



華中師範大學
CENTRAL CHINA NORMAL UNIVERSITY



Nuclear Science
Computing Center at CCNU

QCD Chiral Crossover Line from Lee-Yang Edge Singularities

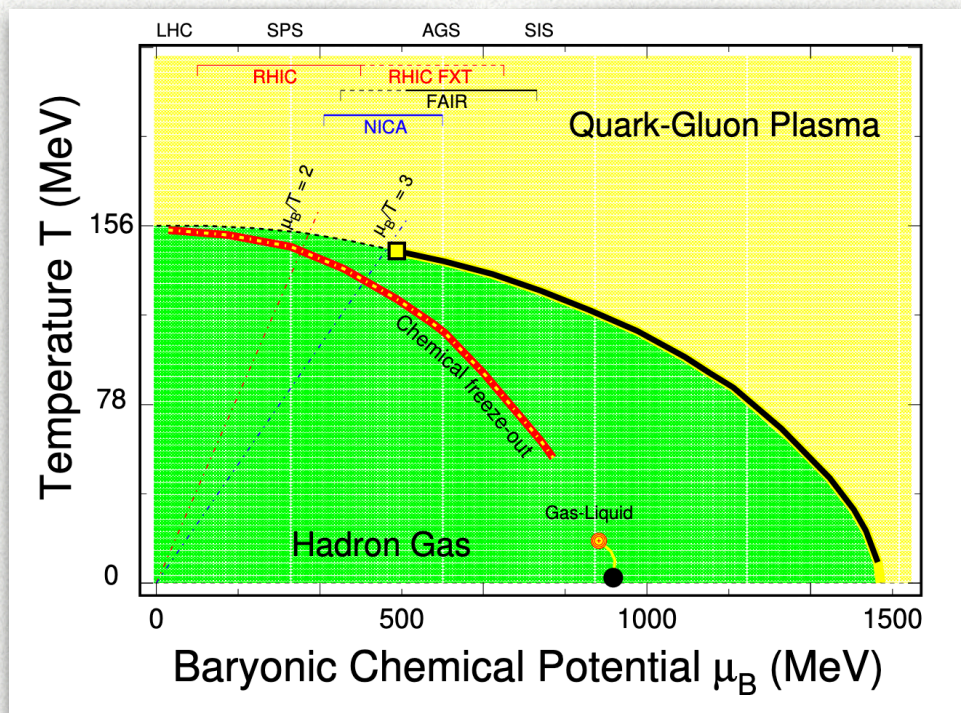
叶凯帆

合作者: 丁亨通、Swagato Mukherjee、Peter Petreczky

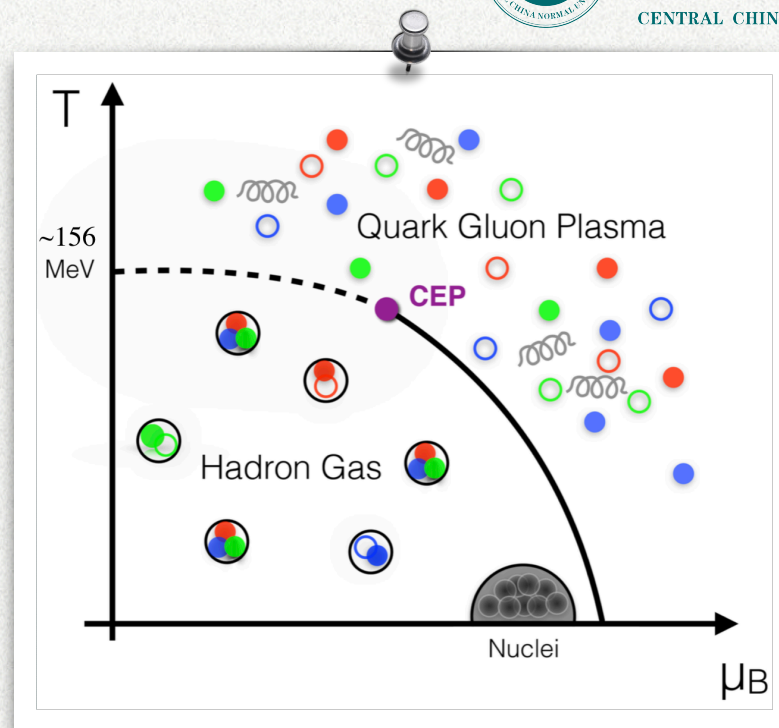
基于 arXiv: 2608.05752 [hep-lat], Phys. Lett. B 审稿中

第六届中国格点量子色动力学研讨会, 2026 年 10 月 8-11 日, 深圳

寻找 QCD 临界现象



QCD Phase Structure at Finite Baryon Density,
HTD, W. J. Fu, F. Gao, M. Huang, X. G. Huang, F. Karsch, J. F. Liao, X. F. Luo, B. Mohanty, T. Nonaka, P. Petreczky, K. Redlich, C. D. Roberts & N. Xu, Springer book chapter 1



HTD, F. Karsch, S. Mukherjee, arXiv:1504.05274

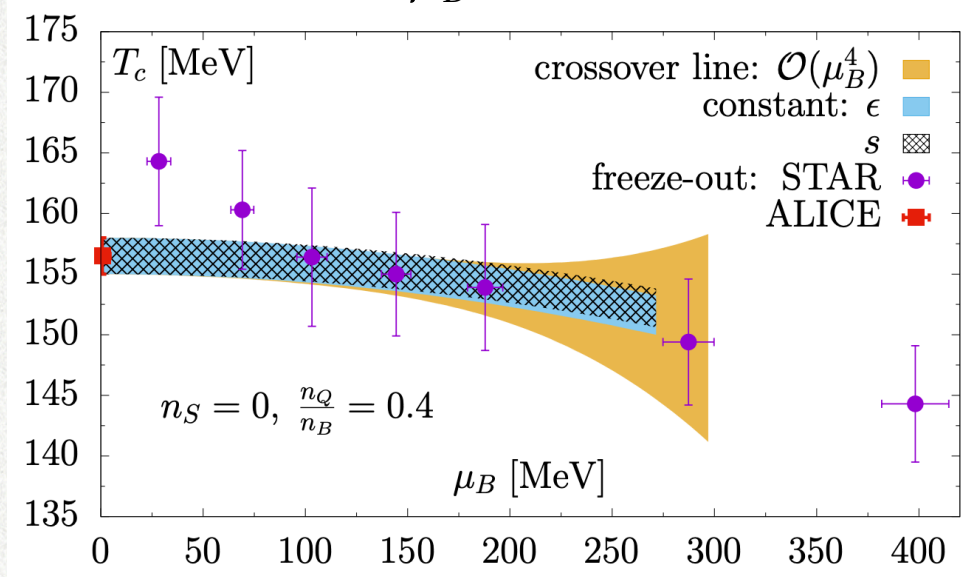
实 $\mu_B \neq 0$ 时的符号问题

零 μ_B 泰勒展开、纯虚 μ_B 解析延拓方法，
复朗之万方法、Lefschetz 陡降流形方法等

有限 μ_B 下的 QCD 手征膺临界线



从零 μ_B 泰勒展开



手征凝聚 $\langle \bar{\psi}\psi \rangle$ 的拐点 / 手征磁化率 χ 的峰

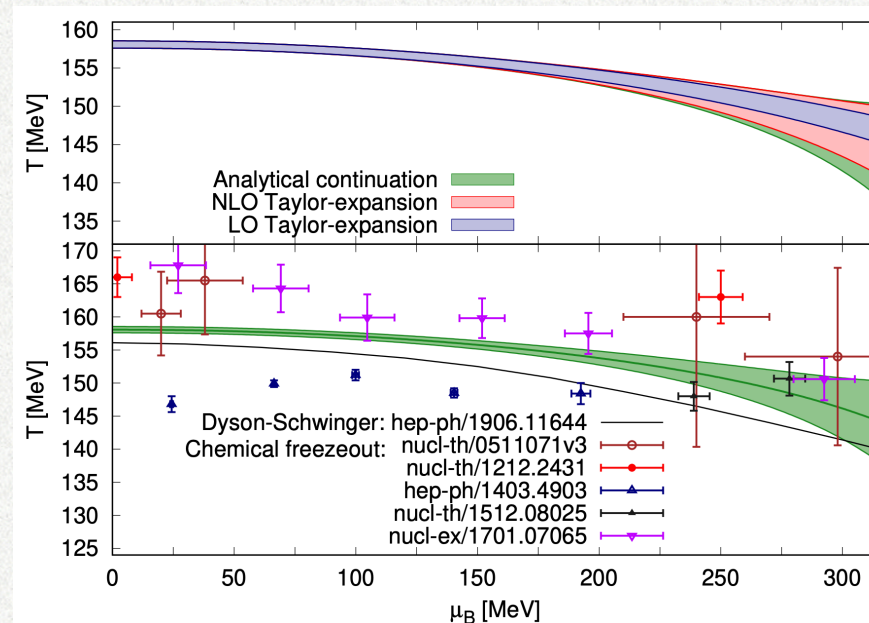
$$T_{pc}(0) = 156.5(1.5) \text{ MeV}$$

$$\kappa_2 = 0.016(6), \kappa_4 = 0.001(7)$$

A. Bazavov et al., Phys. Lett. B 795, 15 (2019).

$$\frac{T_{pc}(\mu_B)}{T_{pc}(0)} = 1 - \kappa_2 \left(\frac{\mu_B}{T_{pc}(0)} \right)^2 - \kappa_4 \left(\frac{\mu_B}{T_{pc}(0)} \right)^4 + \dots$$

从纯虚 μ_B 解析延拓



手征磁化率 $\chi_{\langle \bar{\psi}\psi \rangle}$ 的峰

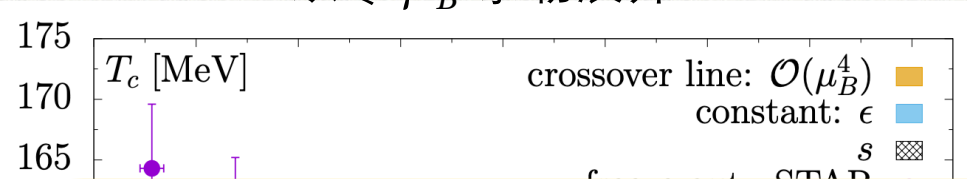
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$$\kappa_2 = 0.0153(18), \kappa_4 = 0.00079(68)$$

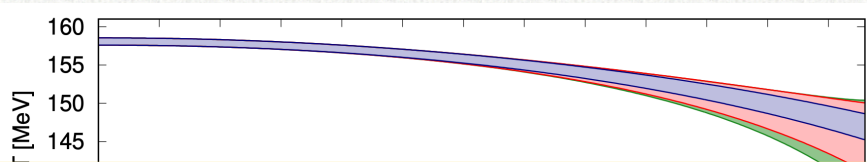
S. Borsanyi et al., Phys. Rev. Lett. 125, 052001 (2020).

WB 的 κ_4 已换算为 HotQCD 的约定。

从零 μ_B 泰勒展开



从纯虚 μ_B 解析延拓



我们提出了新的方法，结合李-杨边缘奇点与普适手征标度
重建 QCD 膺临界线



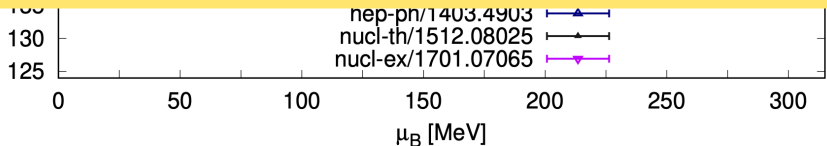
手征凝聚 $\langle \bar{\psi}\psi \rangle$ 的拐点 / 手征磁化率 χ 的峰

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赝临界线的定义



手征临界线:

$$T_c(\mu_B) = T_c^0 \left[1 - \tilde{\kappa}_2 \left(\frac{\mu_B}{T_c^0} \right)^2 + \dots \right]$$

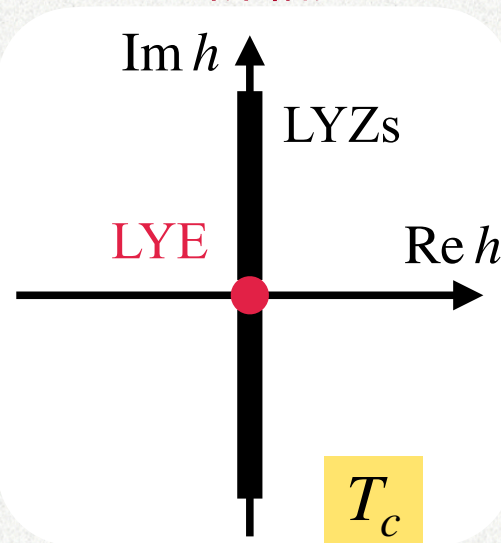
平滑过渡线 / 赝临界线:

$$T_{pc}^{(*)}(\mu_B) = T_{pc}^{(*)}(0) \left[1 - \kappa_2 \left(\frac{\mu_B}{T_{pc}^{(*)}(0)} \right)^2 + \dots \right]$$

手征极限 ($m_l \rightarrow 0$)

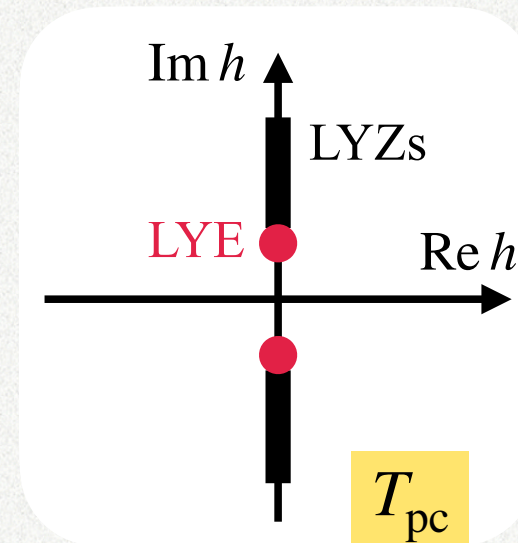
$$h \sim m_l$$

边缘奇点夹逼实轴 $\Rightarrow$
二阶相变



物理质量 (m_l^{phys})

边缘奇点仍位于复平面 $\Rightarrow$
平滑过渡



pseudo-critical

热力学极限, $T \rightarrow T_c^+$

C. N. Yang and T. D. Lee, Phys. Rev. 87, 404 (1952)

T. D. Lee and C. N. Yang, Phys. Rev. 87, 410 (1952)

赅临界线的定义



赅临界线：

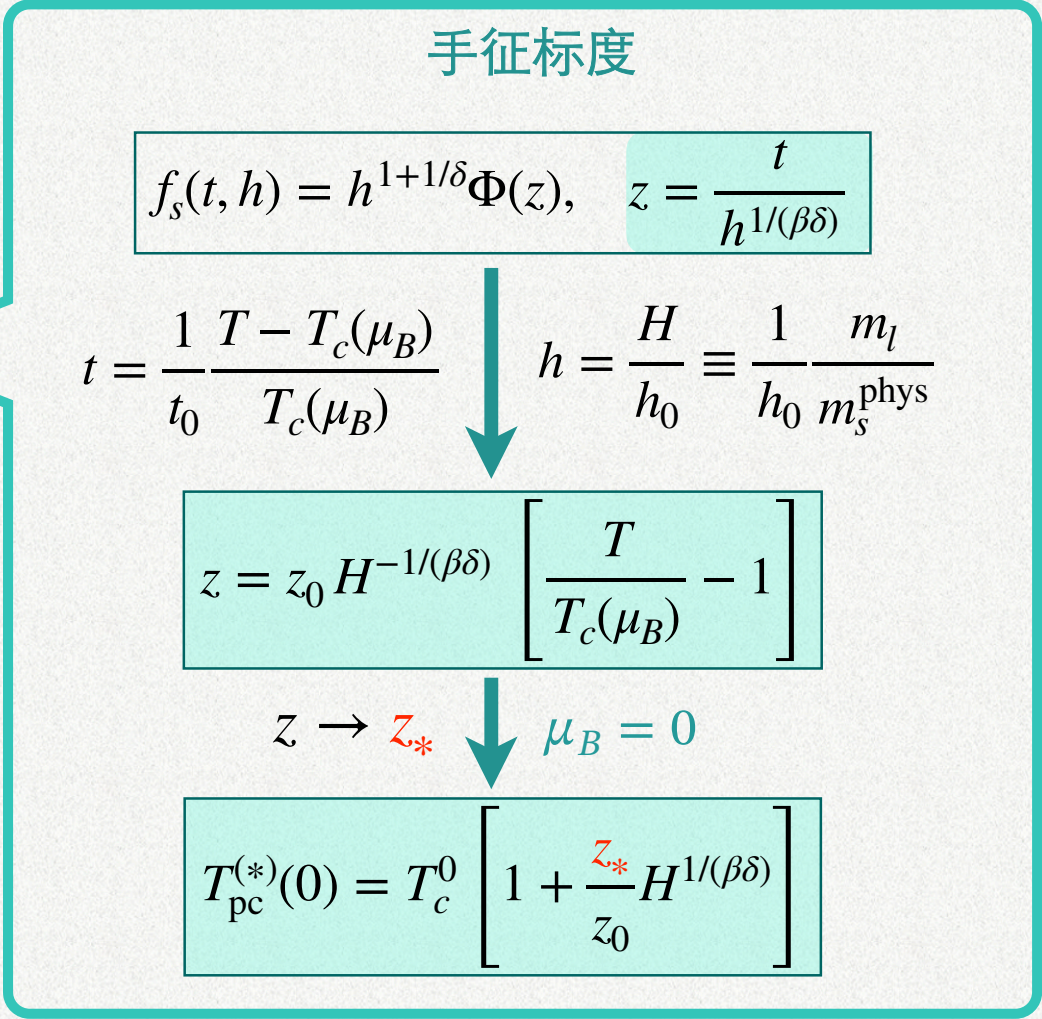
$$T_{pc}^{(*)}(\mu_B) = T_{pc}^{(*)}(0) \left[1 - \kappa_2 \left(\frac{\mu_B}{T_{pc}^{(*)}(0)} \right)^2 + \dots \right]$$

↓

$$T_{pc}^{(*)}(\mu_B) = T_c^0 \left[1 + \frac{z_*}{z_0} H^{1/(\beta\delta)} \right] \left[1 - \kappa_2 \left(\frac{\mu_B}{T_{pc}^{(*)}(0)} \right)^2 + \dots \right]$$

- 此标度形式具有普适性
- z_0 与 T_c^0 为非普适参数

手征标度



赝临界线的定义



赝临界线:

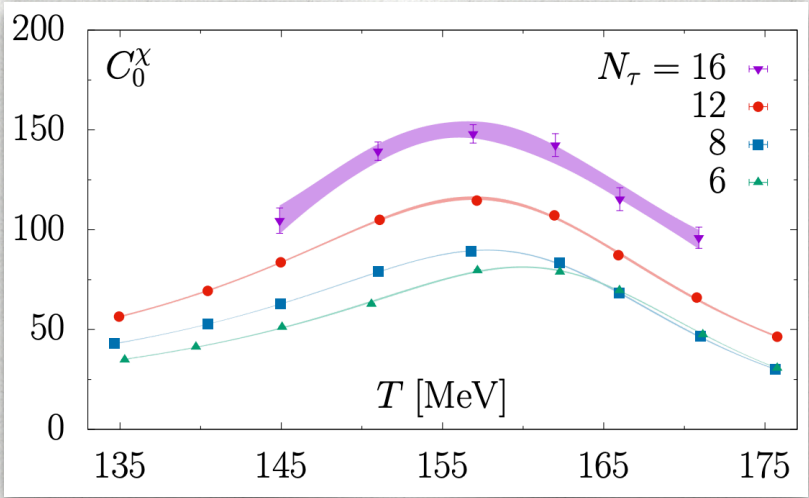
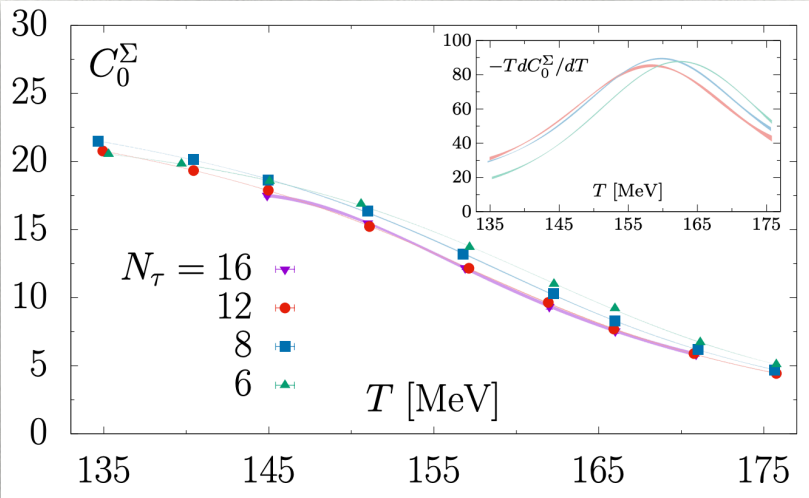
$$T_{pc}^{(*)}(\mu_B) = T_c^0 \left[1 + \frac{z_*}{z_0} H^{1/(\beta\delta)} \right] \left[1 - \kappa_2 \left(\frac{\mu_B}{T_{pc}^{(*)}(0)} \right)^2 + \dots \right]$$

$T_{pc}^{(*)}(0)$

z_* 的选择决定 $T_{pc}^{(*)}(0)$ 的定义

z_* 的几种选择:

- 手征凝聚 Σ 的拐点: $f_G''(z_*) = 0$
- 手征磁化率 χ^Σ 的峰: $f_\chi'(z_*) = 0$



手征临界线:

$$T_c(\mu_B) = T_c^0 \left[1 - \tilde{\kappa}_2 (\tilde{x})^2 + \dots \right], \tilde{x} \equiv \frac{\mu_B}{T_c^0}$$



$$T_c(\mu_B) = T_c(0) \tilde{g}(\tilde{x})$$

赝临界线:

$$T_{pc}(\mu_B) = T_{pc}(0) \left[1 - \kappa(x)^2 + \dots \right], x \equiv \frac{\mu_B}{T_{pc}(0)}$$



$$T_{pc}(\mu_B) = T_{pc}(0) g(x)$$

映射函数

$$g(x) = \exp[-F(x)]$$

$g(x)$ 需满足:

- $g(0) = 1$
- 电荷共轭对称性: $g(-x) = g(x)$
- 单调不增

✓ M1: $F(x) = a^2 x^2 + b^2 x^4$

$a, b \in \mathbb{R}$; M2、M3 检验参数化依赖

赝临界线的边缘奇点约束方程



实 μ_B 轴上的标度变量 z

$$T_{\text{pc}}^{(*)}(\mu_B) = T_{\text{pc}}^{(*)}(0) g(\mu_B)$$



$$T_{\text{pc}}^{(*)}(\mu_B) = T_c^0 \left[1 + \frac{z_*}{z_0} H^{1/(\beta\delta)} \right] g(\mu_B)$$

复 μ_B 平面中的标度变量 z

边缘奇点约束方程

$$z_c = z_0 H^{-1/(\beta\delta)} \left[\frac{T}{T_c^0 g(\mu_{B,c})} - 1 \right]$$

$$z = z_0 H^{-1/(\beta\delta)} \left[\frac{T}{T_c(\mu_B)} - 1 \right]$$

z 平面中的普适边缘奇点

$$f_s(t, h) = h^{1+1/\delta} \Phi(z), \quad z = \frac{t}{h^{1/(\beta\delta)}}$$

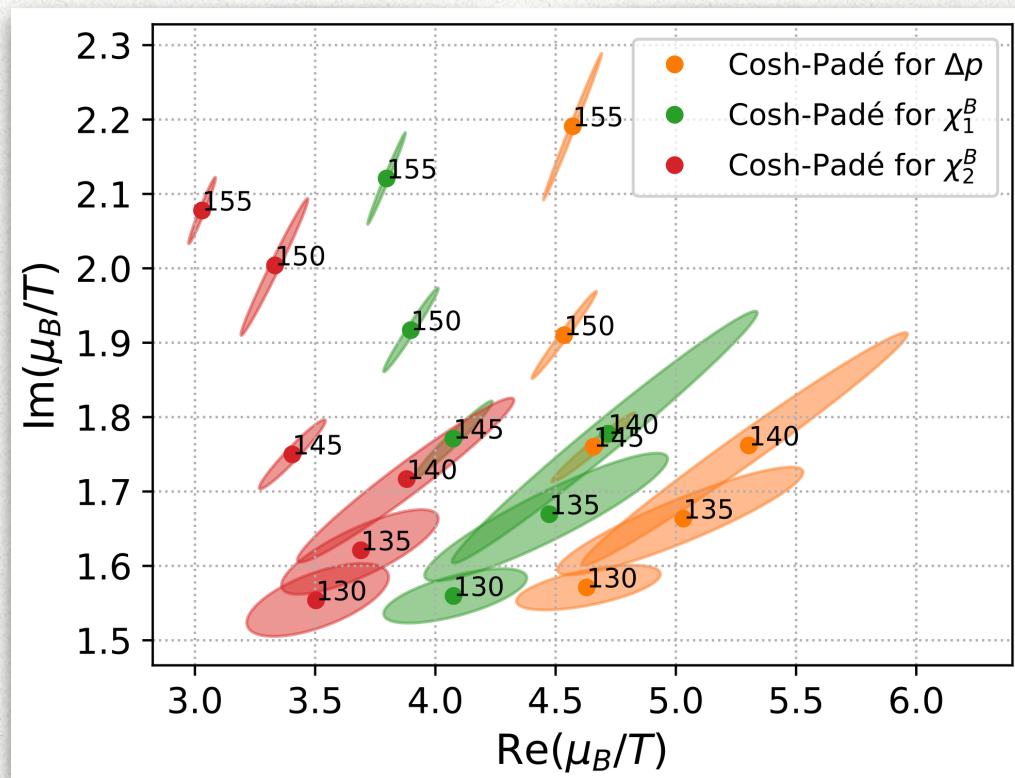
李-杨边缘奇点是
普适标度函数 $\Phi(z)$ 的分支点

$$z_c = |z_c| \exp\left(\frac{i\pi}{2\beta\delta}\right)$$

$$O(2): |z_c| \simeq 1.95(7), \quad \beta\delta \simeq 1.6664(5)$$

输入: z_c 、 $\mu_{B,c}(T_i)$ 确定: $g(\mu_B)$ 、 T_c^0 、 z_0

Karsch, F. et al., Phys. Rev. D 109, 014508 (2024).



A. Adam et al., Phys. Rev. D 113, 074525 (2026)

格点输入

- 在 $16^3 \times 8$ 格子上进行纯虚 μ_B 模拟
- 对 $\chi_1^B - \chi_4^B$ 作有理函数近似，提取主导零点
- 以主导零点近似边缘奇点

施加边缘奇点约束方程

$$z_c = z_0 H^{-1/(\beta\delta)} \left[\frac{T_i}{T_c^0 g(\mu_{B,c})} - 1 \right]$$

$$\mu_{B,c}(T_i), T_i \in \{135, 140, 145, 150\} \text{ MeV}$$

给出 4 个复方程

拟合 4 个参数 $\{T_c^0, z_0, a, b\}$

求解候选根

给定 $\{T_c^0, z_0, a, b\}$

候选根: $\{\mu_{B,c}^{(k)}(T_i)\}_{k=1}^{N_i}$

拟合参数

$$\chi_s^2 = \sum_i (\mu_{i,\text{pred}}^{(s)} - \mu_i^{(s)})^\top \mathbf{C}_i^{-1} (\mu_{i,\text{pred}}^{(s)} - \mu_i^{(s)})$$

分支追踪

随温度 T 变化,
连续追踪同一支根

$$\frac{d}{dT} z(T, \mu_{B,c}^{(k)}(T)) = 0$$

施加边缘奇点约束方程

$$z_c = z_0 H^{-1/(\beta\delta)} \left[\frac{T_i}{T_c^0 g(\mu_{B,c})} - 1 \right]$$

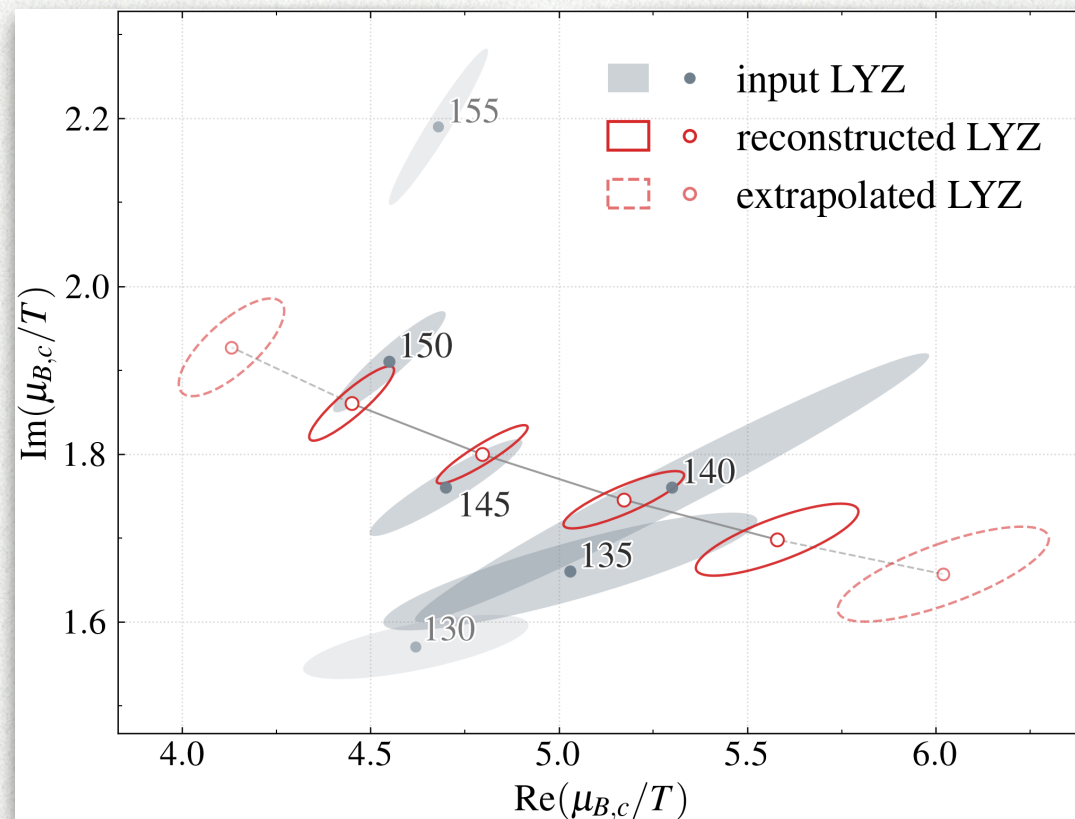
求解候选根

给定 $\{T_c^0, z_0, a, b\}$

候选根: $\{\mu_{B,c}^{(k)}(T_i)\}_{k=1}^{N_i}$

拟合参数

$$\chi_s^2 = \sum_i (\mu_{i,\text{pred}}^{(s)} - \mu_i^{(s)})^T \mathbf{C}_i^{-1} (\mu_{i,\text{pred}}^{(s)} - \mu_i^{(s)})$$



基准设置: C4T + M1

拟合: 135, 140, 145, 150 MeV

外推: 130, 155 MeV

赝临界线

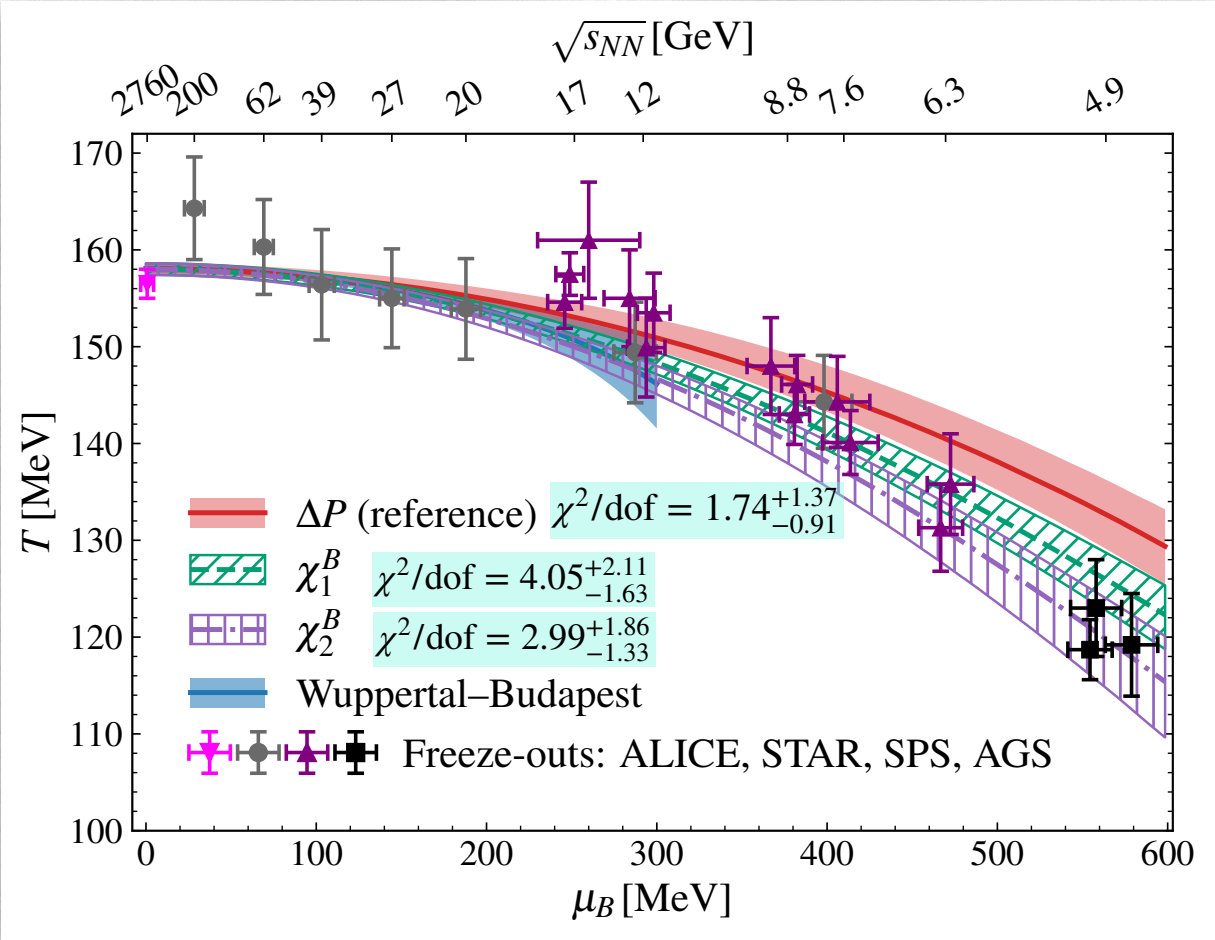
$$T_{pc}(\mu_B) = T_{pc}(0) g(\mu_B)$$

$T_{pc}(0) = 158.01(61) \text{ MeV}$ S. Borsanyi et al., Phys. Rev. Lett. 125, 052001 (2020).

基准设置:

M1: $g(x) = \exp[-F(x)]$, $F(x) = a^2x^2 + b^2x^4$

★ 与小 μ_B 连续极限格点结果相容



基准设置: C4T + M1

拟合: 135, 140, 145, 150 MeV

拟合参数: T_c^0 与曲率

LYZ input	T_c^0 [MeV]	κ_2	κ_4 [10^{-4}]	χ^2/dof	z_0
ΔP	$143.6^{+8.8}_{-14.8}$	$0.012^{+0.002}_{-0.004}$	$0.33^{+2.64}_{-1.25}$	$1.74^{+1.37}_{-0.91}$	$0.72^{+0.20}_{-0.13}$
χ_1^B	$145.3^{+0.6}_{-4.7}$	$0.017^{+0.002}_{-0.002}$	$-1.45^{+0.89}_{-0.39}$	$4.05^{+2.11}_{-1.63}$	$0.78^{+0.08}_{-0.10}$
χ_2^B	$136.8^{+2.7}_{-6.6}$	$0.020^{+0.003}_{-0.002}$	$-1.42^{+2.45}_{-0.93}$	$2.99^{+1.86}_{-1.33}$	$0.80^{+0.11}_{-0.11}$

基准拟合

手征标度分析: $T_c^0 = 143.7(2) \text{ MeV } (N_\tau = 8)$ H.-T. Ding, et al., Phys. Rev. D 109, 114516 (2024).

连续极限格点结果:

HotQCD: $\kappa_2 = 0.016(6), \kappa_4 = 0.001(7)$	A. Bazavov, et al., Phys. Lett. B 795, 15 (2019).
Wuppertal-Budapest: $\kappa_2 = 0.0153(18), \kappa_4 = 0.00079(68)$	S. Borsanyi, et al., Phys. Rev. Lett. 125, 052001 (2020).

WB 的 κ_4 已换算为 HotQCD 的约定。

系统误差检查：温度窗口



基准设置：C4T，M1

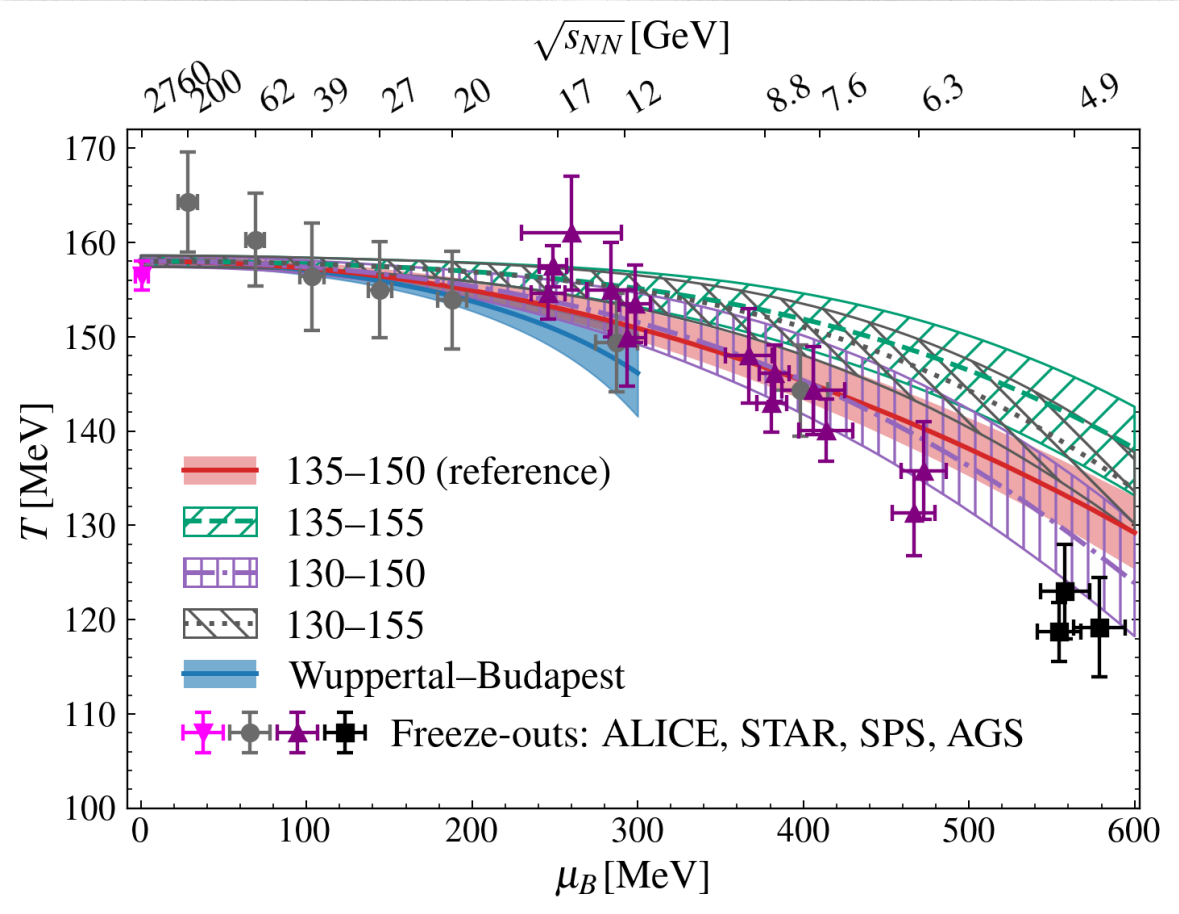
$T_i \in \{130, 135, 140, 145, 150, 155 \text{ MeV}\}$

手征标度分析：

$T_c^0 = 143.7(2) \text{ MeV} \quad (N_\tau = 8)$ H.-T. Ding, et al., PRD 109, 114516 (2024).

- 135-150（基准） $\chi^2/\text{dof} = 1.74^{+1.37}_{-0.91}$
- 全部六个温度 $\chi^2/\text{dof} = 4.42^{+1.50}_{-1.28}$
- 去掉最高温度 $\chi^2/\text{dof} = 3.79^{+1.62}_{-1.33}$
- 去掉最低温度 $\chi^2/\text{dof} = 2.68^{+1.37}_{-1.08}$

★ 扩展窗口： T_c^0 降低， χ^2/dof 增大



系统误差检查：映射函数



基准设置：C4T, M1

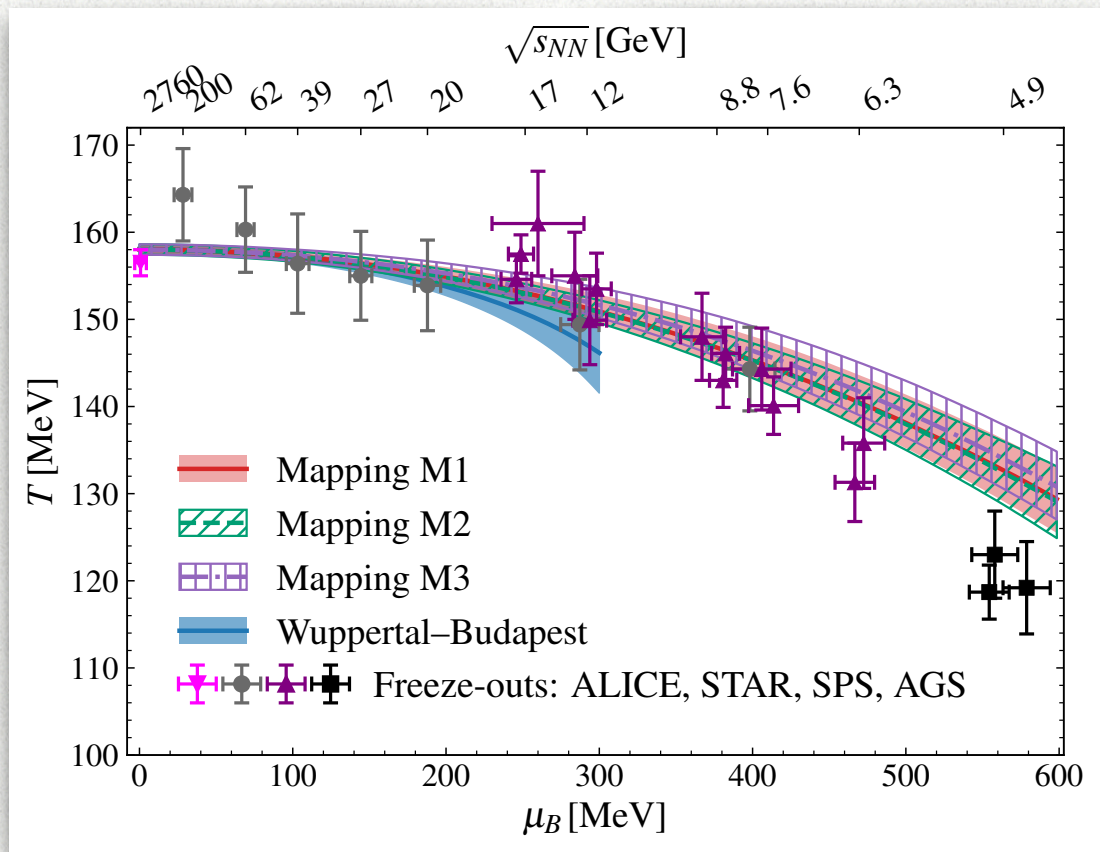
$$T_{\text{pc}}(\mu_B) = T_{\text{pc}}(0) g(\mu_B)$$

$$g(x) = \exp[-F(x)]$$

✓ M1: $F(x) = a^2x^2 + b^2x^4$

✓ M2: $F(x) = a^2x^2 - abx^4 + \frac{b^2}{3}x^6$

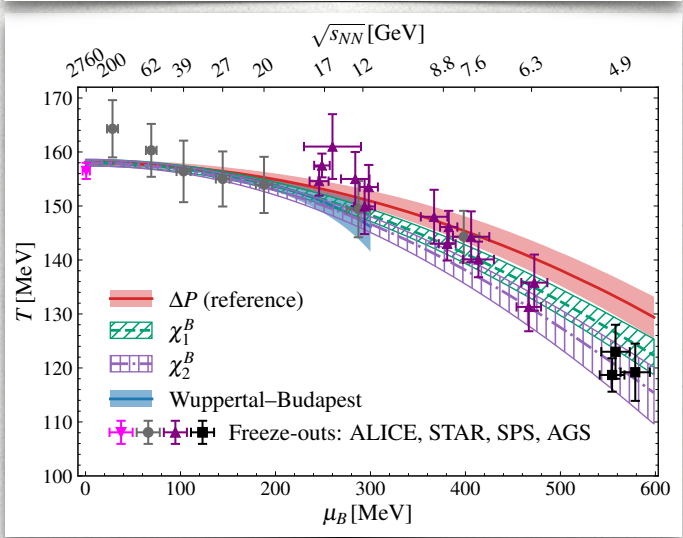
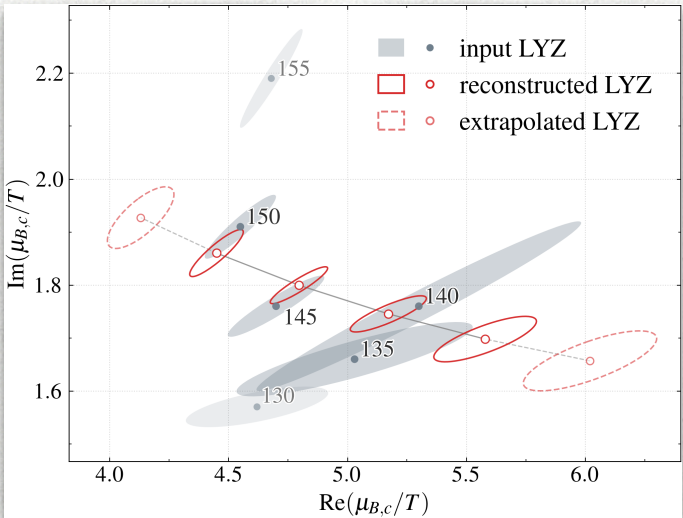
✓ M3: $F(x) = \frac{a^2x^2 + b^2x^4}{1 + c^2x^2}$



当前误差范围内，M1-M3 给出的误差带相容

- 结合李-杨边缘奇点与普适手征标度，提出了重建 QCD 赝临界线的新方法。
- 135-150 MeV 的零点输入可由同一套标度映射描述。
- 手征临界温度与已有手征标度分析相容。
- 重建曲率与连续极限格点 QCD 结果相容。
- 更大体积、更细格距的李-杨零点位置将进一步约束结果。

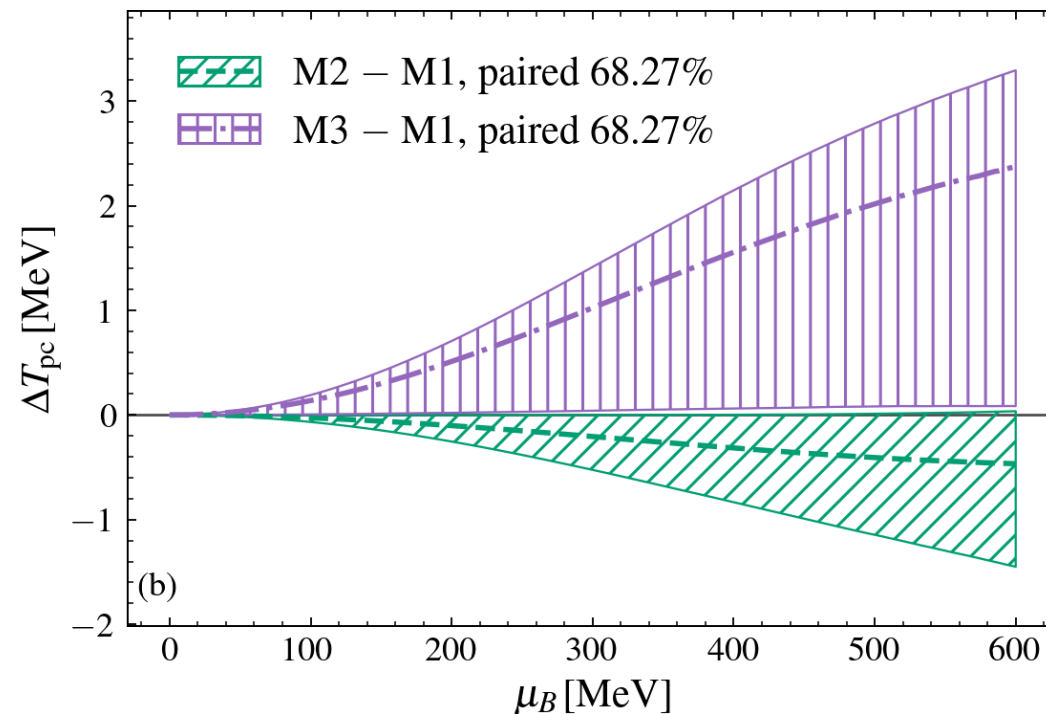
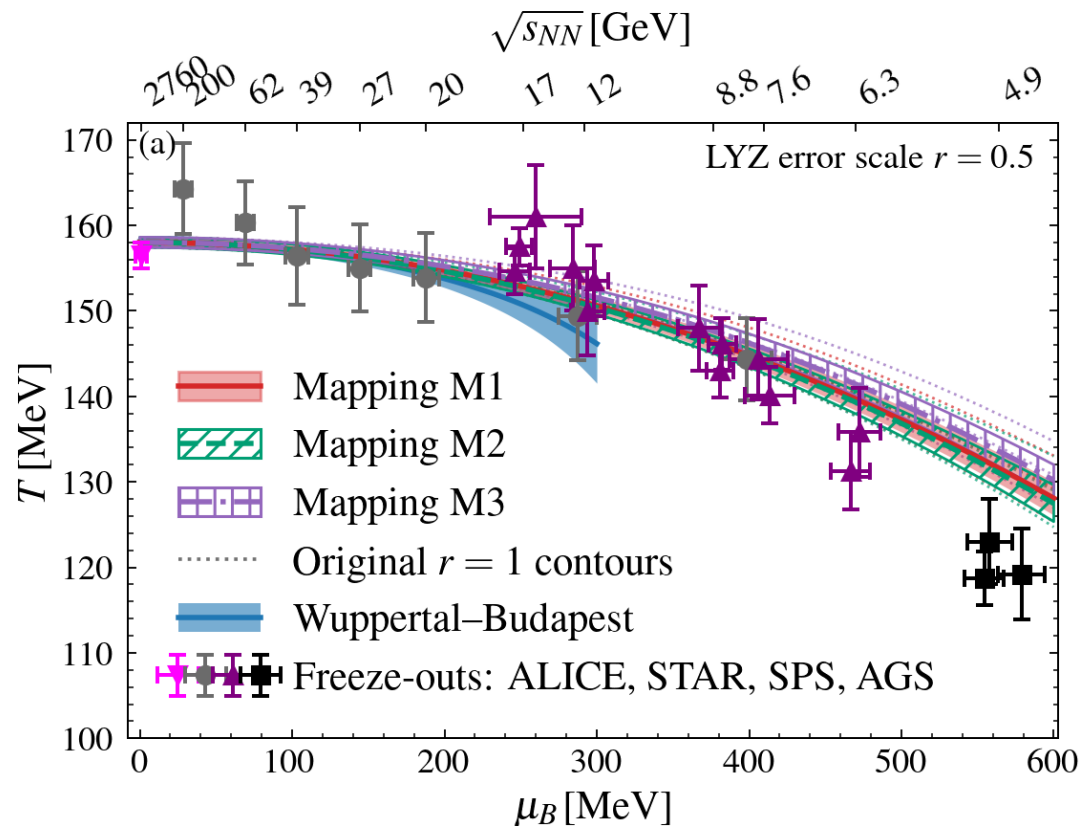
$$z_c = z_0 H^{-1/(\beta\delta)} \left[\frac{T_i}{T_c^0 g(\mu_{B,c})} - 1 \right]$$



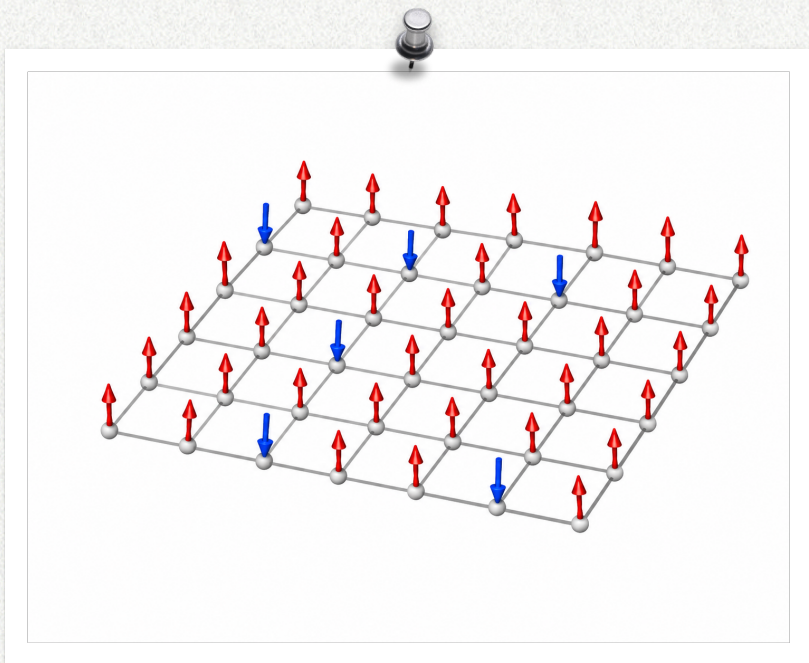


备份幻灯片

假设性精度检验：将零点输入误差缩至原来的 0.5 倍



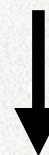
铁磁伊辛模型中的李-杨零点



◦ 配分函数:

$$Z_N(T, h) = \sum_{\{\sigma\}} \exp \left(\sum_{\langle ij \rangle} K_{ij} \sigma_i \sigma_j + h \sum_i \sigma_i \right), \quad K > 0$$

近邻自旋相互作用 磁场耦合



$$e^{h \sum_i \sigma_i} = e^{h(N-2n)} = e^{Nh} (e^{-2h})^n$$

◦ 多项式:

$$Z_N(K, h) = e^{Nh} P_N(z) \quad z = e^{-2h}$$



$$P_N(z) = \sum_{n=0}^N A_n(K) z^n, \quad A_n(K) = \sum_{\{\sigma\}: n(\sigma)=n} \exp \left(\sum_{\langle ij \rangle} K_{ij} \sigma_i \sigma_j \right)$$

$$\odot \quad Z_N(K, h) = e^{Nh} A_N \prod_{j=1}^N (z - z_j) \quad \text{李-杨零点: } z_j$$

$$\odot \quad f(T, h) = -\frac{T}{V} \log Z_N = f_{\text{reg}} - \frac{T}{V} \sum_j \log(z - z_j)$$

奇异部分

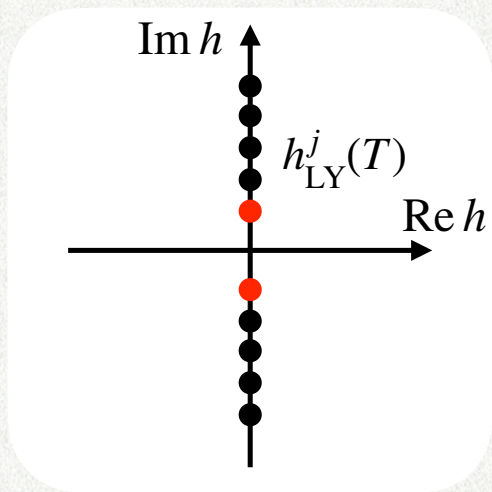
C. N. Yang and T. D. Lee, Phys. Rev. 87, 404 (1952)
T. D. Lee and C. N. Yang, Phys. Rev. 87, 410 (1952)

李-杨边缘奇点中的相结构信息



有限体积

$$\begin{aligned} \textcircled{\bullet} \quad Z_N(T, h) &= e^{Nh} A_N \prod_{j=1}^N \left(e^{-2h} - e^{-2h_{\text{LY}}^j(T)} \right) \\ \textcircled{\bullet} \quad f(T, h) &= -\frac{T}{V} \log Z_N = f_{\text{reg}} - \frac{T}{V} \sum_j \log \left(e^{-2h} - e^{-2h_{\text{LY}}^j(T)} \right) \end{aligned}$$

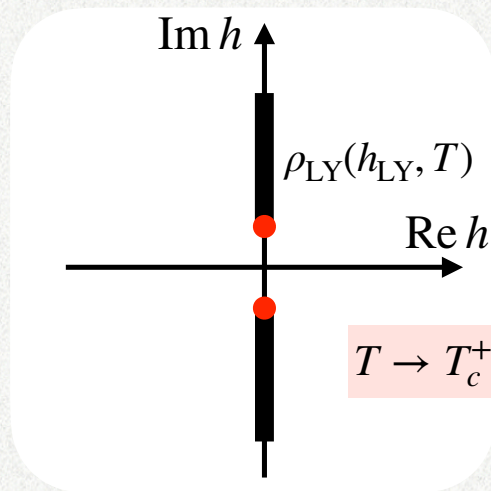


黑点：李-杨零点
红点：主导李-杨零点

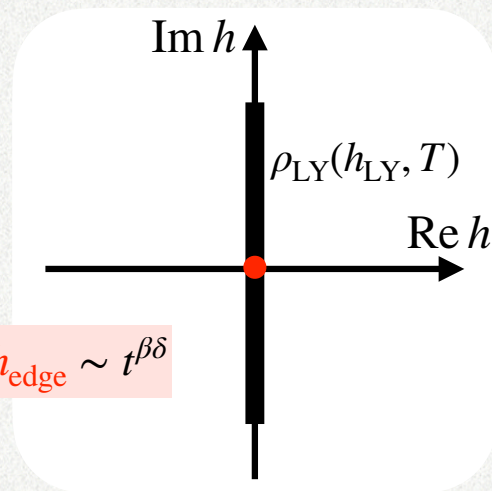
热力学极限

$$f_s(T, h) \sim - \int_{\mathcal{C}} dh_{\text{LY}} \rho_{\text{LY}}(h_{\text{LY}}, T) \log(e^{-2h} - e^{-2h_{\text{LY}}(T)})$$

红点：李-杨边缘奇点



平滑过渡



真正的相变

一阶： $\rho_{\text{LY}}(h_{\text{LY}} = 0) > 0$
二阶： $\rho_{\text{LY}}(h_{\text{LY}} = 0) = 0$

$$T \rightarrow T_c^+, h_{\text{edge}} \sim t^{\beta\delta}$$

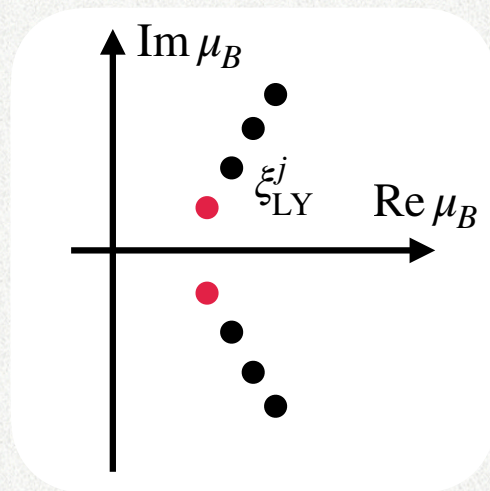
C. N. Yang and T. D. Lee, 1952
T. D. Lee and C. N. Yang, 1952

李-杨边缘奇点中的相结构信息



有限体积

$$\begin{aligned} \bullet Z &= C \xi_B^{-N_{\max}} \prod_j (\xi_B - \xi_j), \text{ 其中 } \xi_B = e^{\mu_B/T} \\ \bullet f &= -\frac{T}{V} \log Z = f_{\text{reg}} - \frac{T}{V} \sum_j \log (\xi_B - \xi_{\text{LY}}^j(h_{\text{LY}}, T, V)) \end{aligned}$$



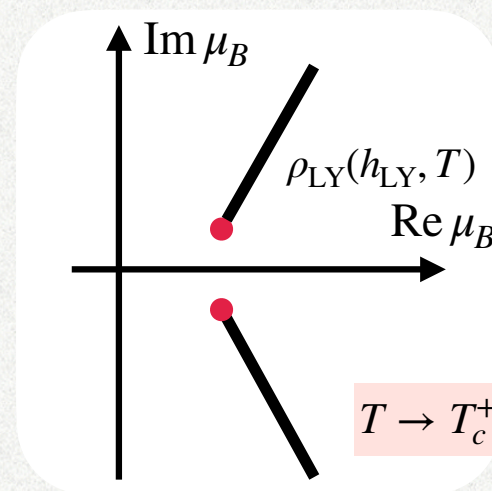
$$\xi_B = e^{\mu_B/T}$$

黑点：李-杨零点
红点：主导李-杨零点

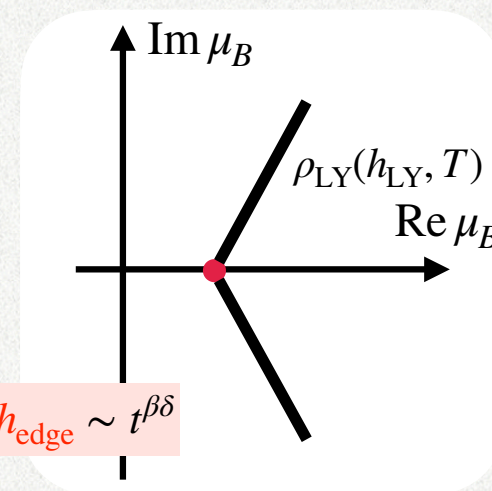
热力学极限

$$f_s(\xi_B) \sim - \int_{\mathcal{C}} dh_{\text{LY}} \rho_{\text{LY}}(h_{\text{LY}}, T) \log(\xi_B - \xi_{\text{LY}}(h_{\text{LY}}, T))$$

红点：李-杨边缘奇点



平滑过渡

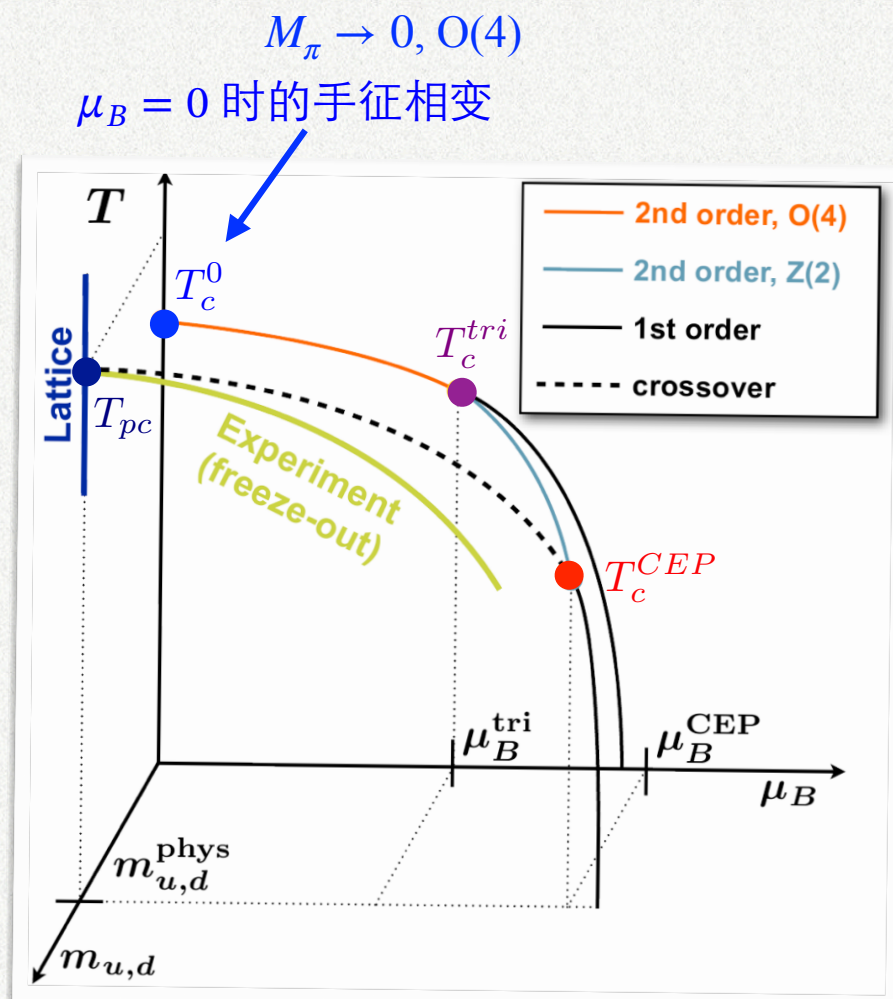


真正的相变

一阶： $\rho(h=0) > 0$
二阶： $\rho(h=0) = 0$

$$T \rightarrow T_c^+, h_{\text{edge}} \sim t^{\beta\delta}$$

C. N. Yang and T. D. Lee, 1952
T. D. Lee and C. N. Yang, 1952



- 连续极限与手征极限外推

$$T_c^0(\mu_B = 0) = 132_{-6}^{+3} \text{ MeV}$$

HTD et al.[HotQCD], PRL 123 (2019) 062002

Wilson 费米子也给出相容结果:

Kotov et al., PLB 823 (2021) 136749

- 格点结果: $T_c^0(\mu_B)$ 随 μ_B 增大而降低

P. Hegde & HTD, PoS LATTICE2015 (2016) 141

O. Kaczmarek et al., PRD83 (2011) 014504