

Collins-Soper Kernel & Intrinsic Soft Function from Lattice QCD



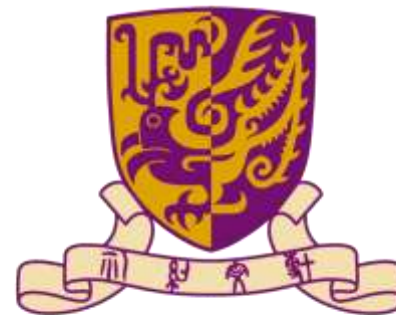
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10th Oct. @CLQCD2026

In Collaboration with

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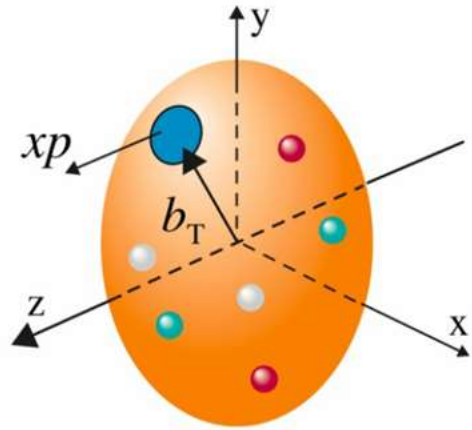


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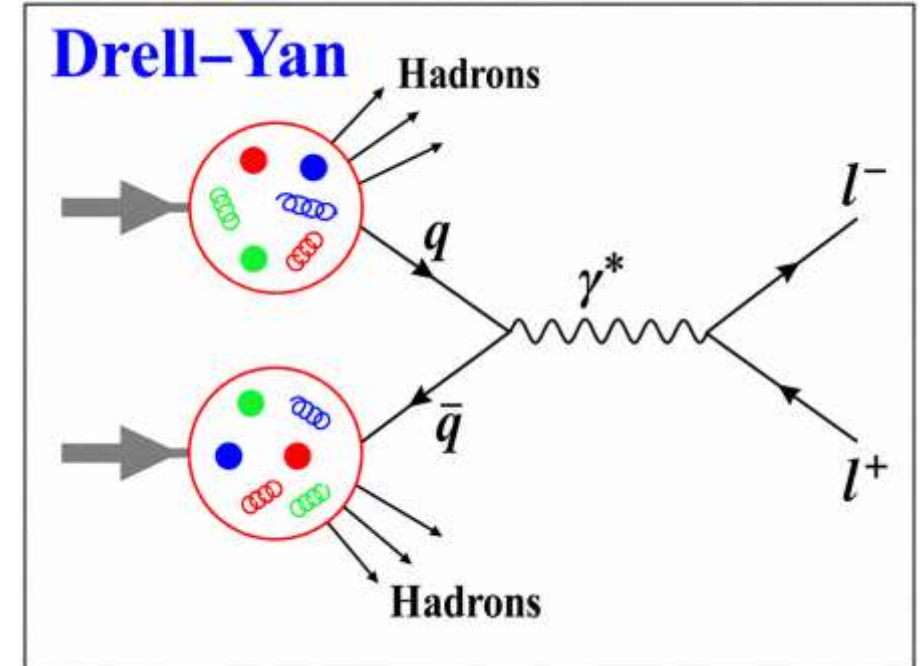
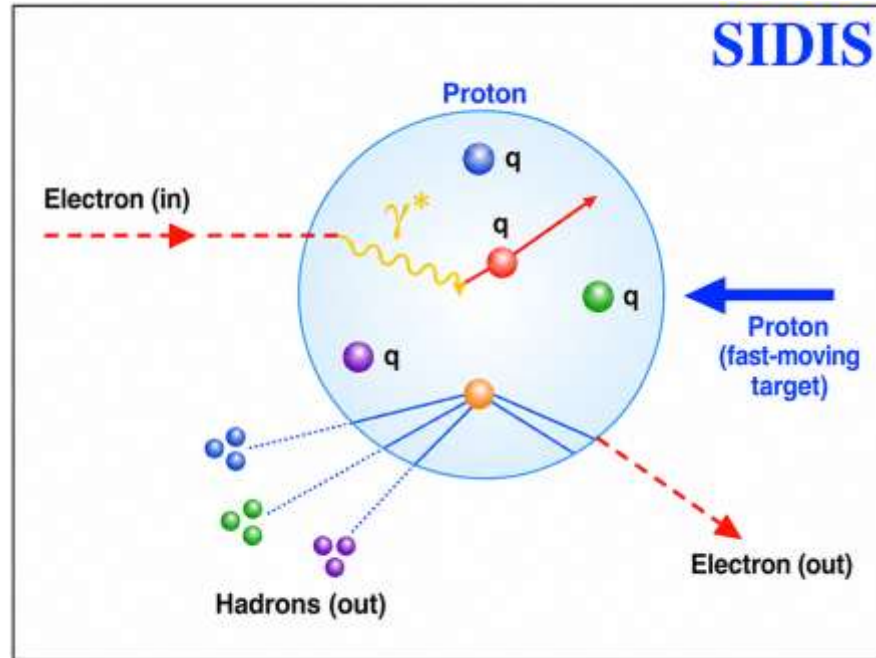
The Chinese University of Hong Kong, Shenzhen

- Motivation and Theoretical Framework
- Lattice Calculation
- Summary and Outlook

TMDs and 3D Hadron Structure



TMDs



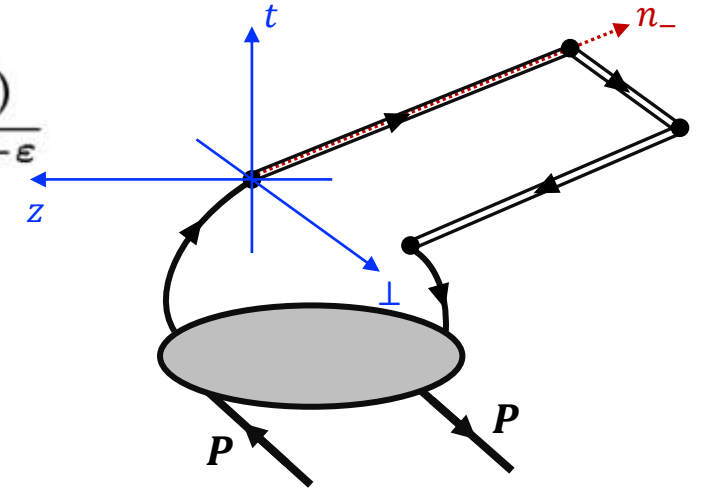
TMDs are key inputs for SIDIS and Drell-Yan process!

Definitions

➤ Rapidity divergence :

$$I_{\text{div}} \sim \int dk^+ dk^- \frac{f(k^+ k^-)}{(k^+ k^-)^{1+\varepsilon}} = \frac{1}{2} \int \frac{d(k^- / k^+)}{k^- / k^+} \int dk^+ dk^- \frac{f(k^+ k^-)}{(k^+ k^-)^{1+\varepsilon}}$$

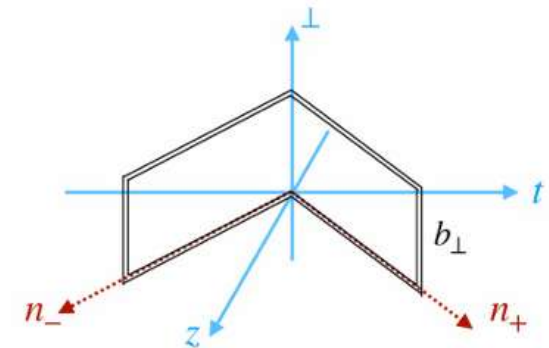
$$y_k = \frac{1}{2} \ln \frac{k^+}{k^-} \quad y_k \rightarrow \pm\infty: \text{rapidity divergence}$$



➤ Rapidity divergence subtracted by **soft function**:

$$S(b_\perp, \mu, Y) = \frac{1}{N_c} \text{Tr} \langle 0 | W_n(\vec{b}_\perp; Y) W_p^\dagger(\vec{b}_\perp; Y) | 0 \rangle$$

Y is the rapidity regulator.



Definitions

Ji et al., PRD 71, 034005(2005);

Collins, Vol. 32(Cambridge University Press, 2011)

- The **intrinsic/reduced soft function** is defined from

$$S(b_{\perp}, \mu, Y) = S_I(b_{\perp}, \mu) \exp [2Y K(b_{\perp}, \mu)]$$

- Rapidity scale evolution** controlled by :

$$\frac{\partial}{\partial Y} \ln S(b_{\perp}, \mu, Y) = 2K(b_{\perp}, \mu) \longrightarrow \text{Collins-Soper kernel}$$

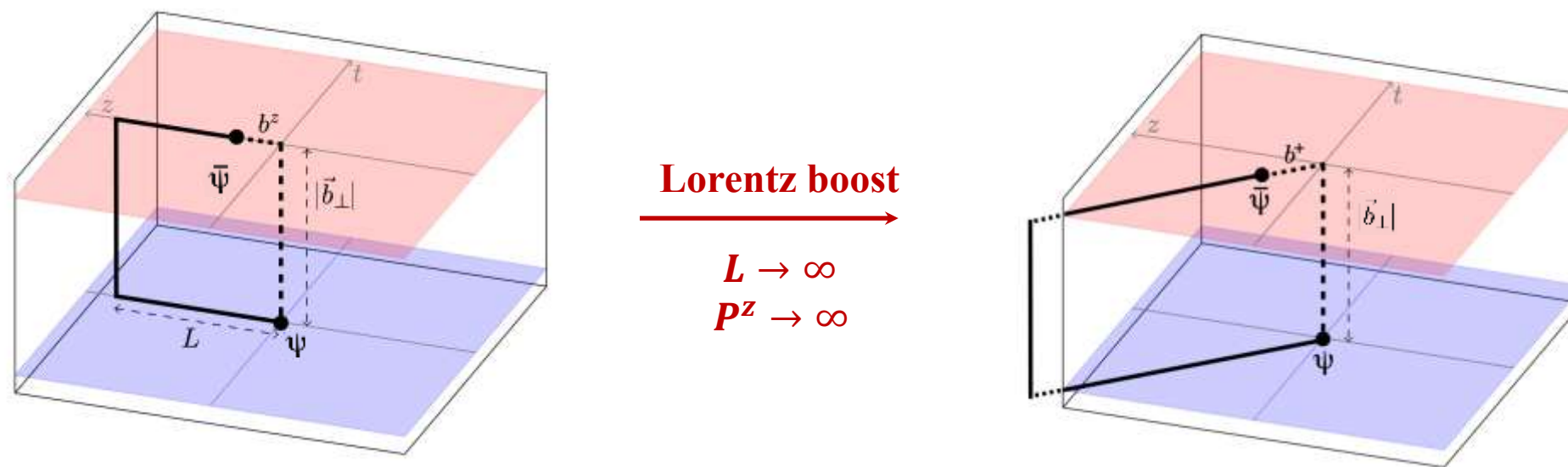
- Collins-Soper equation:

$$\frac{\partial}{\partial \ln \sqrt{\zeta}} \ln f_{\text{TMD}}(x, b_{\perp}, \mu, \zeta) = K(b_{\perp}, \mu)$$

CS kernel and S_I are essential in describing the rapidity dependence of TMDs.

Lattice & LaMET

LaMET gives a bridge between light-cone quantities and lattice-calculable quantities.



For TMDs

Perturbative matching kernel

Light-cone TMDs

$$\tilde{f}_F(x, b_\perp, \mu, \zeta^z) S_I^{\frac{1}{2}}(b_\perp, \mu) = H_F(\zeta^z, \mu) \exp \left[\frac{1}{2} K(b_\perp, \mu) \ln \frac{\zeta^z - i\epsilon}{\zeta} \right] f(x, b_\perp, \mu, \zeta) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{\zeta^z}, \frac{M^2}{(P^z)^2}, \frac{1}{b_\perp^2 \zeta^z}\right)$$

Quasi-TMDs

Lattice & LaMET

For CS kernel:

$$K(b_{\perp}, \mu) = \frac{1}{\ln(P_1^Z/P_2^Z)} \ln \left(\frac{H_{\Gamma}(\zeta_2^Z, \mu) \tilde{f}_{\Gamma}(x, b_{\perp}, \zeta_1^Z, \mu)}{H_{\Gamma}(\zeta_1^Z, \mu) \tilde{f}_{\Gamma}(x, b_{\perp}, \zeta_2^Z, \mu)} \right)$$

Quasi-TMDs

For a better x -plateau, quasi-TMDWFs are used here.

Chu et al.(LPC), JHEP 08 (2023) 172

For intrinsic soft function

$$S_I(b_{\perp}, \mu) = \frac{F(b_{\perp}, P_1, P_2, \mu)}{\int dx_1 dx_2 H(x_1, x_2) \tilde{\Phi}^{\dagger}(x_2, \zeta_2^z, b_{\perp}, \mu) \tilde{\Phi}(x_1, \zeta_1^z, b_{\perp}, \mu)}$$

Form factor

Ji et al. , Nucl.Phys.B 955 (2020)

matching kernel

Quasi-TMDWF

Review on Intrinsic Soft Function

<i>Authors/ Collaboration</i>	<i>Pion mass</i>	<i>Continuum</i>	<i>Matching Kernel</i>	<i>Largest P^z</i>	<i>Reliable $b_{\perp}^{\max}$</i>	<i>Systematic Uncertainty</i>
LPC PRL125, 2020	547 MeV	No	LO	2.11 GeV	0.7 fm	No
ETMC PRL128, 2022	350-827 MeV	No	LO	3.3 GeV	0.7 fm	No
LPC JHEP08, 2023	670 MeV	No	NLO	2.58 GeV	0.6 fm	No
ANL/BNL PRD112, 2025	300 MeV	No	NLO	2.58 GeV for FF, 4.3 GeV for qWF	1.1 fm	No
ETMC PRD113,2026	640 MeV	No	NLO	3.33 GeV	0.5 fm	No
LPC In progress	135.5 MeV	Yes	NLO	3GeV	1 fm	Yes

Adapted from Yong Zhao

Review on CS Kernel

<i>Authors/ Collaboration</i>	<i>Pion mass</i>	<i>Continuum</i>	<i>Matching Kernel</i>	<i>Largest P^Z</i>	<i>Reliable $b_{\perp}^{\max}$</i>	<i>Systematic Uncertainty</i>
SWZ PRD102, 2020	1.2 GeV	No	LO	2.58 GeV	0.8 fm	No
LPC PRL125, 2020	547 MeV	No	LO	2.11 GeV	0.4 fm	No
SVZES JHEP08, 2021	422 MeV	No	--	2.27 GeV	0.5 fm	Estimated but not included
SWZ PRD104, 2021	538 MeV	No	NLO	1.5 GeV	0.5 fm	No
PKU/ETMC PRL128, 2022	827 MeV	No	LO	3.3 GeV	0.3 fm	No
LPC PRD106, 2022	670 MeV	No	NLO	2.58 GeV	0.6 fm	Yes
S⁴VWEY PRD108, 2023	686 MeV	No	--	1.36 GeV	0.4 fm	No
LPC JHEP08, 2023	220 MeV	No	NLO	3.16 GeV	0.6 fm	Yes
ASWZ PRD108, 2023	148.8 MeV	No	uNNLL	2.15GeV	0.8 fm	Yes
ASWZ PRL132, 2024	148.8 MeV	Model	uNNLL	2.3GeV	0.8 fm	Yes
BGHMZ PRD112, 2025	300 MeV	No	NLO	4.3 GeV	-- (Large err.)	Yes
ETMC PRD113,2026	640 MeV	No	NLO	3.33GeV	0.5 fm	No
LPC PRD113,2026	135.5 MeV	Yes	uNLO	3.0 GeV	1.0 fm	Yes

Adapted from Yong Zhao

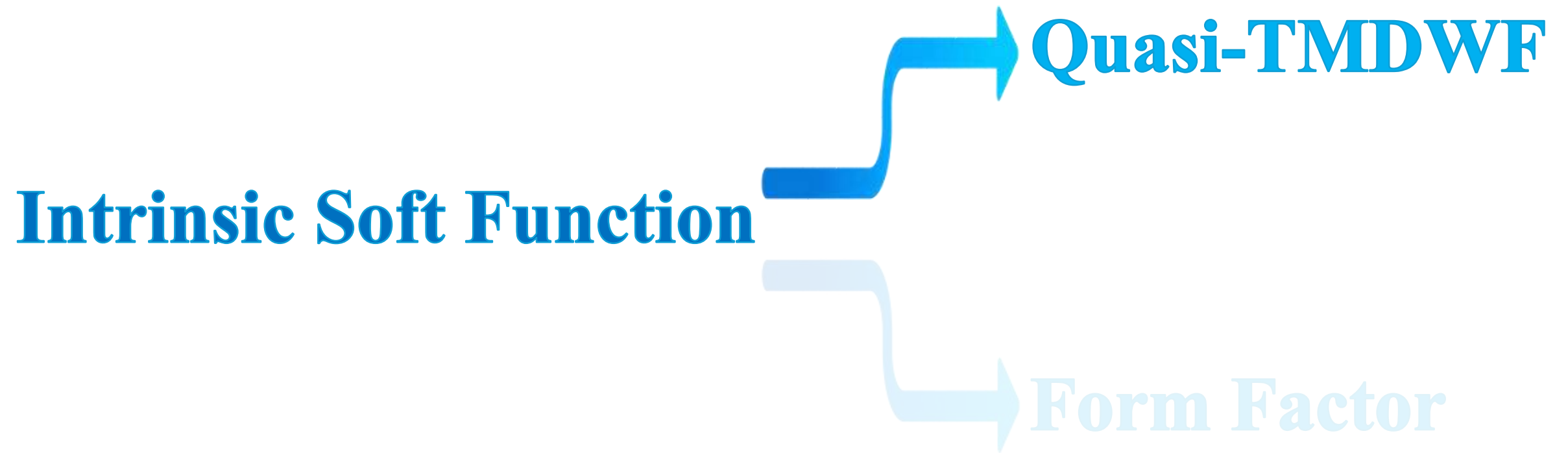
Content

- Motivation and Theoretical Framework
- **Lattice Calculation**
- Summary and Outlook

Lattice Setup

	a/fm	m_π/MeV	P^z/GeV	$n_{cfg} \times n_{meas}$	
				CS Kernel	S_I
C32P29		292.4	1.47, 1.84, 2.21	984×4	984×2
C32P23	0.10530	228.0	1.47, 1.84, 2.21	448×10	448×24
C48P14		135.5	0.98, 1.22, 1.47	204×24	$204 \times 8 \times 12$
F32P30	0.07746	303.2	2.00, 2.50, 3.00	1153×6	1153×6
G36P29	0.06826	295.1	2.02, 2.52, 3.03	656×6	656×6
H48P32	0.05187	317.2	2.00, 2.50, 3.00	550×8	550×8

Ensembles with multi lattice spacings and pion masses are used.



Quasi-TMD Wave Function

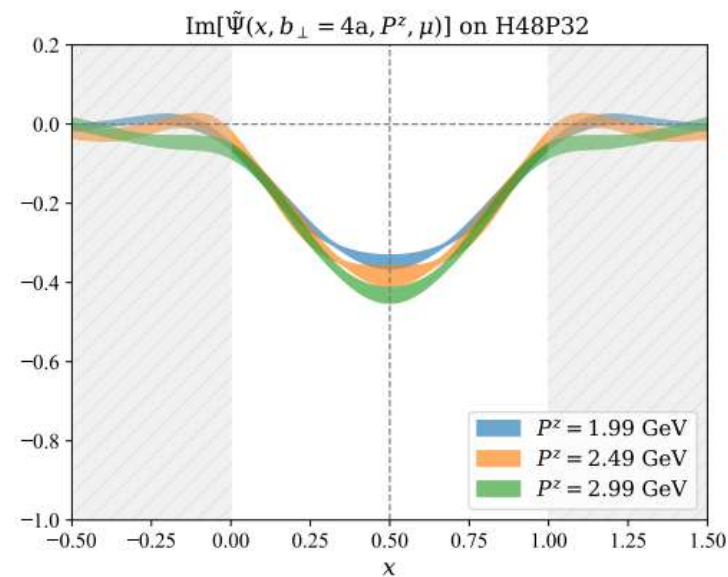
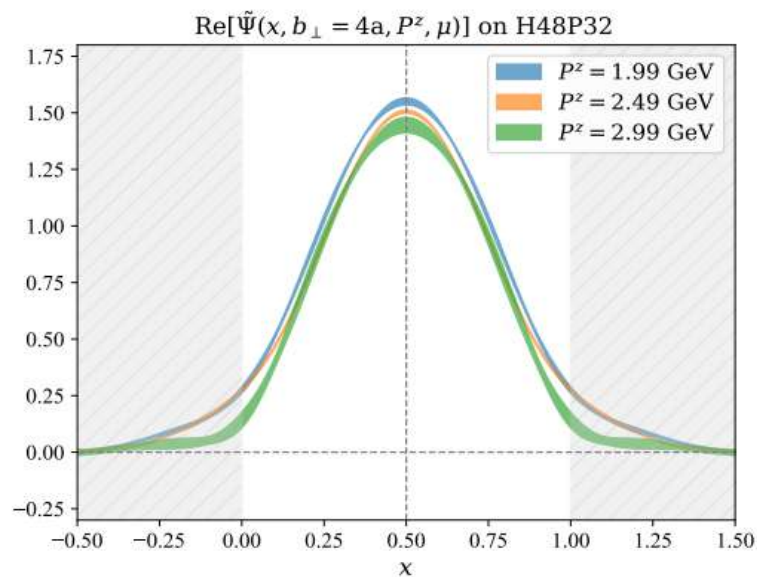
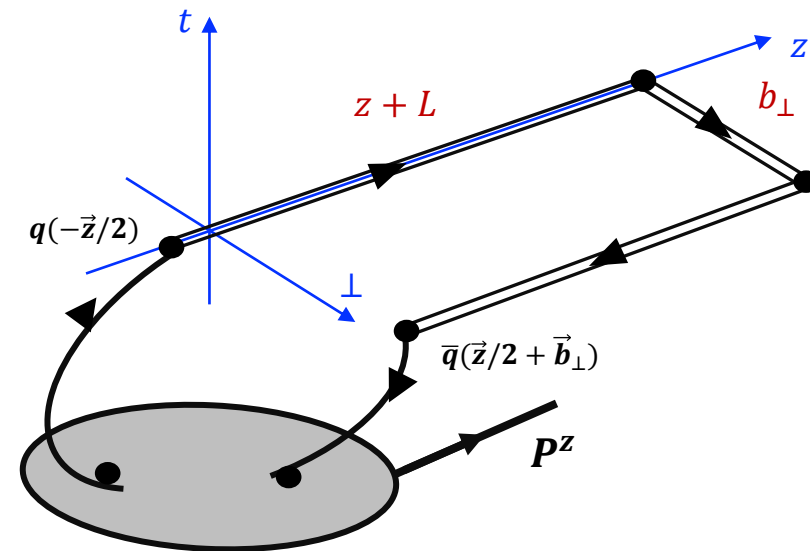
Common inputs for intrinsic soft function and CS kernel.

➤ Bare quasi-TMDWF in coordinate space

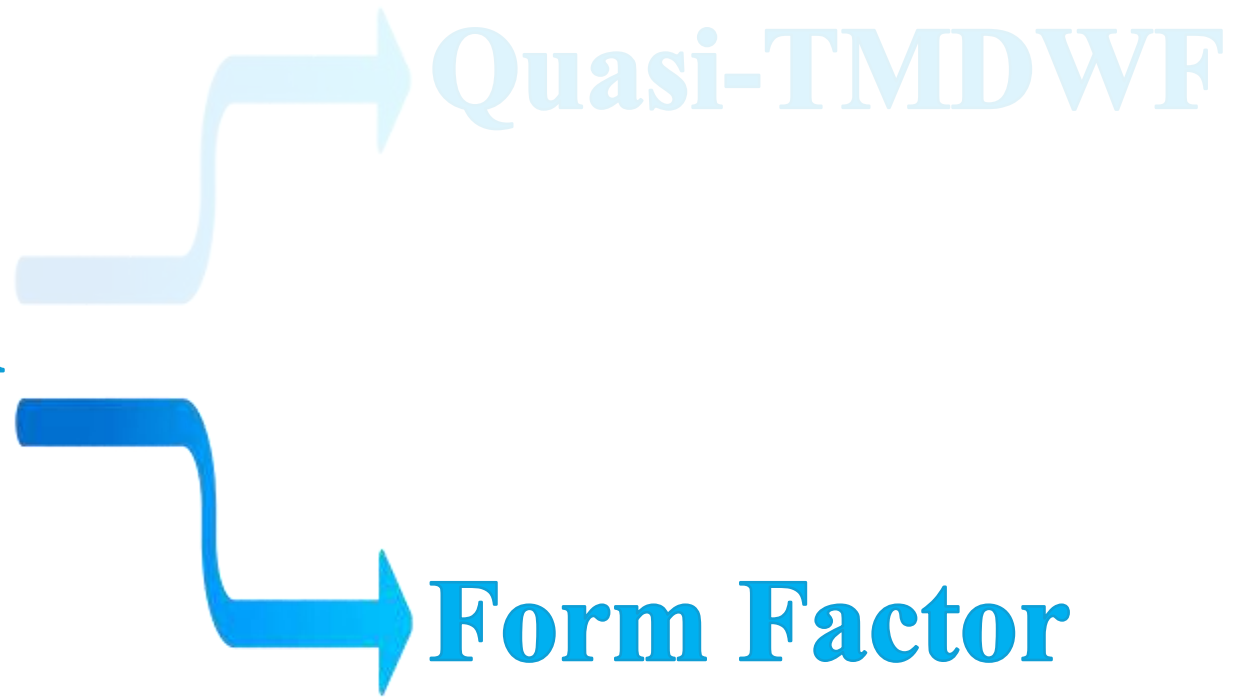
$$\tilde{\Phi}^{\pm 0}(z, b_{\perp}, P^z, L) = \langle 0 | \bar{q}(z\hat{n}_z/2 + b_{\perp}\hat{n}_{\perp}) \Gamma U_{\square, \pm}(L, z, b_{\perp}) q(-z\hat{n}_z/2) | P^z \rangle.$$

➤ Subtracted quasi-TMDWF in momentum space

$$\tilde{\Psi}^{\pm}(x, b_{\perp}, \mu, \zeta_z) = \lim_{L \rightarrow \infty} \int \frac{dz P^z}{2\pi} e^{ixzP^z} \frac{\tilde{\Phi}^{\pm 0}(z, b_{\perp}, P^z, L)}{Z_O(\mu, a) \sqrt{Z_E(2L + z, b_{\perp}, \mu)}},$$



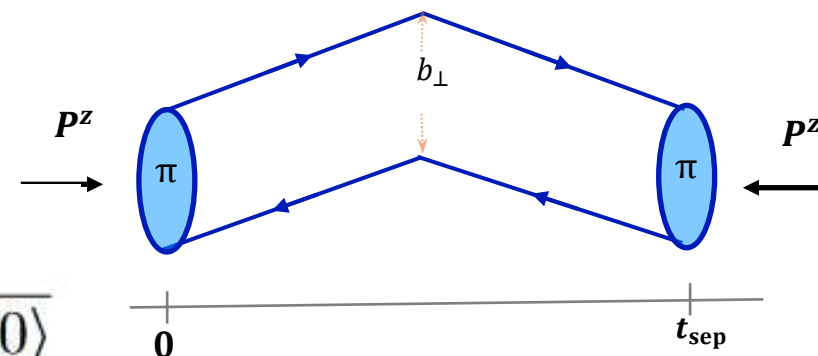
Intrinsic Soft Function



Four Quark Form Factor

➤ Four-quark form factor:

$$F(b_{\perp}, P_1, P_2, \mu, \Gamma) = \frac{\langle P_2 | \bar{q}(b_{\perp}) \Gamma q(b_{\perp}) \bar{q}(0) \Gamma q(0) | P_1 \rangle}{\langle 0 | \bar{q}(0) \gamma^{\mu} \gamma^5 q(0) | P_1 \rangle \langle P_2 | \bar{q}(0) \gamma_{\mu} \gamma^5 q(0) | 0 \rangle}$$



➤ Leading twist:

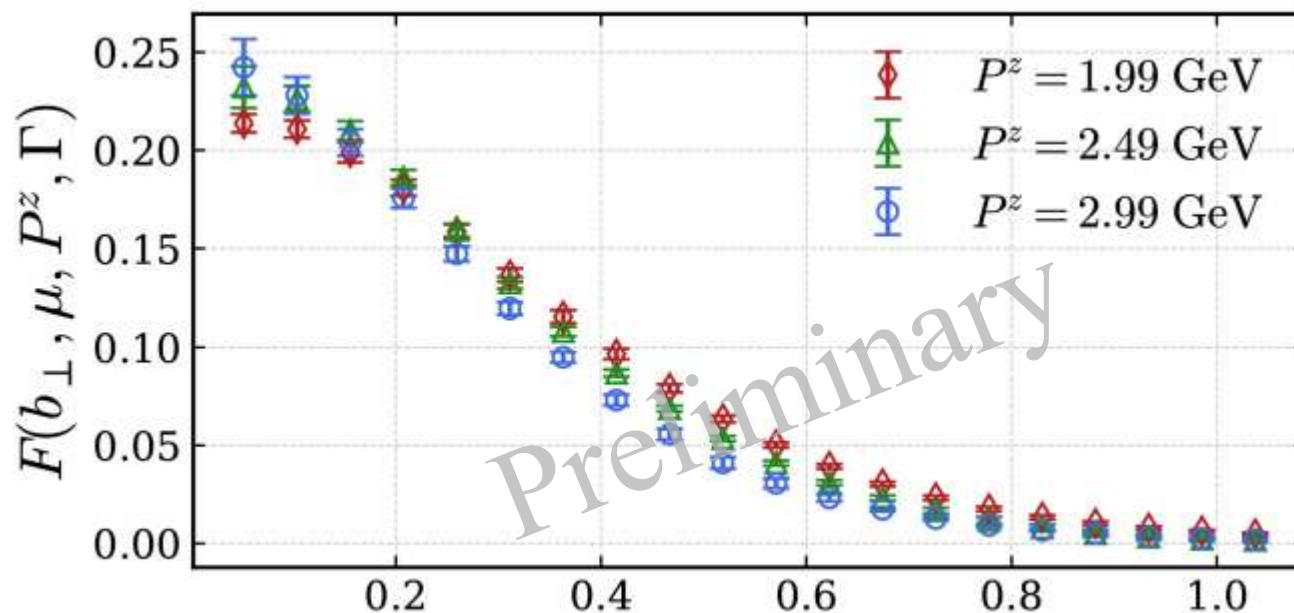
$$F(\Gamma = \gamma_{\perp}) + F(\Gamma = \gamma_{\perp} \gamma_5)$$

Deng et al., JHEP 09 (2022).

➤ Kinetically enhanced interpolator:

$$\bar{u} \gamma_5 d \rightarrow \bar{u} \gamma_{\mu} \gamma_5 d$$

Zhang et al., Phys.Rev.D 112 (2025)



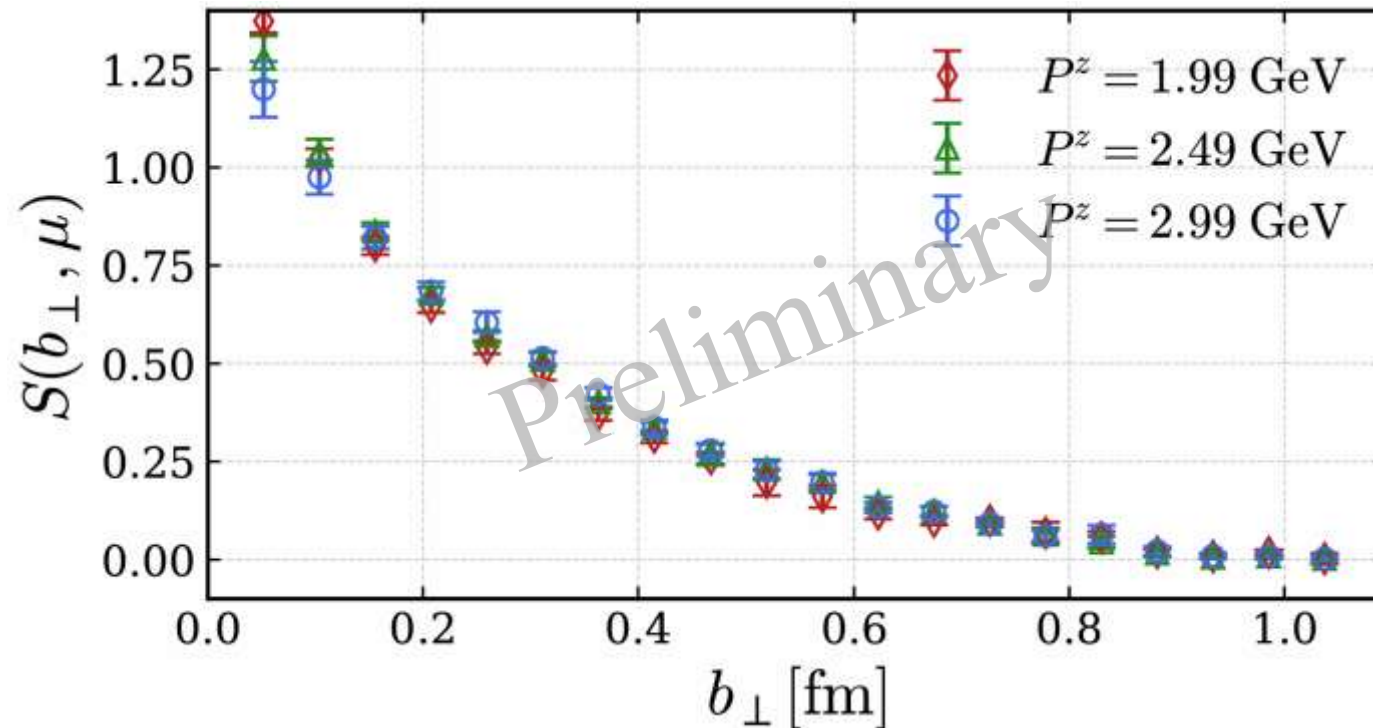
Form factor on ensemble H48P32.

Ji et al., Nucl.Phys.B 955 (2020)

Determine Intrinsic Soft Function

Form Factor $\xrightarrow{\text{NLO matching + quasi-TMDWFs}}$ S_I

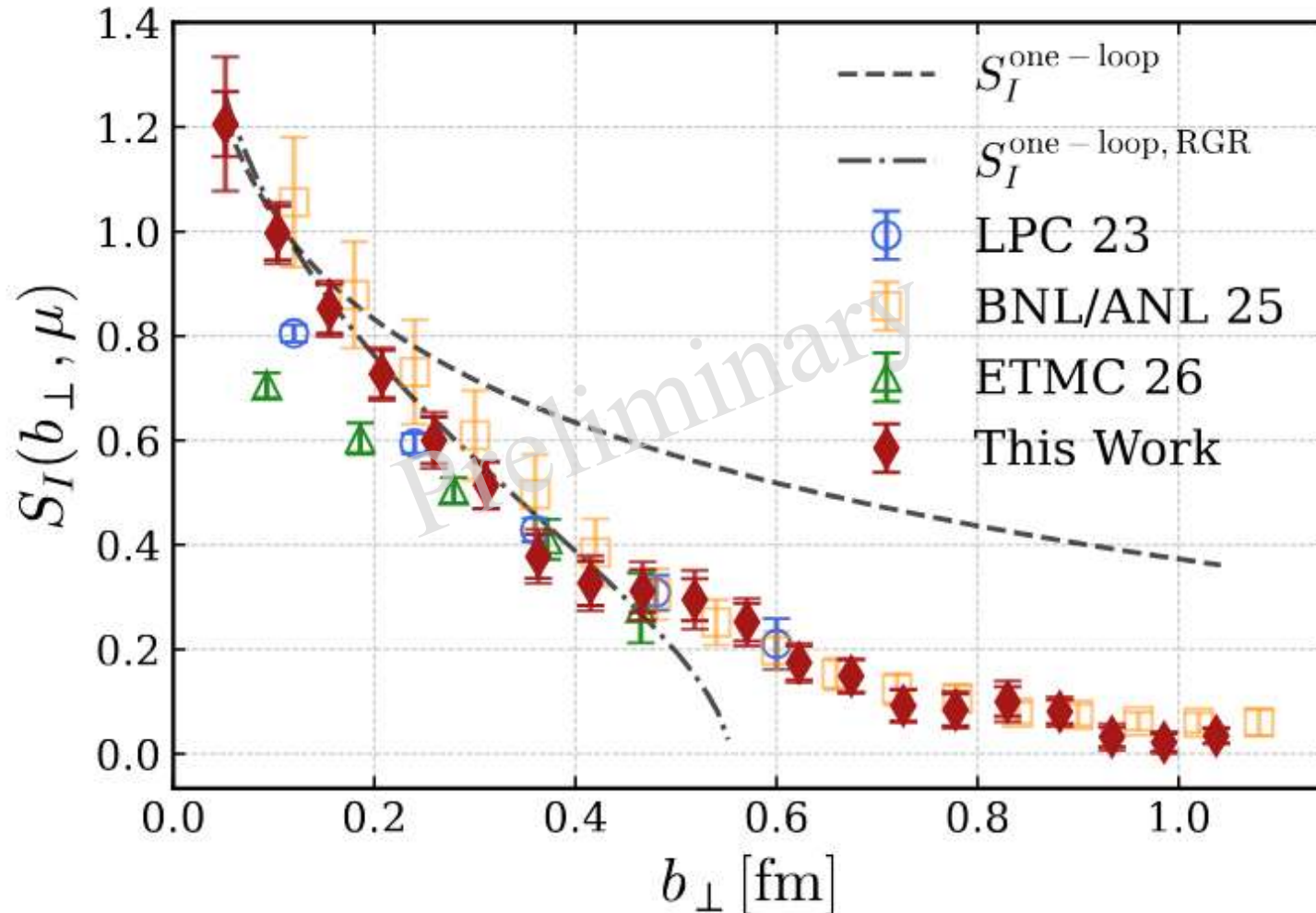
$$S_I(b_\perp, \mu) = \frac{F(b_\perp, P_1, P_2, \mu)}{\int dx_1 dx_2 H(x_1, x_2) \tilde{\Phi}^\dagger(x_2, \zeta_2^z, b_\perp, \mu) \tilde{\Phi}(x_1, \zeta_1^z, b_\perp, \mu)}$$



Intrinsic soft function on ensemble H48P32.

Result of Intrinsic Soft Function

$$S_I(b_\perp, \mu) = S_{\text{lim}}(b_\perp, \mu) + \frac{A}{(P^z)^2} + Ba^2 + C(m_{\text{latt}}^2 - m_{\text{phy}}^2)$$

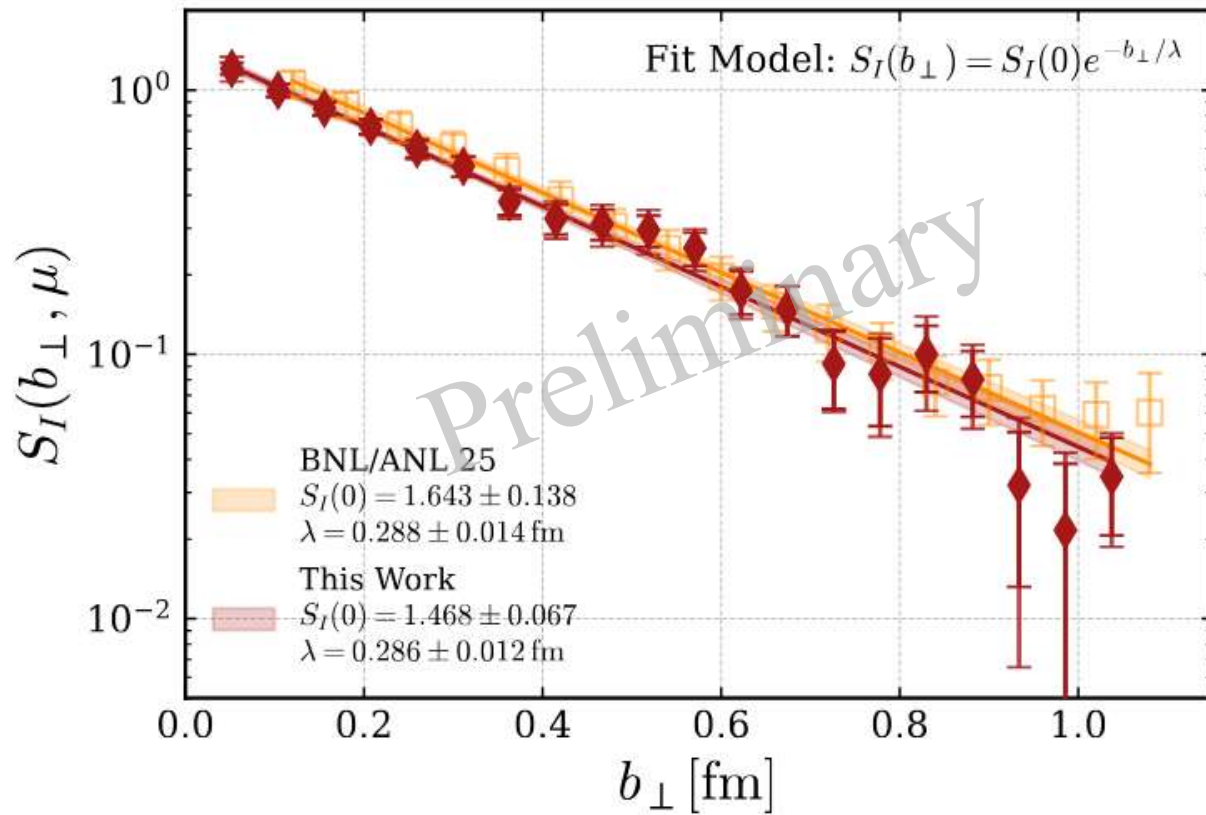


- ✓ Continuum limit;
- ✓ Physical mass;
- ✓ $b_\perp = 1\text{fm}$

*Chu et al. (LPC), JHEP 08 (2023);
 Bollweg et al., Phys.Rev.D 112 (2025);
 Alexandrou et al., Phys.Rev.D 113 (2026)*

Rapidity independent soft contribution of TMDs.

Behavior of Intrinsic Soft Function



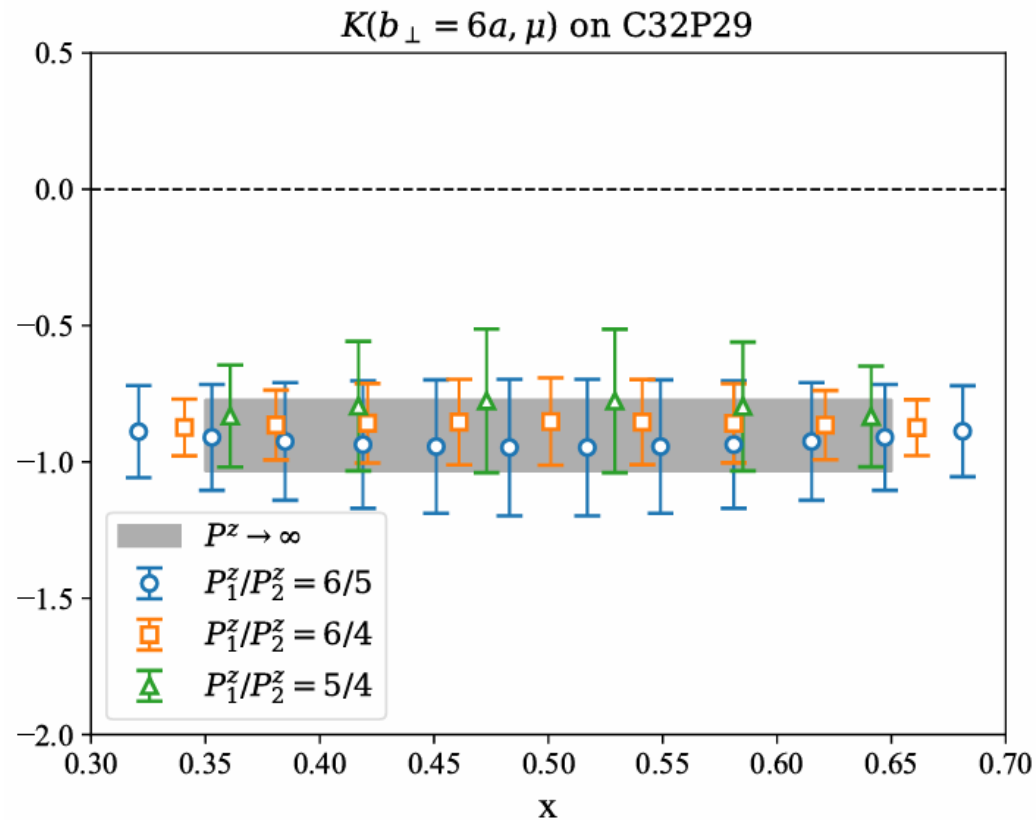
Fit range [b/a]	χ^2/dof	$\lambda [\text{fm}]$	$1/\lambda [\text{GeV}]$
[1, 20]	0.405	0.286 ± 0.012	0.689 ± 0.030
[3, 20]	0.432	0.282 ± 0.014	0.701 ± 0.036
[5, 20]	0.481	0.277 ± 0.019	0.711 ± 0.049
[7, 20]	0.556	0.277 ± 0.027	0.713 ± 0.070

**Possible connection to the nonperturbative
Wilson-line observables or confinement?**

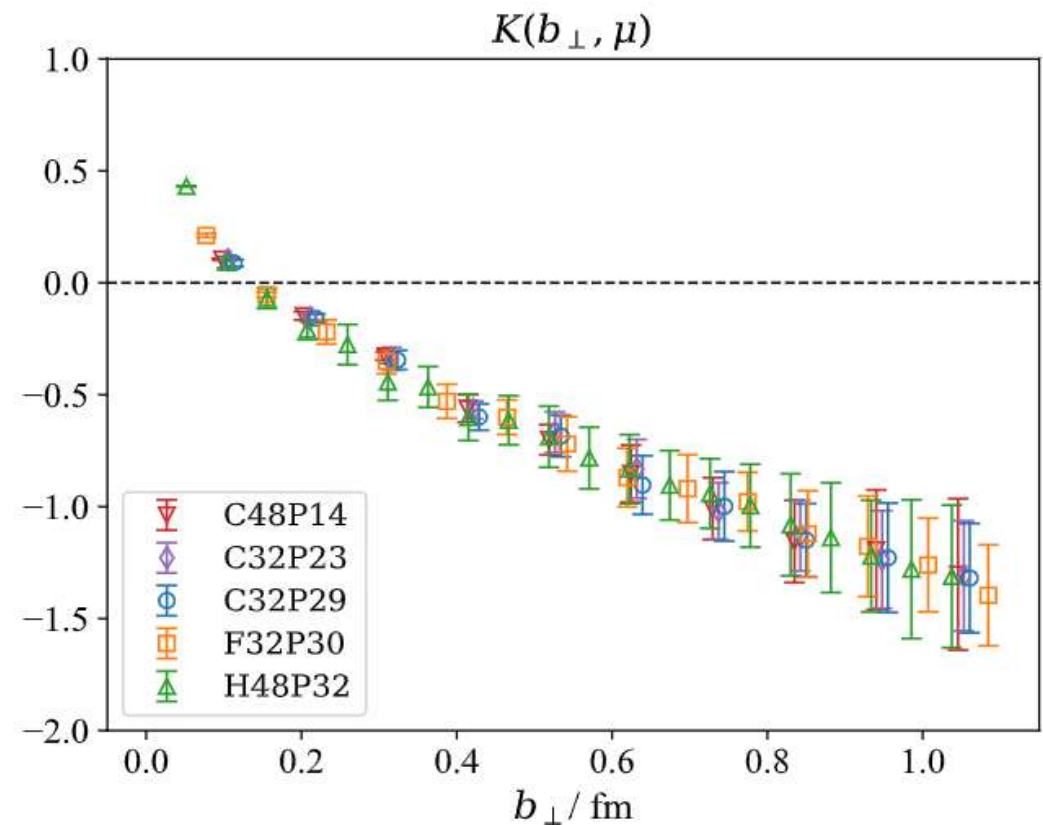
CS Kernel

CS Kernel Extraction

$$K(b_{\perp}, \mu) = \frac{1}{\ln(P_1^z/P_2^z)} \ln \left[\frac{H_{\Gamma}(\zeta_2^z, \mu) \tilde{\Phi}(x, b_{\perp}, \zeta_1^z, \mu)}{H_{\Gamma}(\zeta_1^z, \mu) \tilde{\Phi}(x, b_{\perp}, \zeta_2^z, \mu)} \right]$$

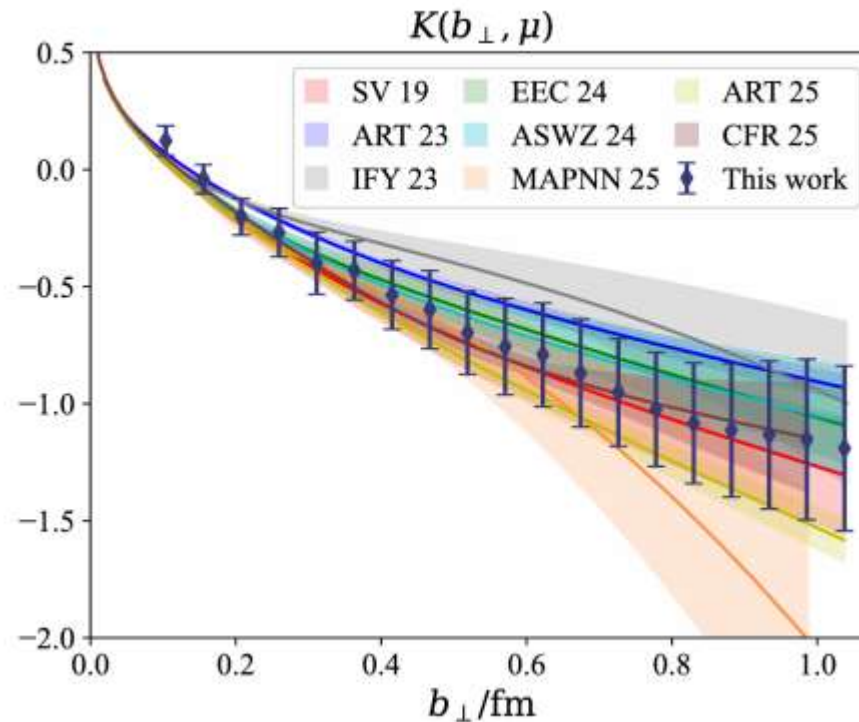
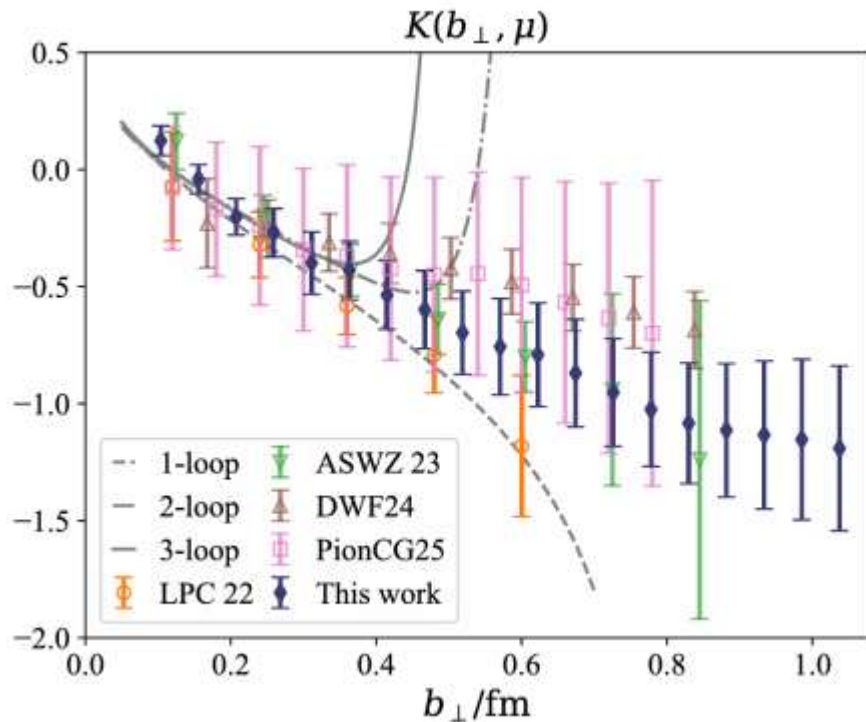


Extraction and large momentum extrapolation.



Results on different ensembles are highly consistent!

Results of CS Kernel



- ✓ Continuum limit;
- ✓ Physical mass;
- ✓ $b_{\perp} = 1\text{fm}$

Reliable description on rapidity evolution of TMDs.

Lattice:

Chu et al. (LPC), JHEP 08, 172 (2023);
Avkhadiev, et al., PRD 108, 114505 (2023);
Bollweg, et al., PRB 852, 138617 (2024);
Avkhadiev et al., PRL. 132, 231901 (2024);
Bollweg, et al., PRD 112, 034501(2025).

Phenomenological:

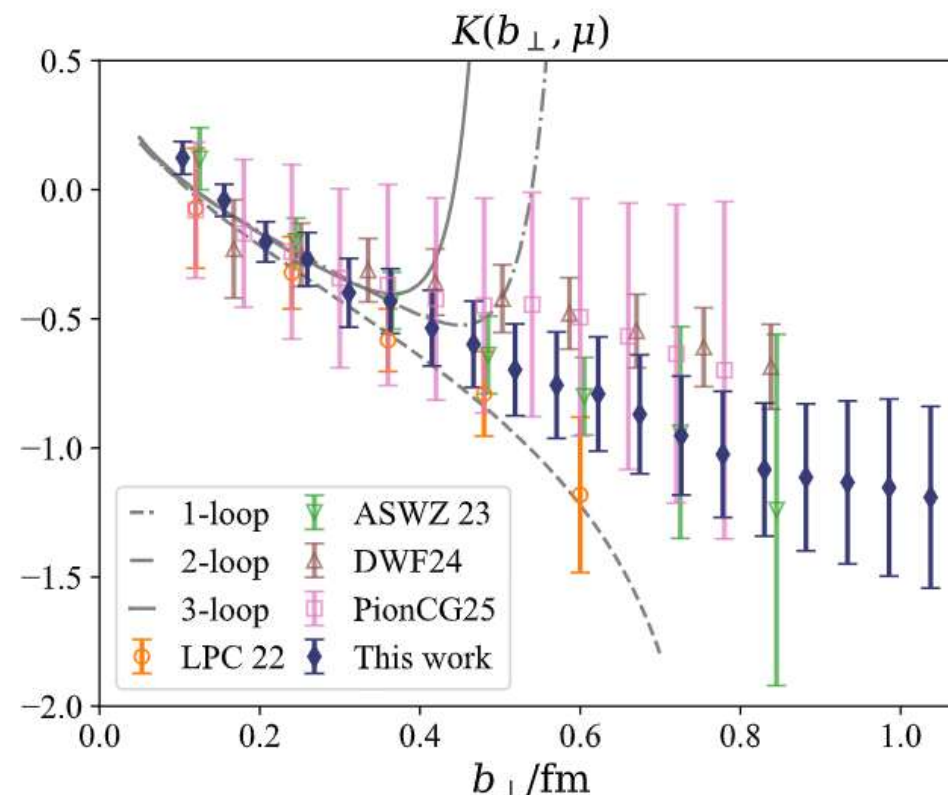
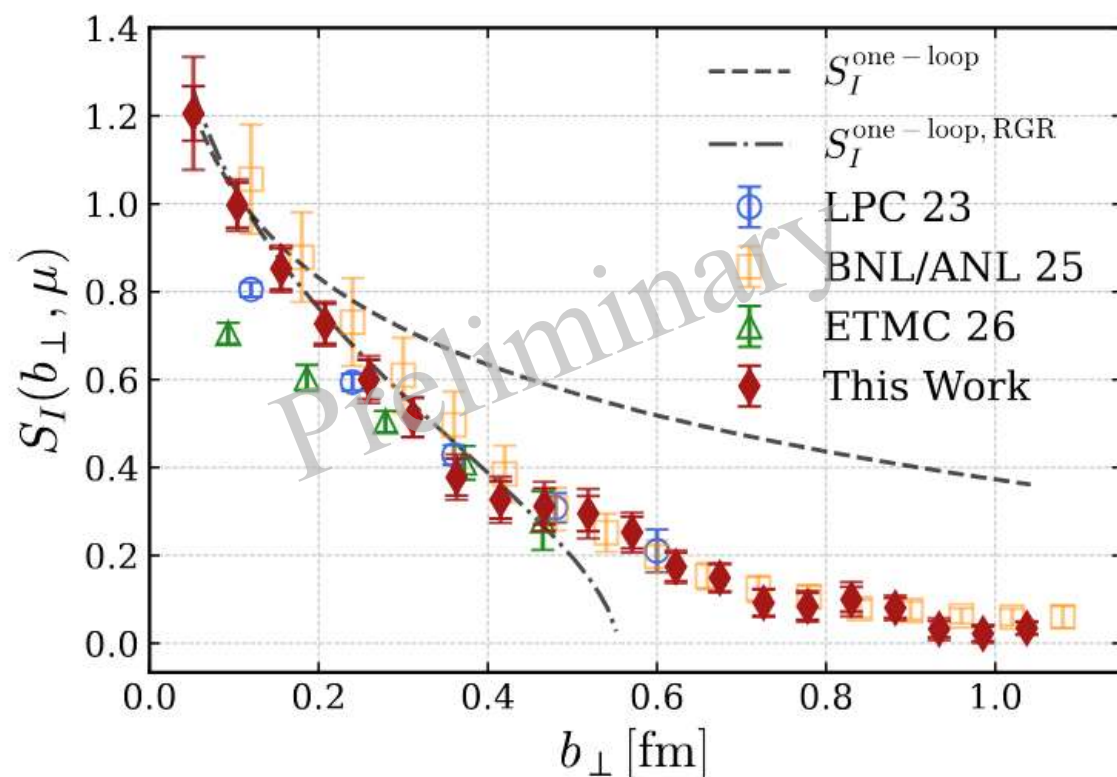
Scimemi et al., JHEP 06, 137 (2020);
Moos et al., JHEP 05, 036 (2024);
Isacson et al., PRD 110, 073002 (2024);
Kang et al., arXiv:2410.21435;

Bacchetta et al., PRL. 135, 021904 (2025);
Moos et al., arXiv:2503.11201;
Camarda et al., arXiv:2508.06201.

- Motivation and Theoretical Framework
- Lattice Calculation
- **Summary and Outlook**

Summary & Outlook

- Collins-Soper kernel and the intrinsic soft function are essential nonperturbative inputs for TMDs;
- Both results are extrapolated to the **continuum limit and the physical pion mass**, and reach $b_{\perp} \sim 1\text{fm}$ with high accuracy;
- Give a **first-principal description** for the rapidity behavior of TMDs.



Thank you!

Back up

Form Factor

$$\frac{C_3(b_\perp, \Gamma, t_{\text{seq}}, t, P^z)}{2 [C_2(0, 0, 0, t_{\text{seq}}/2, P^z)]^2} = \boxed{F(b_\perp, \Gamma, P^z)} \frac{1 + c_1(e^{-\Delta E t} + e^{-\Delta E(t_{\text{seq}} - t)})}{1 + c_2 e^{-\Delta E t_{\text{seq}}/2}}$$

➤ Leading twist:

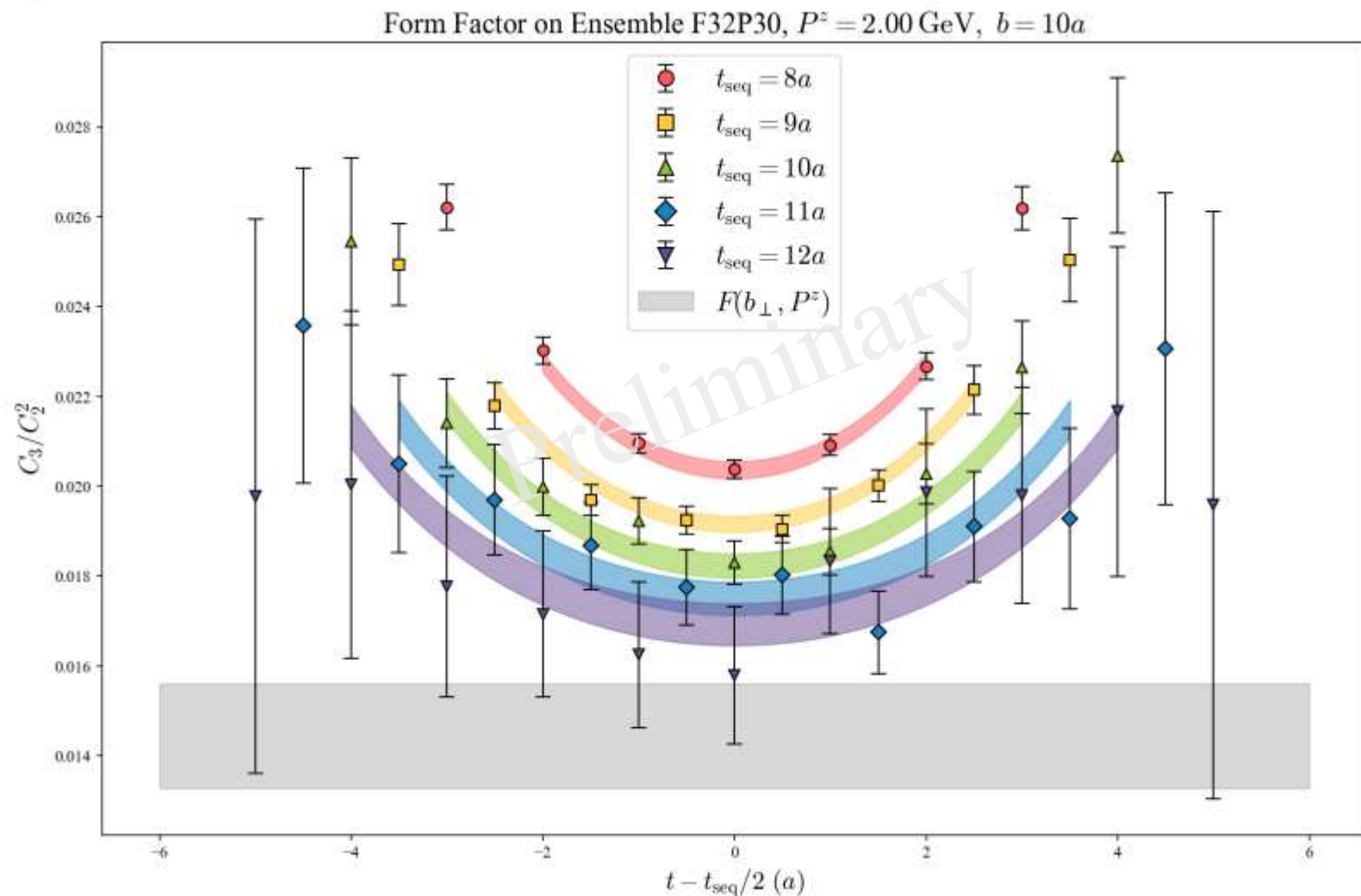
$$F(\Gamma = \gamma_\perp) + F(\Gamma = \gamma_\perp \gamma_5)$$

Deng et al., JHEP 09 (2022).

➤ Kinetically enhanced interpolator:

$$\bar{u} \gamma_5 d \rightarrow \bar{u} \gamma_\mu \gamma_5 d$$

Zhang et al., Phys.Rev.D 112 (2025)

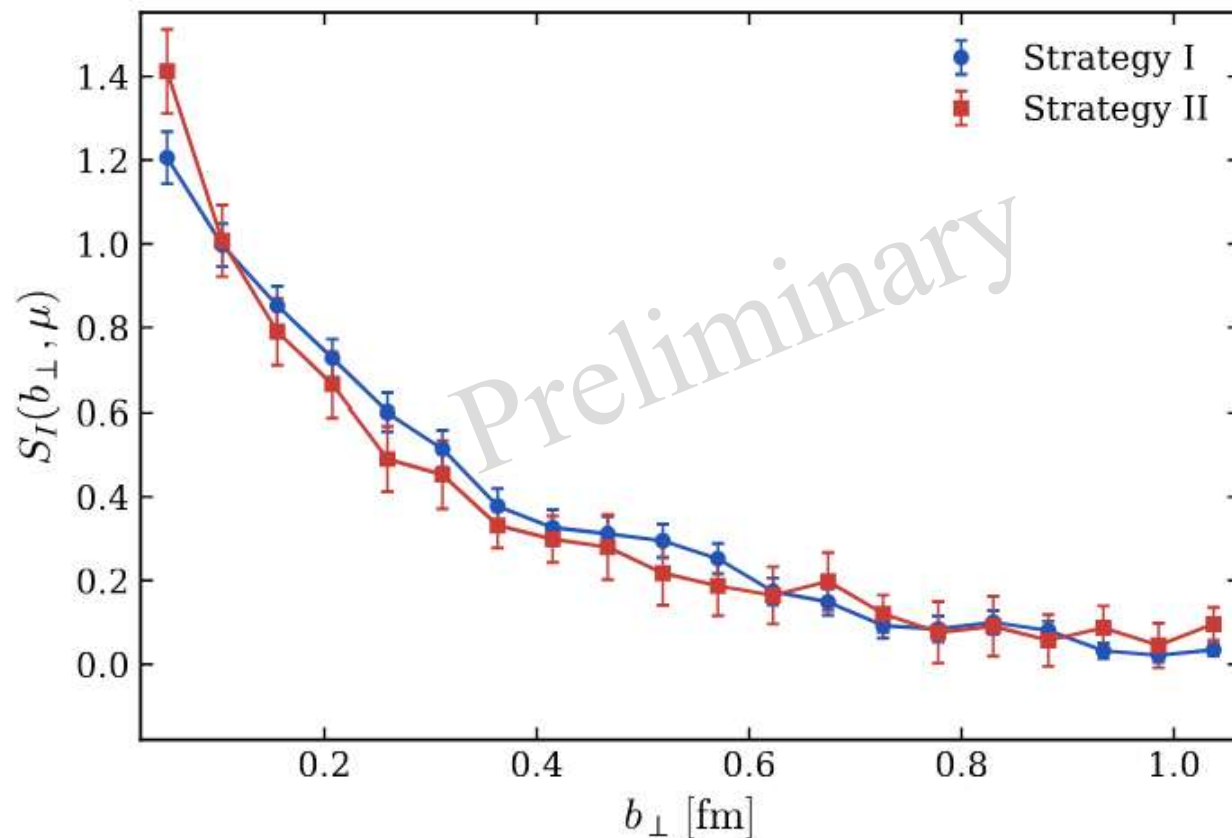


Extrapolations

$$S_I(b_\perp, \mu) = S_{\text{lim}}(b_\perp, \mu) + \frac{A}{P_z^2} + B(m_{\pi, \text{lat}}^2 - m_{\pi, \text{phy}}^2) + \Delta_{\text{disc}}(a)$$

Strategy I: $\Delta_{\text{disc}}(a) = Ca^2$

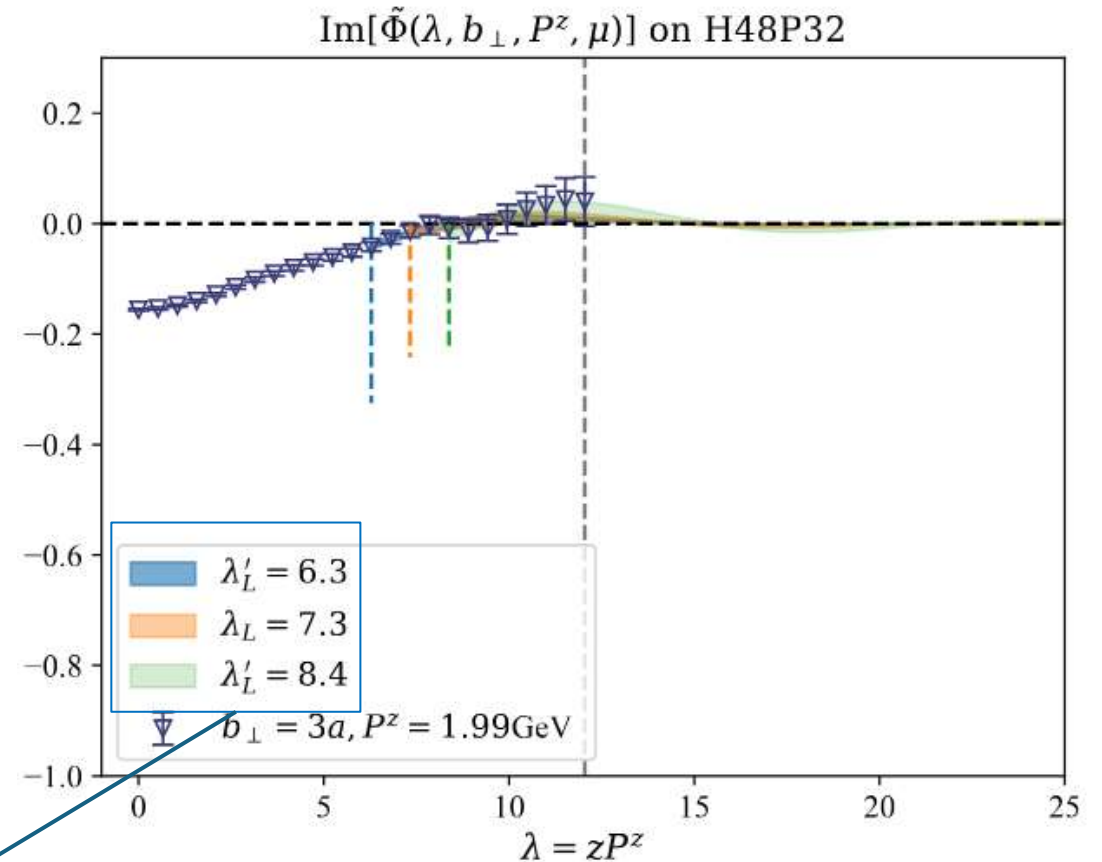
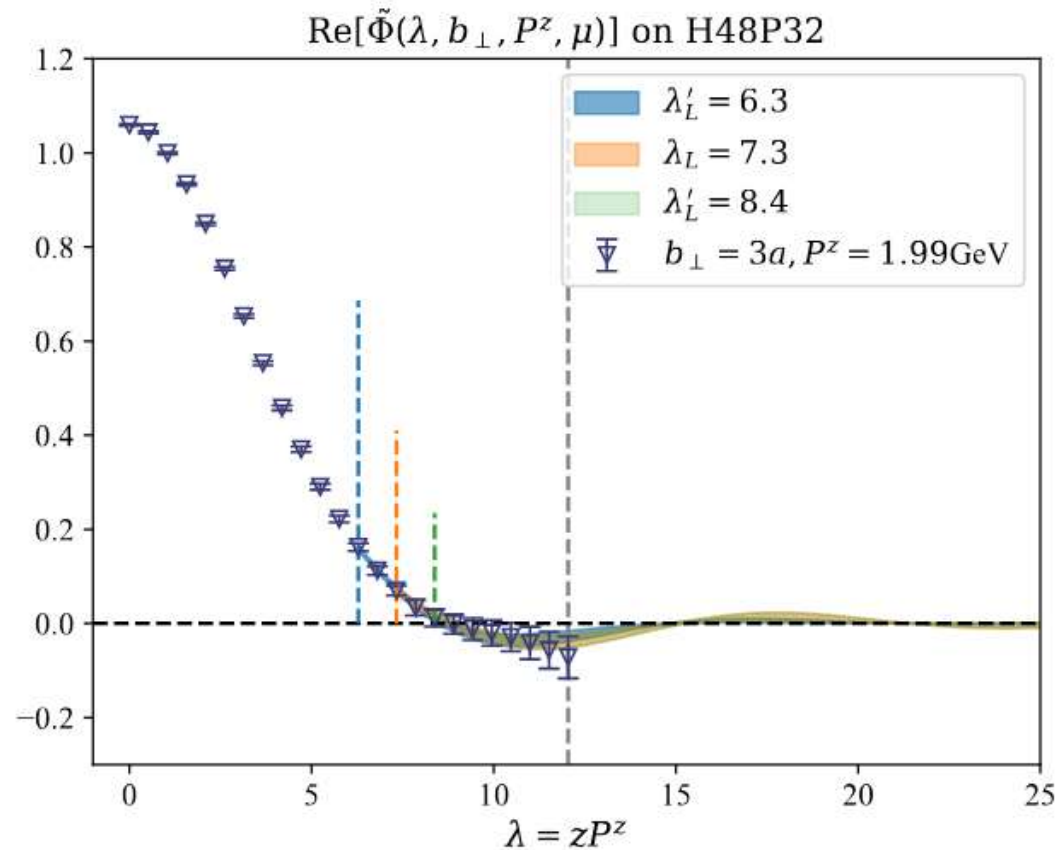
Strategy II: $\Delta_{\text{disc}}(a) = C_0 a^2 + C_1 a$



Different discretization ansatz give consistent results for S_I .

Large- λ extrapolation and Fourier Transformation

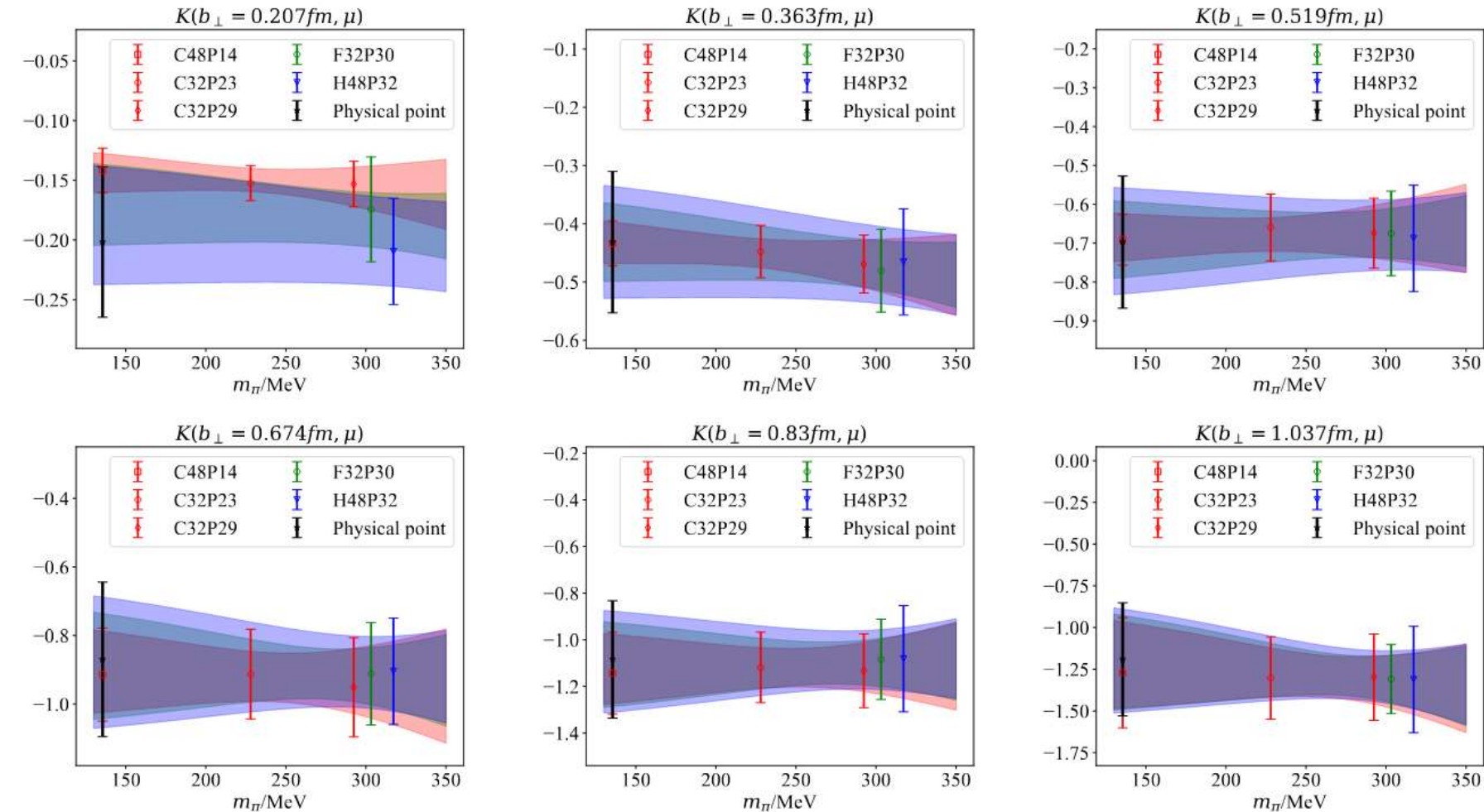
$$\lim_{\lambda \rightarrow \infty} \tilde{\Phi}(z, b_{\perp}, P^z, a, L) = [f_1(b_{\perp}) + if_2(b_{\perp})] \left[\frac{c_1}{(-i\lambda)} + e^{i\lambda} \frac{c_2}{(i\lambda)^d} \right] e^{\frac{\lambda}{\lambda_0}}$$



Take different selections of λ as **systematic uncertainty**.

Continuum and Physical Pion Mass Extrapolation

$$K(b_{\perp}, \mu; a, m_{\pi}) = K(b_{\perp}, \mu) + a^2 B(b_{\perp}, \mu) + (m_{\pi}^2 - m_{\pi,phy}^2) C(b_{\perp}, \mu)$$



Short-distance region is more sensitive to discretization.

Final CS kernel are obtained after extrapolation.