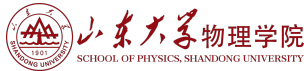


Matching Axial and Topological Currents from RI(S)MOM to $\overline{\text{MS}}$ through $\mathcal{O}(\alpha_s^3)$ and Beyond

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How Much Proton Helicity Comes From Quarks & Gluons?

Nucleon spin & matrix element of axial current

$$J_{\text{QCD}} = S_q + L_q + J_g \quad (\text{Ji-scheme})$$

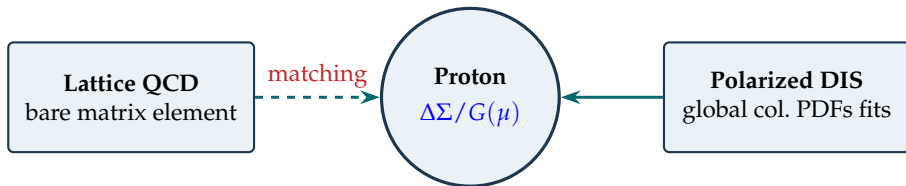
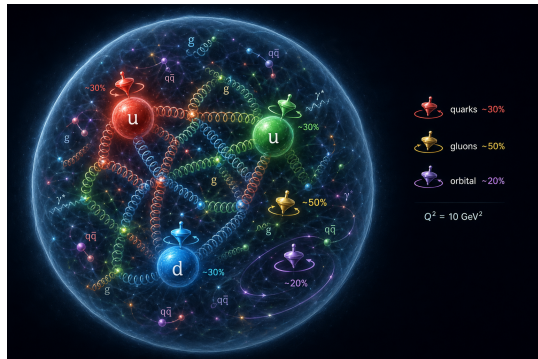
$$J_{\text{QCD}} = S_q + L_q + S_g + L_g \quad (\text{JM-scheme})$$

$$S_q = \frac{1}{2} \Delta\Sigma(\mu) = \frac{1}{2} (g_A^u + g_A^d + g_A^s)$$

$$\langle P, S | \left[\sum_f \bar{q}_f \gamma^\mu \gamma_5 q_f \right]_R | P, S \rangle = 2M_N S^\mu \Delta\Sigma(\mu)$$

$\Delta G(\mu)$ can be similarly connected to $\langle P, S | [K^\mu]_R | P, S \rangle$

[→See talks by Ji, Tan, Zhong]



Why Singlet Axial Current Operator and Why Special

- Connection to Quark Spin:

$$\frac{u^\dagger(p, S) \Sigma^\alpha u(p, S)}{u^\dagger(p, S) u(p, S)} = \frac{\bar{u}(p, S) \gamma^\alpha \gamma_5 u(p, S)}{u^\dagger(p, S) u(p, S)} = \frac{m}{E} S^\alpha$$

Spin is *indifferent* to the flavor (and charge etc) of the quarks $\Rightarrow$ *flavor-singlet* axial current operator

- **Axial anomaly** calls for additional operator renormalization

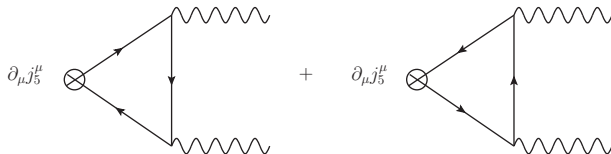
$$\partial_\mu \sum_f \bar{\psi}_f \gamma^\mu \gamma_5 \psi_f = \sum_f 2m_f \bar{\psi}_f i \gamma_5 \psi_f - \frac{\alpha}{4\pi} n_f T_F \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

classical mass term + quantum anomaly

$$\langle P, S | [J_{A,s}^\mu]_R | P, S \rangle = 2M_N S^\mu \Delta\Sigma(\mu)$$

Proper $\overline{\text{MS}}$ with non-anticommuting γ_5 in DR

- γ_5 -issue in DR with Axial Anomaly



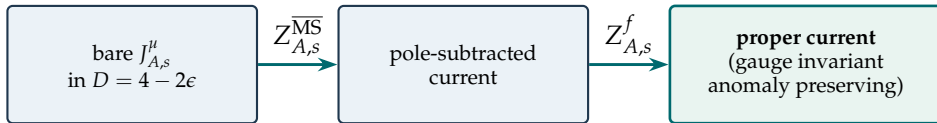
“**vanishes**” with translational invariant loop integrals **and** an anticommuting γ_5 , due to perturbative pinch identity

$$(\not{p}_1 + \not{p}_2) \gamma_5 = \underbrace{(I + \not{p}_1 - m) \gamma_5 + \gamma_5 (I - \not{p}_2 - m)} + 2m \gamma_5 + \Delta_5^{ac}(I)$$

- HV/BM** [72, 79] NAC γ_5 prescription:

$$\gamma_5 = -\frac{i}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \quad \gamma_\mu \gamma_5 \rightarrow \frac{1}{2} (\gamma_\mu \gamma_5 - \gamma_5 \gamma_\mu), \quad \Delta_5^{ac} \equiv \{\gamma^\mu, \gamma_5\} \sim \mathcal{O}(\epsilon) \neq 0.$$

“Symmetrization” needed to have a **hermitian** axial-current [Akyeampong, Delbourgo 77; Fujii, Ohta, Taniguchi 81]

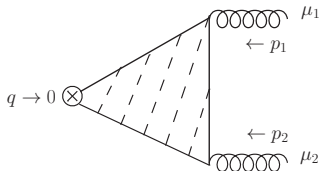


Proper $\overline{\text{MS}}$ with non-anticommuting γ_5 in DR

- Larin's variant [Larin, Vermaseren 91; Larin 93] of HVBM prescription $[J_5^\mu]_R \equiv Z_5^f Z_5^{ms} \bar{\psi}_B \frac{-i}{3!} \epsilon^{\mu\nu\rho\sigma} \gamma_\nu \gamma_\rho \gamma_\sigma \psi_B$
 - ▶ $\epsilon^{\mu\nu\rho\sigma}$ treated outside R -operation formally in D dimensions [Larin, Vermaseren 91; Zijlstra, Neerven 92]
 - ▶ Determine $Z_s \equiv Z_s^f Z_s^{ms}$ by maintaining ABJ-equation $[\partial_\mu J_5^\mu]_R = a_s n_f [F\tilde{F}]_R$ with the standard $\overline{\text{MS}}$ -renormalized $[F\tilde{F}]_R$ and ϵ -independent Z_s^f
 - ▶ The renormalization of the operators involved: [Adler 69; Espriu, Tarrach 82; Breitenlohner, Maison, Stelle 84; Bos 92; Larin 93 ...]

$$\frac{d}{d \ln \mu^2} \begin{pmatrix} [\partial_\mu J_5^\mu]_R \\ [F\tilde{F}]_R \end{pmatrix} = \frac{d}{d \ln \mu^2} \begin{pmatrix} Z_J & 0 \\ Z_{FJ} & Z_{F\tilde{F}} \end{pmatrix} \cdot \begin{pmatrix} [\partial_\mu J_5^\mu]_B \\ [F\tilde{F}]_B \end{pmatrix} = \begin{pmatrix} \gamma_s & 0 \\ \gamma_{FJ} & \gamma_{F\tilde{F}} \end{pmatrix} \cdot \begin{pmatrix} [\partial_\mu J_5^\mu]_R \\ [F\tilde{F}]_R \end{pmatrix}$$

- ▶ Larin's original implementation [Larin 93] using AVV-amplitude: **Inefficient!** (L -integral diagram $\rightarrow Z_s @ \mathcal{O}(\alpha_s^{L-1})$)



$\overline{\text{MS}}$ Renormalization of $F\tilde{F}$ and K-current

The axial-anomaly (*topological-charge density*) operator $F\tilde{F}$ with the Chern-Simons (*topological-charge*) current K^μ

$$\begin{aligned} F\tilde{F} &= \partial_\mu K^\mu \\ &= \partial_\mu \left(-4\epsilon^{\mu\nu\rho\sigma} \left(A_\nu^a \partial_\rho A_\sigma^a + g_s \frac{1}{3} f^{abc} A_\nu^a A_\rho^b A_\sigma^c \right) \right) \end{aligned}$$

by the virtue of total antisymmetry of $\epsilon^{\mu\nu\rho\sigma}$ [Bardeen 74].

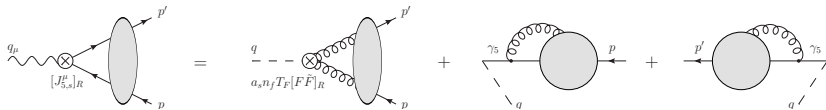
- Unlike J_5^μ and $F\tilde{F}$, infinitesimal gauge variation of K^μ is NOT zero, but linear in the total derivatives of A_ν^a
(vanishing if only perturbative configurations are considered at zero-momentum insertion)

•

$$\begin{pmatrix} [J_5^\mu]_R \\ [K^\mu]_R \end{pmatrix} = \begin{pmatrix} Z_J & 0 \\ Z_{FJ} & Z_{F\tilde{F}} \end{pmatrix} \cdot \begin{pmatrix} [J_5^\mu]_B \\ [K^\mu]_B \end{pmatrix}$$

with “non-Abelian” non-renorm. $Z_{F\tilde{F}} = Z_{a_s}$ verified to four QCD loops [Ahmed, LC, Czakon 21] and later proved [Lüscher, Weisz 21]

A Convenient Prescription for Calculating $Z_s^f Z_s^{ms}$



Combining the off-shell Axial WTIs for both singlet and non-singlet currents

$$q_\mu \Gamma_{5,s}^\mu(p', p) = a_s n_f T_F q_\mu \Gamma_K^\mu(p', p) + \gamma_5 \hat{S}^{-1}(p) + \hat{S}^{-1}(p') \gamma_5,$$

$$q_\mu \Gamma_{A,ns}^{a,\mu}(p', p) = t^a \gamma_5 \hat{S}^{-1}(p) + \hat{S}^{-1}(p') \gamma_5 t^a$$

and by **first differentiating in q and then taking the forward limit**, we now attempt

$$\boxed{\Gamma_{A,s}^\mu(p, p) = a_s n_f T_F \Gamma_K^\mu(p, p) + \Gamma_{A,ns}^\mu(p, p)}$$

- Makes the determination of the proper $\overline{\text{MS}}$ -renormalization $Z_{A,s} = Z_{A,s}^f \bar{Z}_{A,s}$ as **straightforward as for the non-singlet counterpart** (quasi-current-level relation)
- An observation: **Only when restricted to the 4-dimensional limit, then follows the same *universal gauge-independent* (proper) $\overline{\text{MS}}$ -renormalization $Z_{A,s}$ for $J_{A,s}^\mu$**
(beyond which gauge and subtraction-point dependent differences start to appear)
- Had Larin used this, $Z_{A,s}$ @ α_s^3 would have been published 30 years ago! (along with $Z_{A,ns}$)

$\overline{\text{MS}}$ -subtraction in Lattice Calculation ?!

How can the $\overline{\text{MS}}$ subtraction prescription [Bardeen 78] defined originally with Dimensional Regularization [t Hooft, Veltmann 72] be implemented in Lattice Regularization?

Feasible with the aid of a *pivot*: [Martinelli et al. 95]

Regularization-independent (mass-independent) momentum-subtraction scheme

Schematically, the extraction of the matrix element of the $\overline{\text{MS}}$ -subtracted operator $\overline{\mathcal{O}}$ from Lattice calculation:

$$\begin{aligned} \langle \psi_{\text{RI}} | \overline{\mathcal{O}} | \psi_{\text{RI}} \rangle \Big|_{\text{ampu.}} &= \langle \psi_{\text{RI}} | \overline{Z}_{\mathcal{O}} \hat{\mathcal{O}}_B | \psi_{\text{RI}} \rangle \Big|_{\text{ampu.}} = \frac{\overline{Z}_{\mathcal{O}}}{Z_{\text{RI}}^{\text{RI}}} \langle \psi_{\text{RI}} | Z_{\mathcal{O}}^{\text{RI}} \hat{\mathcal{O}}_B | \psi_{\text{RI}} \rangle \Big|_{\text{ampu.}} \\ &\stackrel{\text{4-dim cont.}}{=} R_{\text{RI}2\overline{\text{MS}}} \langle \psi_{\text{RI}} | Z_{\mathcal{O}}^{\text{RI}} \hat{\mathcal{O}}_B | \psi_{\text{RI}} \rangle \Big|_{\text{ampu.}} \end{aligned}$$

$$\overline{Z}_{\mathcal{O}}^{\text{lat}}(a) \equiv R_{\text{RI}2\overline{\text{MS}}} Z_{\text{RI}}^{\text{lat}}(a) = \frac{\overline{Z}_{\mathcal{O}}(\epsilon)}{Z_{\text{RI}}(\epsilon)} \Big|_{\epsilon \rightarrow 0} Z_{\text{RI}}^{\text{lat}}(a).$$

Both $\langle \psi_{\text{RI}} | Z_{\mathcal{O}}^{\text{RI}} \hat{\mathcal{O}}_B | \psi_{\text{RI}} \rangle$ and $\frac{\overline{Z}_{\mathcal{O}}(\epsilon)}{Z_{\text{RI}}(\epsilon)} \Big|_{\epsilon \rightarrow 0}$ are **independent** of UV-regularization.

$\overline{\text{MS}}$ -subtraction in Lattice Calculation

To compute the targets

$$\langle P, S | [J_A^\mu]_R | P, S \rangle \text{ and } \langle P, S | [K^\mu]_R | P, S \rangle$$

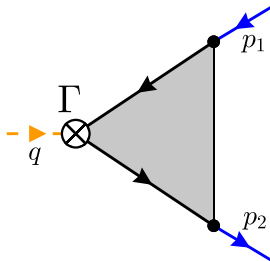
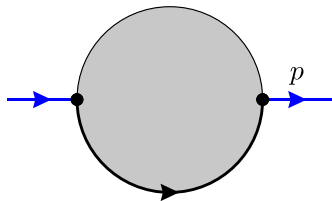
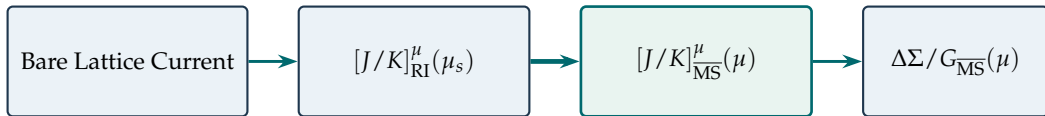
with bare operator matrix elements available, the remaining task is to get $\overline{Z}_O^{lat}(a)$ for J_A^μ and K^μ .

To this end, keep in mind:

- $\overline{Z}_O(\epsilon)$ for J_A^μ and K^μ are well-known to be independent of masses, momenta and QCD gauge-fixing (perturbatively determined to 4-loops) [Breitenlohner, Maison, Stelle 84; Larin etc 91-93; LC, Czakon 21-22; Lüscher, Weisz 21, Pang, Zhang 24]
- $\overline{Z}_O^{lat}(a)$ is **independent** of the particular Z_O^{RI} (and $R_{\text{RI}2\overline{\text{MS}}}$) which may depends on subtraction (momenta) scales and QCD gauge-fixing
- In principle, any intermediate *pivoting* RI would do, provided implementable in lattice QCD with sufficient accuracy
- However, different choice of the intermediate *pivoting* Z_O^{RI} lead to the $R_{\text{RI}2\overline{\text{MS}}}$ with different perturbative convergence behaviors!

Bridge from Lattice RI(S)MOM to $\overline{\text{MS}}$

Finite Matching Factors R_{ij}

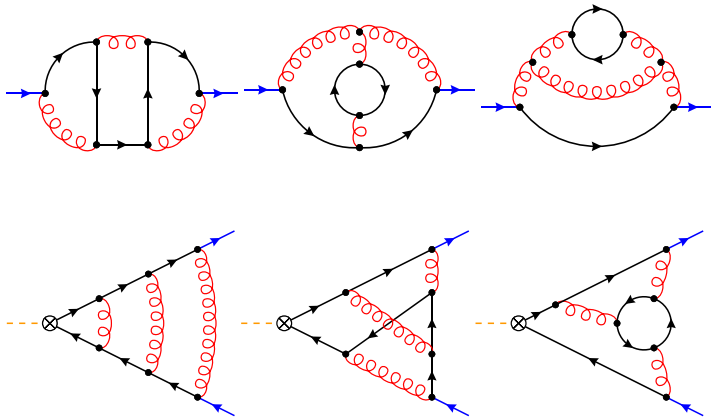


$$Z_O \Gamma_{\alpha\beta}^{\mu} \hat{\mathcal{P}}_{\mu}^{\alpha\beta} \Big|_{p^2=\mu_s^2} = [\Gamma_{\text{tree}}]_{\alpha\beta}^{\mu} \hat{\mathcal{P}}_{\mu}^{\alpha\beta} \Big|_{p^2=\mu_s^2}; \quad \text{RI/MOM} : \gamma^{\mu}; \quad \text{RI'/MOM} : \gamma^{\mu} - \not{p} p^{\mu} / p^2$$

- **RIMOM** [Martinelli et al. 95] : $p^2 = p_1^2 = p_2^2 = -\mu_s^2 < 0$ and $q = 0$
- **RISMOM** [Aoki 08; Sturm et al. 10] : $p^2 = p_1^2 = p_2^2 = q^2 = -\mu_s^2 < 0$ and $q = p_2 - p_1$

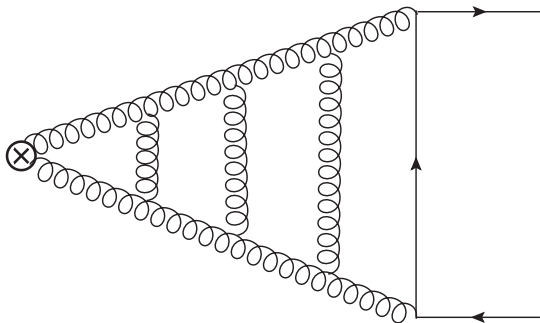
Generate Feynman Diagram Representations

- Feynman diagrams generated using DiaGen
- Lorentz and Dirac algebra done using FORM
- Full ζ -dependence in Green functions kept at off-shell kinematics
- Non-vanishing momentum through the operator in SMOM leads to very complicated loop integrals



Technical Challenge

A representative Feynmann diagram out of 9622 four-loop diagrams



On top of the **multiple triple-gluon vertices**, additional combinatorial inflation:

- Generic ξ -gauge (Landau gauge necessarily): $2^{11} = 2048$
- $\epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon^{\nu_1 \nu_2 \nu_3 \nu_4} \longrightarrow 24$ terms
- Latest FORM 5 with manual tweaks for improving efficiency (thanks to J.Davies)

A Short-cut Formula for R_A

Combining the RI/MOM R.C. $Z_{\mathcal{O}} \Gamma_{\alpha\beta}^{\mu} \hat{\mathcal{P}}_{\mu}^{\alpha\beta} \Big|_{p^2=\mu_s^2} = [\Gamma_{\text{tree}}]_{\alpha\beta}^{\mu}$ with the previously derived

$$\Gamma_{A,s}^{\mu}(p,p) = a_s n_f T_F \Gamma_K^{\mu}(p,p) + \Gamma_{A,ns}^{\mu}(p,p)$$

we obtain the short-cut formula for R_A :

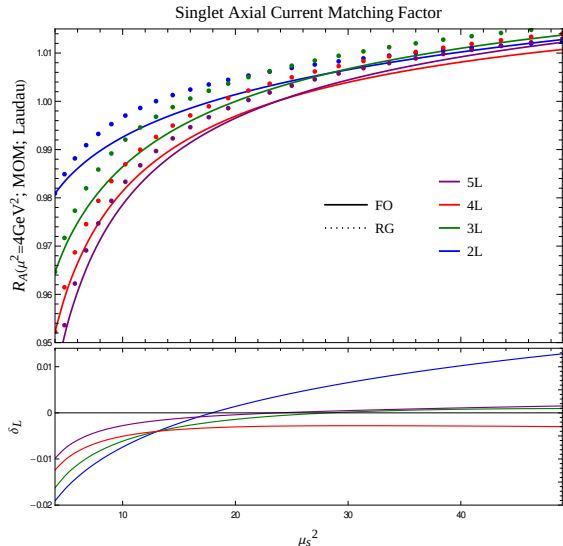
$$R_A \equiv \frac{\overline{Z}_A}{Z_A^{\text{RI}}} \Big|_{\epsilon \rightarrow 0} = 1 + \frac{(a_s n_f T_F Z_q [\Gamma_K]_R)_{\alpha\beta}^{\mu} \hat{\mathcal{P}}_{A,\mu}^{\alpha\beta} \Big|_{p^2=\mu_s^2}}{[\Gamma_A^{\text{tree}}]_{\alpha\beta}^{\mu} \hat{\mathcal{P}}_{A,\mu}^{\alpha\beta} \Big|_{p^2=\mu_s^2}}$$

- Reduced to the matrix element of the **properly $\overline{\text{MS}}$ -renormalized** J_A^{μ} but with RI/MOM-renormalized external quark fields
- Extract the result at $\mathcal{O}(\alpha_s^{N+1})$ based on N -loop computation of the hardest part
- Feasible to extend to other (such as RI'/MOM) but NOT all renormalization schemes

Matching Factor R_A from RI/MOM to $\overline{\text{MS}}$

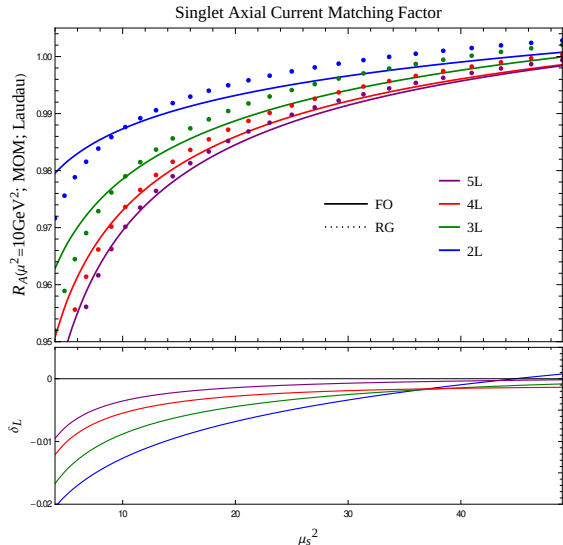
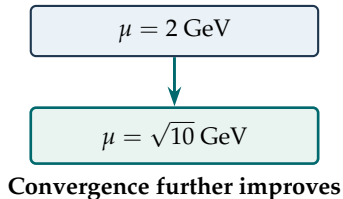
- Perturbative **convergence(!)** through five loops
[LC, Czakon, Davies 26, In preparation]
- Five-loop correction below 1%
- Smaller corrections with increasing μ_s
- Fixed-order and RG approach and agree better at five-loop and more with increasing μ_s

$$R_A(\mu_t/\mu_s) = R_A(\mu_i/\mu_s) \times \exp \left[\int_{\alpha_s(\mu_i)}^{\alpha_s(\mu_t)} \frac{\gamma_s(\alpha_s)}{\beta(\alpha_s)\alpha_s} d\alpha_s \right]$$



Matching Factor R_A from RI/MOM to $\overline{\text{MS}}$

- Perturbative **convergence** through five loops
- Five-loop correction below 1%
- Smaller corrections with increasing μ_s
- Fixed-order and RG approach and agree better at five-loop and more with increasing μ_s and μ



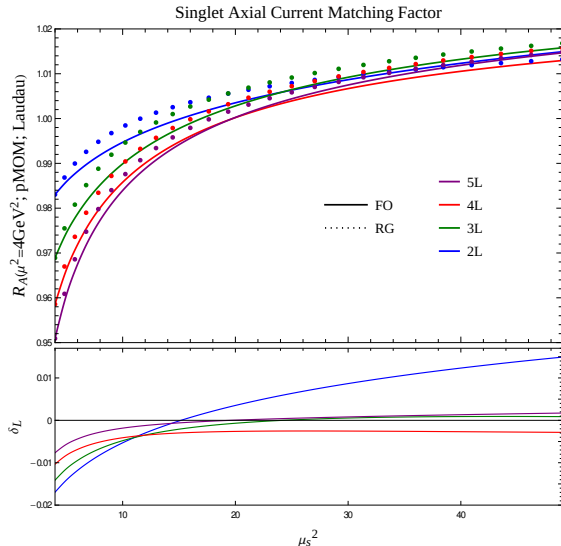
Matching Factor R_A from RI' / MOM to $\overline{\text{MS}}$

- Perturbative **convergence** through five loops
- Five-loop correction at few per-milles
- Smaller corrections with increasing μ_s
- Fixed-order and RG approach and agree better at five-loop and more with increasing μ_s and μ

Better convergence in RI' / MOM v.s. RI / MOM

Transverse structure projector

$$\mathcal{P}_{V,\perp}^\mu = \gamma_{\alpha\beta}^\mu - \not{p}_\alpha \not{p}_\beta / p^2$$

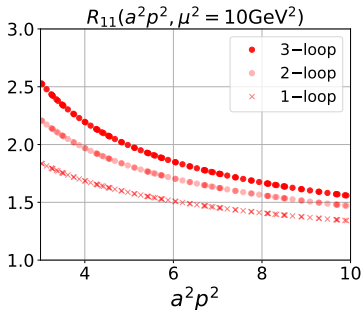
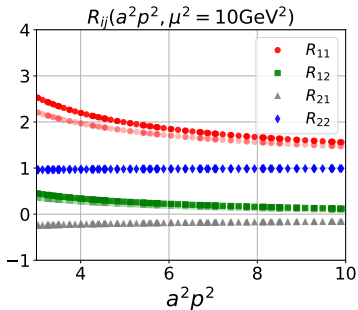


J/K Matching Matrix R_{ij} from RI/MOM to $\overline{\text{MS}}$

- The matching matrix employed in a state-of-the-art Lattice determination of $\Delta G(\mu)$ contribution to Proton spin [CLQCD+LPC-D.J.Zhao etc 25] (2512.24315)

$$\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \begin{pmatrix} 1 + 2.1685 \alpha_s + 5.8066 \alpha_s^2 + 19.003 \alpha_s^3 & 0.6366 \alpha_s + 1.9635 \alpha_s^2 + 5.5974 \alpha_s^3 \\ -0.4775 \alpha_s - 1.0354 \alpha_s^2 - 2.7725 \alpha_s^3 & 1 - 0.3040 \alpha_s^2 - 0.9375 \alpha_s^3 \end{pmatrix}$$

($R_{22} \neq R_A$ due to taking lightcone component when implement RIMOM here)



- Further improvement to be incorporated:** leading logarithmic scale dependence in R_{11} in large β_0 limit can be re-summed into $1/(1 - (\alpha_s \beta_0 (5/3 - \ln(p^2/\mu^2))))$.

J/K Matching Matrix R_{ij} from RI/SMOM to $\overline{\text{MS}}$

What about taking RI/SMOM as the pivot to extract J^μ/K^μ $\overline{\text{MS}}$ -R.C. in Lattice QCD?

- The first precision results:

$$\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \begin{pmatrix} \boxed{1 + 1.2881 \alpha_s + 1.6875 \alpha_s^2 + 2.5876 \alpha_s^3} & 0.3514 \alpha_s + 0.7655 \alpha_s^2 + 1.3999 \alpha_s^3 \\ -0.47747 \alpha_s - 0.6150 \alpha_s^2 - 0.8057 \alpha_s^3 & 1 - 0.1678 \alpha_s^2 - 0.36551 \alpha_s^3 \end{pmatrix}$$

With $\alpha_s(\mu = 3\text{GeV}) \approx 0.25$ the missing $\mathcal{O}(\alpha_s^4)$ correction is estimated to $\mathcal{O}(1\%)$ (based on a geometric power series).

- A significantly improved apparent perturbative convergence over RI/MOM-counterpart up to $\mathcal{O}(\alpha_s^3)$ (to compare with $1 + 2.1685 \alpha_s + 5.8066 \alpha_s^2 + 19.003 \alpha_s^3$)
- The newly obtained 4-loop result for $R_A = R_{22}$ for RI/SMOM:

$$1 - 0.1678 \alpha_s^2 - 0.3655 \alpha_s^3 - 0.6684 \alpha_s^4$$

The $\mathcal{O}(\alpha_s^4)$ correction is reduced to 0.3% at benchmark point $\mu^2 = p^2 = 3^2\text{GeV}^2$.

[LC, Davies 26, In preparation]

Summary and Outlook

Flavor-singlet axial current J_A^μ and topological-charge current K^μ can play a crucial role in determining the $\overline{\text{MS}}$ quark/gluon-helicity contribution to the total helicity of the nucleon.

- ✓ An **efficient variant** of Larin's renormalization condition for J_A^μ in Dimensional Regularization
(renders $Z_{A,s} = Z_{A,s}^f \bar{Z}_{A,s}$ as straightforward as for the non-singlet counterpart)
- ✓ State-of-the-art result for the matching coefficient R_A (for RI/MOM) at $\mathcal{O}(\alpha_s^5)$
 - ▶ Apparent perturbative convergence through 5 loops explicitly **demonstrated**, with the pure 5-loop correction generally less than 1% and better in RI'/MOM
- ✓ The **first** result for the matching matrix R_{ij} for RI/SMOM through $\mathcal{O}(\alpha_s^3)$ (and beyond), exhibiting surprisingly good apparent convergence!
 - ▶ The perturbative truncation error is $\mathcal{O}(1\%)$ for R_{11} and $\mathcal{O}(0.1\%)$ for R_{22} .
- Practical application to improve the accuracy of the Lattice determination of

$$\frac{1}{2}\Delta\Sigma(\mu) = \frac{1}{2} (g_A^u + g_A^d + g_A^s) \text{ and } \Delta G(\mu)$$

Thank you for listening!