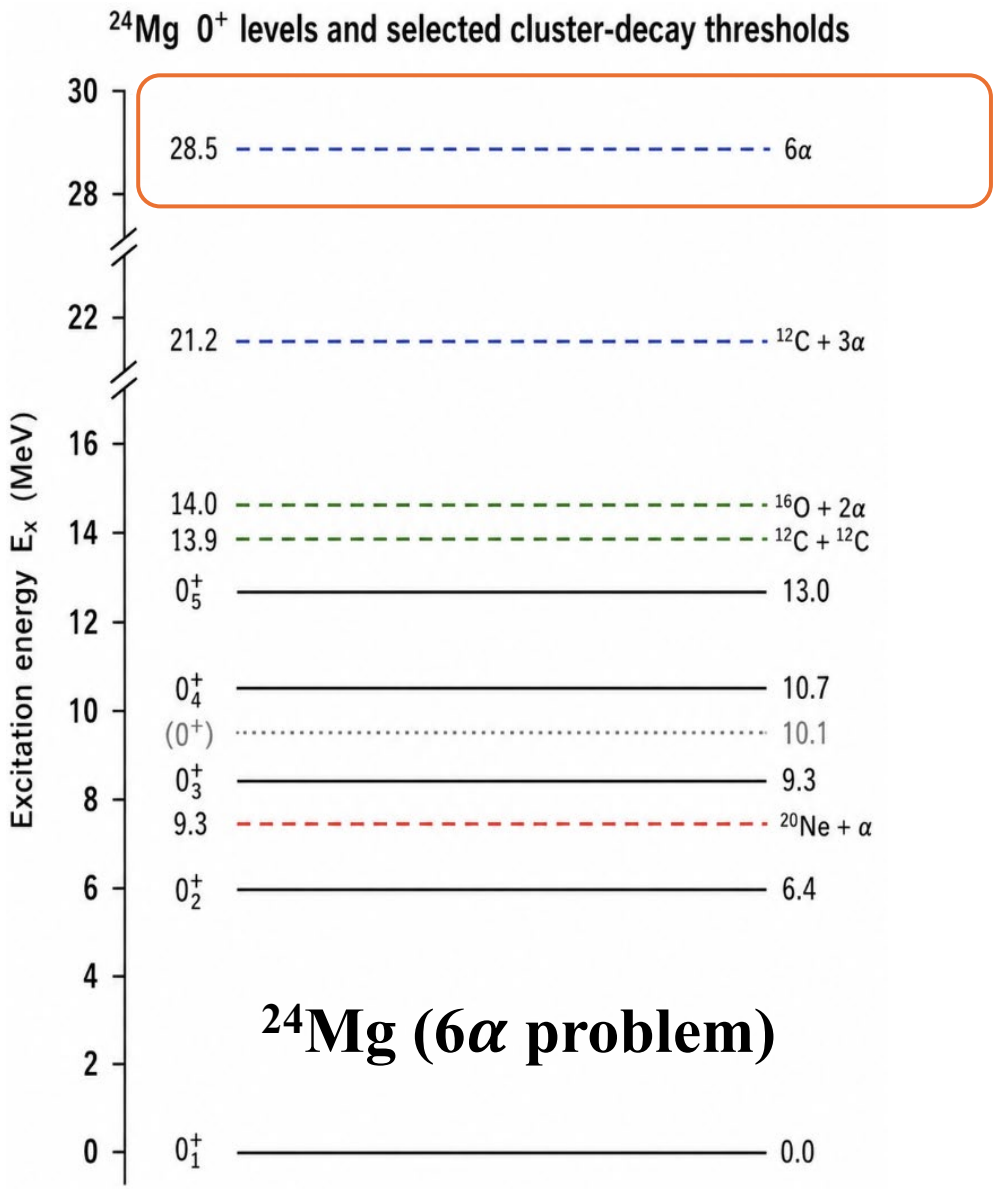
$$y(a) = \sqrt{\frac{20!}{4!16!}} \left\langle \left[\left[\Psi^{0_x^+}_{\mathrm{gcm}}(^{16}\mathrm{O})\varphi_5(\alpha) \right]_{0^+} Y_{00}(\hat{\xi}_4) \right]_{0^+} \frac{\delta(\xi_4 - a)}{\xi_4^2} \middle| \Psi^{0_\lambda^+}_{\mathrm{gcm}}(^{20}\mathrm{Ne}) \right\rangle$$

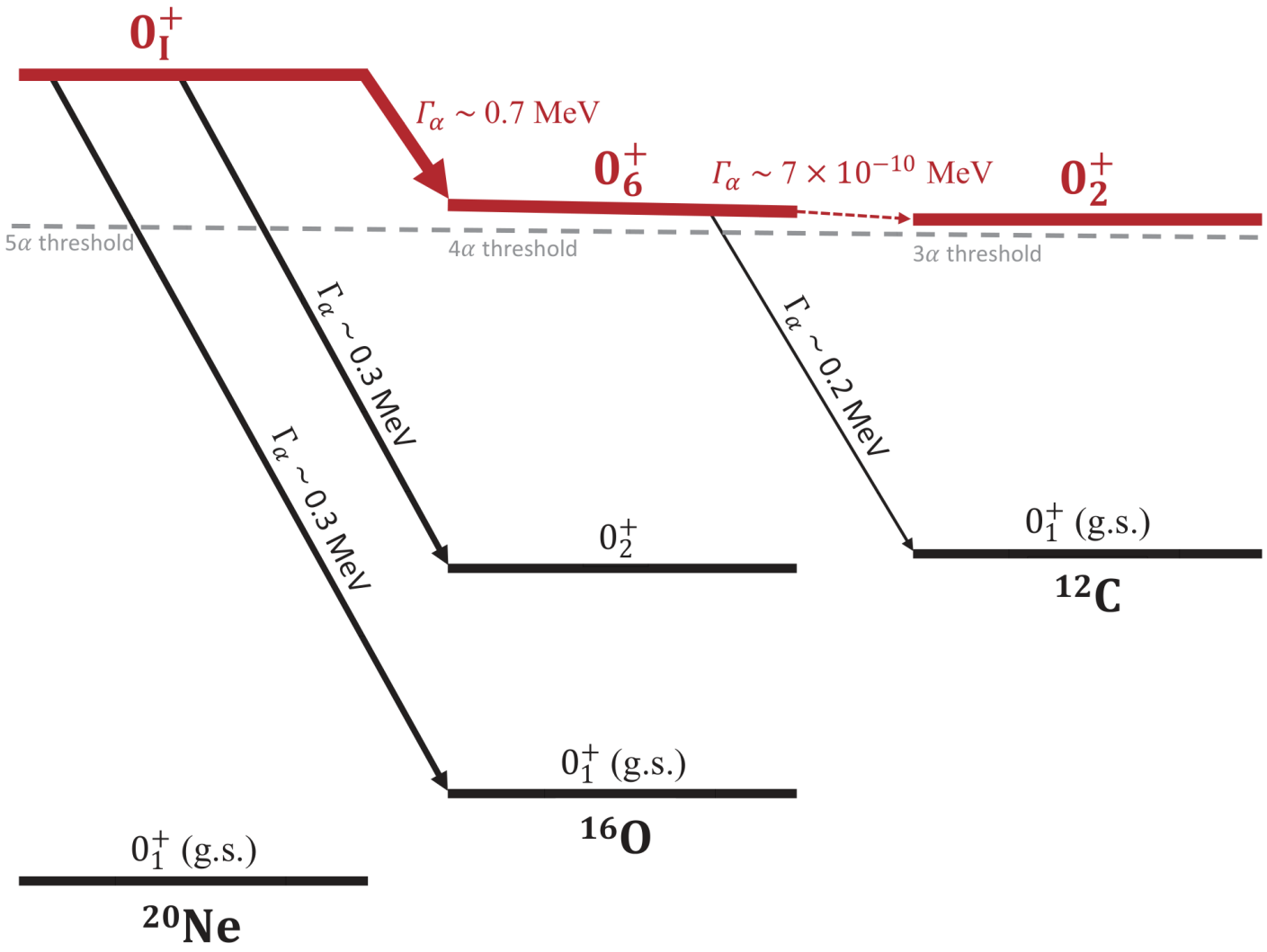
Simple way to confirm condensate state



B. Zhou, et al., *Nat. Commun.* **14**, 8206 (2023).

The decay scheme and connections



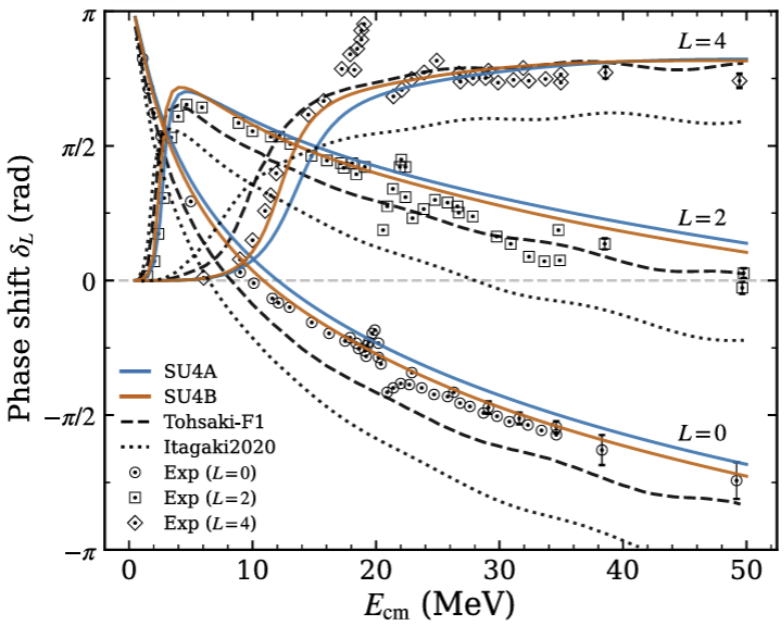


B. Zhou, et al., *Nat. Commun.* **14**, 8206 (2023).

Minimal Wigner-SU(4) Interaction for α-Conjugate Nuclei

$$H = \sum_i T_i - T_{\text{c.m.}} + \sum_{i<j} V_{ij}^{\text{C}} + V_{SU(4)}^{\text{N}}, \quad V_{SU(4)}^{\text{N}} = C \sum_{i<j} \exp\left(-\frac{\Lambda^2 r_{ij}^2}{4\hbar^2 c^2}\right) + D \sum_{i<j<k} \sum_{\text{cyc}} \exp\left(-\frac{\Lambda^2 (r_{ij}^2 + r_{jk}^2)}{4\hbar^2 c^2}\right)$$

$$\Psi_{n\alpha}^{\text{B}}(\{\boldsymbol{R}\}) = \mathcal{A} \prod_{i=1}^{n} \left[\prod_{m=1}^{4} \phi_m(\boldsymbol{r}_k; \boldsymbol{R}_i) \right] = \mathcal{A} \prod_{i=1}^{n} \psi_\alpha(\boldsymbol{R}_i).$$



Scattering phase shifts of the α+α system

	Λ	C	D
SU4A	500	-316.2	395.8
SU4B	400	-180.3	138.5

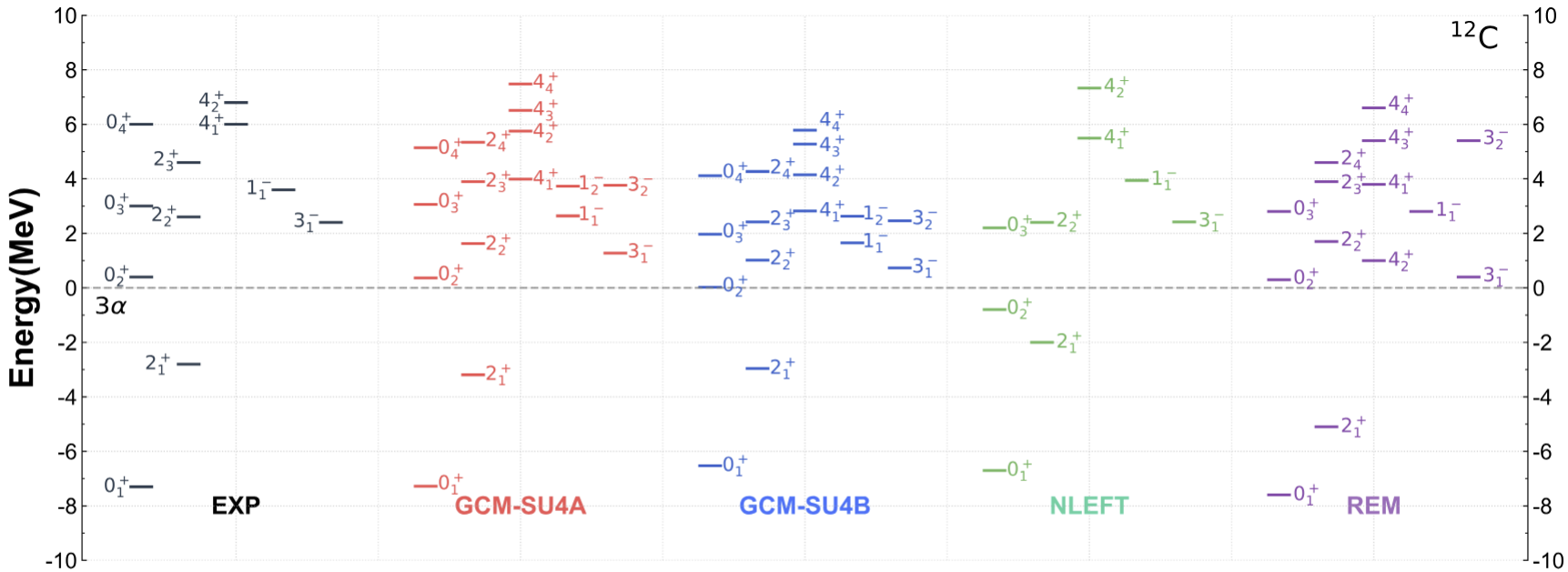


FIG. 2: Energy levels of ^{12}C, measured from the 3α threshold, are presented separately according to their spin-parity J^π. The GCM results utilizing the SU4A and SU4B interactions, NLEFT [36], and REM [50] are compared with experimental data [49].

Guoping Li, et al., arXiv:2605.21228

Minimal Wigner-SU(4) Interaction for α-Conjugate Nuclei

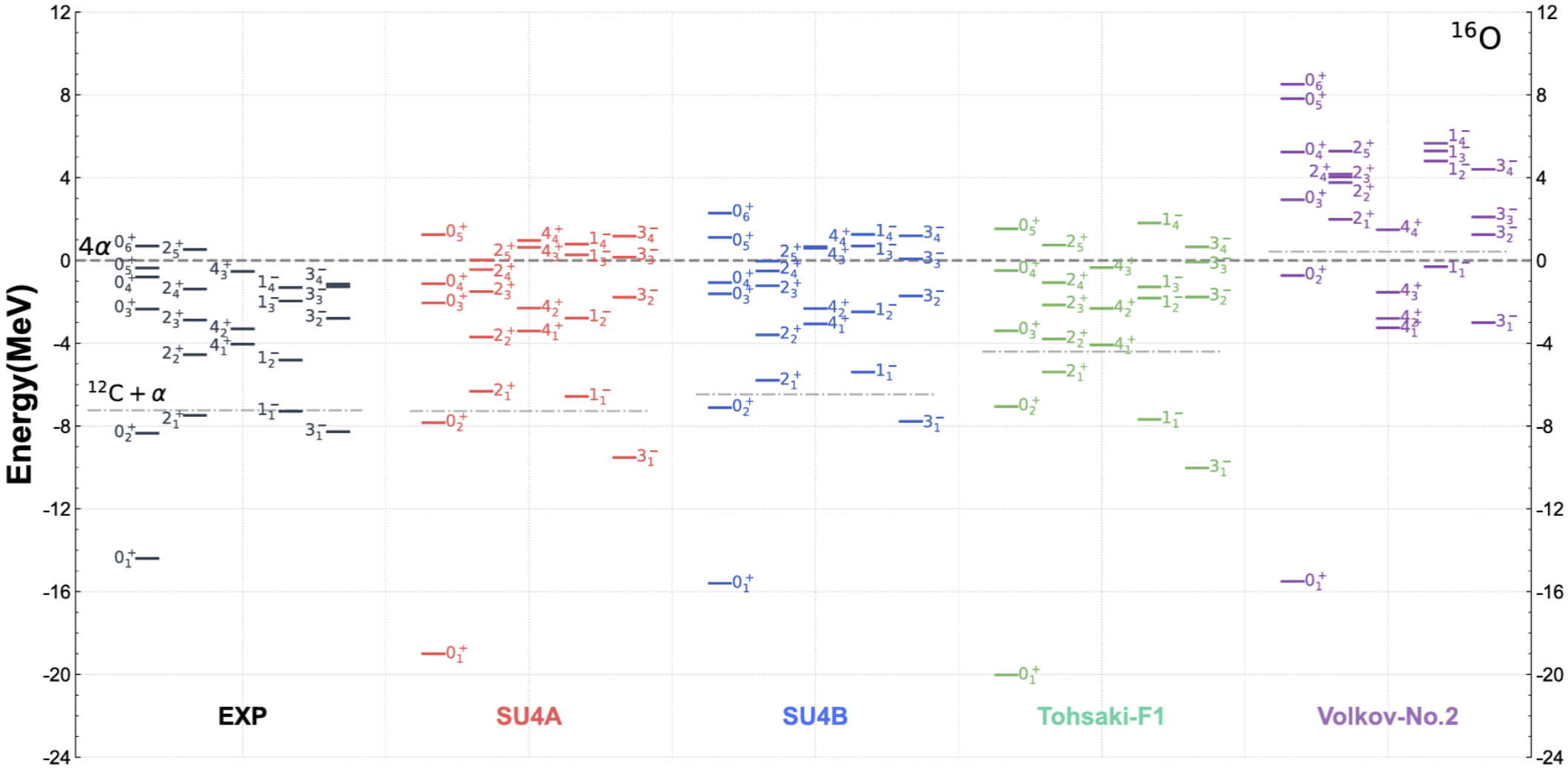
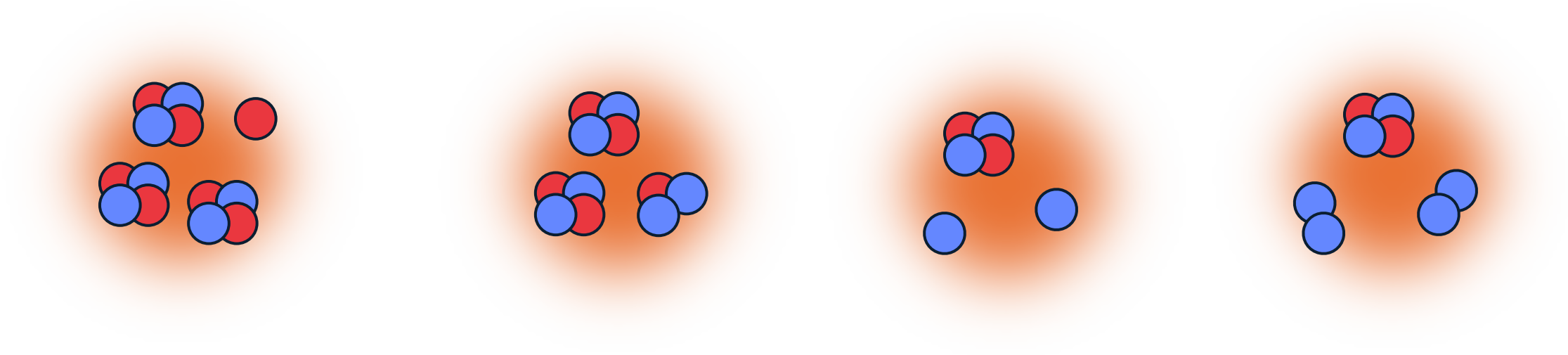


FIG. 3: Energy levels for ^{16}O from experimental data [62, 63], GCM results with SU4A, SU4B, Tohsaki-F1 force [64], and Volkov-No.2 potential (implemented within our GCM framework using parameters from Ref. [65] and verified to reproduce its reported ground state). The gray dashed and dash-dotted lines represent the thresholds of 4α and $^{12}\mathrm{C} + \alpha$, respectively.

Guoping Li, et al., arXiv:2605.21228

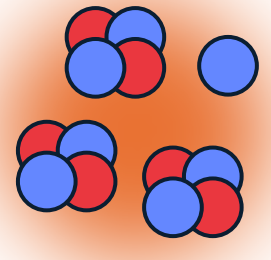
Some examples



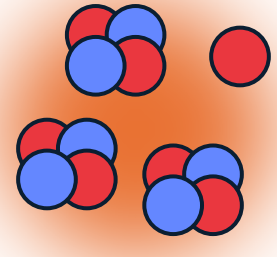
Hoyle analogy states

Hoyle-analog state in ^{13}N

13N(3alpha+proton)



13C(3alpha+neutron)



PHYSICAL REVIEW C **109**, 054308 (2024)

Cluster structure of $3\alpha + p$ states in ^{13}N

J. Bishop,[1,2] G. V. Rogachev,[1,3,4] S. Ahn,[5] M. Barbui,[1] S. M. Cha,[5] E. Harris,[1,3] C. Hunt,[1,3] C. H. Kim,[6] D. Kim,[5] S. H. Kim,[6] E. Koshchiy,[1] Z. Luo,[1,3] C. Park,[5] C. E. Parker,[1] E. C. Pollacco,[7] B. T. Roeder,[1] M. Roosa,[1,3] A. Saastamoinen,[1] and D. P. Scriven[1,3]

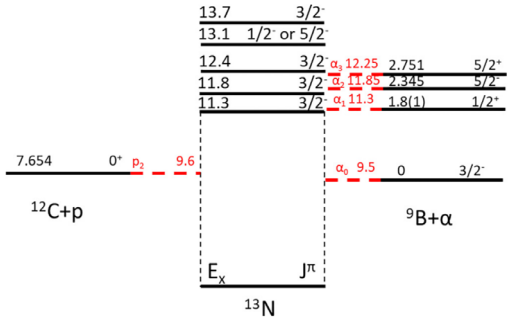
Background: Cluster states in ^{13}N are extremely difficult to measure due to the unavailability of ^{9}B $+\alpha$ elastic-scattering data.

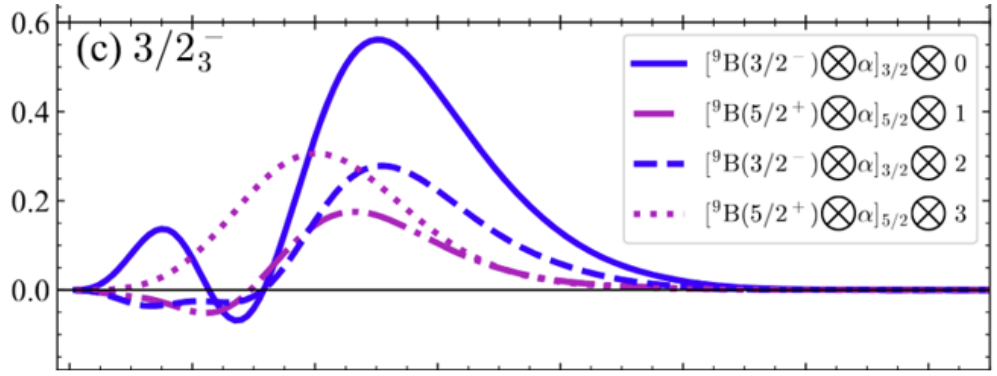
Purpose: Using β-delayed charged-particle spectroscopy of ^{13}O, clustered states in ^{13}N can be populated and measured in the $3\alpha + p$ decay channel.

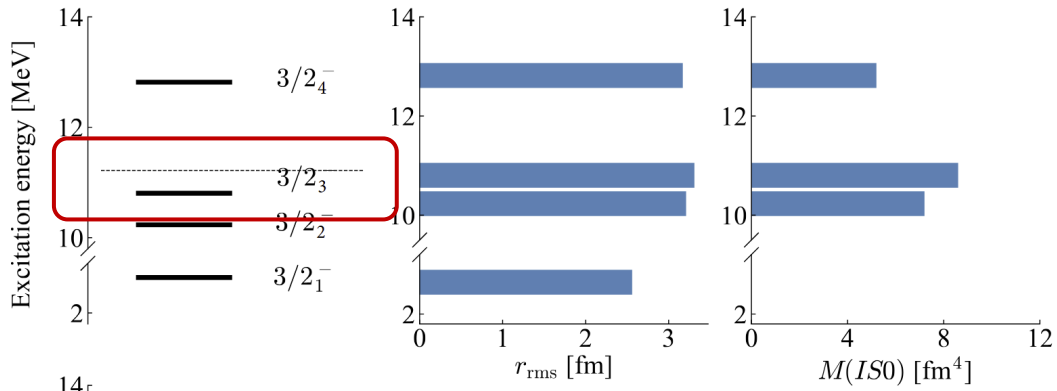
Methods: One-at-a-time implantation and decay of ^{13}O was performed with the Texas Active Target Time Projection Chamber. $149\beta 3\alpha p$ decay events were observed and the excitation function in ^{13}N reconstructed.

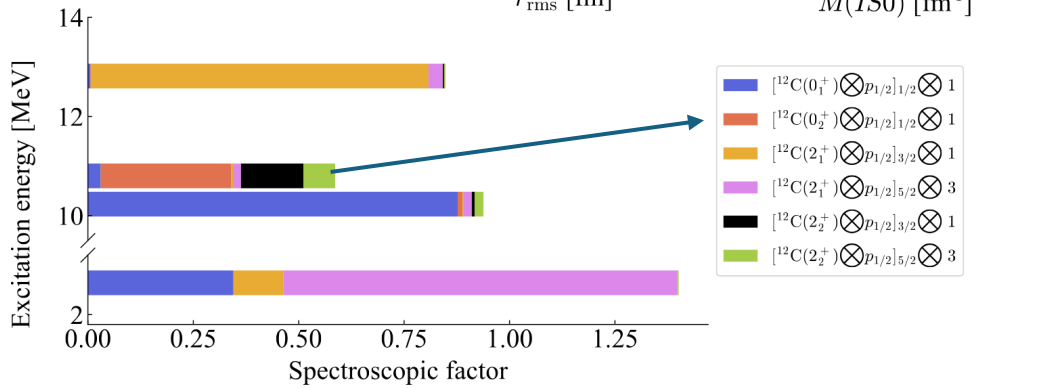
Results: Four previously unknown α-decaying excited states were observed in ^{13}N at an excitation energy of 11.3, 12.4, 13.1, and 13.7 MeV decaying via the $3\alpha + p$ channel.

Conclusions: These states are seen to have a $[^{9}\mathrm{B(g.s)} \otimes \alpha / p + {}^{12}\mathrm{C}(0_2^+)]$, $[^{9}\mathrm{B}(\frac{1}{2}^+) \otimes \alpha]$, $[^{9}\mathrm{B}(\frac{5}{2}^+) \otimes \alpha]$, and $[^{9}\mathrm{B}(\frac{5}{2}^+) \otimes \alpha]$ structure, respectively. A previously seen state at 11.8 MeV was also determined to have a $[p + {}^{12}\mathrm{C(g.s.)} / p + {}^{12}\mathrm{C}(0_2^+)]$ structure. The overall magnitude of the ever, due to the lack of a total width measurement. Clustered states i ersist from the addition of a proton to the highly α-clustered ^{12}C. Ev to be populated by decays from ^{13}N*.





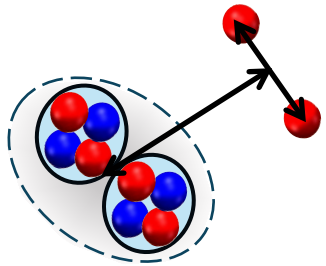




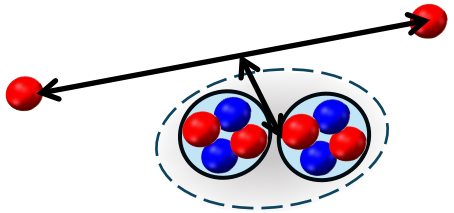
Y.Y Cao, et al., Phys. Rev. C 112, 034313 (2025)

Diproton state in ^{10}C

10C(2alpha+2proton)

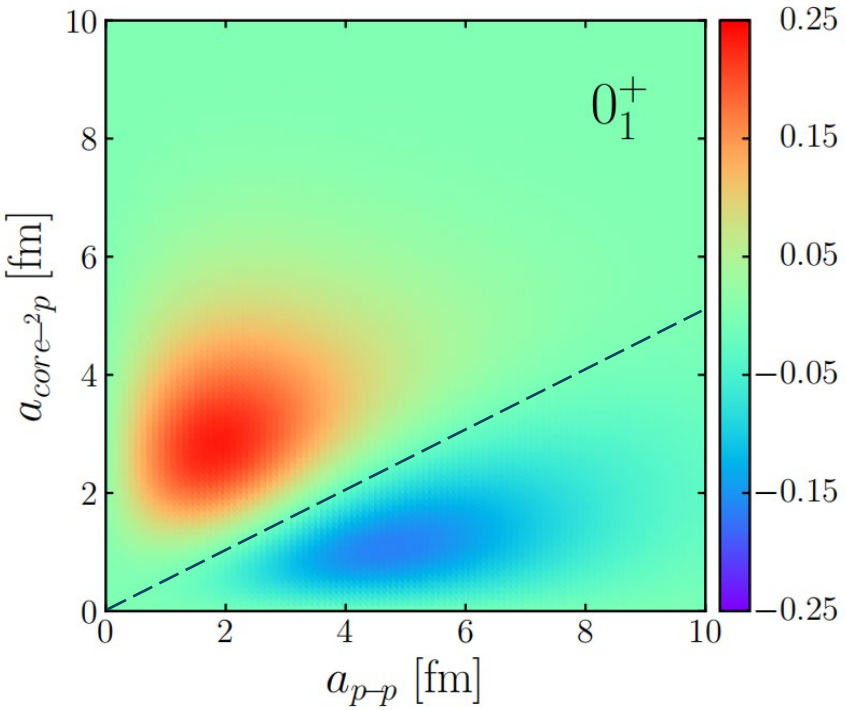


"diproton-like" configuration

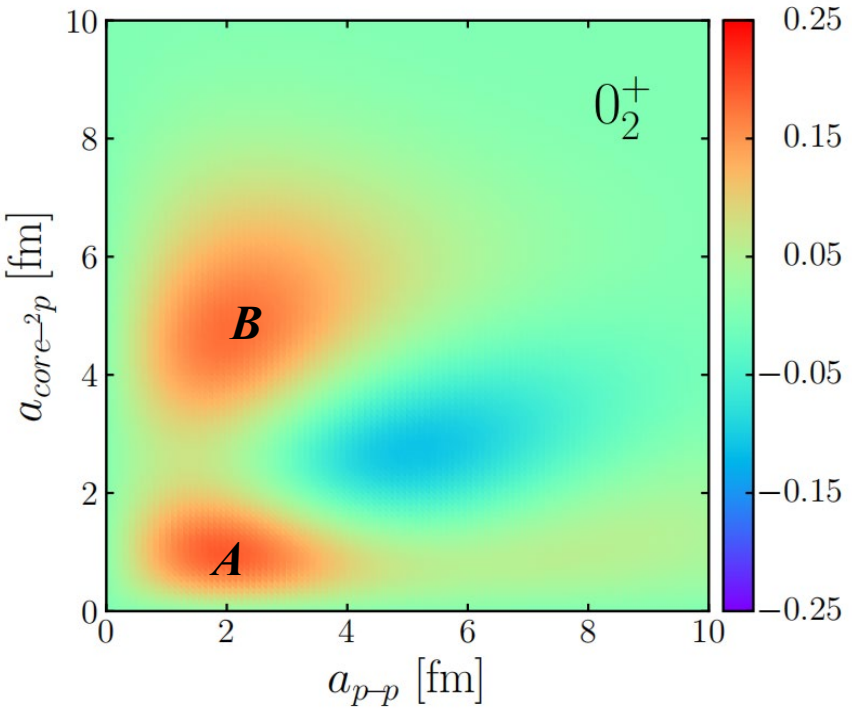


"cigar-like" configuration

- **Compact Structure**

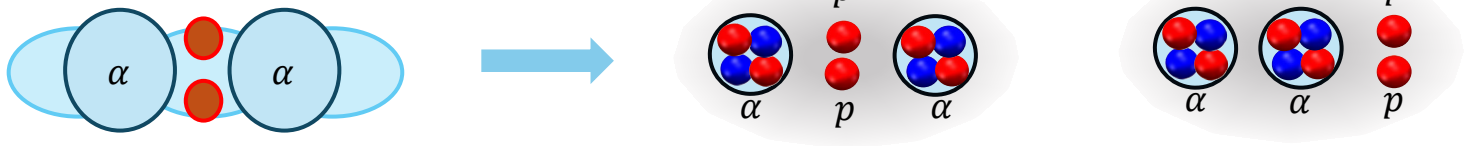


- **Diproton-like Configuration**



- Molecular Orbital $(\sigma^+_{1/2})^2$

A. $a-(pp)-a$ ***B***. $a-a-(pp)$



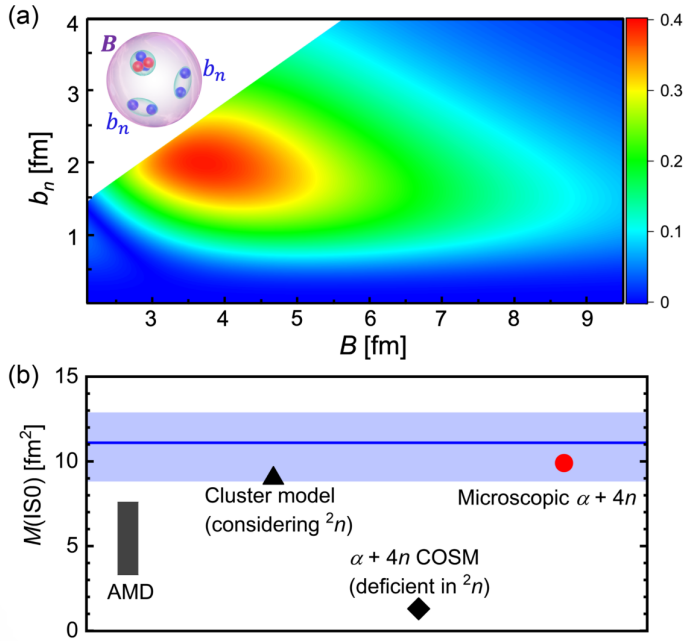
L.-D. Song, et al., Phys. Rev. C 113, 024316 (2026).

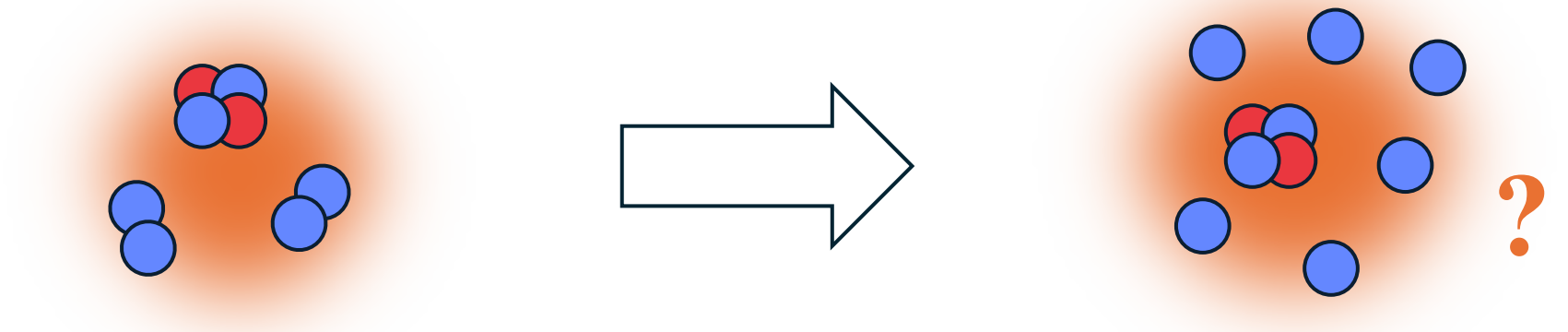
Editors' Suggestion | Featured in Physics

Observation of the Exotic 0_2^+ Cluster State in ^{8}He

Z. H. Yang,[1,2,*,†] Y. L. Ye,[1,*,‡] B. Zhou,[3,4,5] H. Baba,[2] R. J. Chen,[6] Y. C. Ge,[1] B. S. Hu,[1] H. Hua,[1] D. X. Jiang,[1] M. Kimura,[2,5,7] C. Li,[2] K. A. Li,[6] J. G. Li,[1] Q. T. Li,[1] X. Q. Li,[1] Z. H. Li,[1] J. L. Lou,[1] M. Nishimura,[2] H. Otsu,[2] D. Y. Pang,[8] W. L. Pu,[1] R. Qiao,[1] S. Sakaguchi,[2,9] H. Sakurai,[2] Y. Satou,[10] Y. Togano,[2] K. Tshoo,[10] H. Wang,[2,11] S. Wang,[2] K. Wei,[1] J. Xiao,[1] F. R. Xu,[1] X. F. Yang,[1] K. Yoneda,[2] H. B. You,[1] and T. Zheng[1]

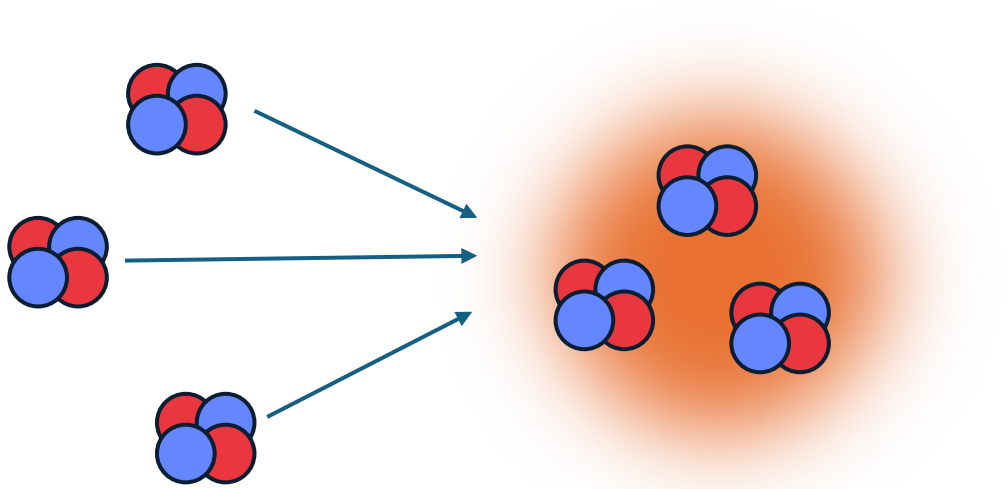
[1]*School of Physics and State Key Laboratory of Nuclear Physics and Technology, Peking University, Beijing 100871, China*
[2]*RIKEN Nishina Center, 2-1 Hirosawa, Wako, Saitama 351-0198, Japan*
[3]*Key Laboratory of Nuclear Physics and Ion-beam Application (MOE), Institute of Modern Physics, Fudan University, Shanghai 200433, China*
[4]*Shanghai Research Center for Theoretical Nuclear Physics, NSFC and Fudan University, Shanghai 200438, China*
[5]*Department of Physics, Hokkaido University, 060-0810 Sapporo, Japan*
[6]*Institute of Modern Physics, Chinese Academy of Science, Lanzhou 730000, China*





$$\Phi(\boldsymbol{B}, b_n) \propto \mathcal{A}\left\{\exp\left[-\frac{4\boldsymbol{\xi}_1^2}{3\boldsymbol{B}^2} - \frac{3\boldsymbol{\xi}_2^2}{2\boldsymbol{B}^2}\right] \times \phi_\alpha(b_\alpha)\phi_{^2_1n}(b_n)\phi_{^2_2n}(b_n)\right\},$$

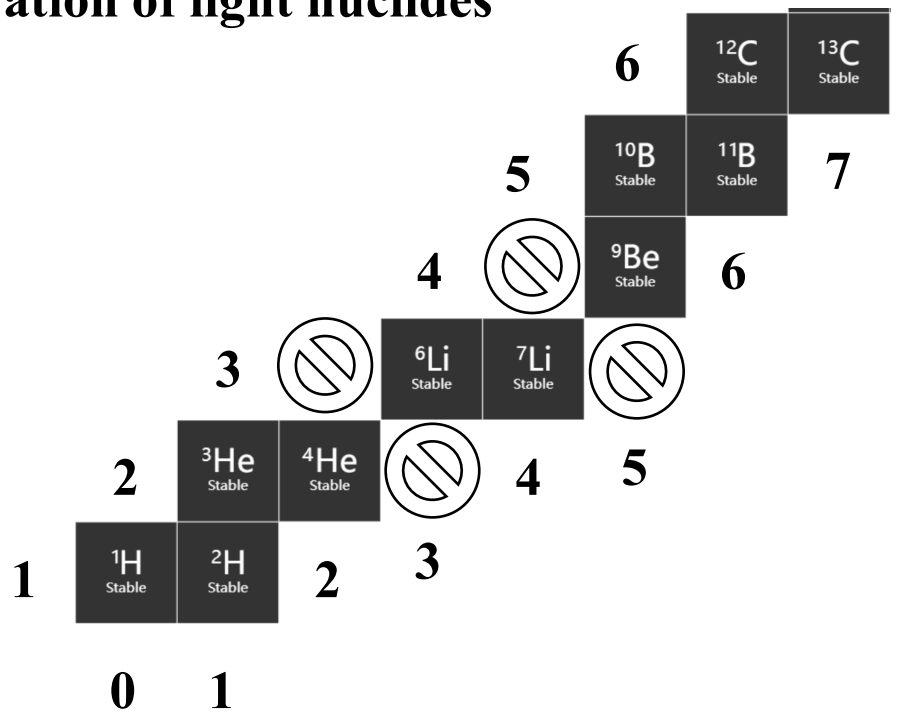
Searching for the *muli-neutron* correlations



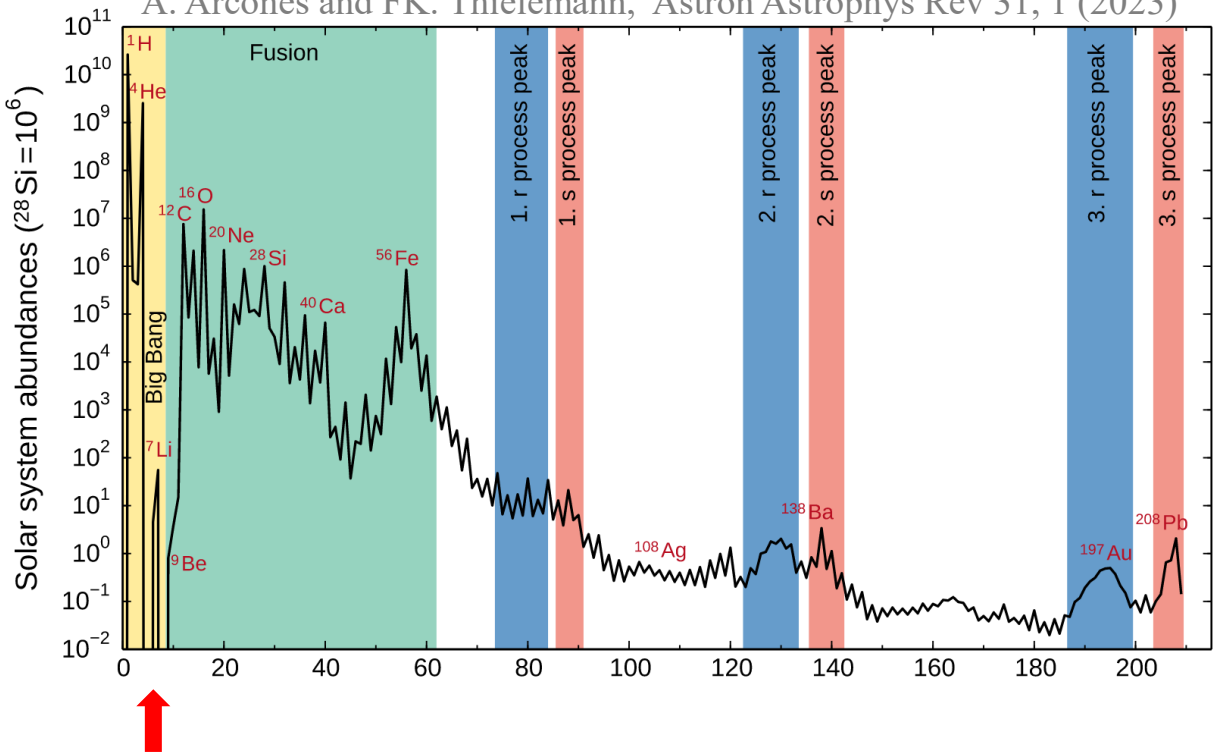
Triple α reaction

Background: Mass Gap

Generation of light nuclides



A. Arcones and FK. Thielemann, Astron Astrophys Rev 31, 1 (2023)



No stable nuclei for A=5, 8 → **Mass Gap** — **Three-body reactions**

Bypass the Gap

$\alpha + \alpha + \alpha \rightarrow {}^{12}\mathrm{C} + \gamma$ α-rich (Sun)

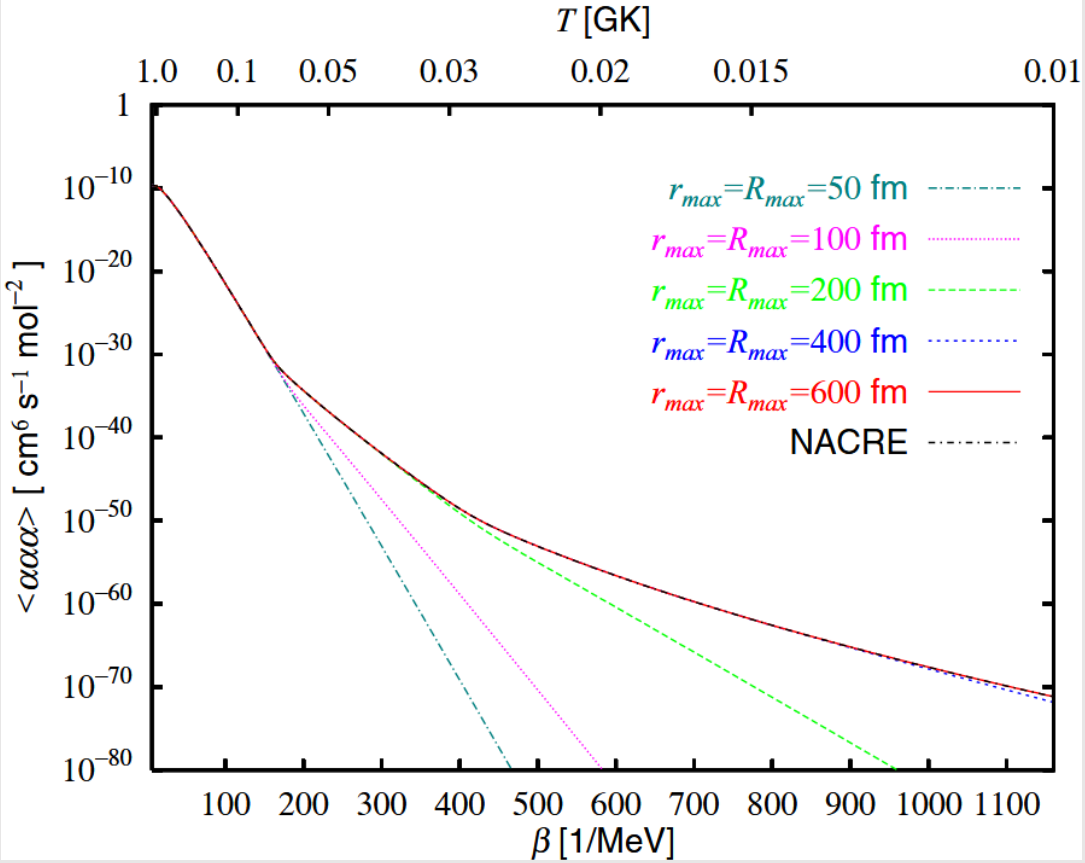
$\alpha + \alpha + n \rightarrow {}^{9}\mathrm{Be} + \gamma$ n-rich (Type-II SN)

$^{4}\mathrm{He}(2n,\gamma)\ ^{6}\mathrm{He}(2n,\gamma)\ ^{8}\mathrm{He}(\beta^{-})\ ^{8}\mathrm{Li}(n,\gamma)\ ^{9}\mathrm{Li}(\beta^{-})\ ^{9}\mathrm{Be}$

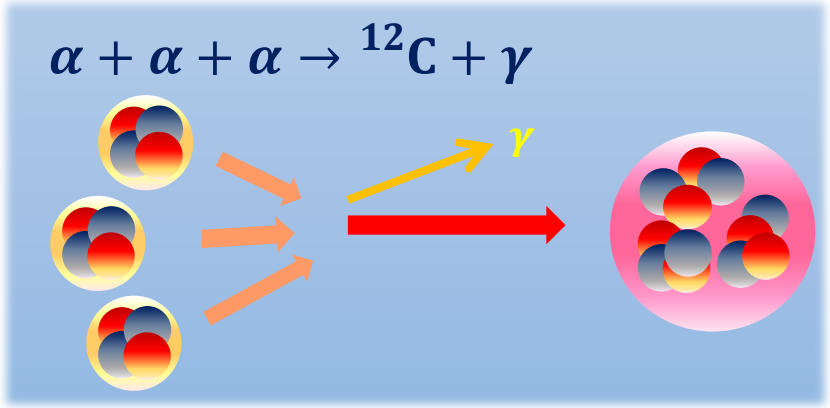
Background: IT method

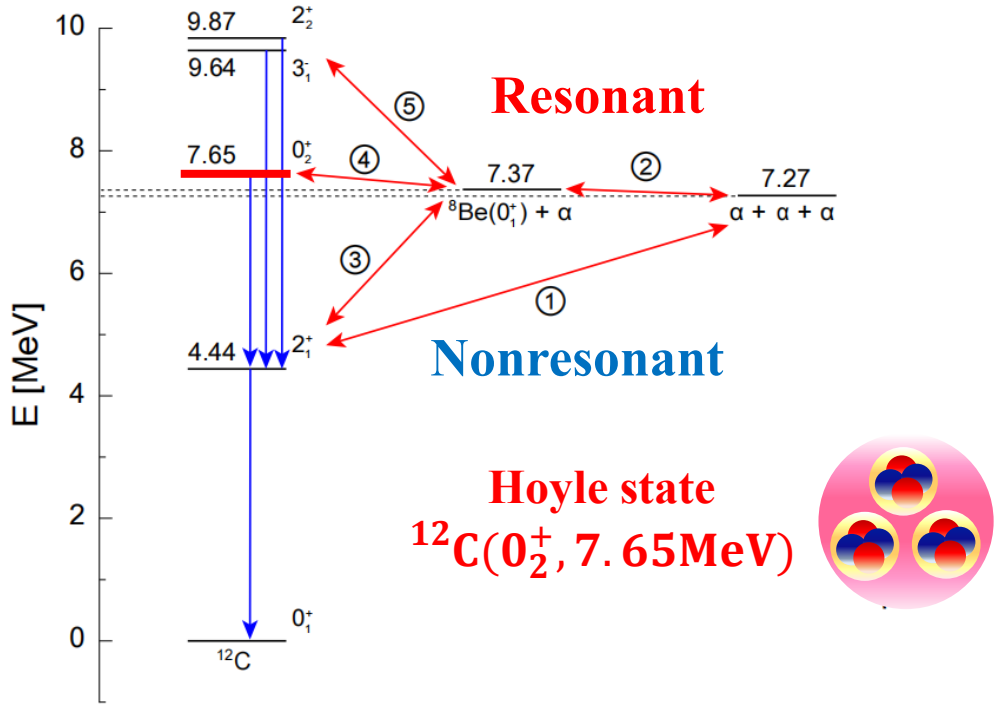
➢ Triple-α process

Previous work: Imaginary-time method for 3α (2015)



T. Akahori, et al., Phys. Rev. C 92, 022801(R) (2015)





Luanhong Pei, et al., Sci Sin-Phys Mech As, 2024, 1674

Framework: IT method

Goal: Remove 3-body scattering states (Φ_i)

Reaction rate from direct picture ($\beta = 1/(k_B T)$)

$$\langle abc\rangle = (1+\Delta_{abc})\frac{1}{\omega_i}\left(\frac{4\pi^2\beta^2\hbar^4}{\mu}\right)^{3/2}\frac{8\pi(\lambda+1)}{\hbar\lambda[(2\lambda+1)!!]^2} \times \sum_{M_f\mu}\sum_i e^{-\beta E_i}\left(\frac{E_i-E_f}{\hbar c}\right)^{2\lambda+1}\left|\langle\Phi_f|M_{\lambda\mu}|\Phi_i\rangle\right|^2,$$

Direct rate formula includes the summation of 'i'

$$\langle\sigma v\rangle \sim \sum_i e^{-\beta E_i}\left(\frac{E_i-E_f}{\hbar c}\right)^{2\lambda+1}\left\langle\Phi_f\left|M_{\lambda\mu}\,|\Phi_i\rangle\langle\Phi_i|M^{+}_{\lambda\mu}\right|\Phi_f\right\rangle$$

Scattering state

Therefore

$$\sum_i e^{-\beta E_i}\left(\frac{E_i-E_f}{\hbar c}\right)^{2\lambda+1}|\Phi_i\rangle\langle\Phi_i| = f(\hat{H})\left(\mathbf{1}-\sum_{n\in\text{bound}}|\Phi_n\rangle\langle\Phi_n|\right)$$

Imaginary-time reaction rate formula

$$\langle abc\rangle = (1+\Delta_{abc})\frac{1}{\omega_i}\left(\frac{4\pi^2\beta^2\hbar^4}{\mu}\right)^{3/2}\frac{8\pi(\lambda+1)}{\hbar\lambda[(2\lambda+1)!!]^2} \times\langle\Phi_f|M_{\lambda\mu}e^{-\beta H}\left(\frac{H-E_f}{\hbar c}\right)^{2\lambda+1}\hat{P}M^{\dagger}_{\lambda\mu}|\Phi_f\rangle, \quad (62)$$

Projection operator $\hat{P} = \mathbf{1} - \sum_{n\in bound}|\Phi_n\rangle\langle\Phi_n|$

Scattering + Bound ⟹ **Bound**

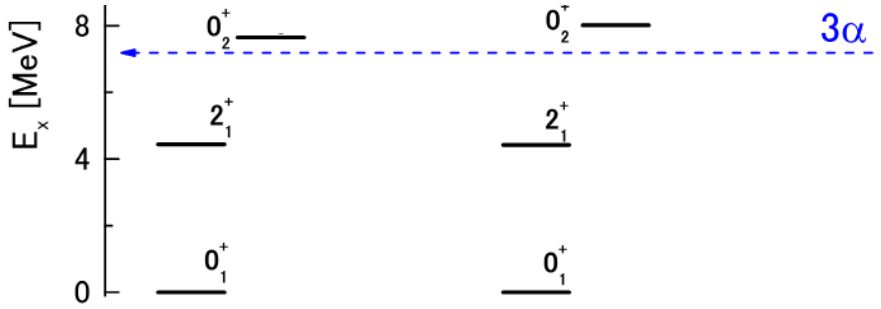
- **Key feature of IT**: Only the bound state wave function Φ_f is required!

K. Yabana et al., Phys. Rev. C 85, 055803 (2012)

Framework: IT-Jacobi

- **IT-Jacobi for 3α**

Spectrum of ^{12}C of OCM

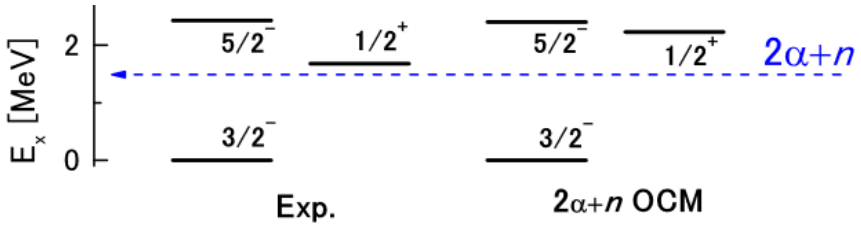


OCM Hamiltonian

$$H = \hat{T}_R + \hat{T}_r + V_{\alpha\alpha} + V_{3\alpha} + V_{\alpha\alpha}(r_{31}) + V_{\alpha\alpha}(r_{12}) + \boxed{V_{pau}}$$

- **IT-Jacobi for $\alpha\alpha n$**

Spectrum of ^{9}Be of OCM



OCM Hamiltonian

$$H = \hat{T}_R + \hat{T}_r + V_{\alpha\alpha} + V_{\alpha\alpha n} + V_{\alpha n}(r_{31}) + V_{\alpha n}(r_{12}) + \boxed{V_{pau}}$$

- Deep potentials (Including PF states)
- Pauli-blocking operator

$$V_{\text{Pauli}} = \lim_{\lambda\to\infty} \lambda \sum_f |u_f\rangle\langle u_f|,$$

However, in IT-Jacobi, **angular-momentum-independent shallow** potentials were used.

- For 3α, Ali-Bodmer version 'd' $V_{2\alpha}^{nucl}(r) = 500e^{-(0.7r)^2} - 130e^{-(0.475r)^2}$
- For $\alpha\alpha n$, THO Hamiltonian: Modified Ali-Bodmer $(\alpha-\alpha)$ + Woods-Saxon $(\alpha-n)$ (Single channel)

$$V_{\alpha\alpha}(r) = (125\widehat{P}_{l=0} + 20\widehat{P}_{l=2})e^{-(r/1.53)^2} - 30e^{-(r/2.85)^2}$$

$$V_{\alpha n}(r) = \frac{-43.0}{1+e^{(r-2.0)/0.7}} + \frac{\mathbf{l}\cdot\mathbf{s}}{r}\left(\frac{d}{dr}\frac{40.0}{1+e^{(r-1.5)/0.35}}\right) + 50.0e^{-(r/2.3)^2}\ \text{MeV},$$

Framework: Limitations of IT-Jacobi

Self-consistent problem

- Different Hamiltonians/Models for **wfs** and **IT evolution**
- The exchange symmetry of three identical bosons in **IT evolution** has not been proven (for 3α)

Multi-channel problem

- Applying angular-momentum-independent potentials
- Single channel or 'fake' multi-channel calculations (0-order Legendre expansion; no LS term in $\alpha\alpha n$)

$$< \mathbf{Rr}|\Phi_f > = \sum_{JLl} \frac{u^J_{Ll}(R,r)}{Rr} \left\{ [Y_L(\hat{R})\, Y_l(\hat{r})]_J \otimes \chi_{1/2} \right\}_{J_{total}, M_f}$$

$$V^{(N)}_{\alpha\alpha}(r) = V_r e^{-\mu_r^2 r^2} - V_a e^{-\mu_a^2 r^2}$$

Table 2: Ali-Bodmer potential.

l	Label	V_r(MeV)	μ_r(fm^{-1})	V_a(MeV)	μ_a(fm^{-1})
$l=0$	a_0	125	0.65	30	0.35
$l=0$	d_0	500	0.7	130	0.475
$l=2$	d_2	320	0.7	130	0.475
$l=4$	d_4	0	–	130	0.475
$l=0$	e_0	1050	0.8	150	0.5
$l=2$	e_2	640	0.8	150	0.5
$l=4$	e_4	0	–	150	0.5

$$\boldsymbol{\rho_{max} = 0}$$

$$V(r_{31}) + V(r_{12}) = V(\boldsymbol{R}\cdot\boldsymbol{r}) = \sum_{\rho=0}^{\infty} V_\rho(R_T, r_T)\left[Y_\rho(\hat{R}) \otimes Y_\rho(\hat{r})\right]_{00}. \quad (82)$$

$$V_\rho(R,r) = 2\pi(-1)^\rho \sqrt{2\rho+1} \int_{-1}^{1} P_\rho(\cos\theta)\{V_{31}(r_{31}(\cos\theta)) + V_{12}(r_{12}(\cos\theta))\}\, d\cos\theta,$$

These issues can all be resolved in IT-HH!

Framework: IT-HH

IT-HH

Imaginary-time method + **Hyperspherical Harmonic basis** → **IT-HH**

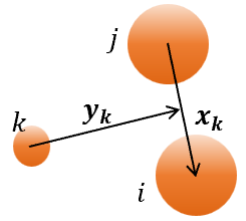
Same IT formalism, different basis representation.

Unchanged IT rate formula

$$\langle abc\rangle = (1+\Delta_{abc})\frac{1}{\omega_i}\left(\frac{4\pi^2\beta^2\hbar^4}{\mu}\right)^{3/2}\frac{8\pi(\lambda+1)}{\hbar\lambda[(2\lambda+1)!!]^2}$$
$$\times\langle\Phi_f|M_{\lambda\mu}e^{-\beta H}\left(\frac{H-E_f}{\hbar c}\right)^{2\lambda+1}\hat{P}M^{\dagger}_{\lambda\mu}|\Phi_f\rangle, \qquad (62)$$

Reformulated in the HH basis

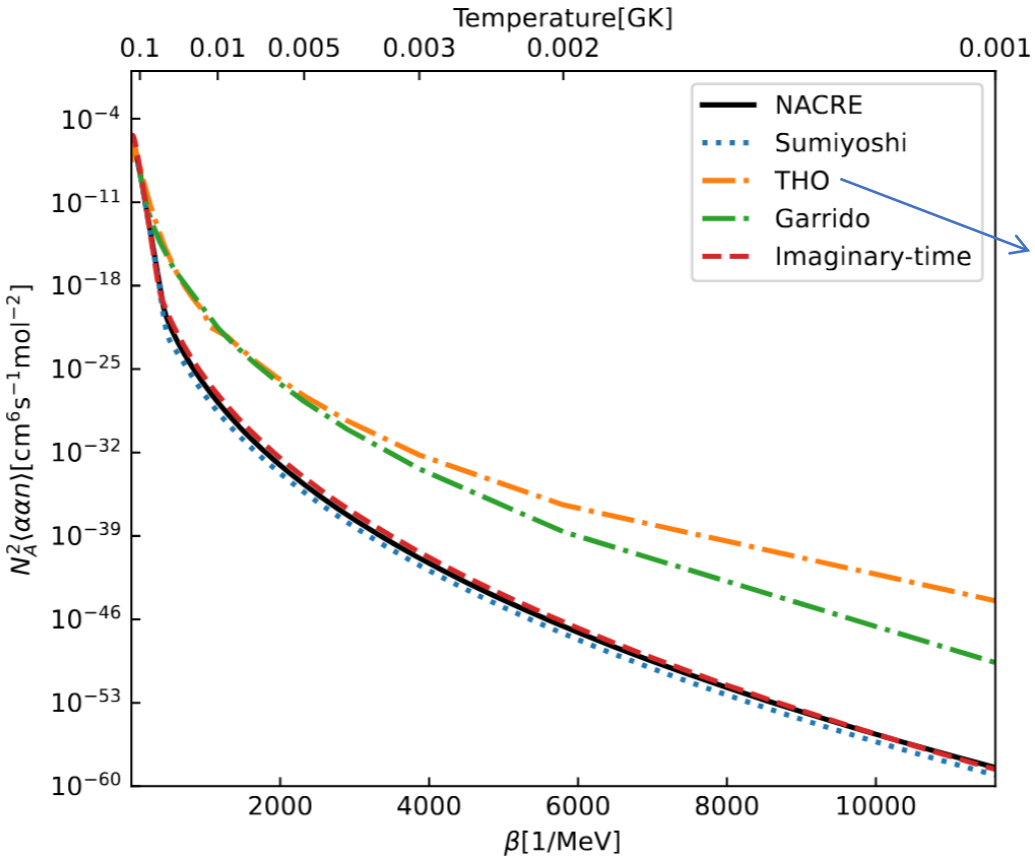
$$H = T_\rho + \frac{\hat{K}^2}{\rho^2} + \sum_{i<j} V_{ij} + V_{3b}(\rho)$$
$$\Psi^{JM} = \rho^{-5/2}\sum_\gamma \chi_\gamma(\rho)\,\mathcal{Y}^{JM}_\gamma(\Omega)$$



Motivation

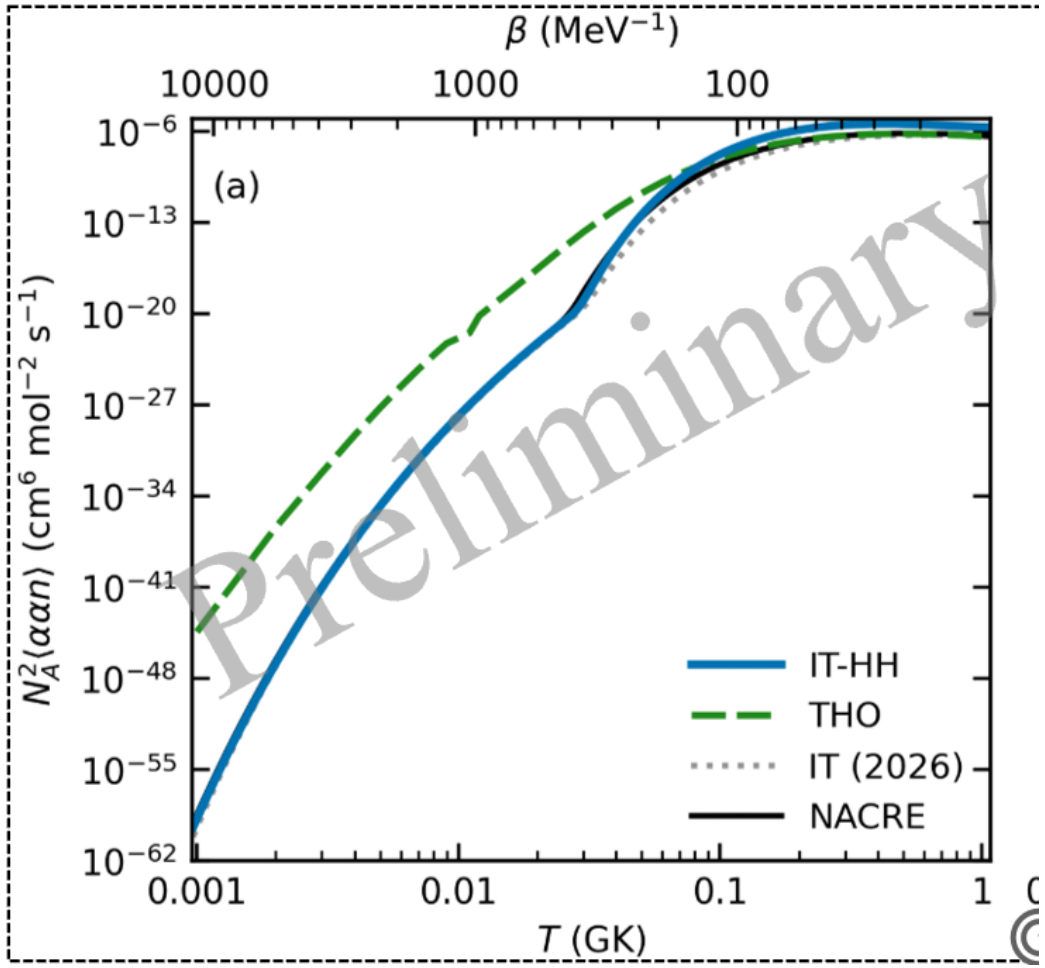
- Establish a more physical and self-consistent framework for 3-body radiative capture reactions
- Diagnose the origin of the low-temperature discrepancy

$\alpha\alpha n$ reaction rates



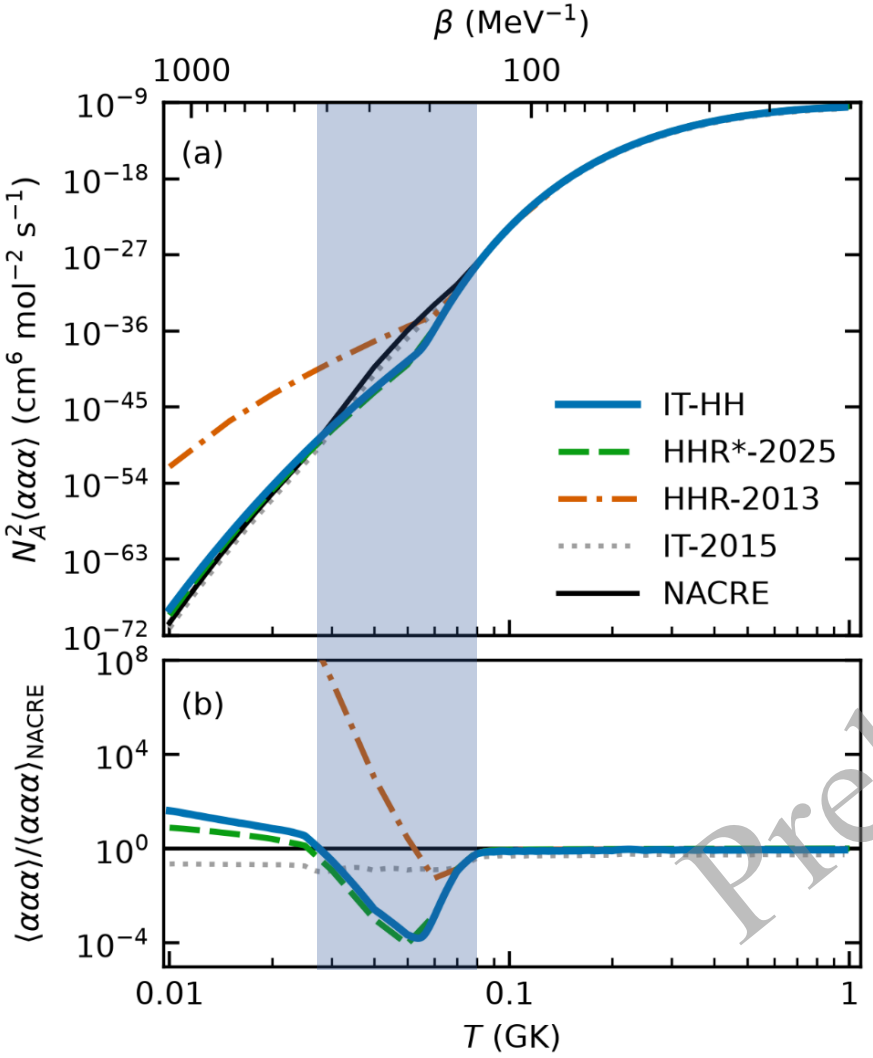
Luanhong Pei, et al., Phys. Lett. B 873, 140196 (2026)

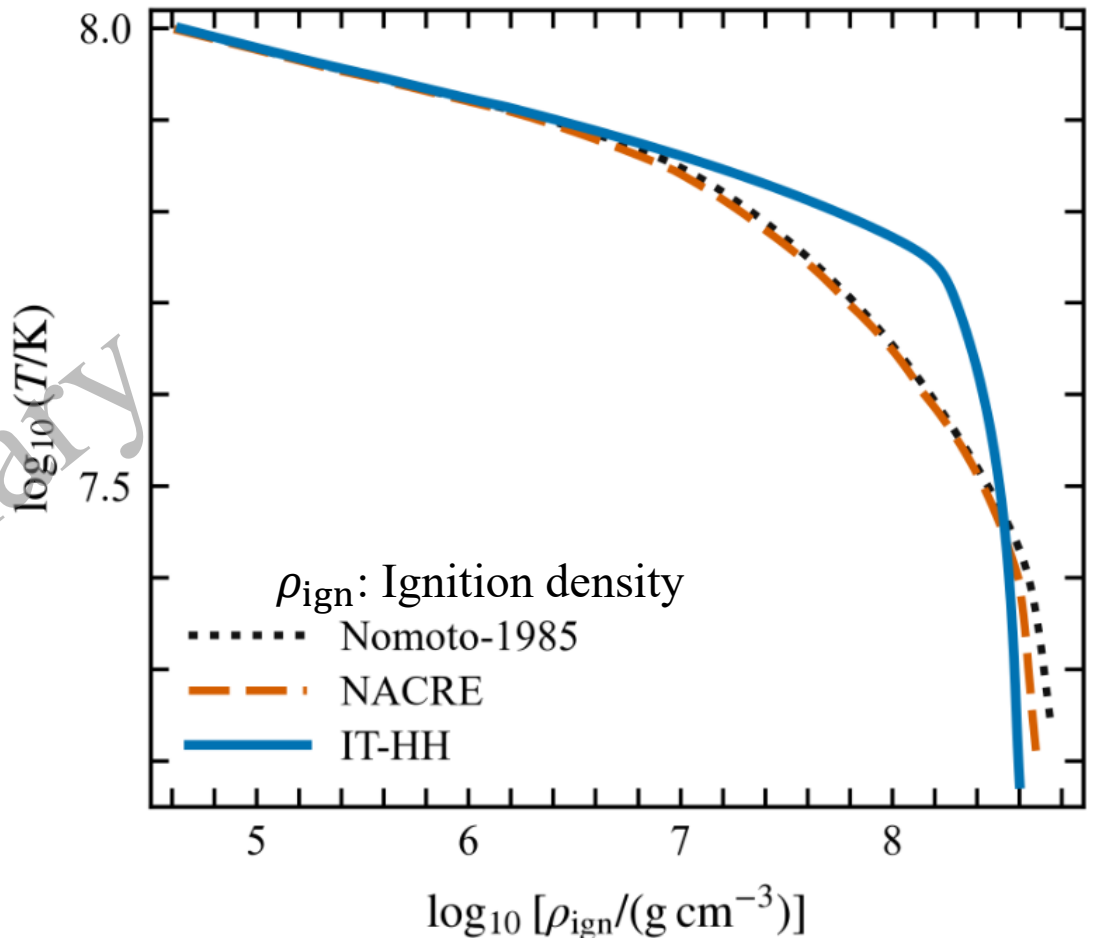
IT-Jacobi



IT-HH

IT-HH for 3α: IT-HH evolution





Preliminary

The rates are suppressed around **T ≃ 0.05–0.1 GK** and enhanced below about 0.03 GK. Thereduced rate in the T ≃ 0.05–0.1 GK window may delay helium ignition and increase ignition masses or recurrence times in helium-rich accreting white dwarfs and AM CVn helium novae.

Luanhong Pei, et al (in preparation)

Summary and Prospect

- Searching for **novel clustering structures** in α-conjugate nuclei. => 6α system
- Develop new microscopic wave functions and **effective interactions** within cluster-model frameworks. => multi-cluster systems
- Triple alpha reaction in IT-HH



Thanks for my collaborators
and your attention.