



Gradient-Flow Renormalization of Nonlocal Quasi Operators in the Lattice QCD

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OUTLINE

1

Renormalizations

Why do Renormalization; Schemes

2

Gradient Flow

Definition; Advantages; Method

3

Linear Divergence (under Flow Scheme)

Different Ways; Wilson Line; Double Ratio; Quasi-MEs

4

Apply the Renorm (on quasi-MEs)

Eliminate the Linear Divergence; Matching, ...

5

Summary

Flow Scheme for LaMET Work



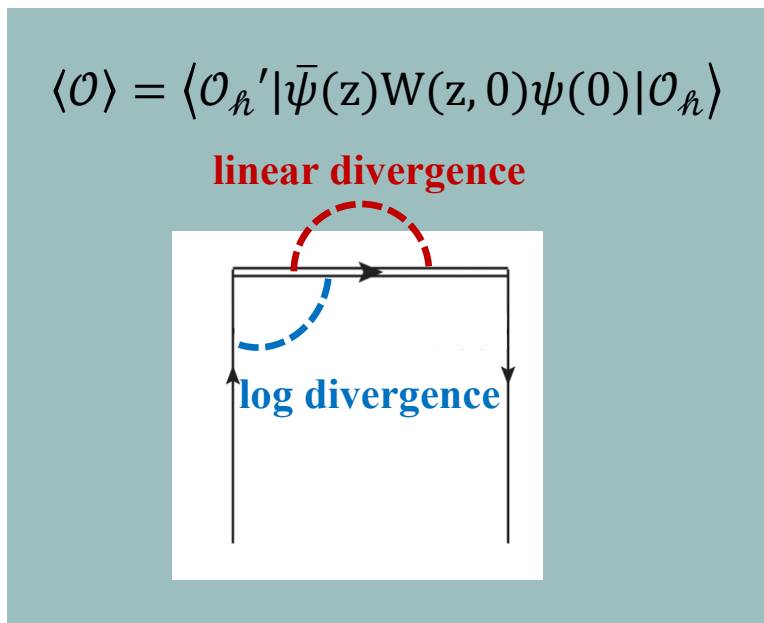
Part 1



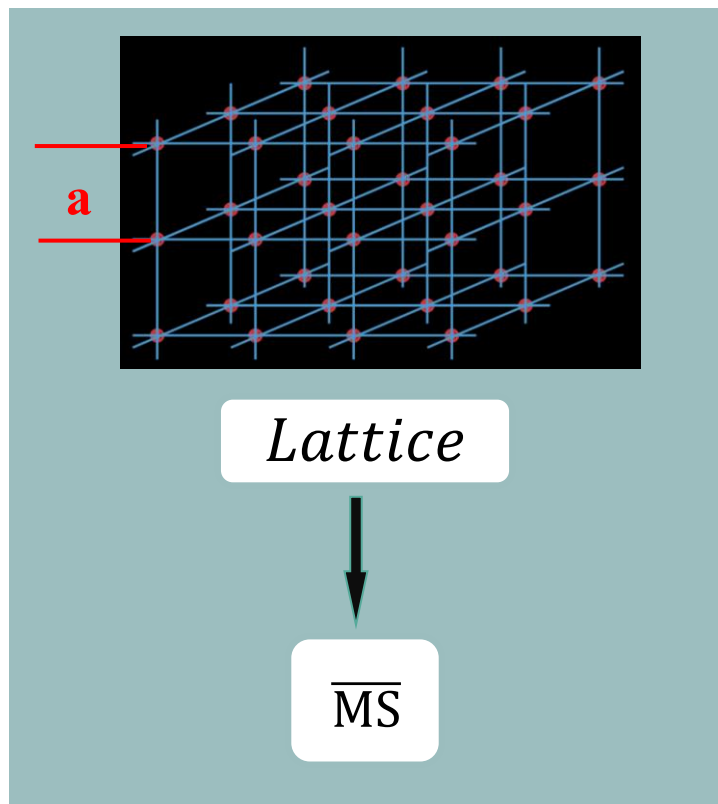
Why Renormalization

1. Why Renormalization(nonlocal matrix element)

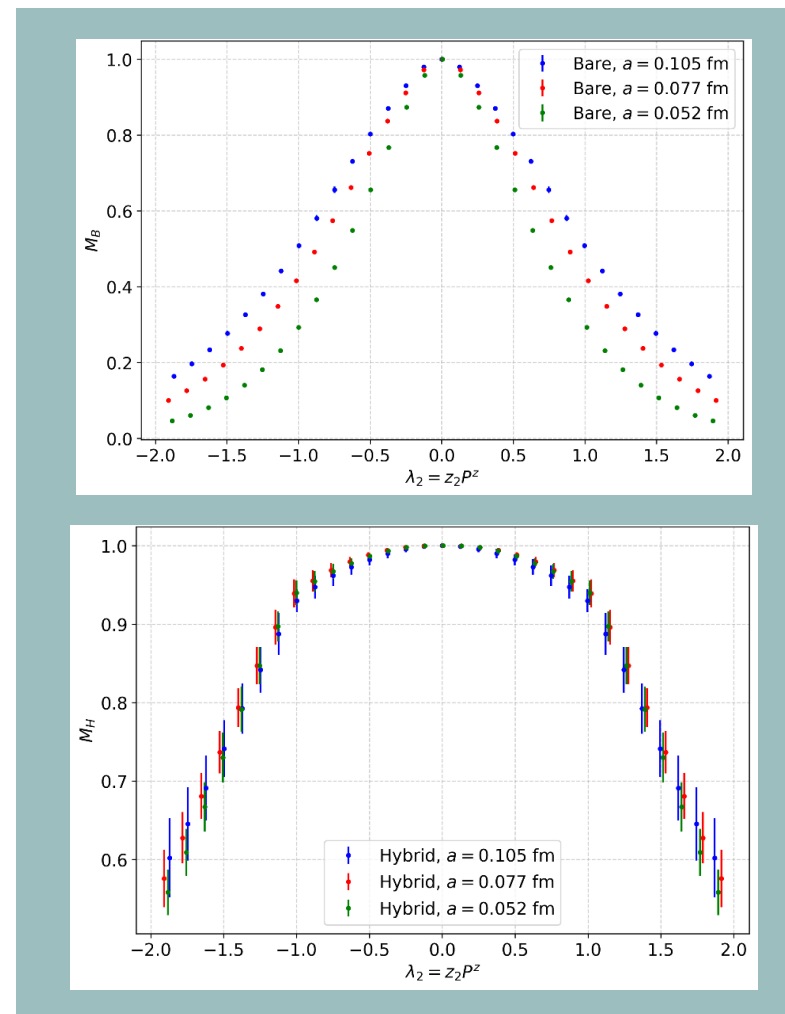
- Nonlocal Matrix element



- Scheme Conversion



- Divergence Subtraction



1) Renormalization requires the ability to **eliminate** the **divergence** caused by the lattice effect.

2) The converted scheme is well-defined for effective **matching** or running ...

2.Comparison of Different Renormalization

➤ RI/MOM、RI/SMOM Scheme

Martinelli et.al, NPB (1994);

Alexandrou et.al, NPB (2017); Stewart, Zhao, PRD (2018)

He et.al, PRD (2022); Bi et.al, PRD (2023)

➤ Ratio Scheme

Radyushkin et.al, PRD (2017)

➤ Self Renormalization (and Hybrid Scheme)

Ji et.al, NPB (2021); Huo et.al, NPB (2021)

Bai et.al, PRD (2025)

➤ Gradient Flow Scheme (also Hybrid Scheme)

M.Lüscher, JHEP 08 (2010) 071

M.Lüscher and P. Weisz, JHEP 02 (2011) 051

N. Brambilla and X. P. Wang, JHEP 06 (2024) 210

Gao et al., arXiv:2609.23379 [hep-lat] (2026)

Pros

Strict subtraction、
Well established、
Widely applied ...

Easy to use
Well subtraction at
short-distance

Well subtraction at
All-distance

Step by step, eliminate the
divergence,
Additional data for
divergence subtraction,
Better **signal-to-noise ratio**
...

Cons

Not suitable for **non-local**
(e.g. Gauge dependence,
Signal problem ...)

Extra IR behavior at
large-distance

Mixed a dependence
Fitting is **subtle**

Generate **multiple**
flow time data



Part 2

What is Gradient Flow renormalization

What is Gradient Flow

Gradient flow equation:

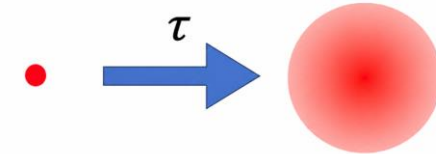
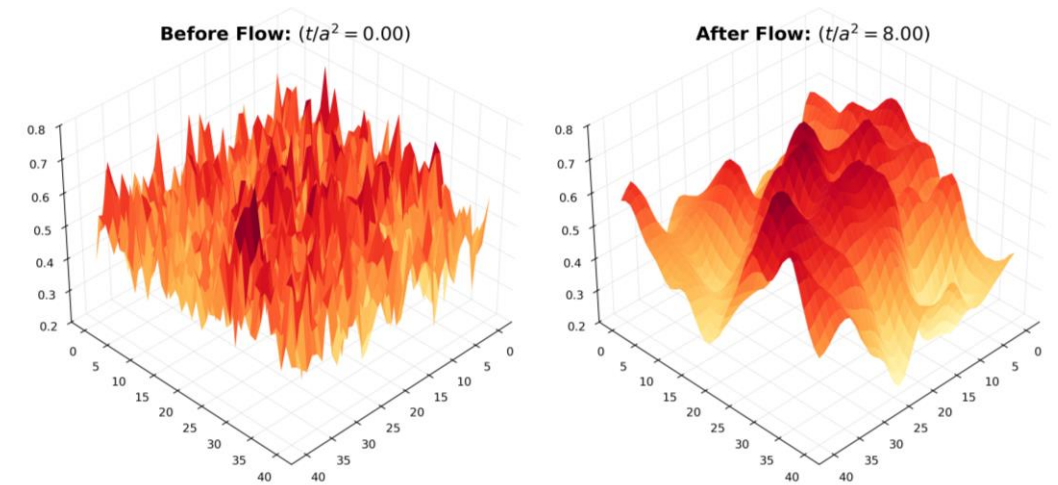
Gauge field: $\partial_\tau B_\mu^a = \mathcal{D}_\nu^{ab} G_{\nu\mu}^b$

Fermion field: $\partial_\tau \chi = \Delta \chi$

Anti-fermion field: $\partial_\tau \bar{\chi} = \bar{\chi} \overleftarrow{\Delta}$

x : spacetime 3+1; τ : an artifact called “flow time”

- 1). **Renormalization scale**: It transforms a quantity dependent on the lattice spacing(a) into one dependent on flow time(τ).
- 2). **Smearing**: For gauge, gradient flow is a continuous version of 4D stout smearing.



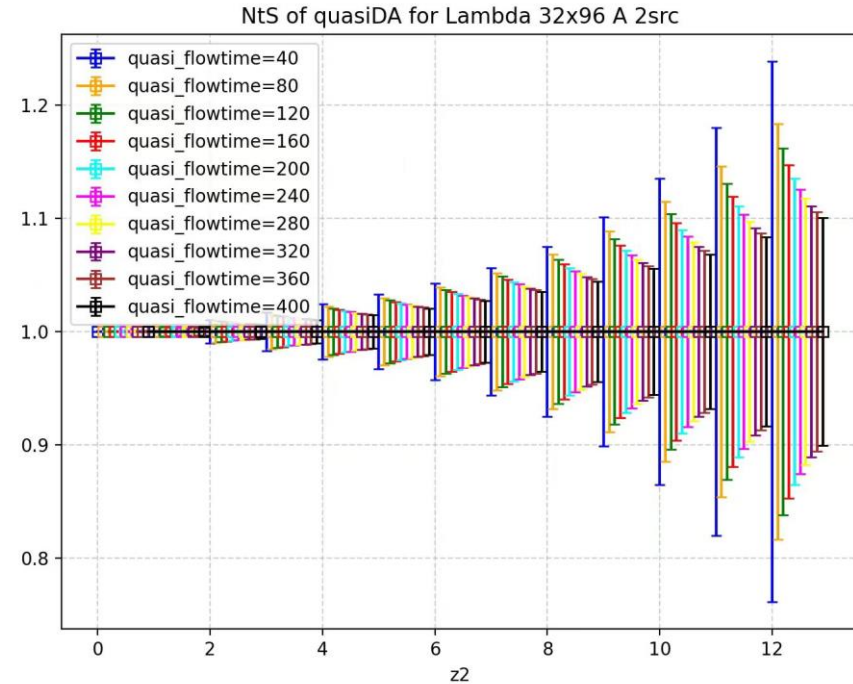
A localized field configuration evolves with flow time τ , spreading into a smooth distribution with effective radius $\sim \sqrt{8\tau}$.

Lattice implementation: Starting from the original fundamental fields, we numerically calculate the gauge fields and quark fields at a given flow time τ via the **3rd-order Runge-Kutta method**.

Why use Gradient Flow

Merits of gradient flow (in theory | on Lattice)

- ◆ **Simpler renormalization for Lattice QCD** (Renormalization scale)
- ◆ **Matching to $\overline{MS}$ scheme:** It admit perturbative calculation and can be matched to $\overline{MS}$ scheme through factorization.
- ◆ **UV renormalon:** It illustrate the interplay between linear renormalons in lattice QCD and continuous scheme
- ◆ **Improved Signal-to-Noise Ratio**(smearing)
It is a continuous version of 4D stout smearing.



How to use Gradient Flow renormalization

1. Extract and eliminate the linear divergence

$$\tilde{q}^{Lat}(z, a, \tau) = \frac{\tilde{q}^{Lat}(z, a, \tau) e^{\frac{\delta m}{\sqrt{8\tau}} z}}{\tilde{q}^{Lat}(0, a, \tau)}$$

A) Use Wilson loop to fit linear divergence

$$\lim_{t \rightarrow \infty} \langle W_{loop}(r, \tau, t) \rangle \sim e^{-V(r, \tau) t}$$

$$V(r, \tau) \sim V_0(\tau) + \sigma r - \frac{\alpha}{r}$$

$$V_0(\tau) \sim \frac{2\delta m}{\sqrt{8\tau}} + B$$

B) Use gauge-fixed Wilson line to fit linear divergence

$$\langle W_{line}(z, \tau) \rangle \sim e^{-\frac{\delta m}{\sqrt{8\tau}} z + Bz}$$

2. Matching

$$\tilde{q}^{\overline{MS}}(z, \tau, \mu) = C(z, \tau, \mu) \tilde{q}^{Lat}(z, \tau)$$

Matching kernel is from perturbation calculus

$$C^{FO}(\tau, \mu) = 1 - \frac{\alpha_s}{4\pi} C_F [3 \log(2\mu^2 \tau e^{\gamma_E}) + 5/2]$$

$$C^{RG}(\tau, \mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu')} \right)^{-\frac{\gamma}{2\beta_0}} C^{FO}(\tau, \mu')$$

1. Lattice Setup: Pure gauge quantity and Nonlocal MEs

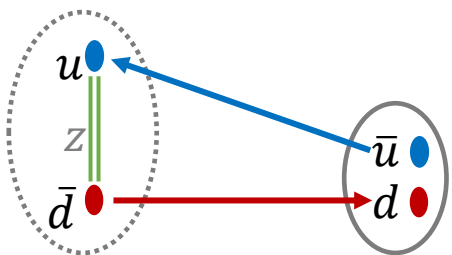
Z.-C. Hu, B.-L. Hu, J.-H. Wang, et al., Phys. Rev. D 109 (2024) 054507.

	a/fm	$n_s^3 \times n_t$	m_π/MeV	n_{cfg}
<i>C24P29</i>	0.10530(18)	$24^3 \times 72$	292.4(1.1)	864
<i>F32P30</i>	0.07746(18)	$32^3 \times 96$	303.2(1.3)	777

	$\sqrt{8\tau}/\text{fm}$	z/a	p^z/a^{-1}
<i>C24P29</i>	0, 0.06 – 0.20	0 – 12 ($\text{max} \sim 1.2\text{fm}$)	0 – 6 ($\text{max} \sim 3.0\text{GeV}$)
<i>F32P30</i>	0, 0.06 – 0.20	0 – 16 ($\text{max} \sim 1.2\text{fm}$)	

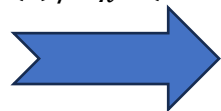
Pure gauge quantity
(Wilson loop | Polyakov loop | Wilson line)

Nonlocal MEs
(pion DA | pion PDF | proton PDF)



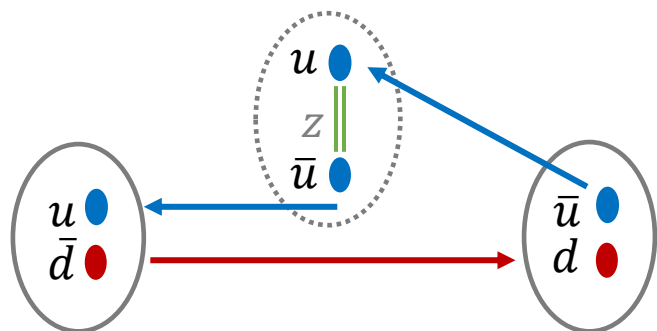
$$C_{\pi}^{2pt}(z, P_z, \tau) = \langle O_{\pi}(z, \tau) | \bar{O}_{\pi}(P_z) \rangle$$

$$C_{\pi}^{2pt}(z)/C_{\pi}^{2pt}(z=0)$$

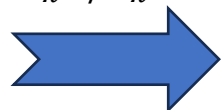


Pion quasi-DA

$$\langle O_{\pi}(P_z, z, \tau) \rangle_{\text{bare}} = \langle 0 | \bar{d}_{\tau}(z) \gamma_t \gamma_5 W_{\tau}(z, 0) u_{\tau}(0) | P_z \rangle$$



$$C_{\pi}^{3pt}/C_{\pi}^{2pt}$$



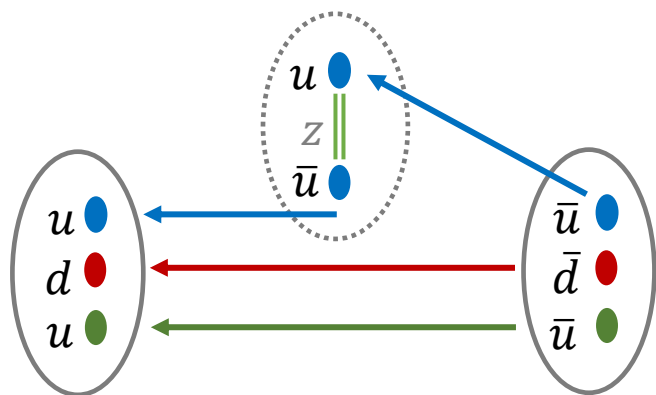
Pion quasi-PDF

$$\langle O_{\pi}(P_z, z, \tau) \rangle_{\text{bare}} = Z_{\chi} \langle P_z | \bar{u}_{\tau}(z) \gamma_t W_{\tau}(z, 0) u_{\tau}(0) | P_z \rangle$$

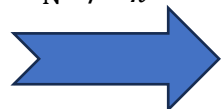
Normalize:

$$\langle O_{\pi}(P_z, z, \tau) \rangle_{\text{norm}} = \langle O_{\pi}(P_z, z, \tau) \rangle_{\text{bare}} / \langle O_{\pi}(P_z, z=0, \tau) \rangle_{\text{bare}}$$

$$C_{\pi}^{3pt}(z, P_z, \tau) = \langle O_{\pi}(P_z) | O_{\Gamma}(z, \tau) | \bar{O}_{\pi}(P_z) \rangle$$



$$C_N^{3pt}/C_{\pi}^{2pt}$$



Proton quasi-PDF

$$\langle O_N(P_z, z, \tau) \rangle_{\text{bare}} = Z_{\chi} \langle P_z | \bar{u}_{\tau}(z) \gamma_t W_{\tau}(z, 0) u_{\tau}(0) | P_z \rangle$$

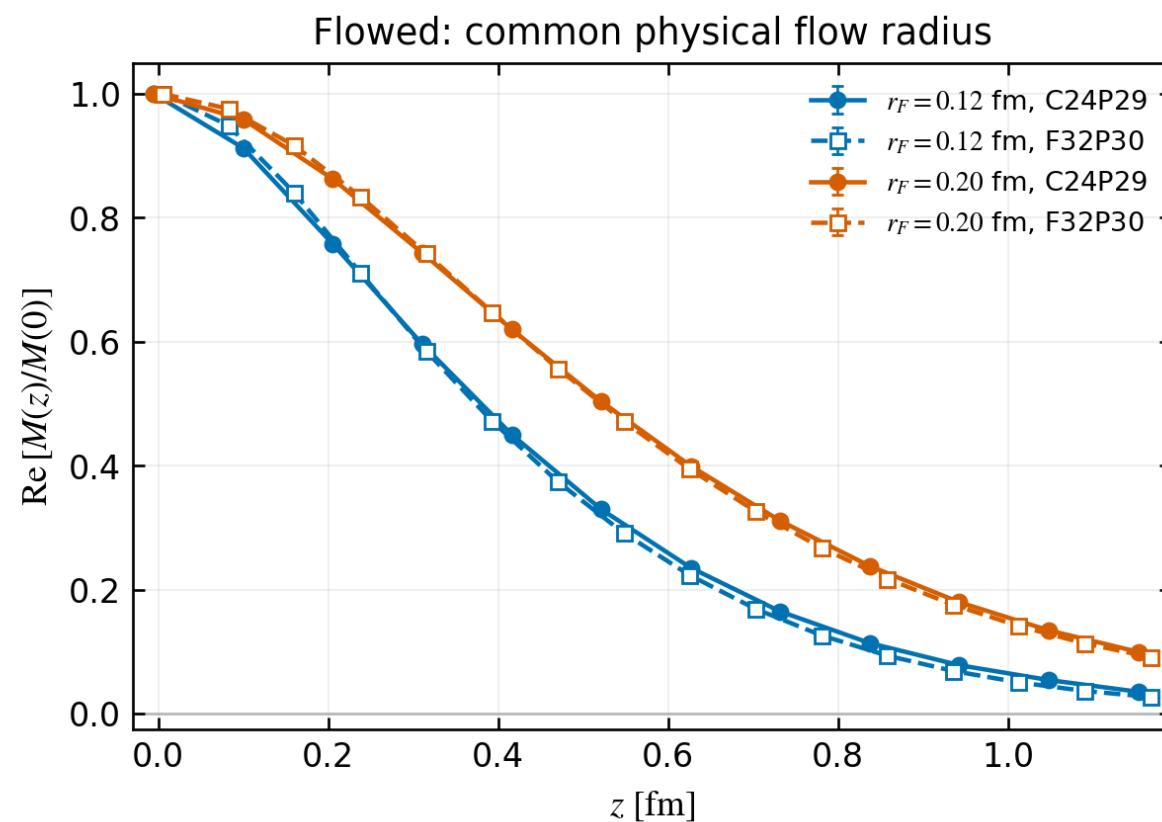
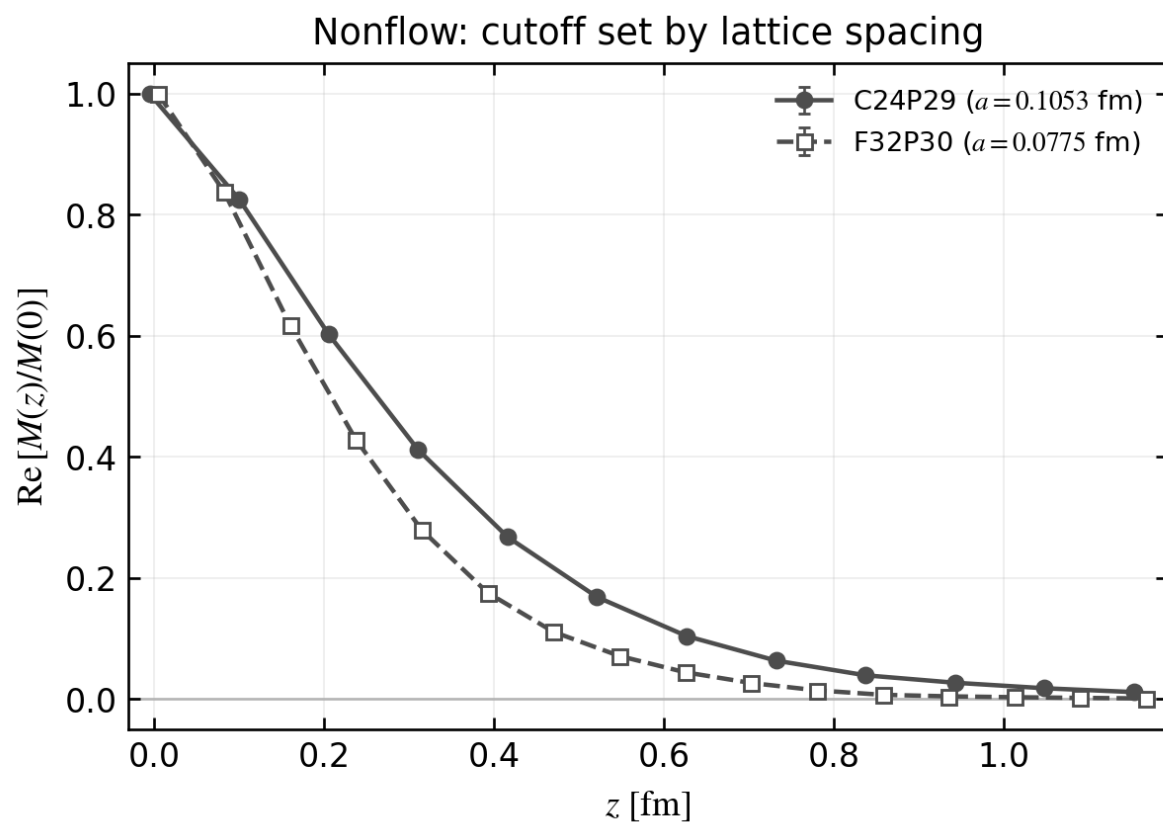
Normalize:

$$\langle O_N(P_z, z, \tau) \rangle_{\text{norm}} = \langle O_N(P_z, z, \tau) \rangle_{\text{bare}} / \langle O_N(P_z, z=0, \tau) \rangle_{\text{bare}}$$

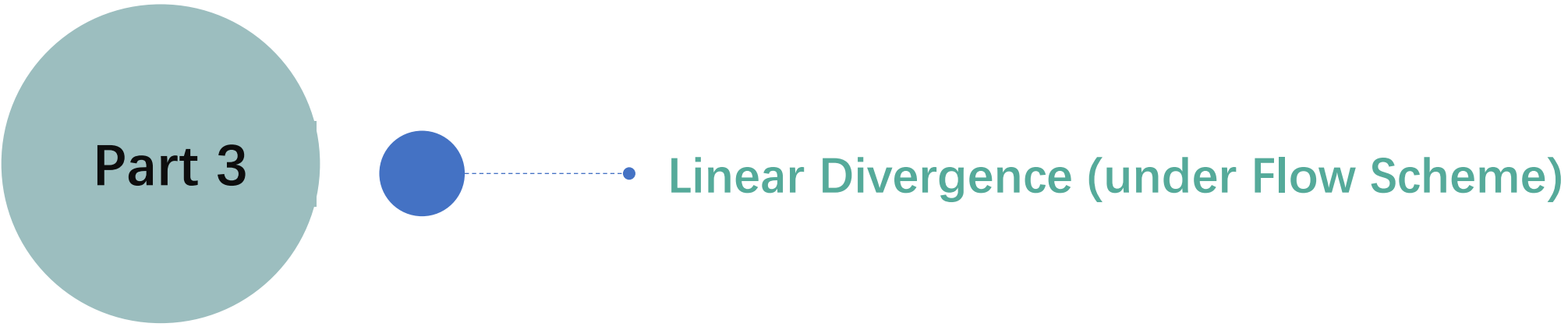
$$C_N^{3pt}(z, P_z, \tau) = \langle O_N(P_z) | O_{\Gamma}(z, \tau) | \bar{O}_N(P_z) \rangle$$

The normalization is introduced to remove the **extra renormalization factor** associated with the flowed quark field.

pion DA: $P_z=0$, real part only



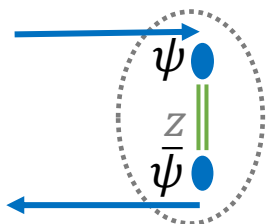
The lattice-spacing dependence of nonlocal matrix elements can **be controlled by the flow time**



Part 3

Linear Divergence (under Flow Scheme)

Linear divergence (in Nonlocal MEs)



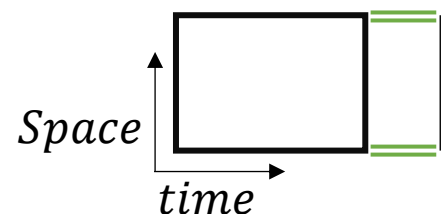
$$\langle \mathcal{O}_{\pi/N}(P_z, z, \tau; a) \rangle_{\text{norm}} = \langle P'_z | \bar{\chi} \Gamma W_\tau(z, 0) \chi | P_z \rangle$$

$$= Z_{R,Flow} * C_{Flow} * M_{\overline{MS}}(z)$$

$$Z_{R,Flow}(z, \frac{1}{\sqrt{\tau}}, a) = \exp\left\{ \frac{\delta_m}{\sqrt{\tau}} z + m_0 z \right\}$$

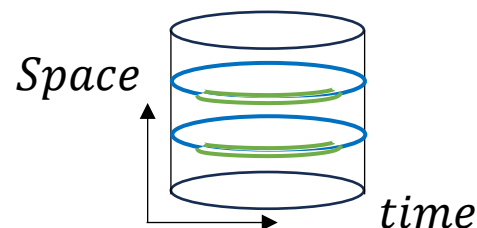
Linear divergence (in pure gauge quantities)

Wilson loop



$$\langle W(\tau, s, t) \rangle = \frac{1}{N_c V} \langle U_\tau(s) U_\tau(t) U_\tau^\dagger(s) U_\tau^\dagger(t) \rangle$$

Polyakov loop



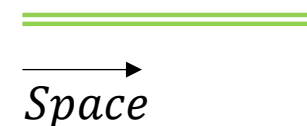
$$\langle P(\tau, s) \rangle = \frac{1}{N_c V} \left\langle \prod_{t=0}^{N_t-1} U_\tau(t, 0) \prod_{t=0}^{N_t-1} U_\tau(t, s) \right\rangle$$

$$\lim_{t \rightarrow \infty} \langle W(\tau, z, t) \rangle, \langle P(\tau, s) \rangle \sim e^{-V(\tau, z)t}$$

$$V(\tau, z) \sim A_0(\tau) + \sigma z + \frac{E(\tau)}{z}$$

$$A_0(\tau) \sim -\frac{\delta_m}{\sqrt{\tau}} + q$$

Wilson line



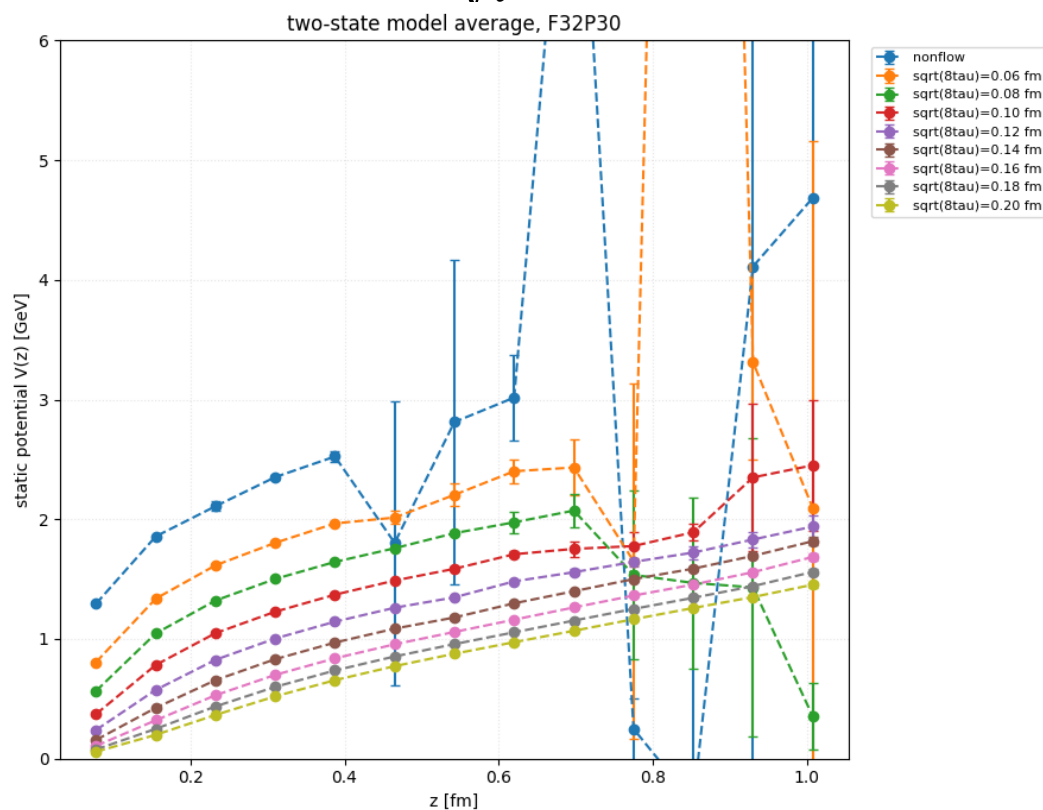
$$\langle L(\tau, s) \rangle = \frac{1}{N_c V} \sum_s \text{ReTr } U_\tau^G(s)$$

$$\langle L(\tau, z) \rangle \sim \exp\left\{ \frac{\delta_m}{\sqrt{\tau}} z + m_0 z \right\}$$

Static potential (Wilson loop)

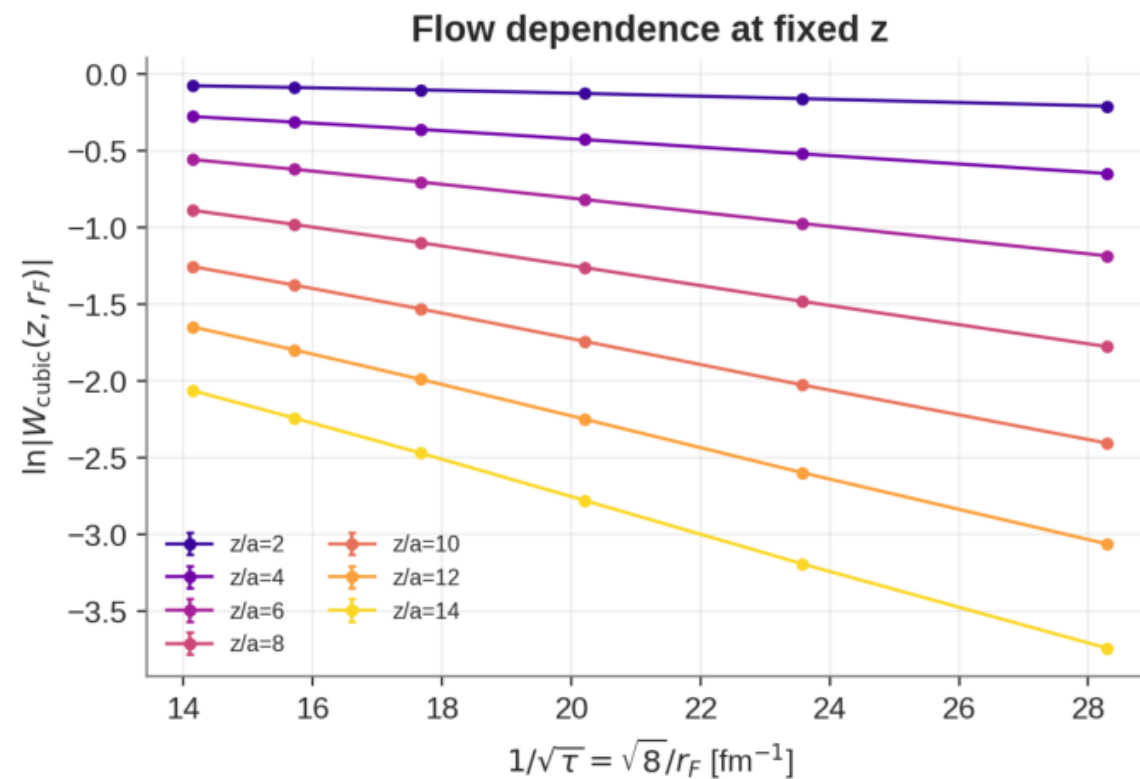
$$V(\tau, z) \sim A_0(\tau) + \sigma z + \frac{E(\tau)}{z}$$

$$A_0(\tau) \sim -\frac{\delta_m}{\sqrt{\tau}} + q$$



Wilson line

$$\langle L(\tau, z) \rangle \sim \exp \left\{ \frac{\delta_m}{\sqrt{\tau}} z + m_0 z \right\}$$



With current statistics, the Wilson line signal is better than the potential signal, so **no robust linear-divergence fit** is possible from the potential; we use the **Wilson line** for extraction.

Direct joint-fit

$$\ln|\langle \mathcal{O}(z, \tau) \rangle| = \delta_m * z * \frac{1}{\sqrt{\tau}} + b(z) \quad \cdots \rightarrow$$

$$M_{\overline{MS}}(z), f(z)a^2, \\ m_0 z \cdots$$

Double ratio

$$\langle \mathcal{R}(z, \tau) \rangle = \frac{\langle \mathcal{O}(z, \tau) \rangle / \langle \mathcal{O}(0, \tau) \rangle}{\langle \mathcal{O}(z, \tau^{\text{ref}}) \rangle / \langle \mathcal{O}(0, \tau^{\text{ref}}) \rangle}$$

$$\ln|\langle \mathcal{R}(z, \tau) \rangle| = \delta_m * z * \left(\frac{1}{\sqrt{\tau}} - \frac{1}{\sqrt{\tau^{\text{ref}}}} \right) + B$$

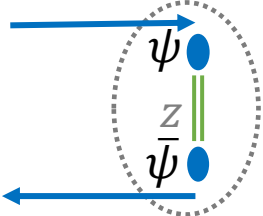
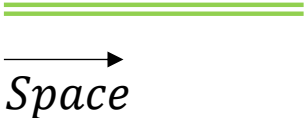
$$\sqrt{8\tau^{\text{ref}}} = 0.20\text{fm}$$

The double-ratio method for extracting the linear divergence is **unique to the gradient flow**.

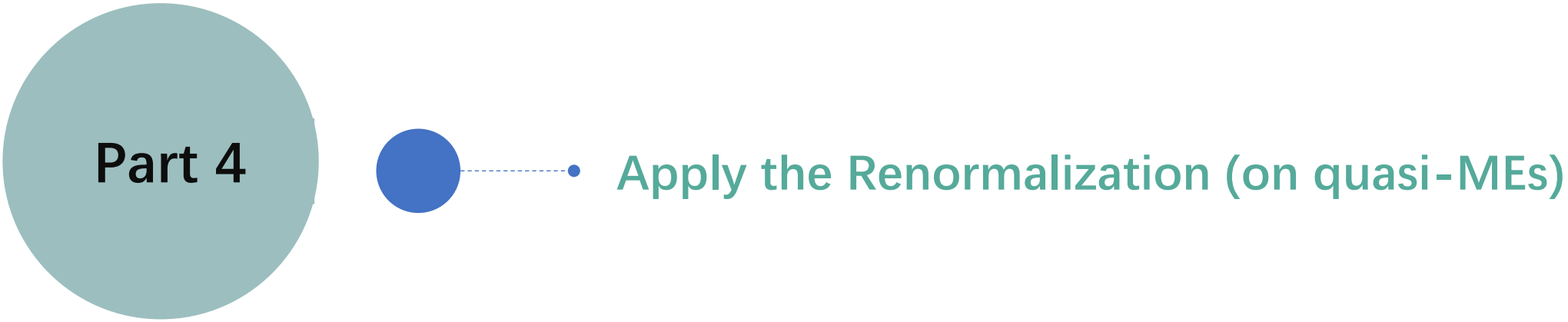
It enables nonlocal MEs to also exhibit a **clean linear divergence** and to be compared with results from other pure-gauge quantities.

Fit function for Linear divergence

$$\ln|\langle \mathcal{R}(z, \tau) \rangle| = \delta_m \left(\frac{1}{\sqrt{\tau}} - \frac{1}{\sqrt{\tau^{\text{ref}}}} \right) z + B(\tau)$$
$$\langle \mathcal{O}(z, \tau) \rangle \in \{ \langle \mathcal{O}_{\pi/N}(z, \tau) \rangle_n, \langle L(\tau, z) \rangle \}, \sqrt{8\tau^{\text{ref}}} = 0.20\text{fm}$$

Linear divergence coefficient δ_m				
quasi MEs		Pion DA	Pion PDF	Proton PDF
		−0.1118 ± 0.0086	−0.1153 ± 0.0020	−0.1150 ± 0.0046
Wilson line			−0.1165 ± 0.0001	

- 1) For different physical quantities, the extracted linear divergences are fairly close.
- 2) At low momentum, the linear divergences of nonlocal MEs for different momenta are relatively close.
- 3) For different lattice spacings, the nonlocal MEs and Wilson line results are roughly comparable.



Part 4

Apply the Renormalization (on quasi-MEs)

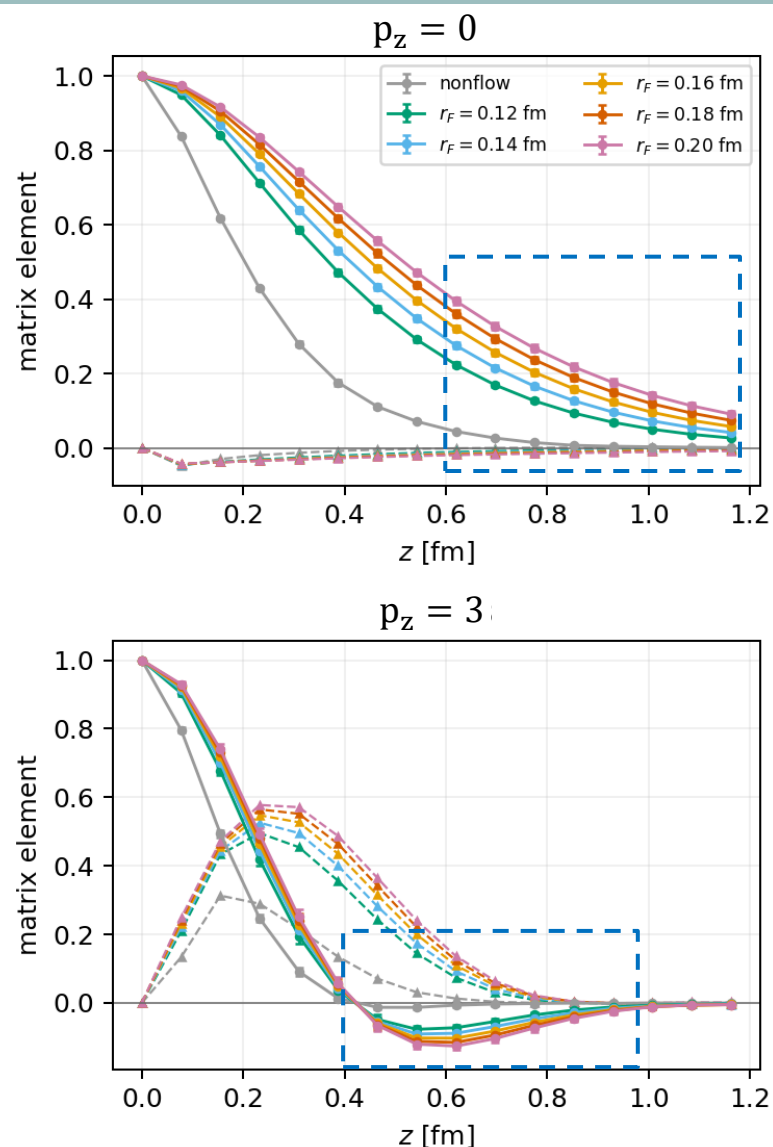
Eliminate the linear divergence

$$\tilde{q}^{Lat}(z, a, \tau) = \tilde{q}_{\text{norm}}^{Lat}(z, a, \tau) e^{\frac{\delta_m}{\sqrt{\tau}} z}$$

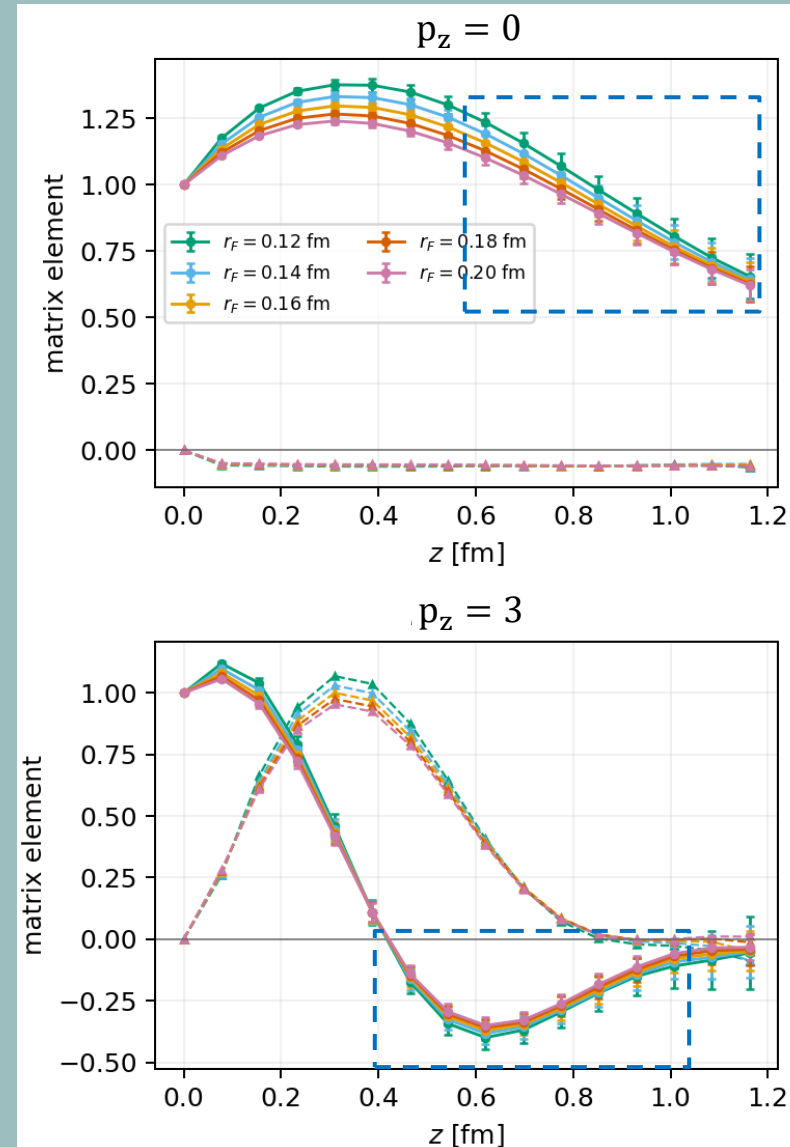
Linear-divergence subtraction largely removes the flow-time dependence, leaving residual differences from **logarithmic divergence**.

F32P30 pion DA

Before eliminate



After eliminate



Flow scheme to MS Matching

$$\tilde{q}^{\overline{MS}}(z, \tau, \mu) = C(z, \tau, \mu) \tilde{q}^{Lat}(z, \tau)$$

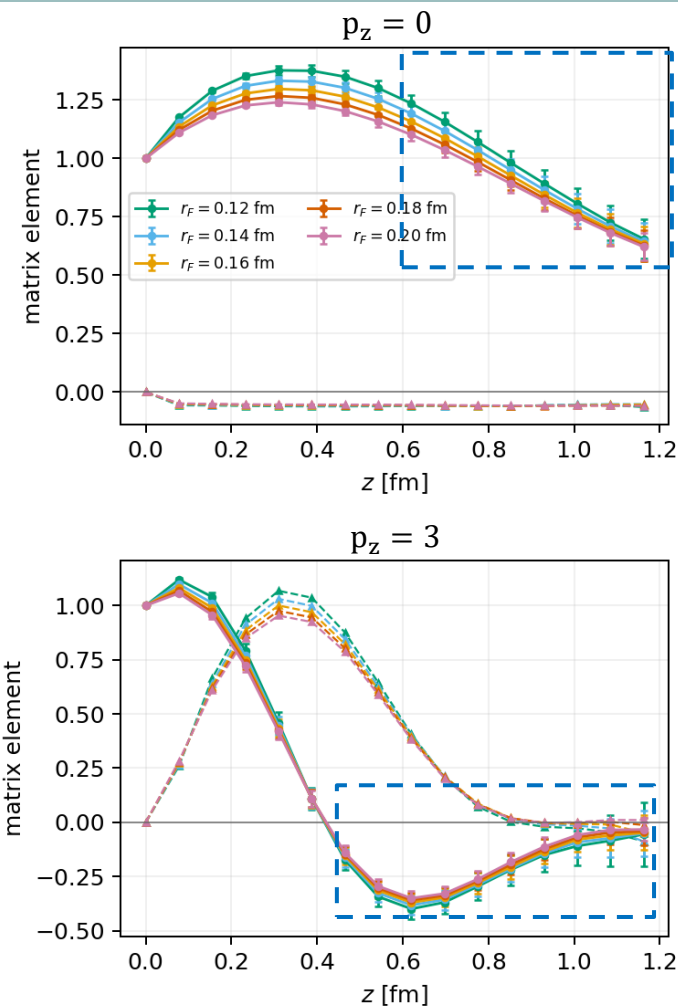
$$C^{FO}(\tau, \mu) = 1 - \frac{\alpha_s}{4\pi} C_F [3 \log(2\mu^2 \tau e^{\gamma_E}) + 5/2]$$

$$C^{RG}(\tau, \mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu')} \right)^{-\frac{\gamma}{2\beta_0}} C^{FO}(\tau, \mu')$$

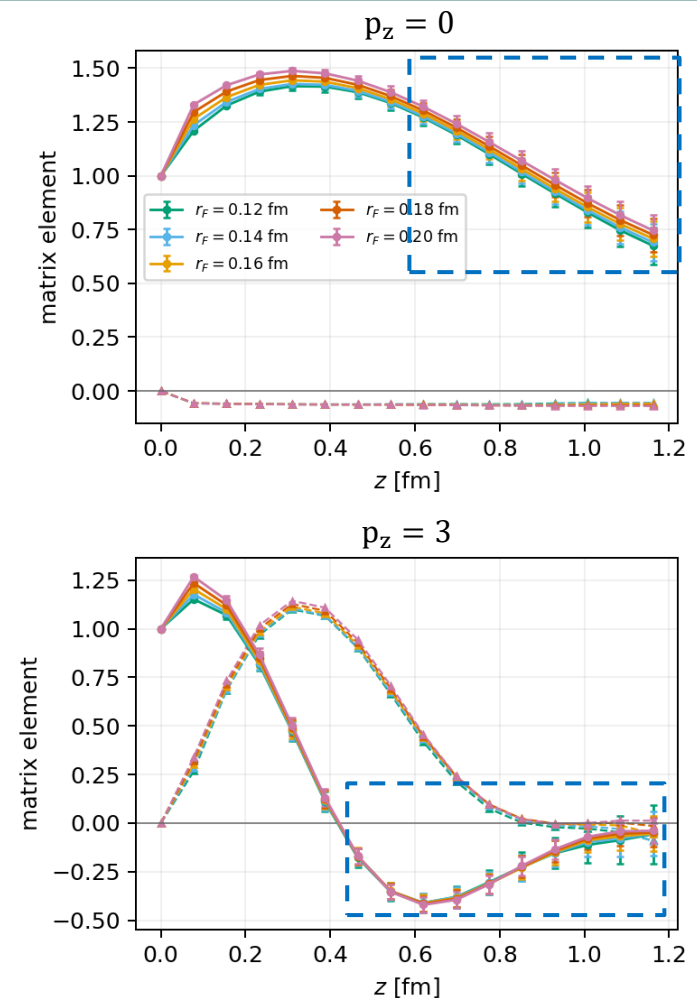
After matching, the logarithmic divergence is removed. Within current precision, the pion DA curves at different flow times can be regarded as **consistent**.

F32P30 pion DA

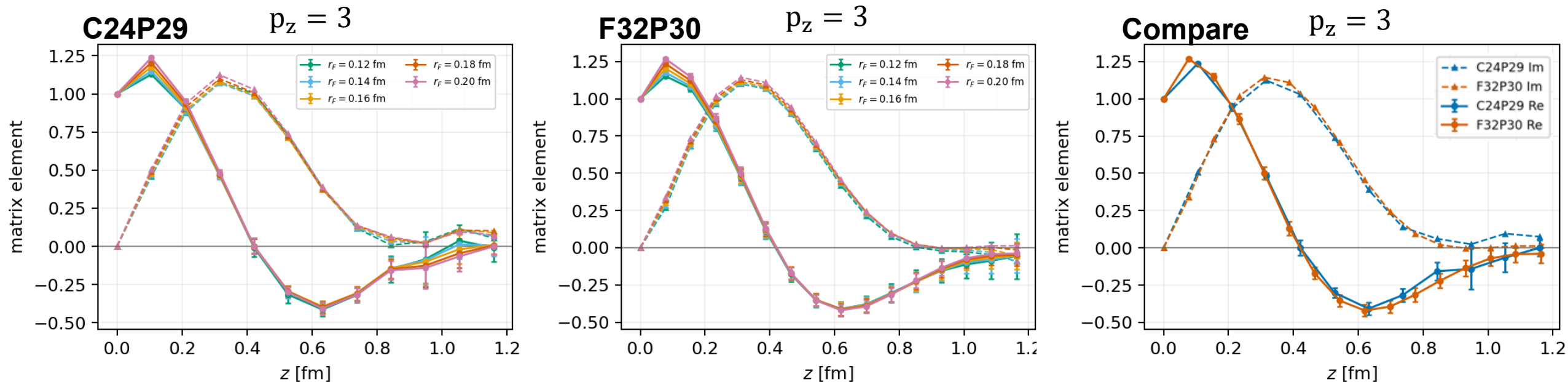
Before matching



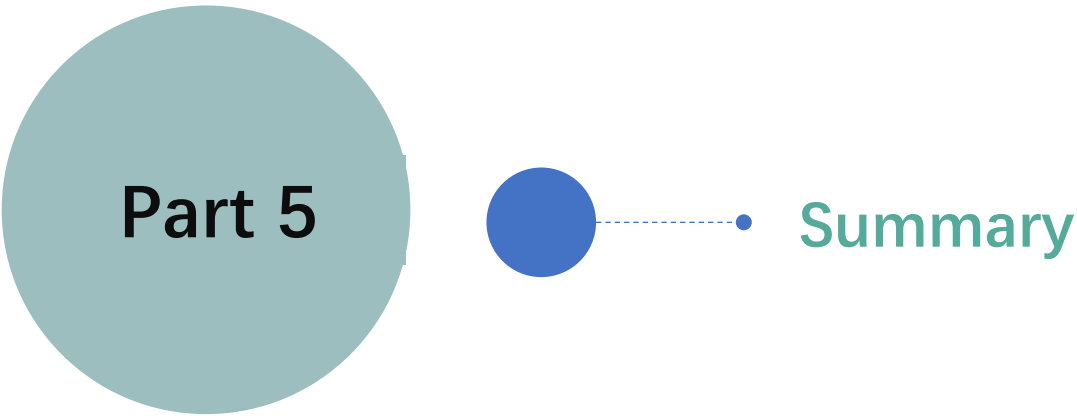
After matching



Compare across Ensembles (pion DA)



- 1) At different lattice spacings, after the subtraction of the **linear divergence and matching**, the curves remain consistent, indicating that the subtraction in these two steps is **quite clean**.
- 2) When the curves from different lattice spacings are put together, they also agree reasonably well, indicating that the **discretization effects are relatively small** and can be eliminated in the continuum extrapolation.



Part 5

Summary

1. Method

- Static potential & **Wilson line** extraction
- Direct joint-fit → **Double ratio** method
- Linear divergence subtraction + **matching to $\overline{MS}$ scheme**

2. Results

- Linear divergences are consistent across **different quantities, momenta, and lattice spacings**.
- After matching, pion DA curves agree at different flow times.
- Discretization effects are relatively small.

3. Conclusion

- Gradient flow is feasible for nonlocal ME renormalization. Applicable to a broader class of **Wilson-line quark bilinear operators**.

Thank you for listening!

backup

How to use Gradient Flow renormalization

1. Use Wilson loop to fit linear divergence

$$\lim_{t \rightarrow \infty} \langle W_{loop}(r, \tau, t) \rangle \sim e^{-V(r, \tau)t}$$

$$V(r, \tau) \sim V_0(\tau) + \sigma r - \frac{\alpha}{r}$$

$$V_0(\tau) \sim \frac{2\delta m}{\sqrt{8\tau}} + B$$

2. Use gauge-fixed Wilson to fit linear divergence

$$\langle W_{line}(z, \tau) \rangle \sim e^{-\frac{\delta m}{\sqrt{8\tau}}z + Bz}$$

3. eliminate the linear divergence and do normalization

$$\tilde{q}^{Lat}(z, a, t) = \frac{\tilde{q}^{Lat}(z, a, t) e^{\frac{\delta m}{\sqrt{8\tau}}z}}{\tilde{q}^{Lat}(0, a, t)}$$

4. extrapolation: $a \rightarrow 0$

$$\tilde{q}^{Lat}(z, t) = \lim_{a \rightarrow 0} \tilde{q}^{Lat}(z, a, t)$$

5. matching

$$\tilde{q}^{\overline{MS}}(z, t, \mu) = C(z, t, \mu) \tilde{q}^{Lat}(z, t)$$

6. extrapolation: $t \rightarrow 0$

$$\tilde{q}^{\overline{MS}}(z, \mu) = \lim_{t \rightarrow 0} \tilde{q}^{\overline{MS}}(z, t, \mu)$$

7. fit and eliminate $m_0 z$

$$\tilde{q}^{\overline{MS}}(z, \mu) = e^{-m_0 z} \tilde{q}^{pert, LRR}(z, \mu)$$