

Non-Perturbative Hamiltonian Framework for Finite-Volume Energy of N-Body Systems

Kang Yu

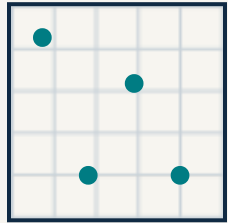
University of Chinese Academy of Sciences (UCAS)

with Jia-Jun Wu, Anthony W. Thomas, Derek B. Leinweber, Guang-Juan Wang and Zhi Yang

Based on [arXiv:2603.07205](#)

Finite-volume energy and infinite-volume scattering amp.

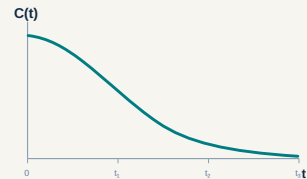
FINITE VOLUME: LATTICE QCD



Finite lattice box

Periodic boundary conditions discretize the momentum and energy and reduce the symmetry from continuous $SO(3)$ to point group.

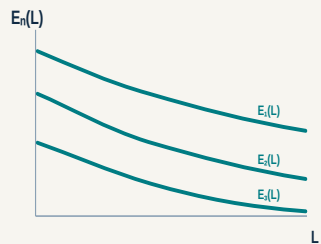
+



Correlation function

$$\langle \pi^+ \pi^+ \pi \pi \rangle$$

↓



Finite-volume energy levels

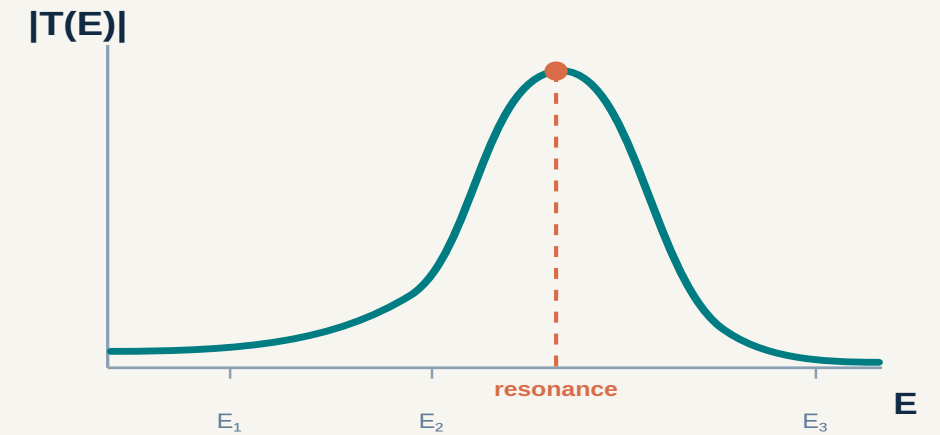
Volume dependence of E encodes dynamic interaction.

FINITE-VOLUME FORMALISM



relate $E_n(L)$ to $T(E)$

INFINITE VOLUME: HADRON DYNAMICS

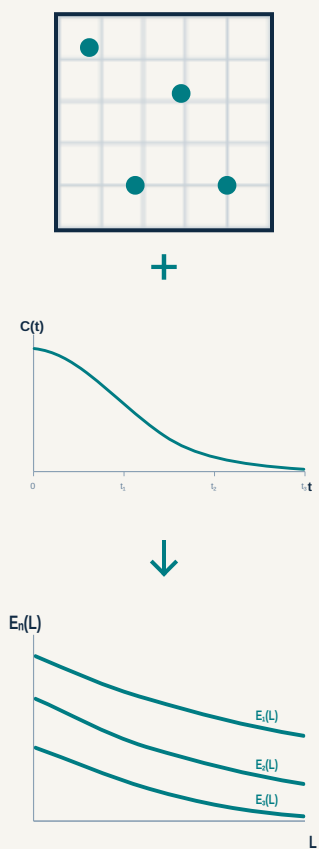


Scattering amp. $T(E)$ at $E_n(L)$

Physics targets: line shapes and resonance poles

Finite-volume energy and infinite-volume scattering amp.

FINITE VOLUME: LATTICE QCD



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$$\langle \pi^+ \pi^+ \pi \pi \rangle$$

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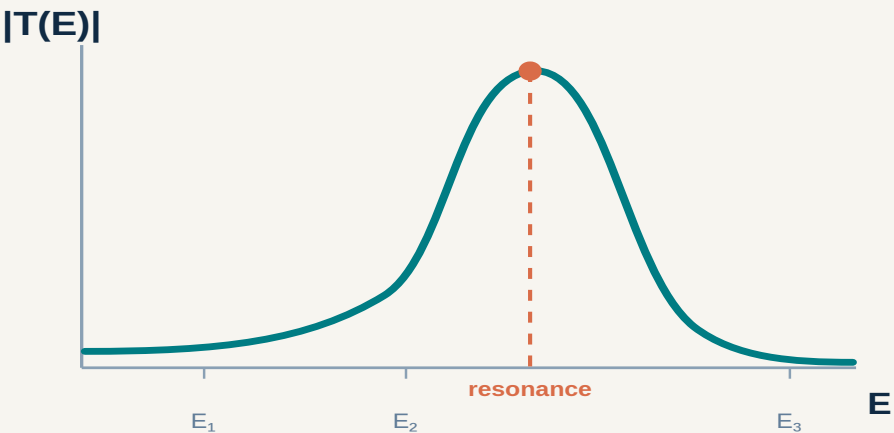
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Physics targets: line shapes and resonance poles

Lüscher Quantization Condition (for simplest case)

$$\delta(k)$$

scattering phase shift

+

$$\varphi(q)$$

geometric function encoding finite-volume effects

$$= n\pi$$

k = on-shell momentum at finite-volume energy

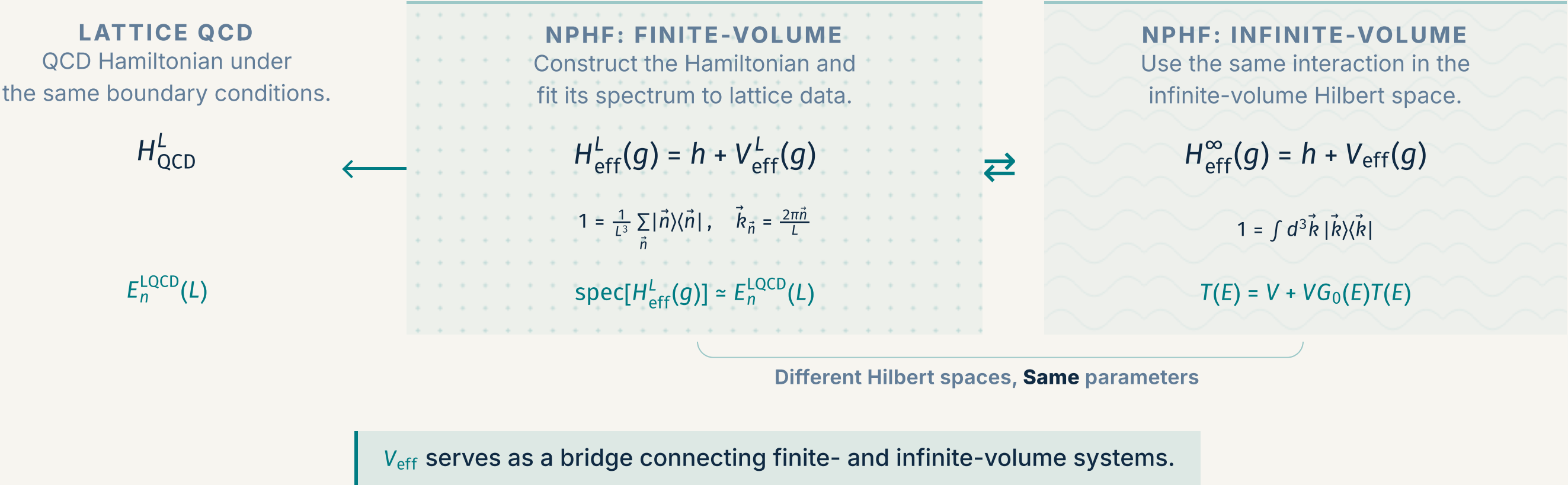
$$q = kL/2\pi$$

Spirit of NPHF

At fixed quantum numbers, lattice spectra extracted from correlation functions are eigenvalues of H_{QCD}^L .

$$C^L(t) = \langle \phi^\dagger(t) \phi(0) \rangle \sim \sum_n \langle \phi^\dagger(0) e^{-H_{\text{QCD}}^L} | n \rangle \langle n | \phi(0) \rangle \sim \sum_n Z_n e^{-E_n t}$$

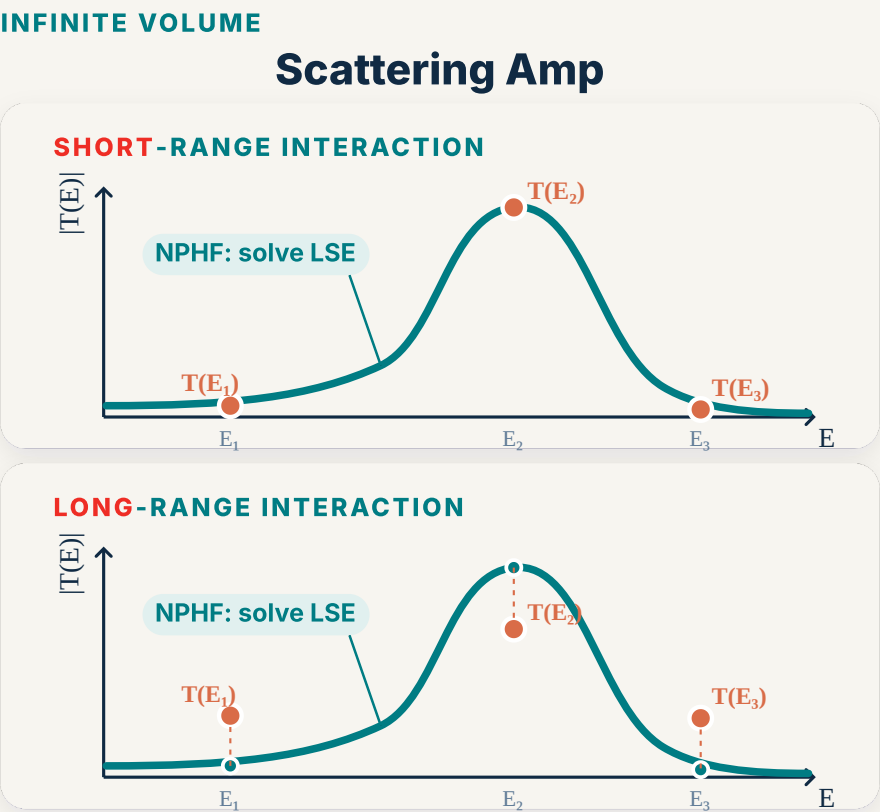
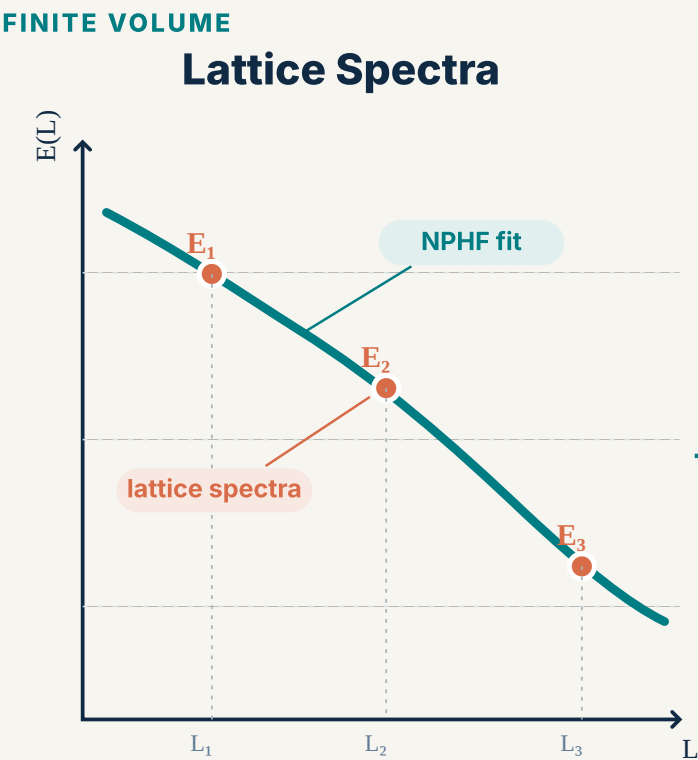
From the perspective of EFT, we don't need to work directly with full QCD Hamiltonian, ...



NPHF and Lüscher formalism for two-body system

For **short-range interaction**, two formalism are proved to be equivalent.

For **long-range interaction**, the discrepancies are observed.



NPHF PREVIOUS TWO-BODY WORK

$\Delta(1232) \rightarrow N\pi$
PRD **87**, 094510 (2013)

$\Lambda(1405)$ as a $\bar{K}N$ molecule
PRL **114**, 132002 (2015)

$N^*(1535)$ in lattice QCD
PRL **116**, 082004 (2016)

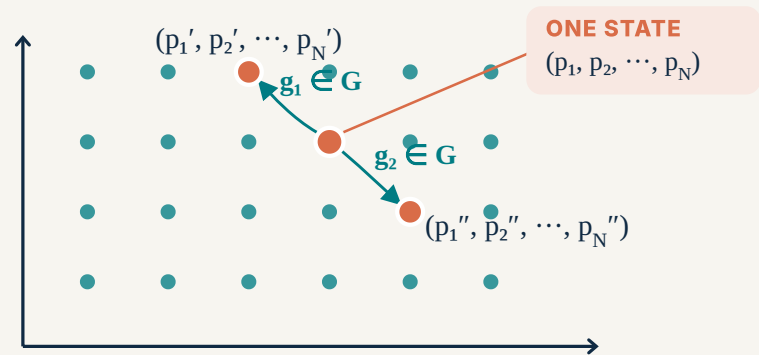
Roper from lattice constraints
PRD **97**, 094509 (2018)

Moving / elongated volume
PRD **103**, 094518 (2021)

Detailed calculation: J.-J. Wu *et al.*, *Phys. Rev. C* **90**, 055206 (2014); L. Meng and E. Epelbaum, *JHEP* **10**, 051 (2021); K. Yu *et al.*, *JHEP* **04**, 108 (2025).

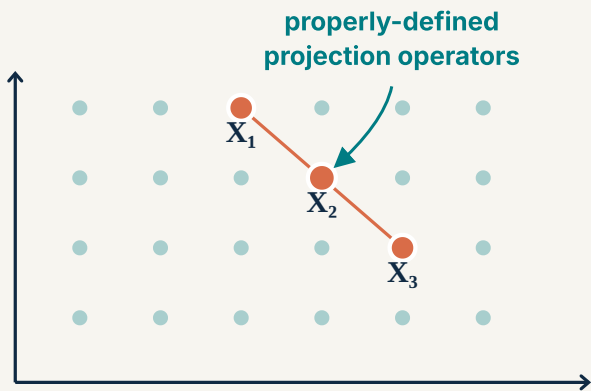
lattice symmetry projection in NPHF

Discrete momentum configurations



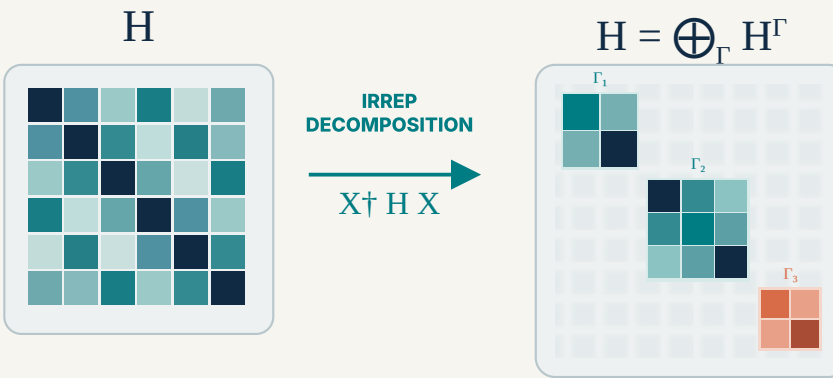
Hilbert space = direct sum of distinct G-orbits.

Construct irrep basis



Given an irrep Γ and a G-orbit, construct an orthonormal irrep basis: $\sum_i X_i |v_i\rangle$

Block diagonalization by symmetry



Each Hamiltonian block corresponds to an irrep.

$$\hat{P}_{\mu r}^\Gamma = \mathcal{N} \sum_{g \in G} \sum_v (c^r)_v D_{\mu v}^{\Gamma*}(g) \hat{g}$$

normalization
(derived analytically)

linear coefficient irrep-matrix-weighted group action

eigenvector associated with the r-th nonzero eigenvalue of the overlap matrix

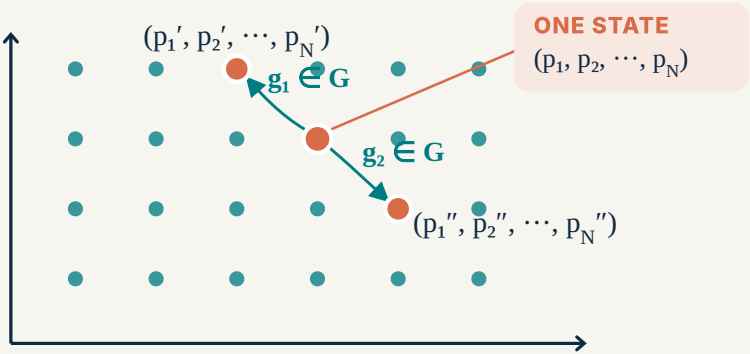
OVERLAP MATRIX I
for the simplest case

$$I_{v'v} = \sum_{g \in G} D_{v'v}^{\Gamma*}(g) \langle v^{\text{ref}} | \hat{g} | v^{\text{ref}} \rangle$$

Encodes all linear information of the G-orbit generated by $|v^{\text{ref}}\rangle$.

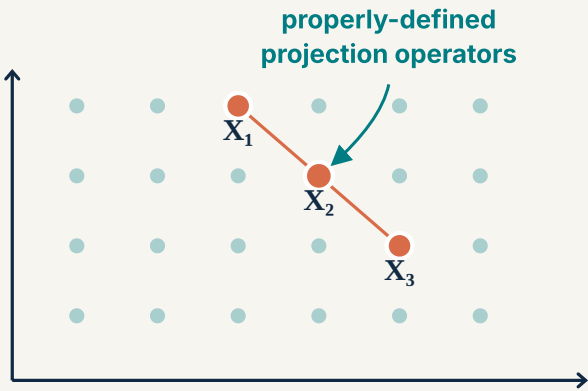
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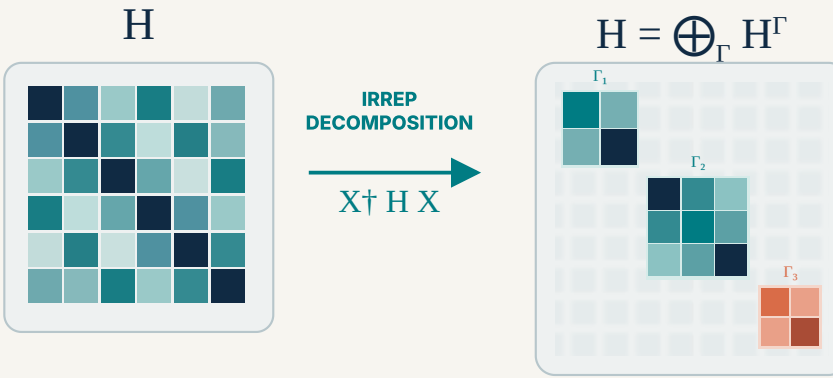
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$$I_{v,v'} = \sum_{g \in G} D_{v,v'}^{\Gamma*}(g) \langle v^{\text{ref}} | \hat{g} | v^{\text{ref}} \rangle$$

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SPIN DEGREES OF FREEDOM

Helicity representation

Under a group action, $\lambda \xrightarrow{g \in G} \pm \lambda$, so helicity is preserved or sign-flipped.

ISOSPIN DEGREES OF FREEDOM

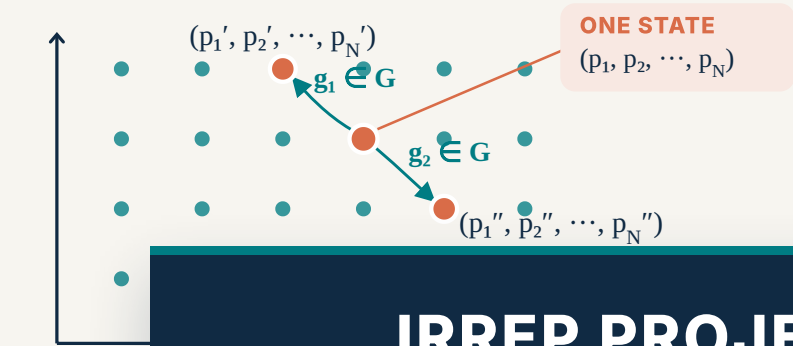
Permutation identification

Include the permutation group S_N and identify physical-equivalent states.

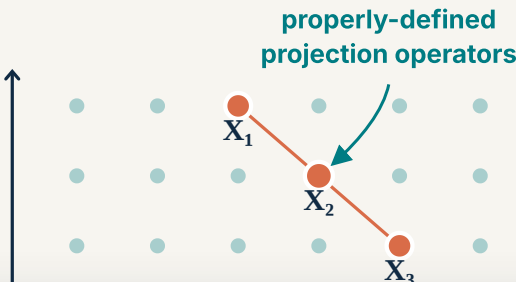
For two particles: $|\vec{p}, -\vec{p}\rangle = \pm |-\vec{p}, \vec{p}\rangle$

lattice symmetry projection in NPHF

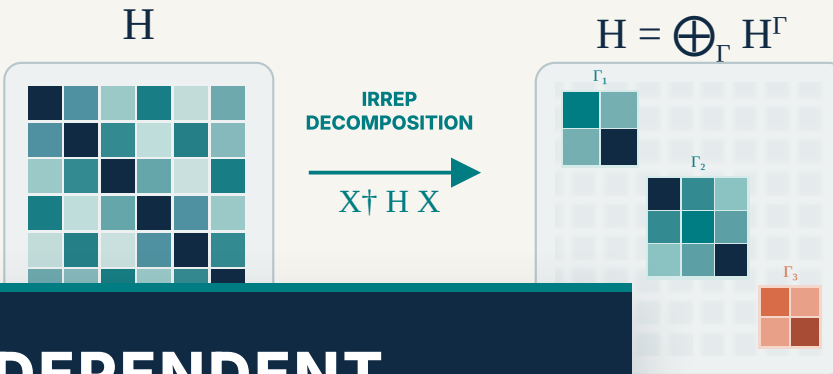
Discrete momentum configurations



Construct irrep basis



Block diagonalization by symmetry



IRREP PROJECTION HERE IS DYNAMICS-INDEPENDENT

Determined solely by kinematics and symmetry

Rest or moving frames · any spin sector · any isospin sector

Hilbert space

to an irrep.

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normalization
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linear coefficient

irrep-matrix-weighted group action

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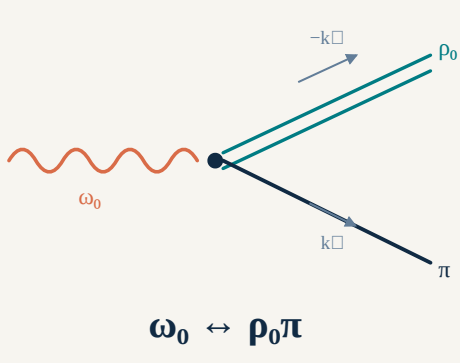
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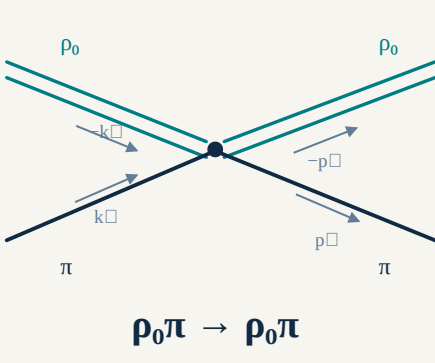
$\omega \rightarrow 3\pi$: a realistic three-body system

Lattice input: finite-volume spectra for the ω - 3π system are already available. *Yan et al., Phys. Rev. Lett.* **133**, 211906 (2024)

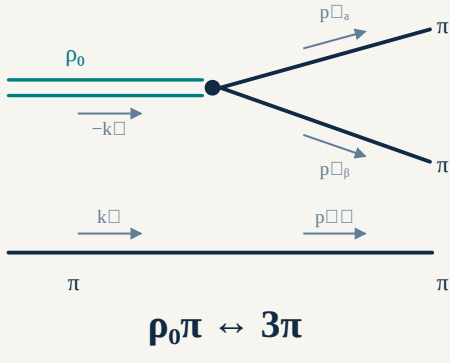
Fock states: ω_0 + $\rho_0\pi$ + 3π



$$\sim g_{\omega\rho\pi} k f(k, \Lambda_\omega)$$



$$\sim c_{22} p k f(p, \Lambda_{\rho_0\pi}) f(k, \Lambda_{\rho_0\pi})$$



$$\sim \sum_{(a,b,c)} \delta^3(\vec{k} - \vec{p}_c) V_{\rho\rightarrow\pi\pi}(\vec{p}_a, \vec{p}_b)$$

2π-system

	ρ_0	2π
ρ_0	0	$V_{\rho\pi\pi}$
2π	$V_{\rho\pi\pi}^\dagger$	0

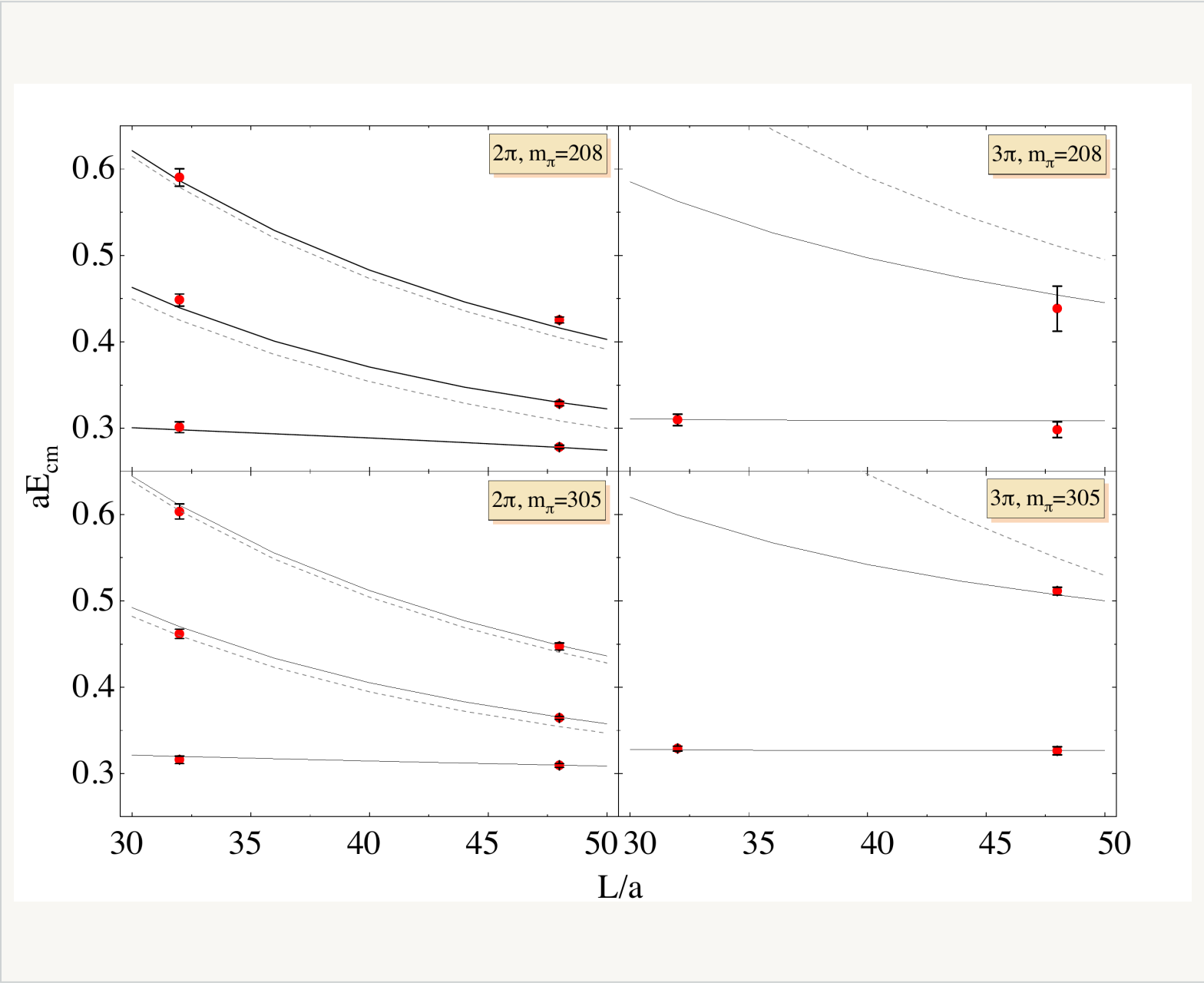
$$V_{\rho\rightarrow\pi\pi}(\vec{p}_a, \vec{p}_b) \sim g_{\rho\pi\pi} p_{\text{cm}} f(p_{\text{cm}}, \Lambda_{\pi\pi})$$

3π-system

	ω_0	$\rho_0\pi$	3π
ω_0	0	$V_{\omega\rho\pi}$	0
$\rho_0\pi$	$V_{\omega\rho\pi}^\dagger$	$C_{\rho\pi}$	$V_{\rho\pi,3\pi}$
3π	0	$V_{\rho\pi,3\pi}^\dagger$	0

Finite-volume spectrum fit

Simultaneous constraint: isovector 2π and isoscalar 3π spectra at $m_\pi = 208$ and 305 MeV. Yan et al., Phys. Rev. Lett. **133**, 211906 (2024)



● Lattice spectra — NPHF eigenvalues --- Non-interacting levels

FIT PARAMETERS AND PION-MASS DEPENDENCE

$m_{\rho_0}^{208}, m_{\rho_0}^{305}, g_{\rho\pi\pi}, g_{\omega\rho\pi}, c_{22}$
 $m_{\omega_0} = m_{\rho_0} + \delta m$, with δm taken independent of m_π .

FIXED PARAMETERS

$\Lambda_{\omega_0} = 1.0$ GeV, $\Lambda_{\pi\pi} = 1.0$ GeV, $\Lambda_{\rho_0\pi} = 0.5$ GeV, δm

FIT SCHEMES

A	B1	B2	B3
$\delta m = 0$	$c_{22} = 0$ $\delta m = 0$	$c_{22} = 0$ $\delta m = 15$ MeV	$c_{22} = 0$ $\delta m = 30$ MeV

scheme B2

quantity	fit result
$\hat{\chi}^2$	1.1
$m_{\rho_0}^{208}$	831(13) MeV
$m_{\rho_0}^{305}$	872(10) MeV
$g_{\rho\pi\pi}$	7.69(0.45)
$g_{\omega\rho\pi}$	30.7(3.1)
c_{22}	0 (fixed)

Infinite volume scattering

LSE FOR $\rho_0\pi \rightarrow \rho_0\pi$

$$T = V + V G_2 T$$

Scattering kernel $V = C + R + Z$

C

Contact interaction



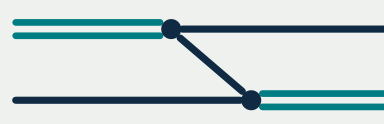
R

Bare- ω_0 exchange



Z

3π -mediated exchange



Dressed $\rho_0\pi$ propagator

$$G_2(p; E) = \left[E - \omega_{\rho_0}(p) - \omega_{\pi}(p) - \Sigma_{\pi\pi}(p; E) \right]^{-1}$$

$\rho_0 \rightarrow \pi\pi \rightarrow \rho_0$ self-energy correction is included.

The **3π unitary branch cut** renders the integral equation substantially more intricate than in a purely two-body system.

Available numerical strategies include contour deformation, interpolation, and related methods.

CONTOUR DEFORMATION FOR ANALYTICAL CONTINUATION

POLE CONDITION

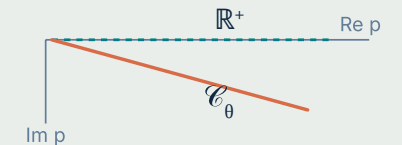
$$\det [1 - V(E) G_2(E)] = 0$$

$\mathbb{R}^+$

↓ contour deformation

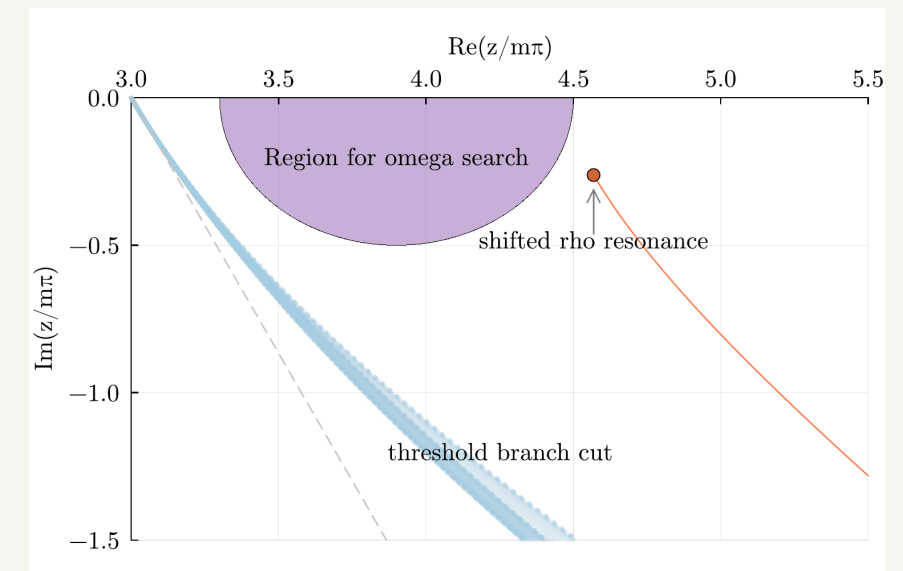
$$\det [1 - V(E) G_2(E)] = 0$$

$\mathcal{C}_\theta$



The self-energy contour in G_2 should be deformed simultaneously.

In the complex energy plane,



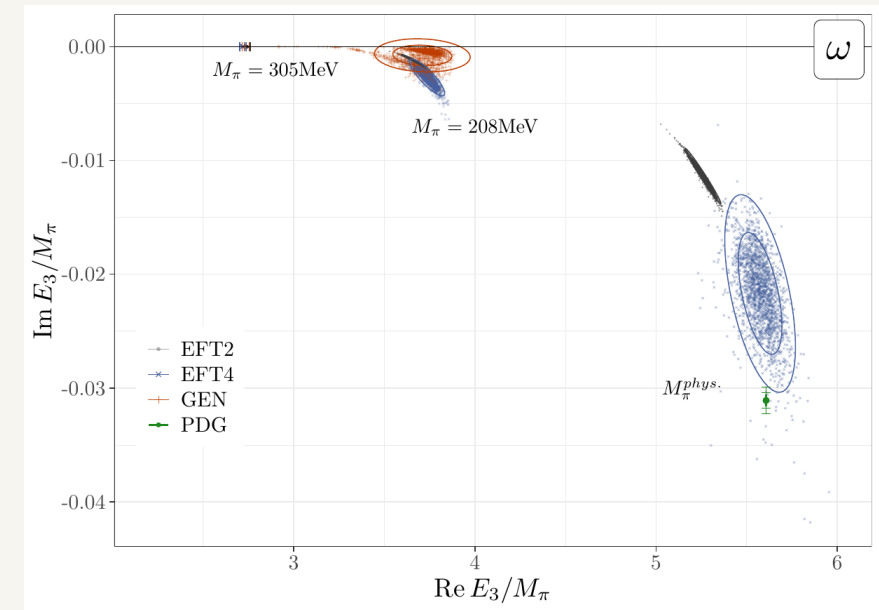
Pole positions

Pole positions across fit schemes

scheme	$m_\pi = 208 \text{ MeV}$		$m_\pi = 305 \text{ MeV}$	
	z_ρ	z_ω	z_ρ	z_ω
A	$742.3(5.3) - i 55.0(5.3)$	$777.9(14.0) - i 0.31(0.09)$	$790.8(4.8) - i 28.1(3.3)$	$823.3(5.8)$
B1	$744.8(5.5) - i 55.3(5.6)$	$779.0(10.0) - i 0.23(0.07)$	$792.0(4.8) - i 27.7(3.3)$	$829.6(5.1)$
B2	$743.1(5.2) - i 54.9(5.1)$	$776.7(10.2) - i 0.30(0.09)$	$791.5(5.0) - i 28.2(3.4)$	$827.2(5.4)$
B3	$743.3(5.3) - i 55.1(5.4)$	$776.3(9.7) - i 0.33(0.07)$	$791.0(4.9) - i 28.2(3.5)$	$826.8(5.3)$

All pole positions are in MeV.

Comparison with FVU Yan et al., Phys. Rev. Lett. 133, 211906 (2024)



At the unphysical pion masses, our results are consistent with Finite Volume Unitarity (FVU) result.

No chiral extrapolation is performed in this work.

CONTOUR DEFORMATION FOR ANALYTICAL CONTINUATION

POLE CONDITION

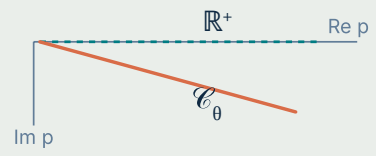
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$\mathbb{R}^+$

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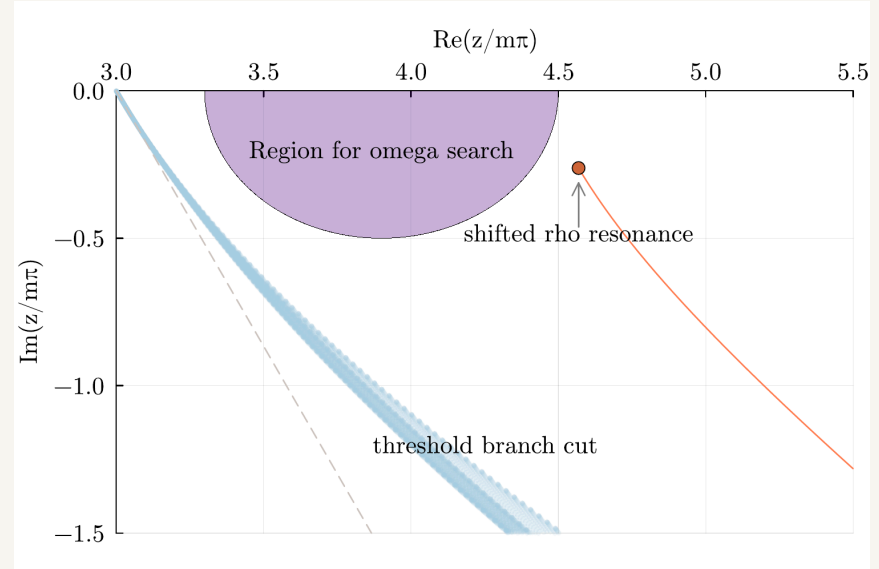
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$\mathcal{C}_\theta$



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Beyond three-body: 4π effects on Fin-Vol Energies of σ

FOCK STATES

σ_0

+

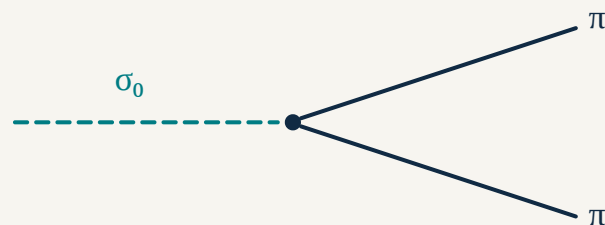
2π

+

4π

A minimal four-body **benchmark** system

TWO-PION COUPLING $\sigma \rightarrow 2\pi$

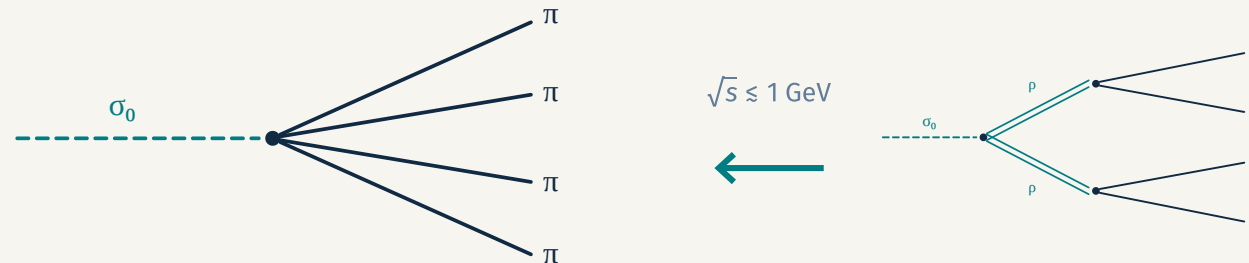


$$\sim g_{\sigma\pi\pi} \left(k^2 + \frac{7}{8} m_\pi^2 \right) f(k, \Lambda_{2\pi})$$

Ensures the correct $\pi\pi \rightarrow \pi\pi$ S -wave isoscalar Adler zero.

$$k^2 + \frac{7}{8} m_\pi^2 \xrightarrow{\text{on-shell}} s - \frac{1}{2} m_\pi^2$$

FOUR-PION COUPLING $\sigma \rightarrow 4\pi$

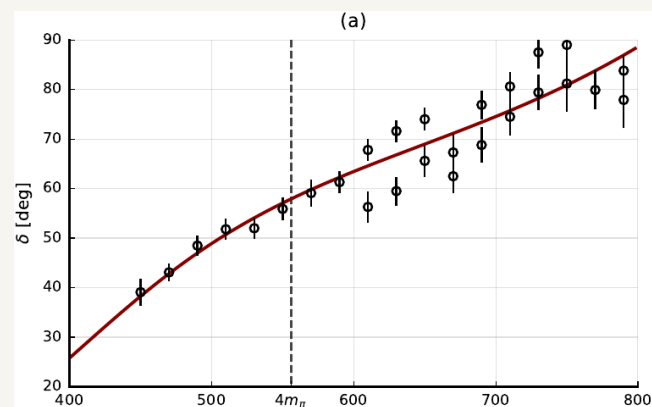


$$\langle \sigma_0 | V | 4\pi, a = 1 \rangle \sim g_{\sigma \rightarrow 4\pi} \left[(\vec{k}_1 - \vec{k}_3) \cdot (\vec{k}_2 - \vec{k}_4) + (\vec{k}_1 - \vec{k}_4) \cdot (\vec{k}_3 - \vec{k}_2) \right] \prod_{i=1}^4 f(k_i, \Lambda_{4\pi})$$

$$\langle \sigma_0 | V | 4\pi, a = 2 \rangle \sim \sqrt{3} \langle \sigma_0 | V | 4\pi, a = 1 \rangle \Big|_{\vec{k}_2 \leftrightarrow \vec{k}_3}$$

$I = 0$ 4π furnishes the two-dimensional irrep $[2, 2]$, hence two independent vertices.

$\Lambda_{4\pi} = 300$ MeV such that it is suppressed at $\sqrt{s} \gtrsim 1$ GeV



σ - 2π **FITTED PARAMETERS**

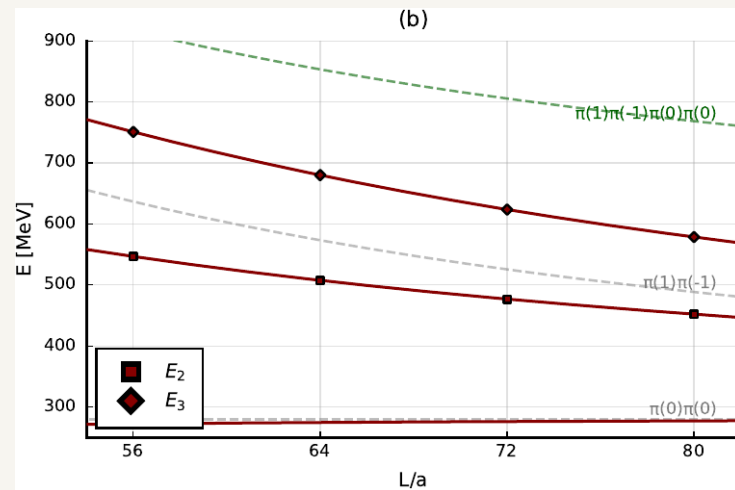
$$g_{\sigma\pi\pi} = 66.9 \quad \Lambda_{2\pi} = 346.0 \text{ MeV} \quad m_{\sigma_0} = 798.0 \text{ MeV}$$

Beyond three-body: 4π effects on Fin-Vol Energies of σ

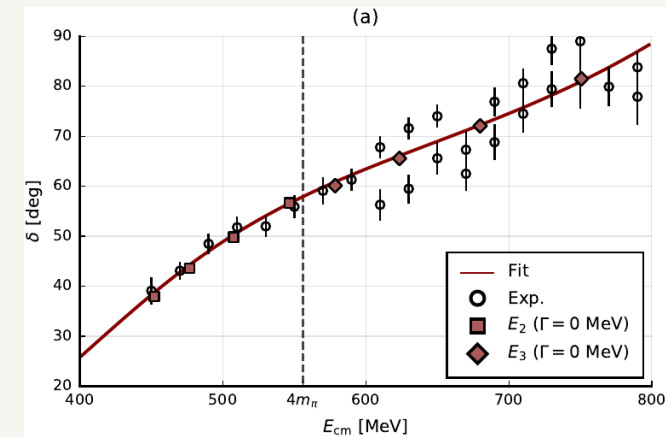
$\Gamma_{\sigma \rightarrow 4\pi}$ determines $g_{\sigma \rightarrow 4\pi}$

$\Gamma_{4\pi} = 0 \Rightarrow g_{\sigma \rightarrow 4\pi} = 0$

Finite-volume σ - 2π spectrum



Lüscher phase shifts from $E_{2,3}(L)$



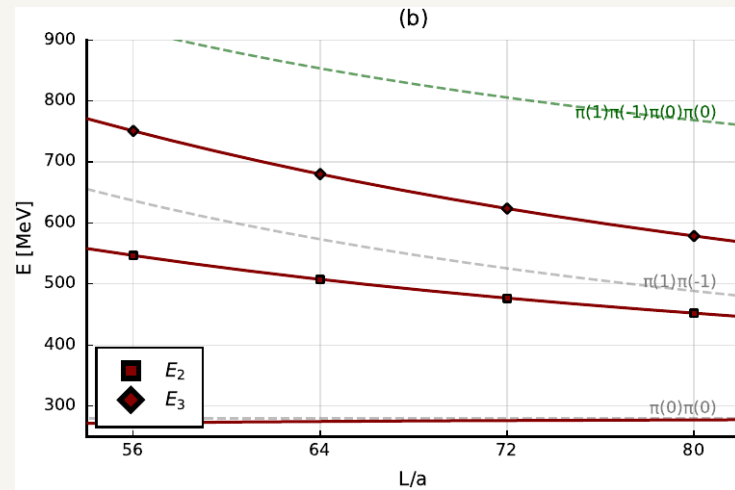
As expected from the equivalence of the two formalisms.

Beyond three-body: 4π effects on Fin-Vol Energies of σ

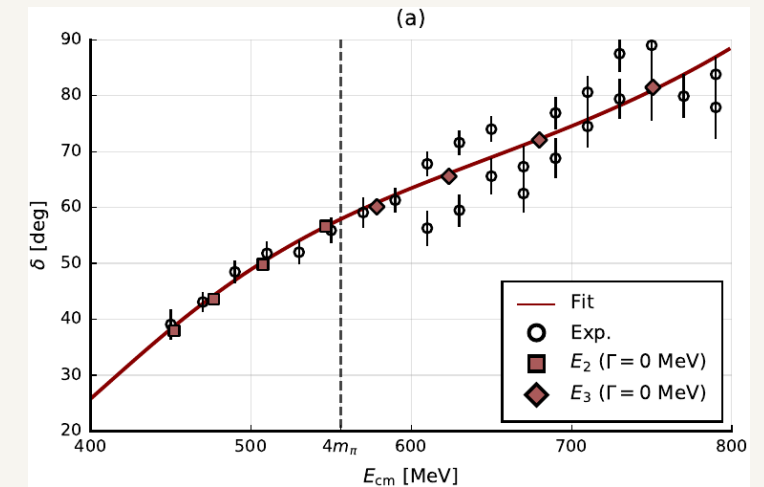
$\Gamma_{\sigma \rightarrow 4\pi}$ determines $g_{\sigma \rightarrow 4\pi}$

$\Gamma_{4\pi} = 1, 5, 20 \text{ MeV} \Rightarrow g_{\sigma \rightarrow 4\pi}^2$ increases linearly

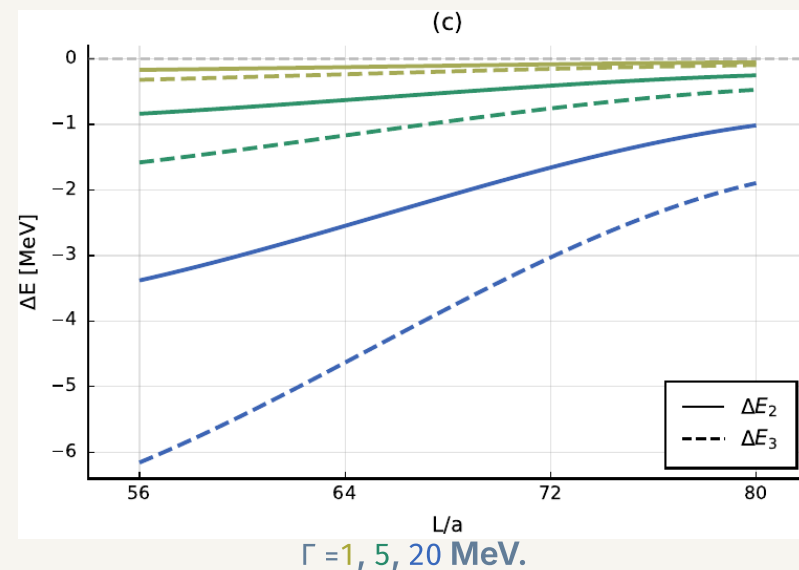
Finite-volume σ - 2π spectrum



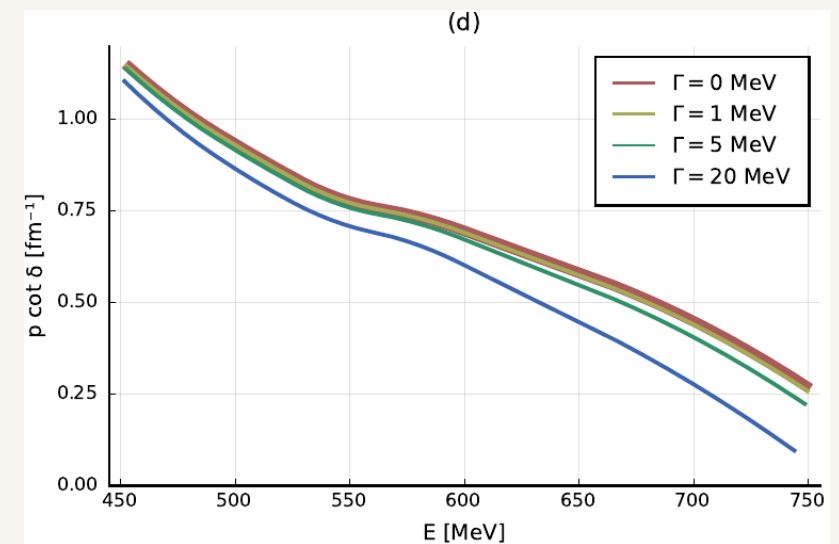
Lüscher phase shifts from $E_{2,3}(L)$



Finite-volume energy shift when switching on $\sigma \rightarrow 4\pi$



$p \cot \delta$ from two-body Lüscher analysis

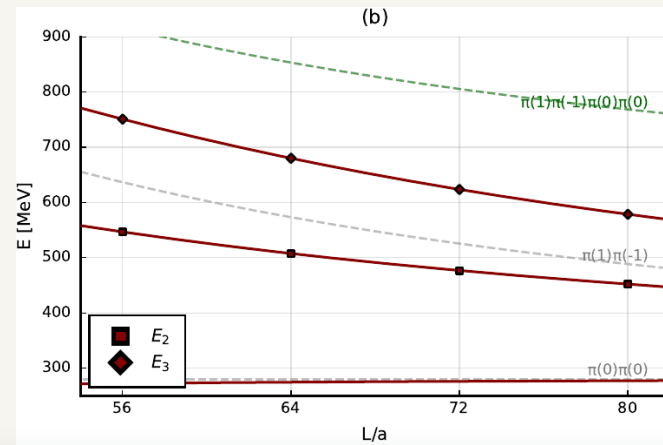


Beyond three-body: 4π effects on Fin-Vol Energies of σ

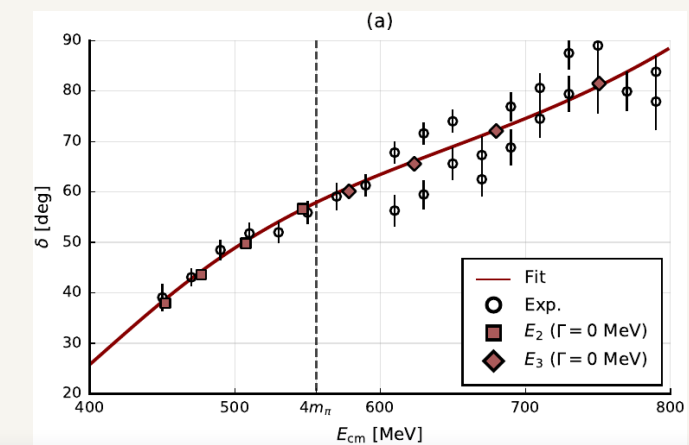
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Finite-volume σ - 2π spectrum

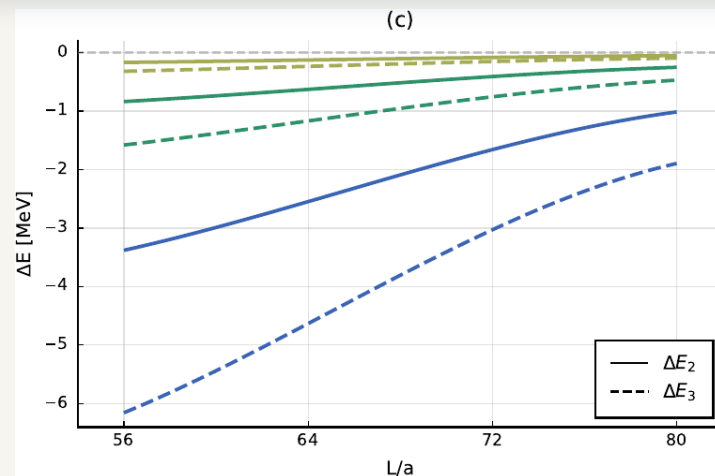


Lüscher phase shifts from $E_{2,3}(L)$



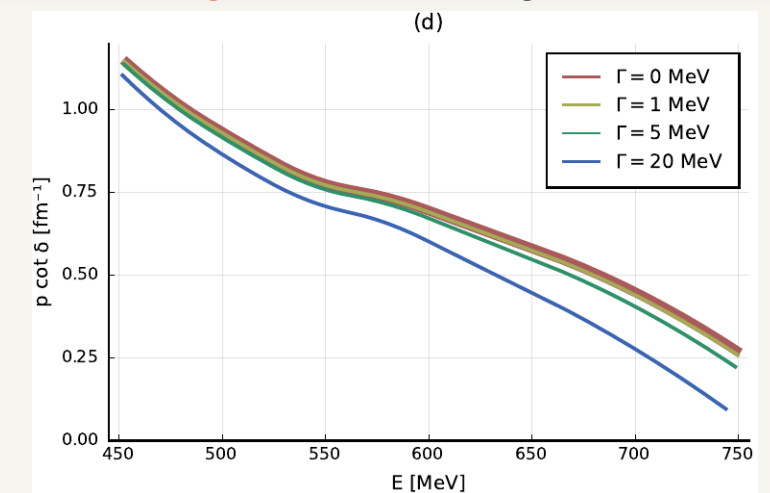
WE CAN INDEED DO A FOUR-BODY FINITE-VOLUME CALCULATION

finite-volume energy shift when switching on $\sigma \rightarrow 4\pi$



$\Gamma = 1, 5, 20 \text{ MeV}$.

$p \cot \delta$ from two-body Lüscher analysis



NPHFforFVE.jl: from interactions to finite-volume spectra

$N \leq 4$ · Julia 1.10+

A Julia package that turns a user-defined N -body Hamiltonian into symmetry-projected finite-volume spectra.

YOU PROVIDE

FockChannel and Quantum Numbers

particle contents, masses, spin, isospin,...

Lattice Config Info

Box size, lattice spacing (only for unit),
irreps,...

User-defined interaction

THE PACKAGE HANDLES

Finite-volume Hamiltonian

Automatically constructs the many-body Hamiltonian from the
supplied Fock sectors and interaction matrix elements.

Irrep projection

Projects the Hamiltonian to the selected finite-volume
irreducible representation.

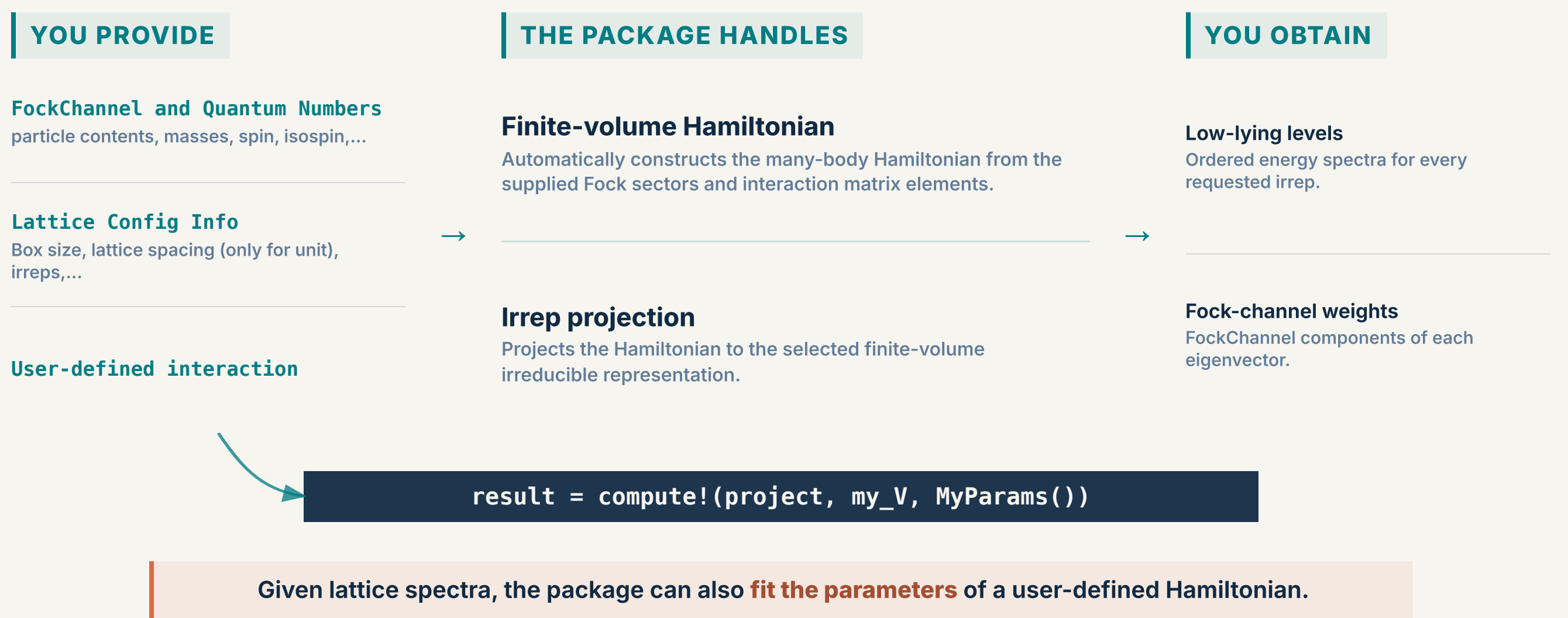
YOU OBTAIN

Low-lying levels

Ordered energy spectra for every
requested irrep.

Fock-channel weights

FockChannel components of each
eigenvector.



```
result = compute!(project, my_V, MyParams())
```

Given lattice spectra, the package can also **fit the parameters** of a user-defined Hamiltonian.

Minimal example: an $I = 1$ $\rho \oplus \pi\pi$ system

Define the system

```
using NPHFforFVE

rho = FockChannel(
    "rho", [1], [:boson],
    [dynamic_mass(:m_rho0)],
    [1//1], [1//1], [-1.0], relativistic,
)

pipi = FockChannel(
    "pipi", [2], [:boson],
    [140.0],                # m $\pi$  [MeV]
    [0//1], [1//1], [-1.0], relativistic,
)

project = Project(D000, 1//1, [rho, pipi], [0, 10])

config = add_config!(project, 48, 0.10, ["T1-"], [6])

generate_potential_template(project, "rho_pipi_potential.jl")
```

Provide the interaction

```
# rho_pipi_potential.jl is generated

@params struct YourParameters
    YOUR_PARAMETERS
end

# Inside the generated potential file
function my_V(..., chA, chB, ..., params)
    if (chA, chB) in ((1,2), (2,1))
        return YOUR_ $\rho \leftrightarrow \pi\pi$  VERTEX
    elseif (chA, chB) == (2,2)
        return YOUR_ $\pi\pi \rightarrow \pi\pi$  INTERACTION
    end
end
```

Compute

```
include("rho_pipi_potential.jl")
result = compute!(project, my_V, YourParameters())
```

QUESTIONS & DISCUSSION

Thank you

NPHF VISION

Toward a more direct **connection** between

**lattice hadron
spectroscopy**
(hep-lat)



**hadron physics
phenomenology**
(hep-ph)

Details of irrep projection

TECHNICAL BACKUP

01

Choose a set of reference states $\{|v_i^{\text{ref}}\rangle\}$ →

02

Generate G -orbits $\mathcal{O}(v_i^{\text{ref}})$ →

04

Projection in each G -orbit →

05

Diagonalize overlap matrix I^Γ

G-orbit equivalence

As a simple example, consider a two-body system in the CM frame with definite total isospin:

$$|\vec{p}, -\vec{p}, \lambda, \lambda'\rangle \xrightarrow{\mathbb{P}} |-\vec{p}, \vec{p}, -\lambda', -\lambda\rangle \stackrel{\text{isospin}}{\propto} |\vec{p}, -\vec{p}, -\lambda, -\lambda'\rangle$$

Due to the rearrangement theorem $gG = G \quad \forall g \in G$,

$$\mathcal{O}(|\vec{p}, -\vec{p}, \lambda, \lambda'\rangle) = \mathcal{O}(|-\vec{p}, \vec{p}, -\lambda', -\lambda\rangle)$$

Analytic form of the overlap matrix

$$I_{bv',av}^\Gamma = \sum_{g \in \text{Per}(\{\vec{n}^r\})} \underbrace{e^{-i\varphi^n(\{\vec{n}^r, \lambda\}, g)}}_{\text{phase factor from helicity}} \times \underbrace{D_{v'v}^{\Gamma*}(g)}_{\text{irrep matrix}} \times \underbrace{\sum_{s \in S_N} \delta(s) \mathcal{R}_{ba}^{[K]}(s) \prod_{i=1}^N \delta_{\vec{n}_{s_i}, g \vec{n}_i} \delta_{\lambda_{s_i}, \epsilon(g) \lambda_i}}_{\text{permutation and statistical factor from isospin symmetry}}$$

nonzero eigenvectors $c^r \rightarrow$ independent orthonormal irrep basis

Scattering kernel

$$U(p, k; z) = C(p, k) + R(p, k; z) + Z(p, k; z)$$

C $\rho_0\pi$ contact interaction

$$C(p, k) \propto \frac{\pi c_1}{3} [pF(p, \Lambda_{\rho_0\pi})] [kF(k, \Lambda_{\rho_0\pi})]$$

R bare ω_0 s -channel exchange

$$R(p, k; z) \propto \frac{1}{z - m_{\omega_0}} \left(\frac{g_{\omega\rho\pi}}{6\sqrt{2}\pi} \right)^2 [pF(p, \Lambda_{\omega_0})] [kF(k, \Lambda_{\omega_0})]$$

Z connected $\rho_0\pi \rightarrow 3\pi \rightarrow \rho_0\pi$ exchange kernel

$$Z(p, k; z) = 4\pi^{3/2} \sum_{\sigma, \sigma'} C_{1\sigma-\sigma'; 1\sigma'}^{1\sigma} C_{10; 1\sigma}^{1\sigma} N_{1, \sigma-\sigma'} \int dx \frac{P_{1, \sigma-\sigma'}(x) v_{\sigma'}^*(-\vec{k}, \vec{p} + \vec{k}) v_{\sigma}(\vec{p} + \vec{k}, -\vec{p})}{z - \omega_{\pi}(k) - \omega_{\pi}(p) - \omega_{\pi}(|\vec{p} + \vec{k}|) + i0^+}$$

$F(k, \Lambda) = \left(\frac{\Lambda^2}{k^2 + \Lambda^2} \right)^2$; $\mathcal{J}$ collects normalization factors; v_{σ} is the $\rho_0 \rightarrow \pi\pi$ vertex. The Z term generates the 3π unitary branch cut.

Contour-deformed pole condition

$$\mathbb{R}^+ \rightarrow \mathcal{C}_{\theta} \arg q_{\theta} = -\theta$$

$$\det_{\mathcal{C}_{\theta}} [\delta_{ij} - U(p_{\theta i}, p_{\theta j}; z) q_{\theta j}^2 w_j G_2(p_{\theta j}; z)] = 0$$

Simultaneously deform the self-energy contour in $\Sigma_{\pi\pi}$.

The effective $\sigma \rightarrow 4\pi$ coupling is calibrated through the spectral-function-averaged partial width.

Spectral-function-averaged $\sigma \rightarrow 4\pi$ partial width

$$\Gamma \equiv \Gamma_{\sigma \rightarrow \pi^+\pi^-\pi^0\pi^0} = \int_{(4m_\pi)^2}^{\infty} \rho(s) \Gamma(s) ds$$

Dressed σ spectral function

$$\rho(s) = \frac{\text{Im } \Pi_{\pi\pi}(s)}{\left[s - m_{\sigma_0}^2 - \text{Re } \Pi_{\pi\pi}(s)\right]^2 + [\text{Im } \Pi_{\pi\pi}(s)]^2}$$

$\Pi_{\pi\pi}(s) = 2m_{\sigma_0} \Sigma'_{\pi\pi}(s)$, generated by the $\sigma \leftrightarrow 2\pi$ self-energy.

Energy-dependent partial width

$$\Gamma(s) = \int d\Phi_4 \left| \mathcal{M}_{\sigma \rightarrow 4\pi}(s) \right|^2$$

$d\Phi_4$ is the four-pion phase-space measure; $\mathcal{M}_{\sigma \rightarrow 4\pi}$ follows from the effective four-pion vertices.

Coupling calibration

$$\Lambda_{4\pi} = 300 \text{ MeV} \quad + \quad \Gamma_{\text{target}} \quad \Longrightarrow \quad g_{\sigma \rightarrow 4\pi}$$

At fixed $\Lambda_{4\pi}$, each chosen averaged width determines the coupling $g_{\sigma \rightarrow 4\pi}$.