Induced Gravitational Waves: From gauge issue to PBH DM

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Institute of Theoretical Physics, Chinese Academy of Sciences

International Conference on Foundamental Physics

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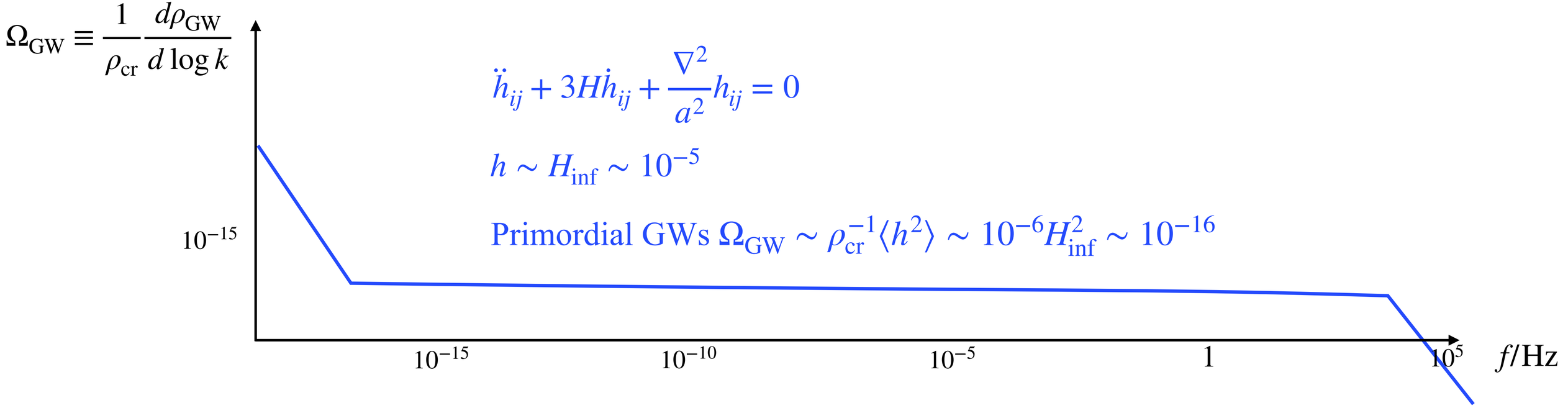
CONTENT

- Introduction: IGW and PBH
- Gauge Issue of IGW
- PBH as dark matter
- Non-Gaussianity

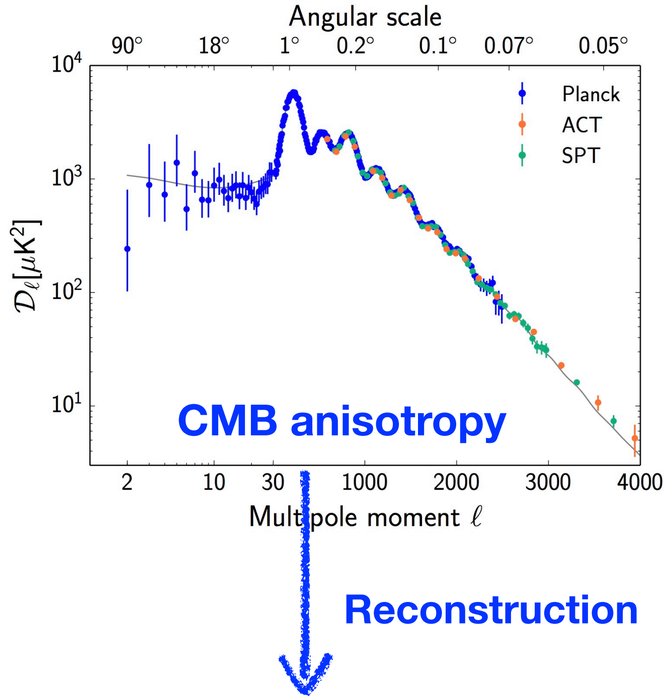
Primordial GW

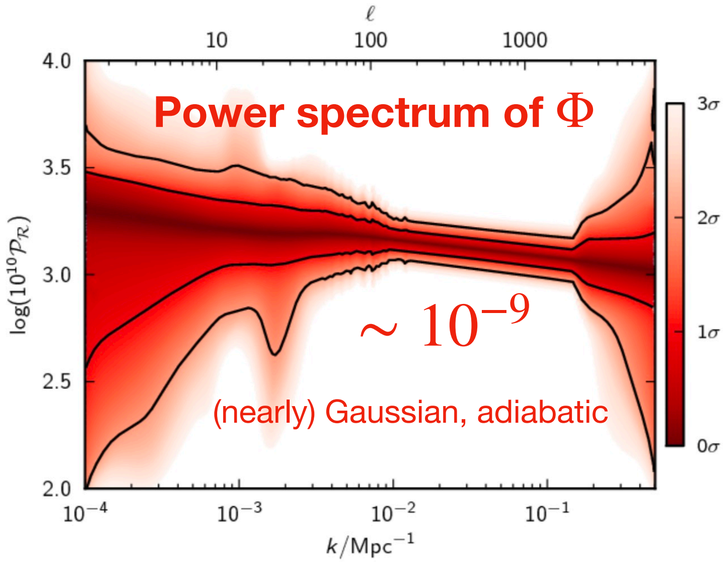
Perturbed metric:

$$ds^2 = a^2(\eta)\left[-(1+2\Psi)d\eta^2 + \left((1+2\Phi)\delta_{ij} + h_{ij}\right)dx^i dx^j\right]$$



Starobinsky 1979

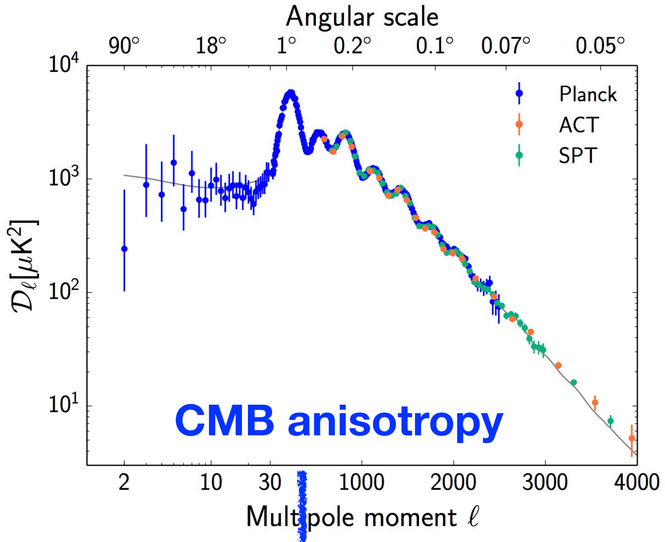




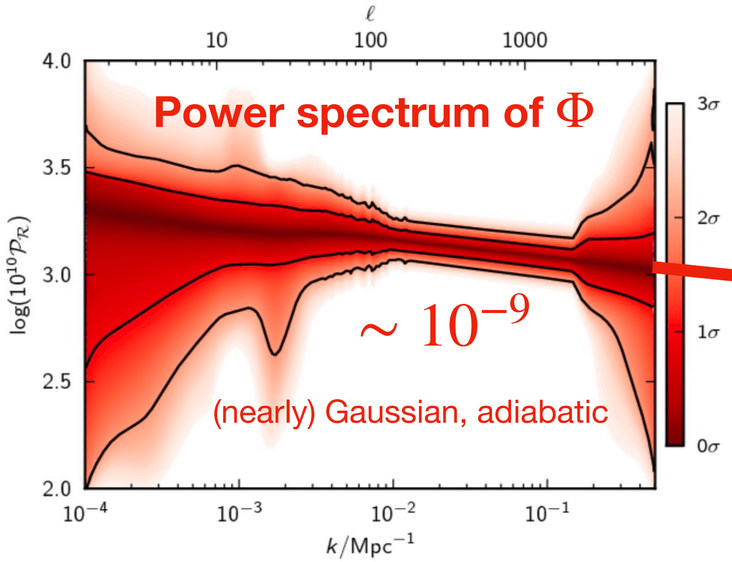
$$\ddot{h}_{ij} + 3H\dot{h}_{ij} + \frac{\nabla^2}{a^2}h_{ij} = \Lambda_{ij}^{lm}\left(2\Phi_{,l}\Phi_{,m} + \frac{\Phi_{,l}\dot{\Phi}_{,m}}{H} + \cdots\right)$$

$$h_{ij} \sim \Phi^2 \sim 10^{-9}$$

Secondary GWs $\langle h_{ij}^2 \rangle \sim 10^{-6}\Phi^4 \sim 10^{-24}$

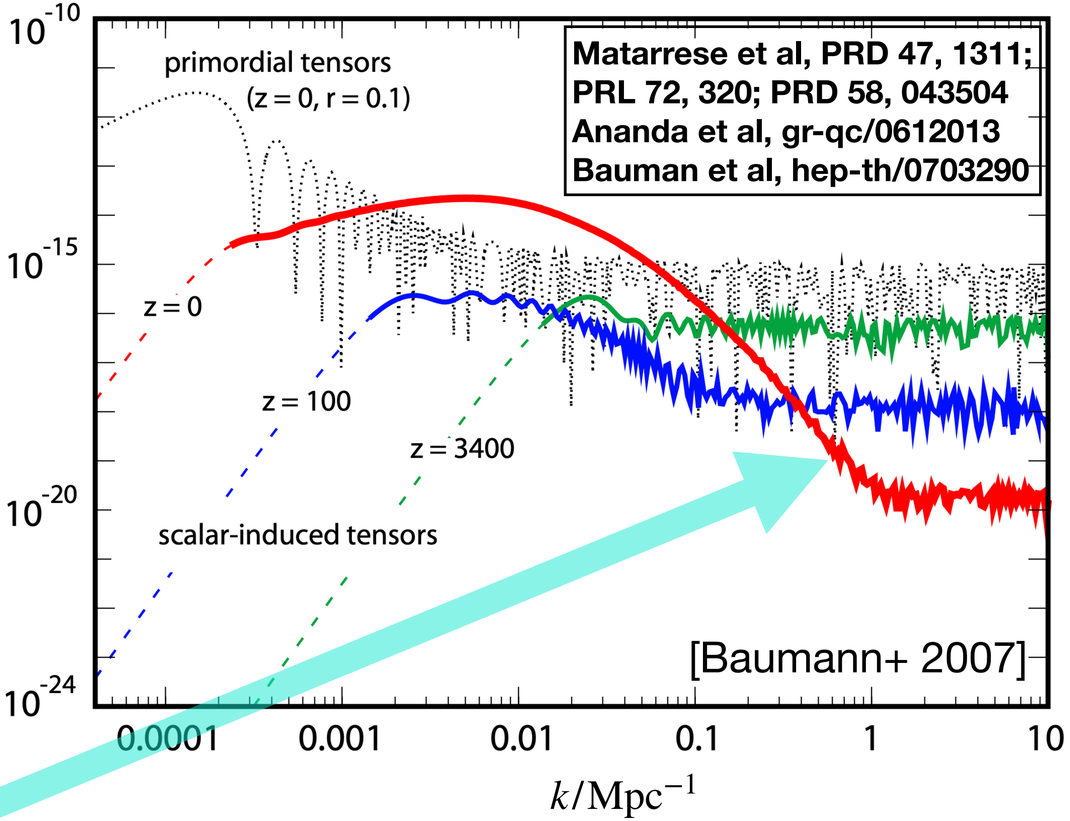


Reconstruction



$h\Phi\Phi$ coupling

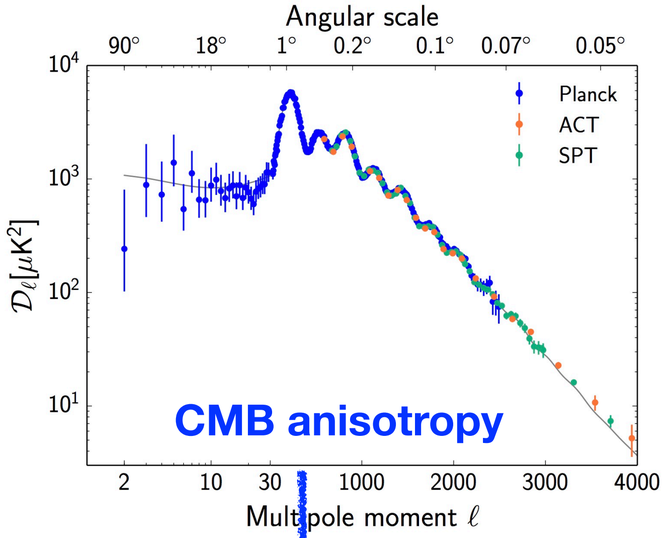
$$\Omega_{\rm GW} \sim 10^{-6}\mathcal{P}_\mathcal{R}^2 \sim 10^{-24}$$



$$\ddot{h}_{ij} + 3H\dot{h}_{ij} + \frac{\nabla^2}{a^2}h_{ij} = \Lambda_{ij}^{lm}\left(2\Phi_{,l}\Phi_{,m} + \frac{\Phi_{,l}\dot{\Phi}_{,m}}{H} + \cdots\right)$$

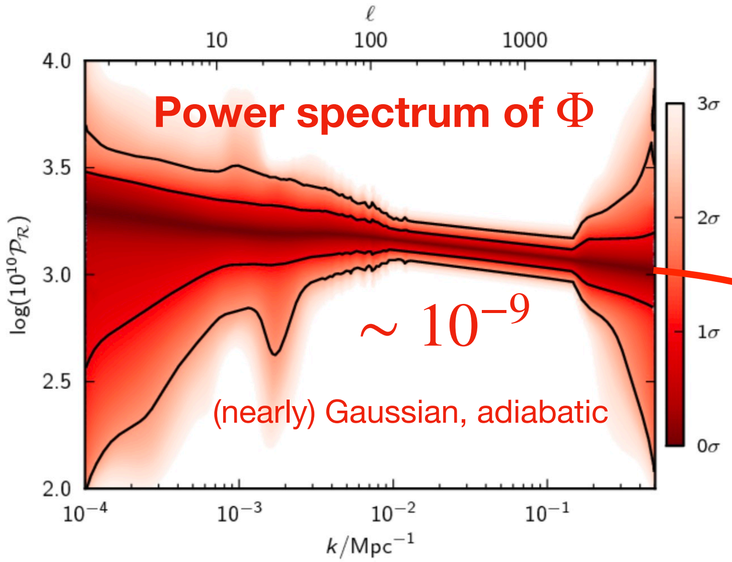
$$h_{ij} \sim \Phi^2 \sim 10^{-9}$$

Secondary GWs $\langle h_{ij}^2 \rangle \sim 10^{-6}\Phi^4 \sim 10^{-24}$



CMB anisotropy

Reconstruction



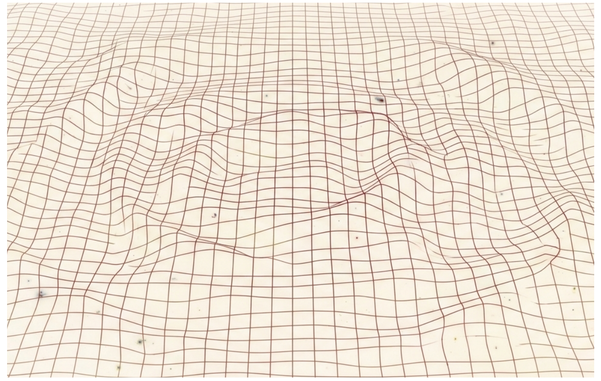
$\sim 10^{-9}$

(nearly) Gaussian, adiabatic

$\sim 10^{-1}$

Gaussian?
adiabatic?

nonlinear perturbation



$\langle h_{ij}^2 \rangle \sim 10^{-6}\Phi^4 \sim 10^{-10}$

Scalar Perturbation Induced GW

Matarrese et al, PRD 47, 1311; PRL 72, 320; PRD 58, 043504
Ananda et al, gr-qc/0612013
Bauman et al, hep-th/0703290

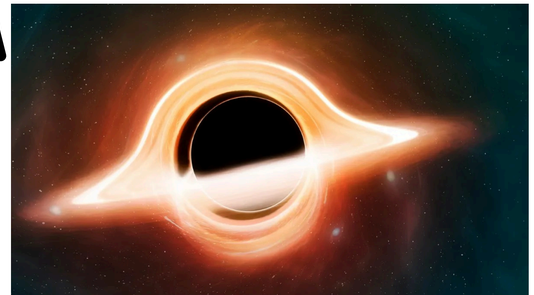
crosscheck

Saito, Yokoyama, 0812.4339
Cai, SP, Sasaki, 1810.11000

gravitational collapse

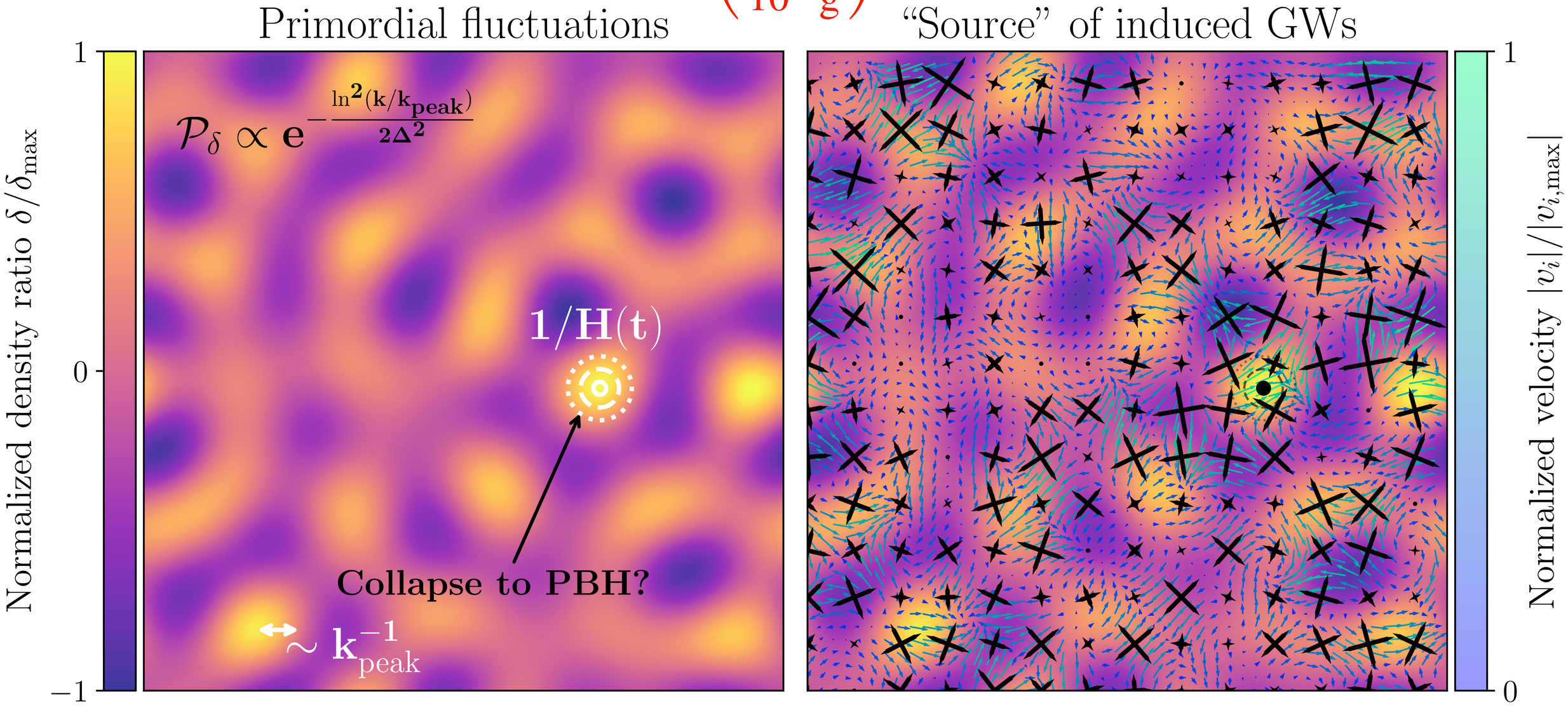
PDF tail $\longrightarrow$ PBH $\sim$ DM

Primordial Black Hole



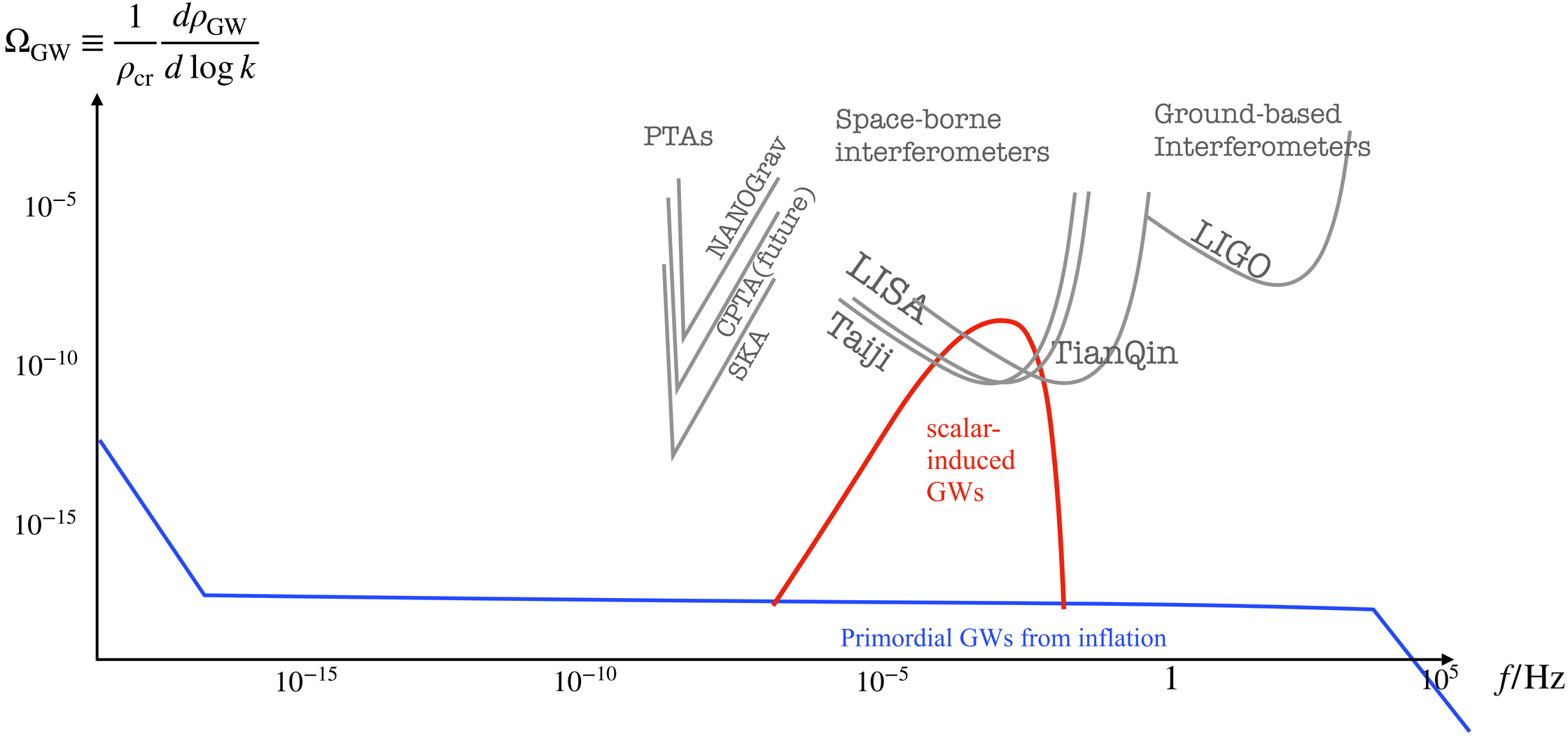
Zeldovich & Novikov 1966
Hawking 1971
Carr & Hawking 1974

$$f_{\rm IGW} \sim 0.003{\rm Hz} \left(\frac{M_{\rm PBH}}{10^{22}{\rm g}}\right)^{-\frac{1}{2}}$$



Domenech, 2402.17388, PBH book, Ch. 17

Induced GW


$\Omega_{\rm GW} \equiv \frac{1}{\rho_{\rm cr}} \frac{d\rho_{\rm GW}}{d \log k}$
PTAs
NANOGrav
CPTA(future)
SKA
Space-borne interferometers
LISA
Taiji
TianQin
Ground-based Interferometers
LIGO
scalar-induced GWs
Primordial GWs from inflation
10^{-5}
10^{-10}
10^{-15}
10^{-15}
10^{-10}
10^{-5}
1
10^{5}
f/Hz

Gauge Issue

Gauge Issue of IGW

Perturbed metric

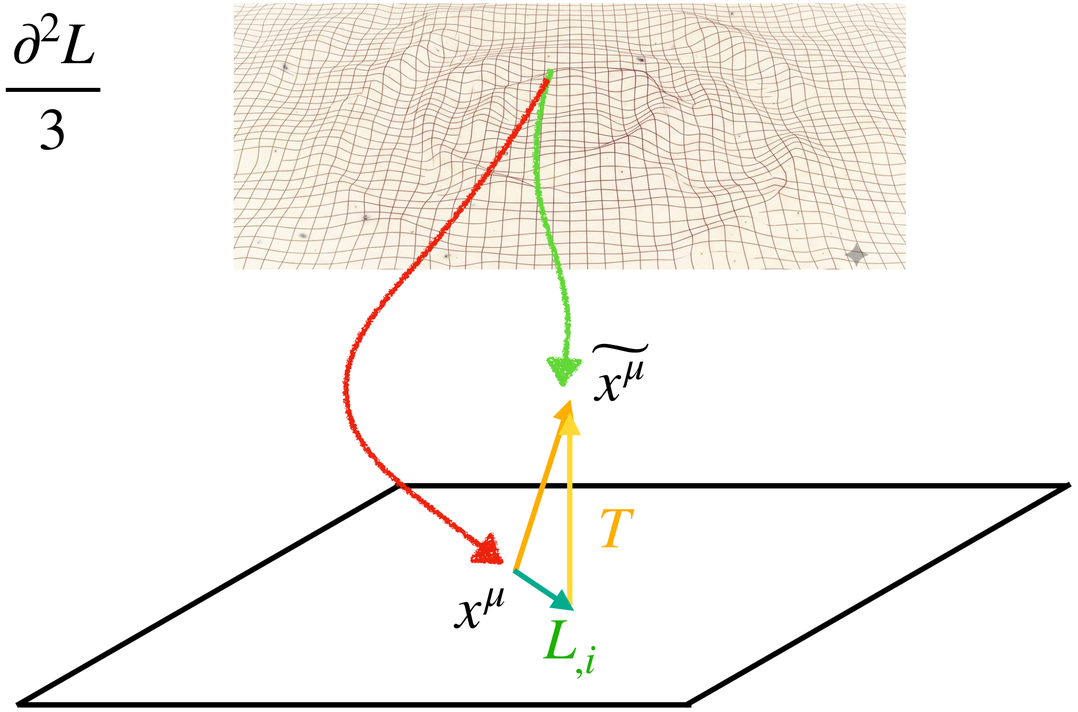
$$\mathrm{d}s^2 = a^2\left(-\left(1+2A\right)\mathrm{d}\eta^2 + 2B_{,i}\mathrm{d}\eta\mathrm{d}x + \left(\left(1+2\left(H_L - \frac{1}{3}\partial^2 H_T\right)\delta_{ij} + 2H_{T,ij} + h_{ij}^{TT}\right)\mathrm{d}x^i\mathrm{d}x^j\right)\right)$$

Gauge (coordinate) transformation in GR: $x^\mu \longrightarrow \tilde{x}^\mu = x^\mu + \xi^\mu$ $\quad [\,\xi^\mu = (T,\ \partial_i L)\,]$

$$A \longrightarrow \tilde{A} = A - \frac{1}{a}\left(aT\right)' \qquad H_L \longrightarrow \widetilde{H}_L = H_L - \frac{a'}{a}T - \frac{\partial^2 L}{3}$$

$$H_T \longrightarrow \widetilde{H}_T = H_T - L \qquad B \longrightarrow \widetilde{B} = B + T - L'$$

$$h_{ij}^{TT} \longrightarrow \widetilde{h}_{ij}^{TT} = h_{ij}^{TT} + \mathcal{O}(\xi^2)$$



Gauge Issue of IGW

Perturbed metric

$$\mathrm{d}s^2 = a^2\left(-\left(1+2A\right)\mathrm{d}\eta^2 + 2B_{,i}\mathrm{d}\eta\mathrm{d}x + \left(\left(1+2\left(H_L - \frac{1}{3}\partial^2 H_T\right)\delta_{ij} + 2H_{T,ij} + h_{ij}^{TT}\right)\mathrm{d}x^i\mathrm{d}x^j\right)\right)$$

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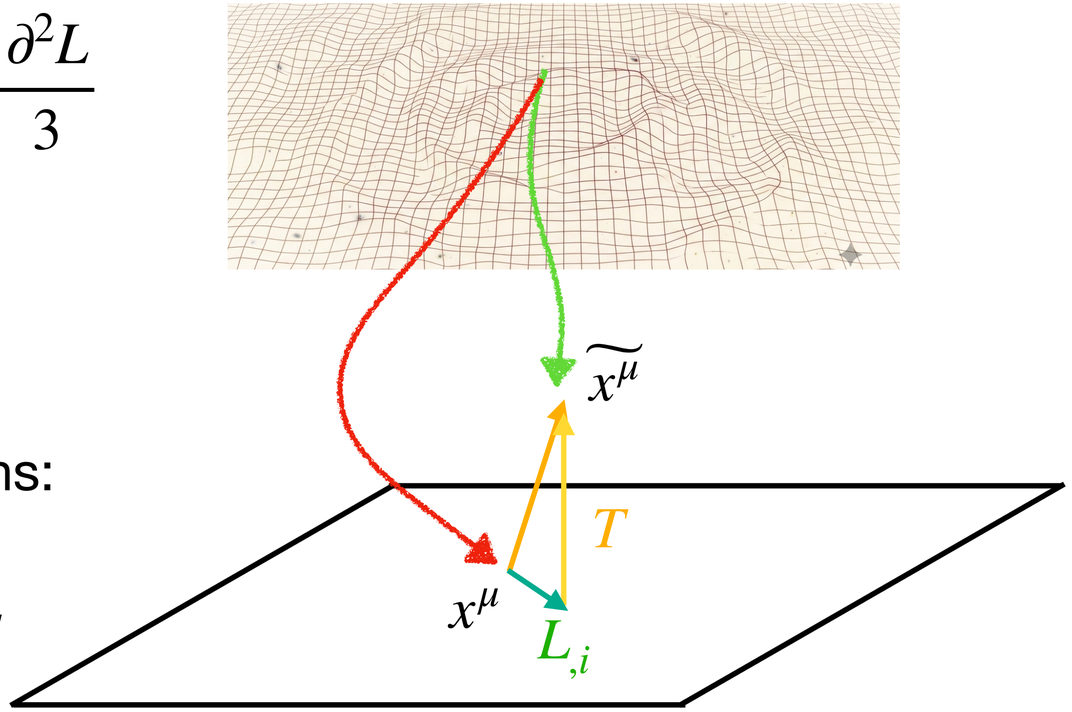
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$$h_{ij}^{TT} \longrightarrow \widetilde{h}_{ij}^{TT} = h_{ij}^{TT} + \mathcal{O}(\xi^2)$$

Write down the infinitesimal shift/translation by perturbations:

$$T = (H_T' - B) - \left(\widetilde{H}_T' - \widetilde{B}\right) \equiv \sigma - \tilde{\sigma} \qquad L = H_T - \widetilde{H}_T$$



Gauge Issue of IGW

- The T and L in the other scalars (A and H_L) can be canceled by σ and H_T, to get

$$\Psi \equiv A + \frac{1}{a}\left(a\sigma\right)' \qquad \Phi \equiv H_L - \frac{1}{3}\partial^2 H_T + \frac{a'}{a}\sigma$$

$$\boxed{\sigma \equiv H_T' - B}$$

- Bardeen potentials. By choosing T such that $\sigma = 0$, we get zero-shear slices. Further choose L such that $H_T = 0$, we get the longitudinal gauge (a.k.a. Newton gauge)

$$ds^2 = a^2(\eta)\left[-(1+2\Psi)d\eta^2 + \left((1+2\Phi)\delta_{ij} + h_{ij}^{TT}\right)dx^i dx^j\right]$$

- The h_{ij}^{TT} above is gauge invariant at linear order. However, it goes at second order as

$$h_{ij}^{TT} \longrightarrow \widetilde{h}_{ij}^{TT} = h_{ij}^{TT} - \Lambda_{ij}{}^{ab}\left(\left(T-2\sigma\right)_{,a}T_{,b} + L_{,ak}\left(L-2H_T\right)_{,bk}\right)$$

Gauge Issue of IGW

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Gauge Issue of IGW

- Similar to the Bardeen potential, we cancel T and L by σ and H_T to get the gauge invariant quantity

$$h_{ij}^{TT} \longrightarrow \widetilde{h}_{ij}^{TT} = h_{ij}^{TT} + \Lambda_{ij}{}^{ab}\Big(\big(\sigma + \widetilde{\sigma}\big)_{,a}\big(\sigma - \widetilde{\sigma}\big)_{,b} + \big(H_T + \widetilde{H}_T\big)_{,ak}\big(H_T - \widetilde{H}_T\big)_{,bk}\Big)$$

$$\widetilde{h}_{ij} + \Lambda_{ij}{}^{ab}\left(\widetilde{\sigma}_{,a}\widetilde{\sigma}_{,b} + \widetilde{H}_{T,ak}\widetilde{H}_{T,bk}\right) = h_{ij} + \Lambda_{ij}{}^{ab}\left(\sigma_{,a}\sigma_{,b} + H_{T,ak}H_{T,bk}\right) \equiv h_{ij}^{N(2)}$$

- This is a gauge-invariant TT tensor, which is invariant at second order. In Newton gauge we have

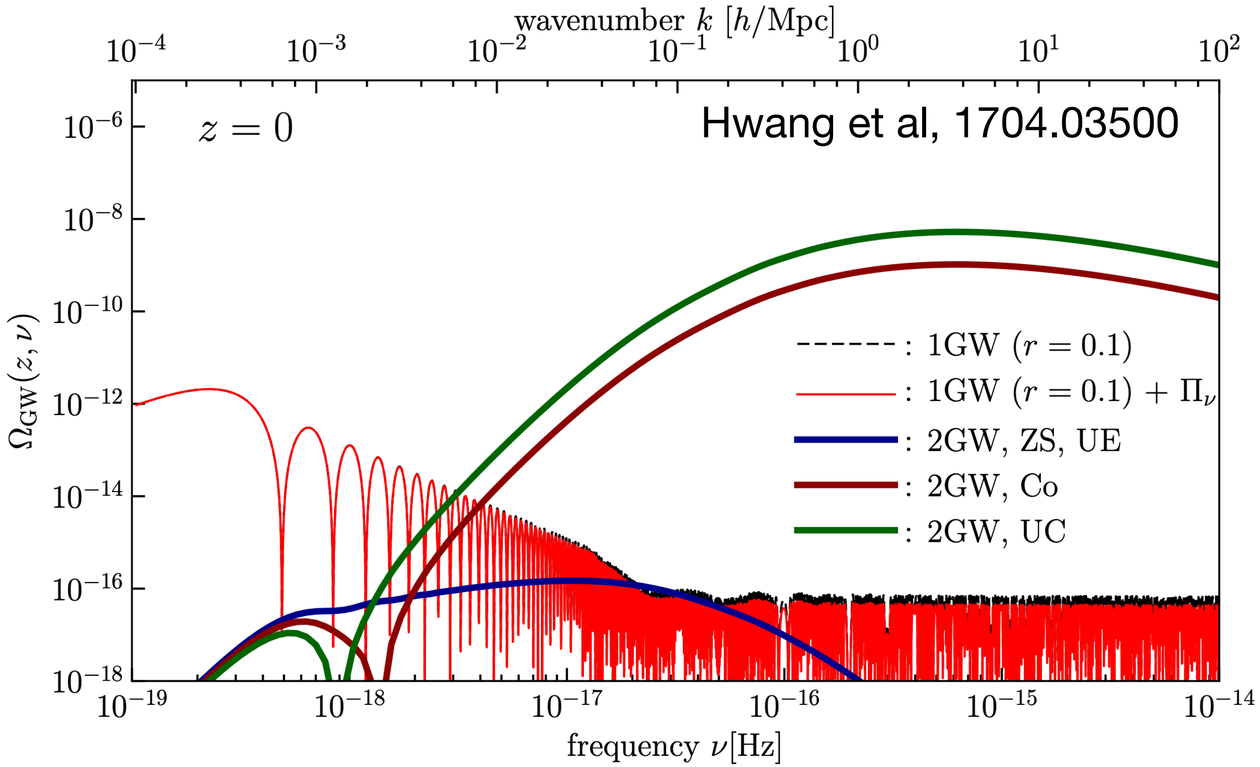
$$ds^2 = a^2(\eta)\Big[-(1+2\Psi)d\eta^2 + \big((1+2\Phi)\delta_{ij} + h_{ij}^{N(2)}\big)dx^i dx^j\Big]$$

- This is usually used to calculate the scalar-induced GW:

$$h_{ij}^{N(2)\prime\prime} + 2\mathscr{H}h_{ij}^{N(2)\prime} - \nabla^2 h_{ij}^{N(2)} = 2\Lambda_{ij}^{lm}\left[2\Psi_{,l}\Psi_{,m} + \left(\Psi + \frac{\Psi'}{\mathscr{H}}\right)_{,l}\left(\Psi + \frac{\Psi'}{\mathscr{H}}\right)_{,m}\right]$$

Gauge Issue of IGW

- The issue is we have infinite ways of constructing such “gauge-invariant metric tensor” in different gauges.
- If we have to calculate in the Newton gauge, where does this priority come from?
- $h_{ij}^{(2)}$ gauge issue: Hwang et al 1704.03500
- 2nd perturbation theory: Domenech, Sasaki, 1709.09804; Wang, Zhang, 1905.03272
- Gauge category: De Luca et al, 1911.09689; Domenech, Sasaki, 2012.14016; Xue, Cao, 2509.26159; ….



Gauge Issue of IGW

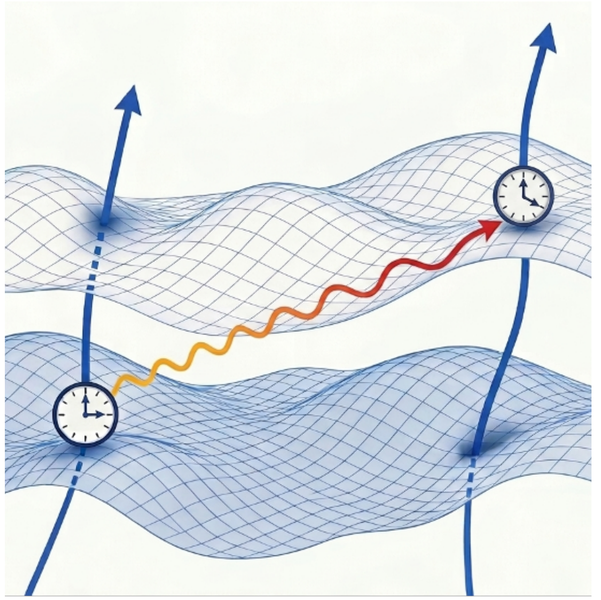
- Q: What do the (space-borne) interferometer and PTA measure?
- A: The change of proper time of a geodesic observer (Fermi observer) when a GW is passing by
- Calculate the change of proper time

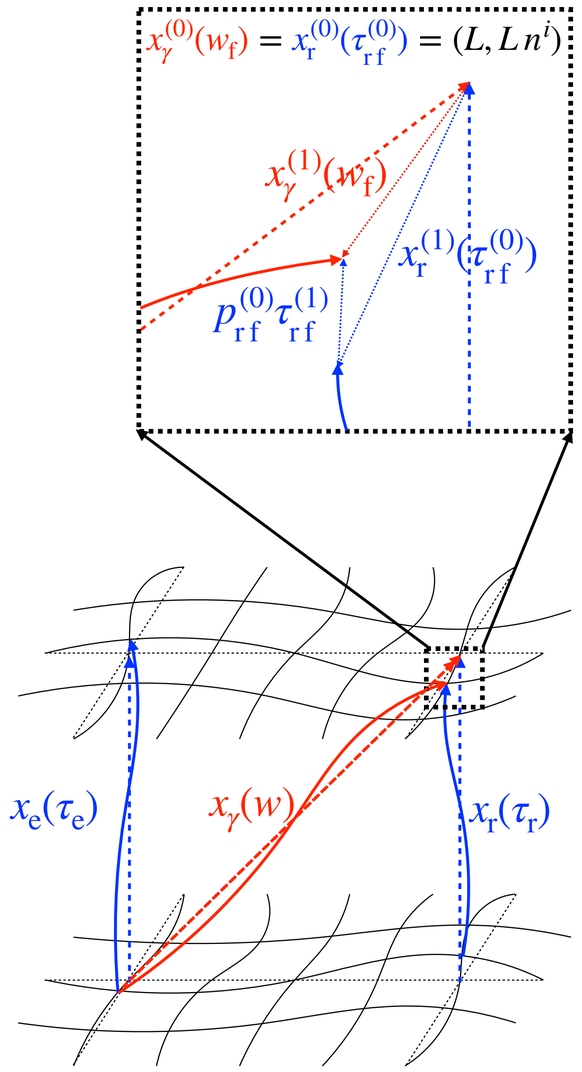
$$\tau^{(1)} = a(\tau)\int_0^L \mathrm{d}\lambda \left(\frac{1}{2}n^i n^j h^{(1)}_{ij} - \Phi^{(1)} - \Psi^{(1)}\right) + \cdots$$

time dilation due to GW

Shapiro delay

detector-motion related





Domenech, SP, Wang, 2512.15704
PRL 136 (2026), 221402

Gauge Issue of IGW

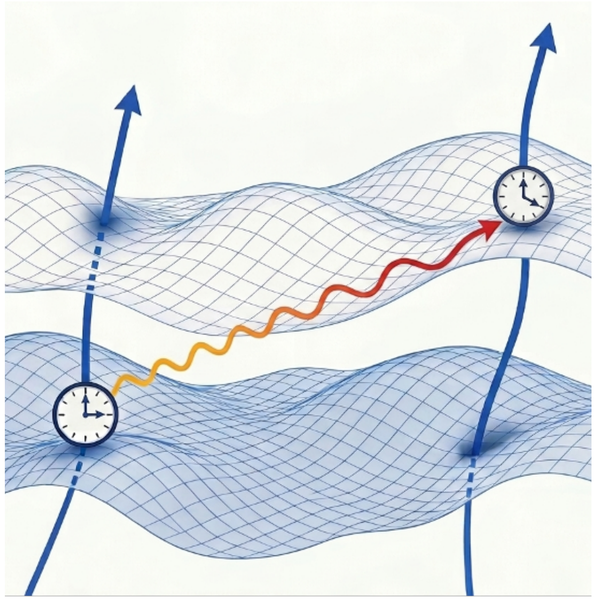
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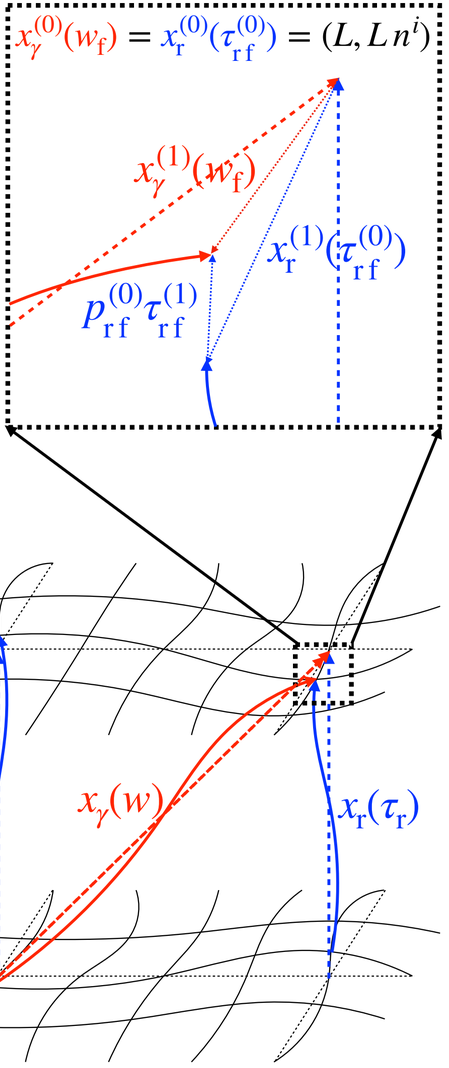
$$\tau^{(2)} \supset a_{\rm rf} n^i n^j \left(\frac{1}{2} \int_0^L {\rm d}\lambda \, h_{ij}^{N(2)} + V_{{\rm rf},i}^{(1)} \int_0^L {\rm d}\eta V_{{\rm r},j} + \cdots \right)$$

time dilation due to GW

detector-motion related

$$h_{ij}^{N(2)} \equiv h_{ij}^{(2)} + \Lambda_{ij}{}^{lk} \left(\sigma_l^{(1)} \sigma_k^{(1)} + E_{,lm}^{(1)} E_{,mk}^{(1)} \right)$$

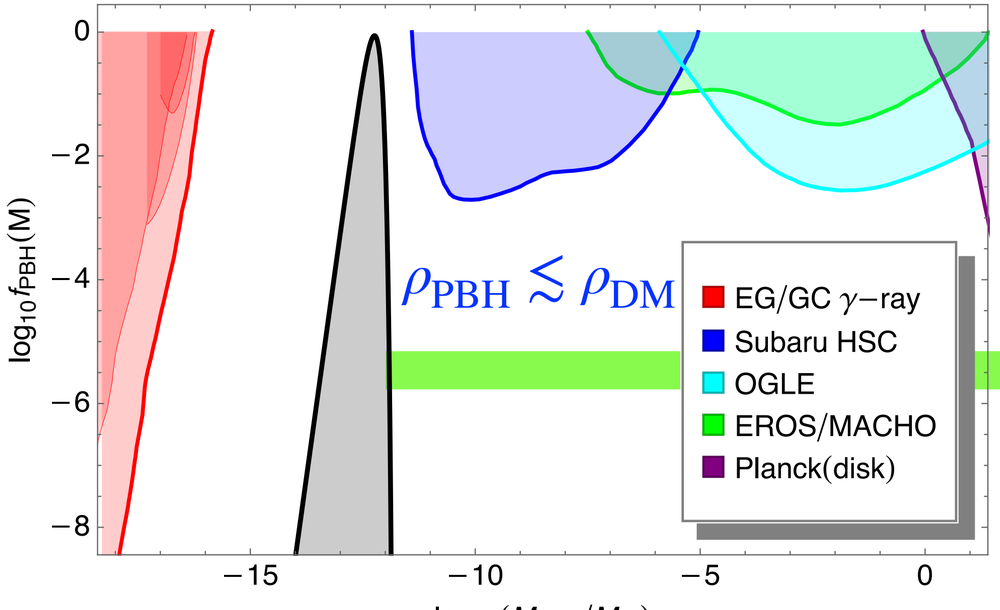




Domenech, SP, Wang, 2512.15704
PRL 136 (2026), 221402

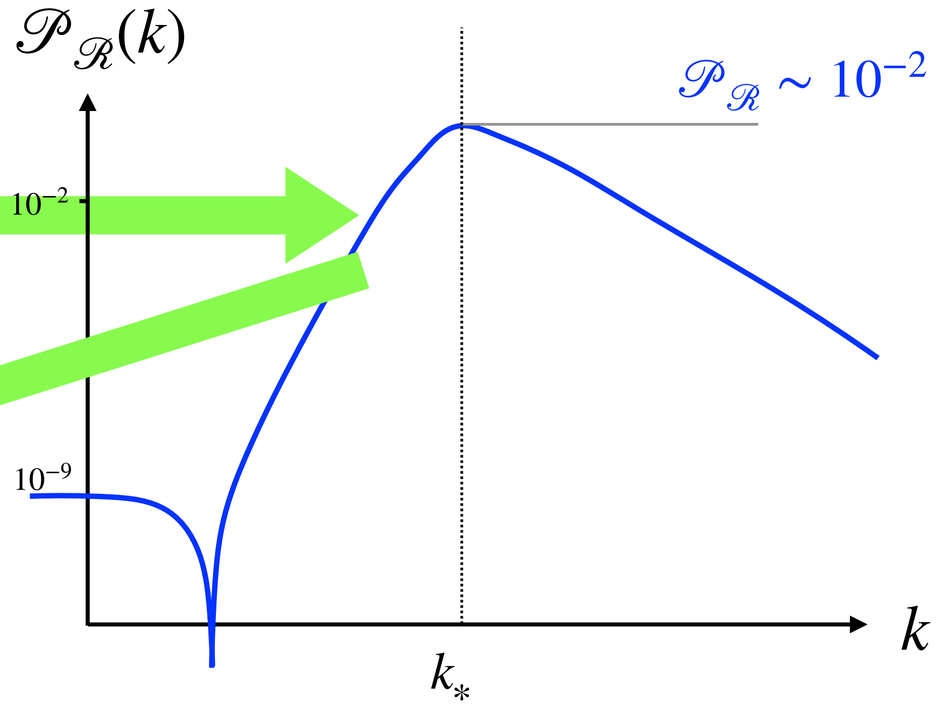
PBH Dark Matter

PBH-IGW crosscheck

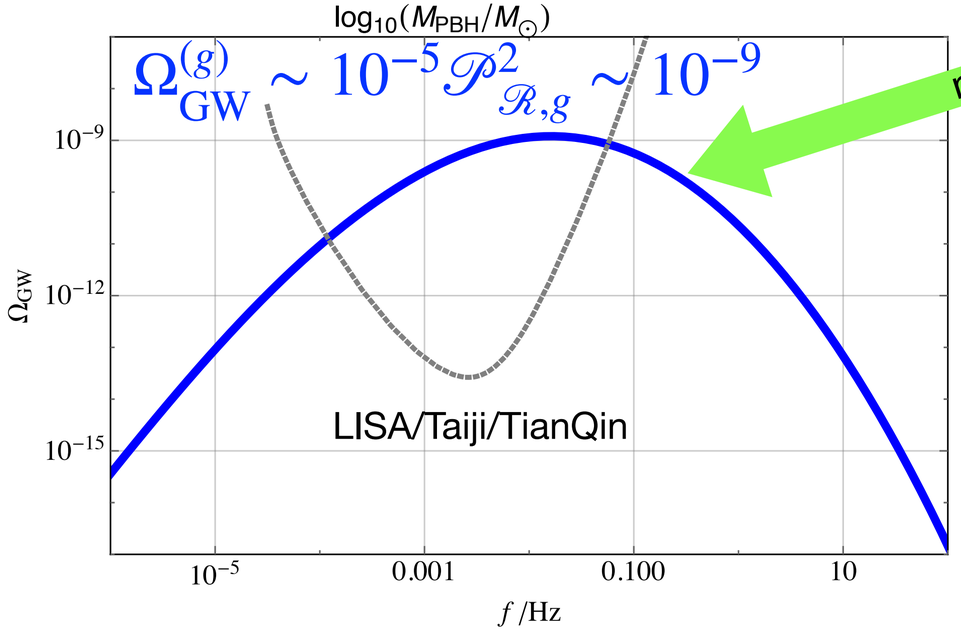


$$f_{\rm IGW} \sim 3{\rm Hz}\left(\frac{M_{\rm PBH}}{10^{16}{\rm g}}\right)^{-\frac{1}{2}}$$

gravitational collapse

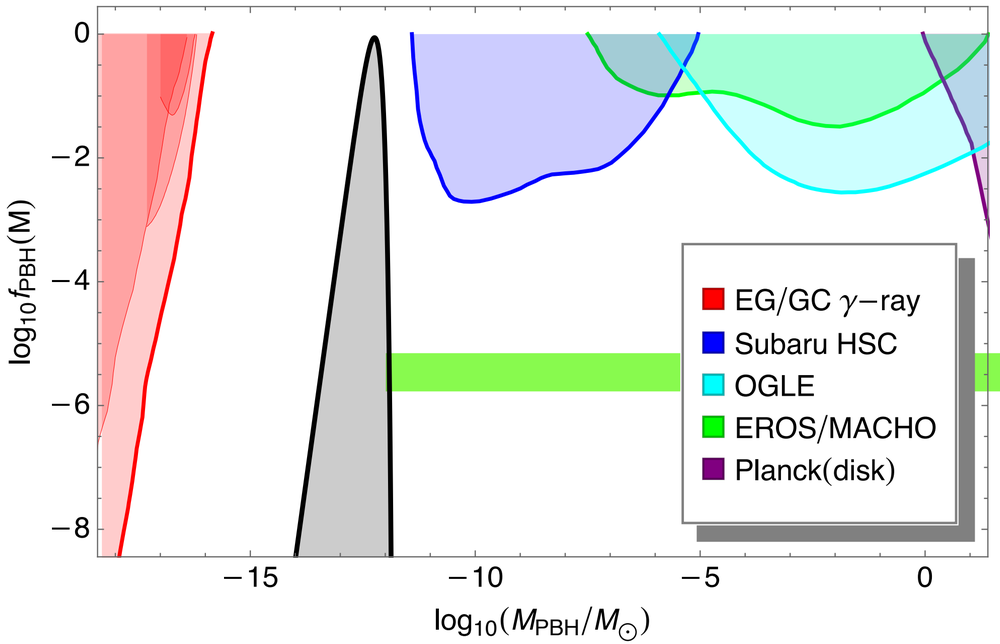


nonlinear coupling $h\mathscr{R}\mathscr{R}$



PBH-IGW Crosscheck
Saito, Yokoyama, 0812.4339

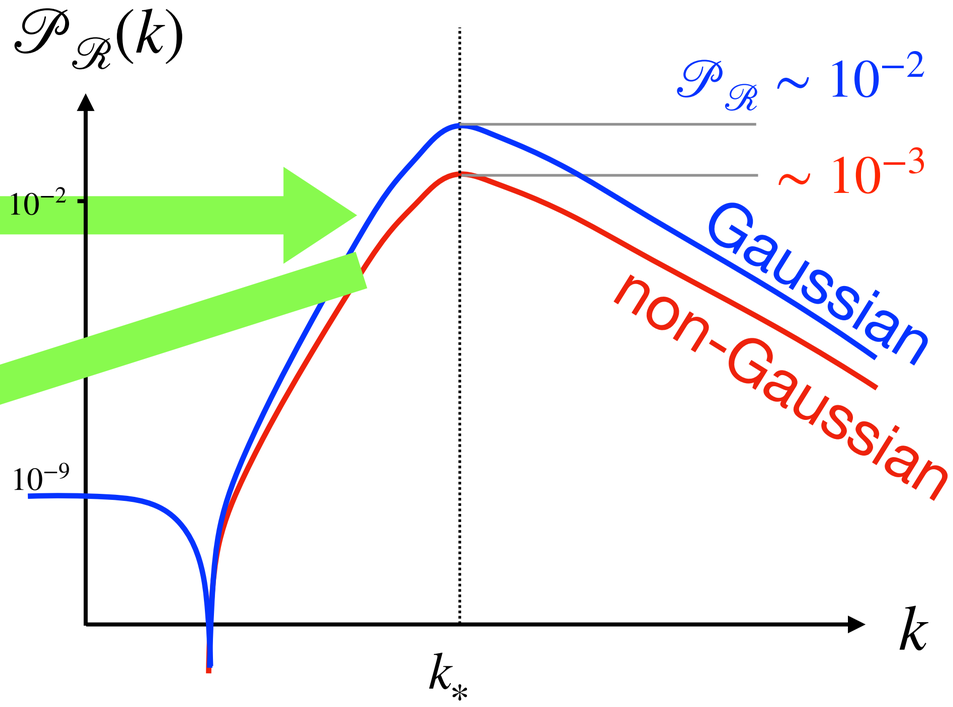
Including non-Gaussianity



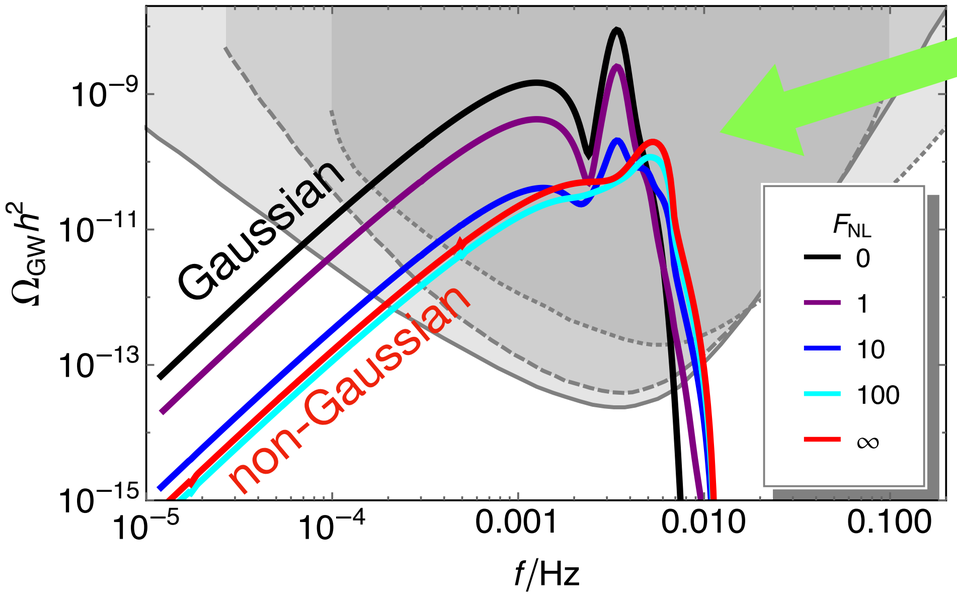
gravitational collapse

$$f_{\rm IGW} \sim 3{\rm Hz}\left(\frac{M_{\rm PBH}}{10^{16}{\rm g}}\right)^{-\frac{1}{2}}$$

$$\mathscr{R} = \mathscr{R}_g + F_{\rm NL}\left(\mathscr{R}_g^2 - \langle \mathscr{R}_g^2 \rangle\right)$$



nonlinear coupling $h\mathscr{R}\mathscr{R}$ with $F_{\rm NL} > 0$



Probing PBH dark matter by induced GW
(even when there is non-Gaussianity)
Cai, SP, Sasaki, 1810.11000, PRL 122, 201101

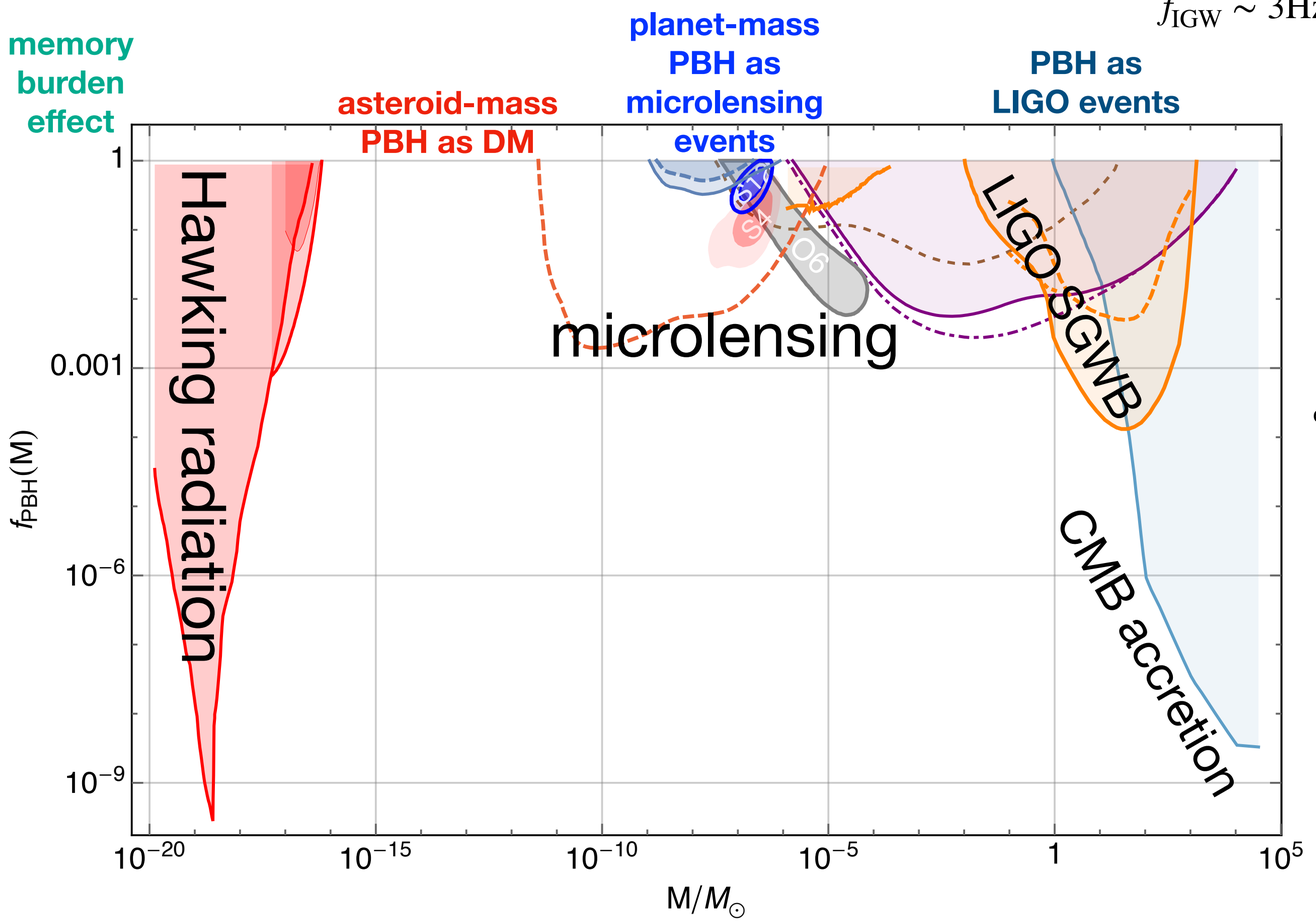
PBH-IGW crosscheck

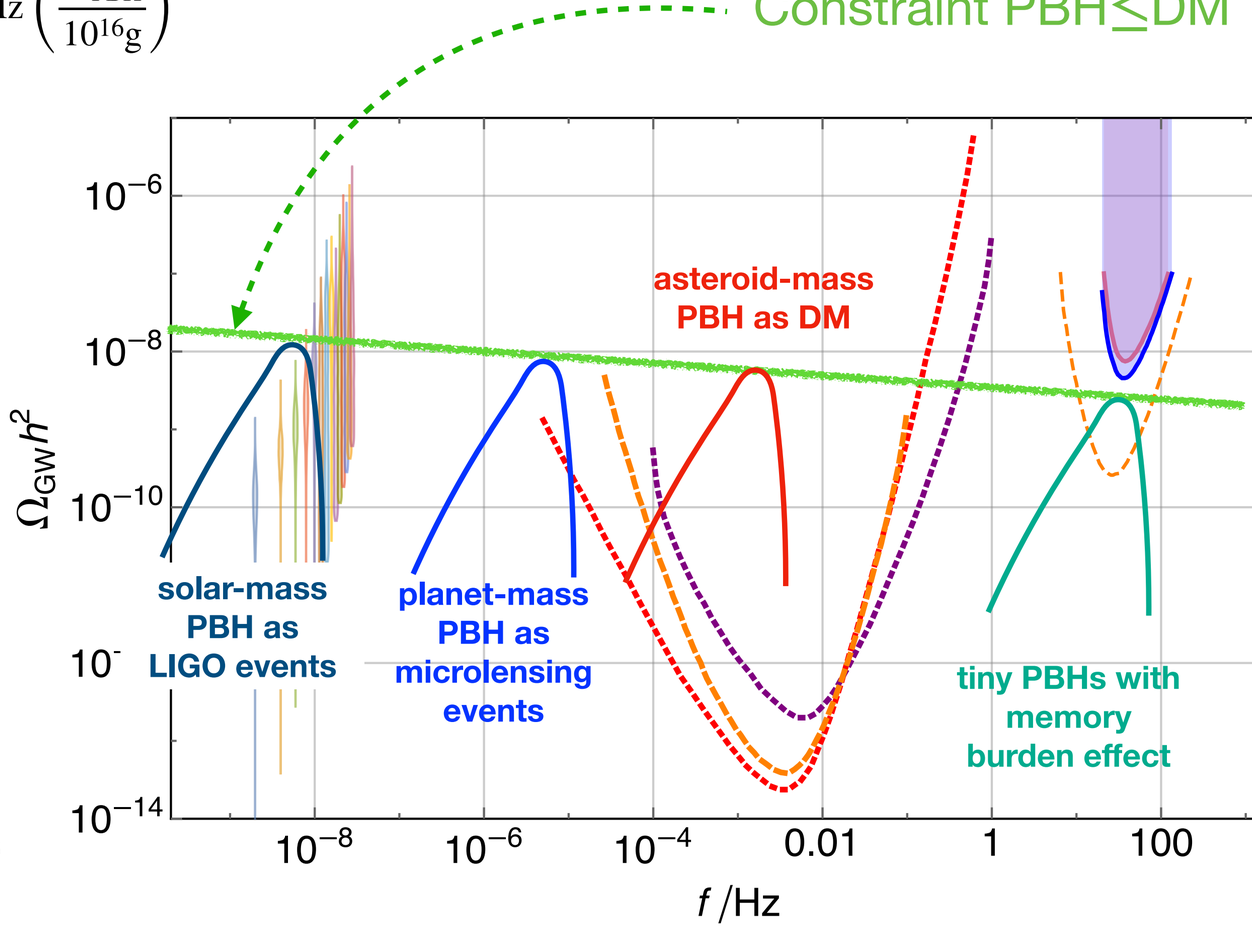
PBH constraints

crosscheck

IGW constraints

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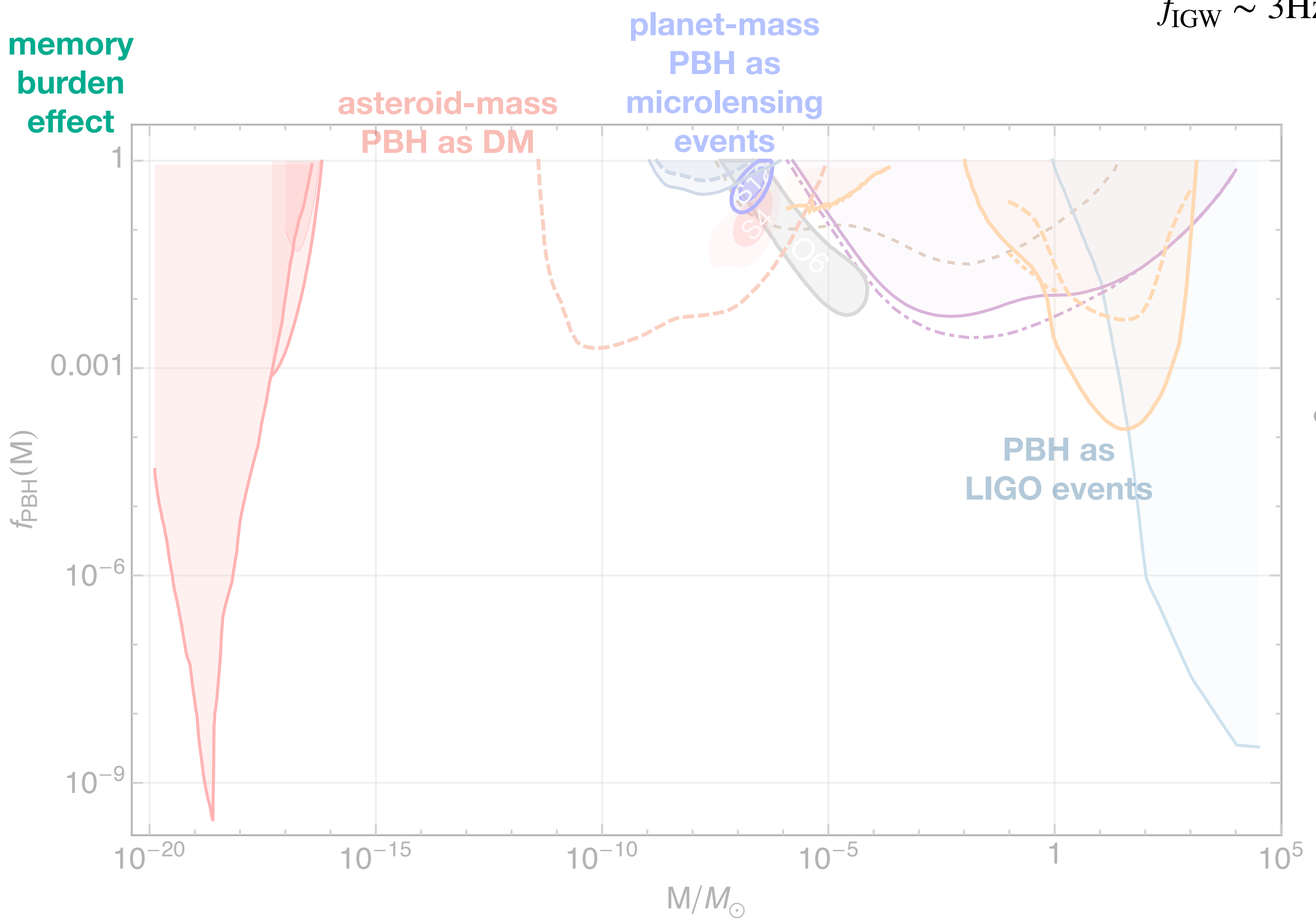
PBH-IGW crosscheck

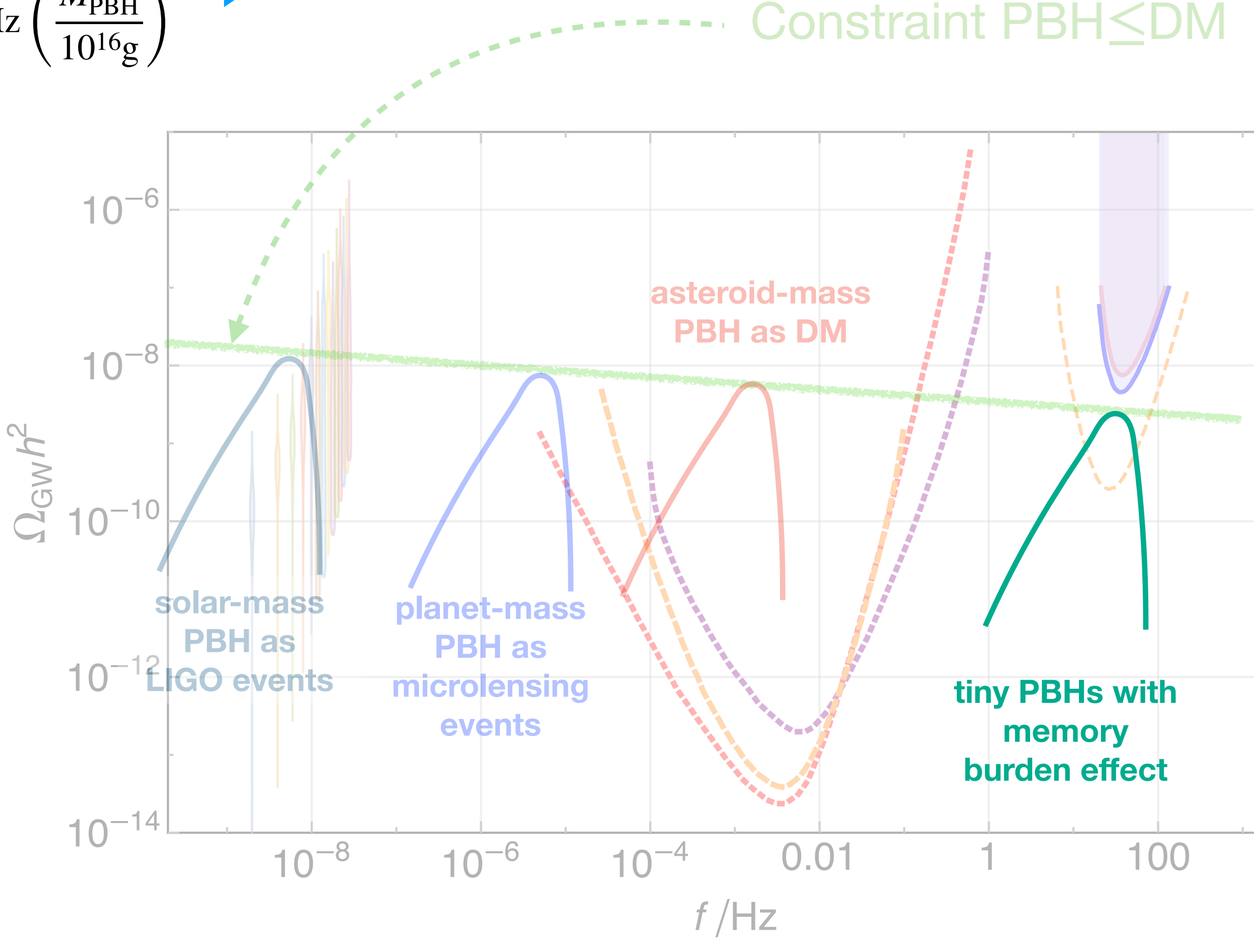
PBH constraints

crosscheck

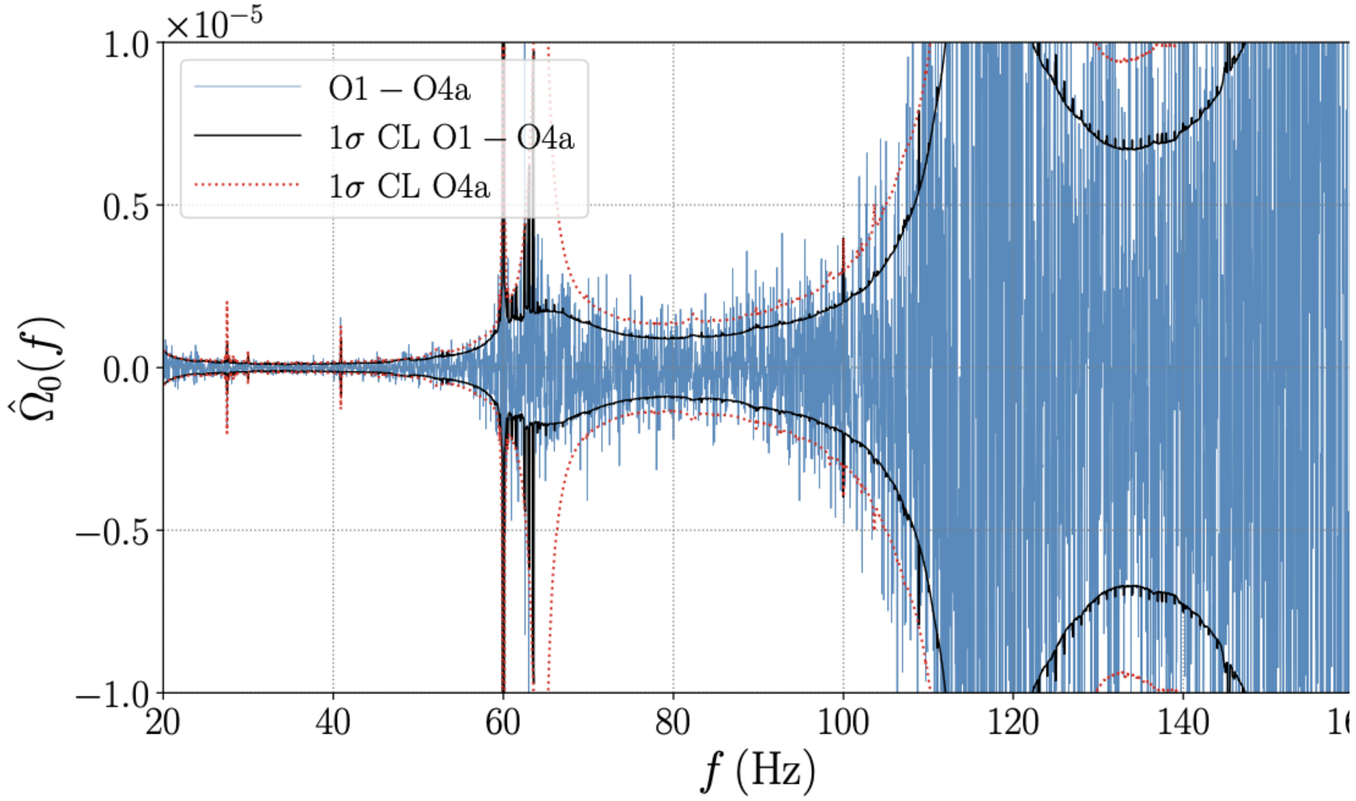
IGW constraints

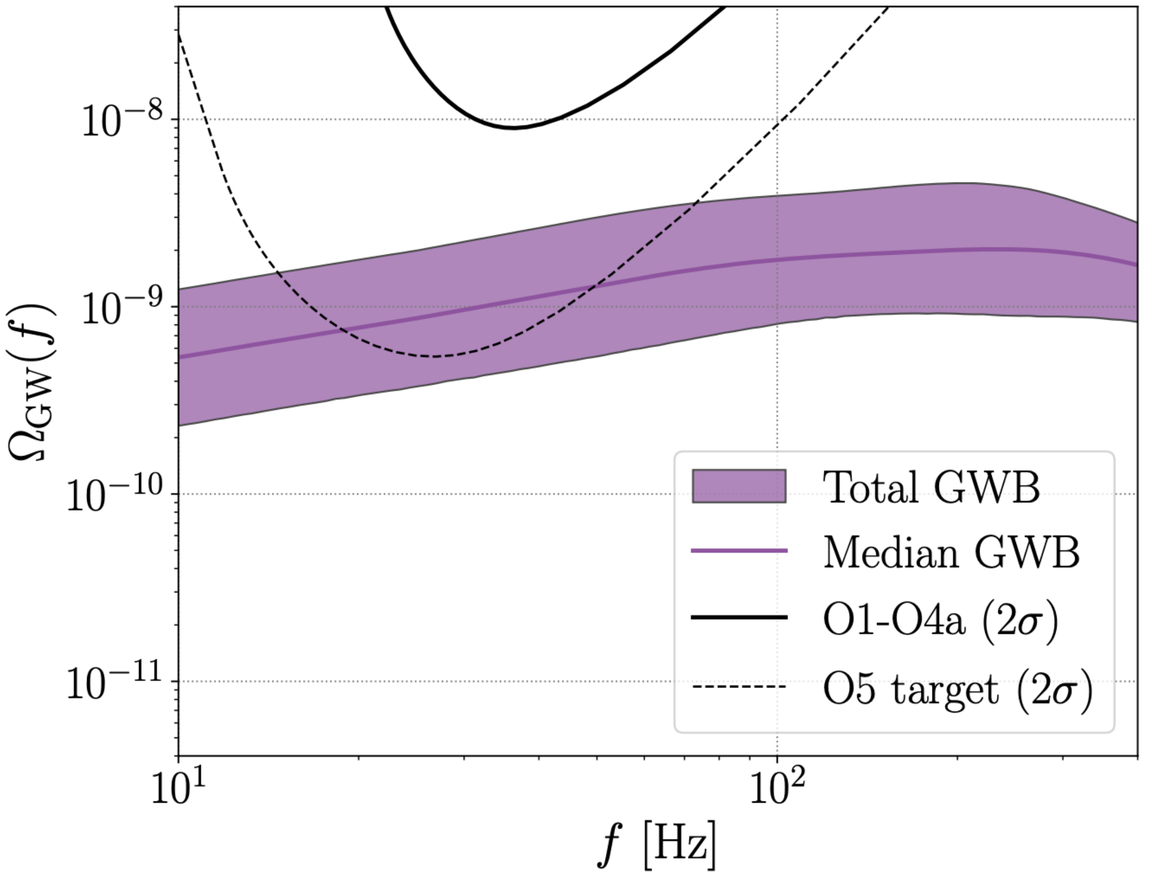
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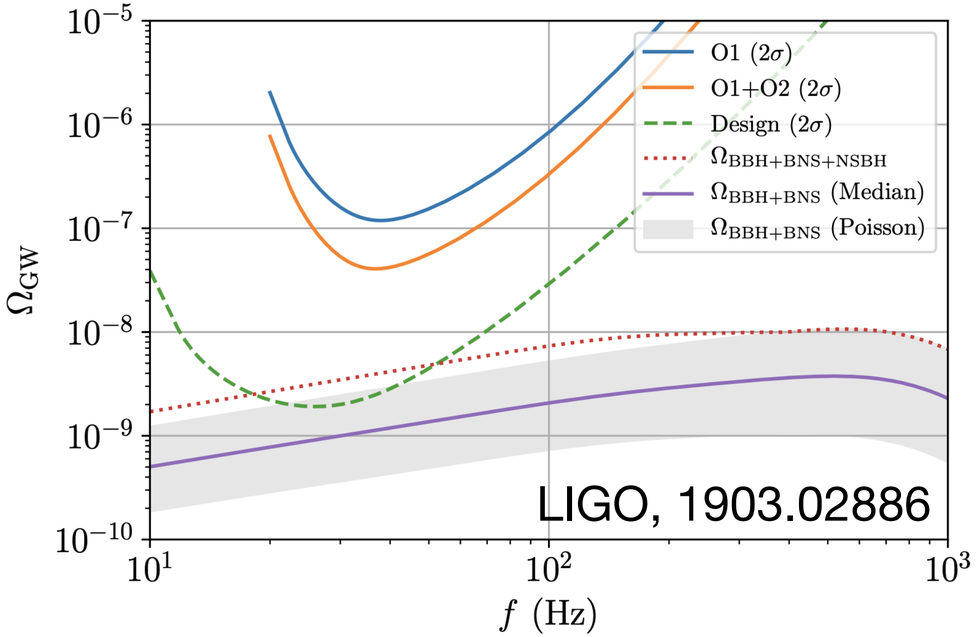
LIGO constraints

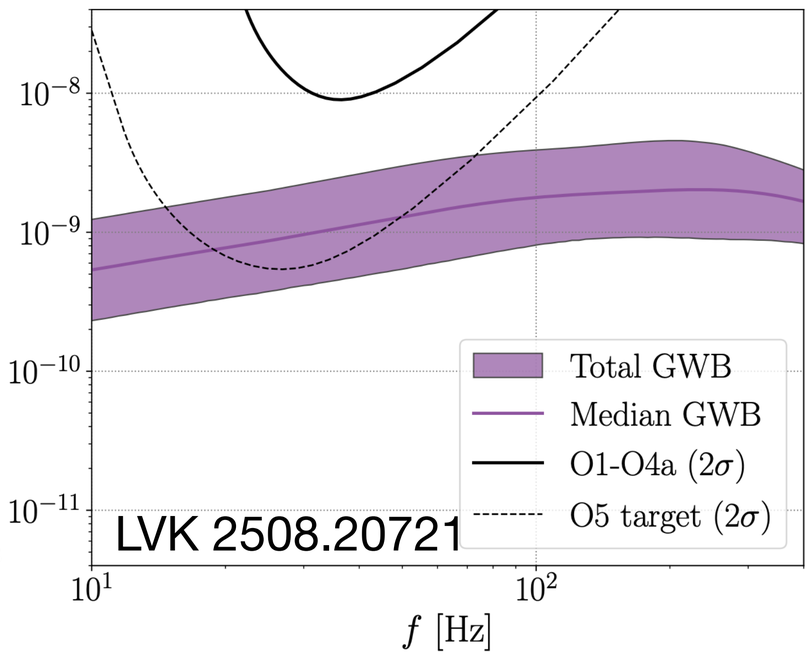


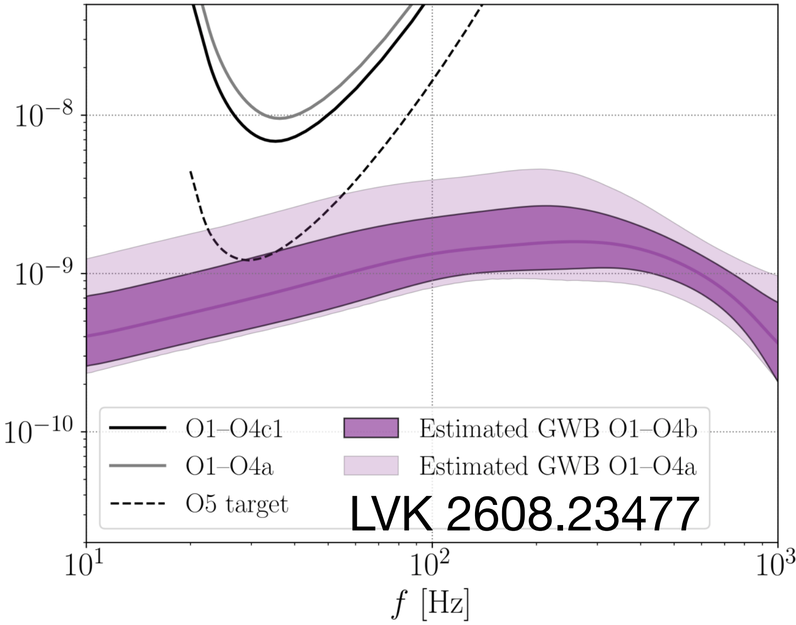


Mairi's Talk

LIGO constraints





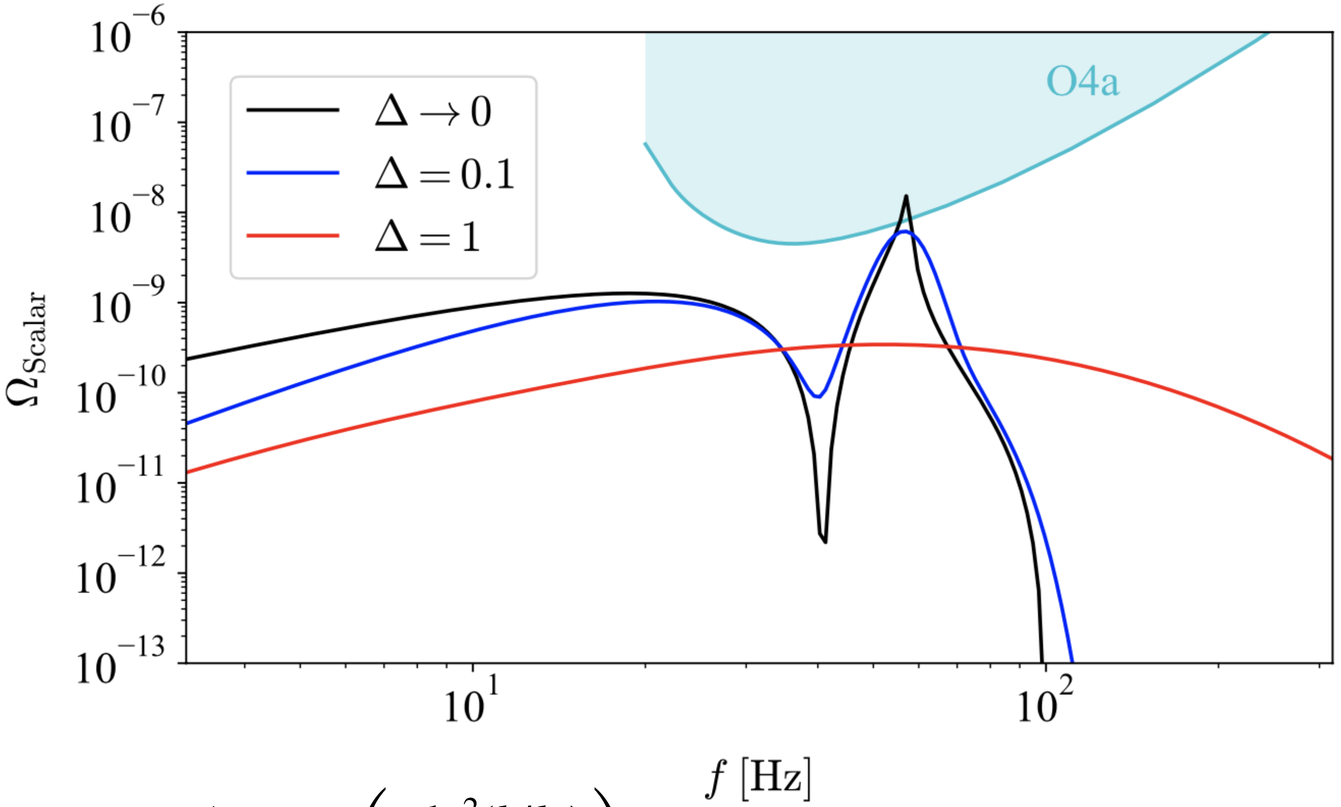


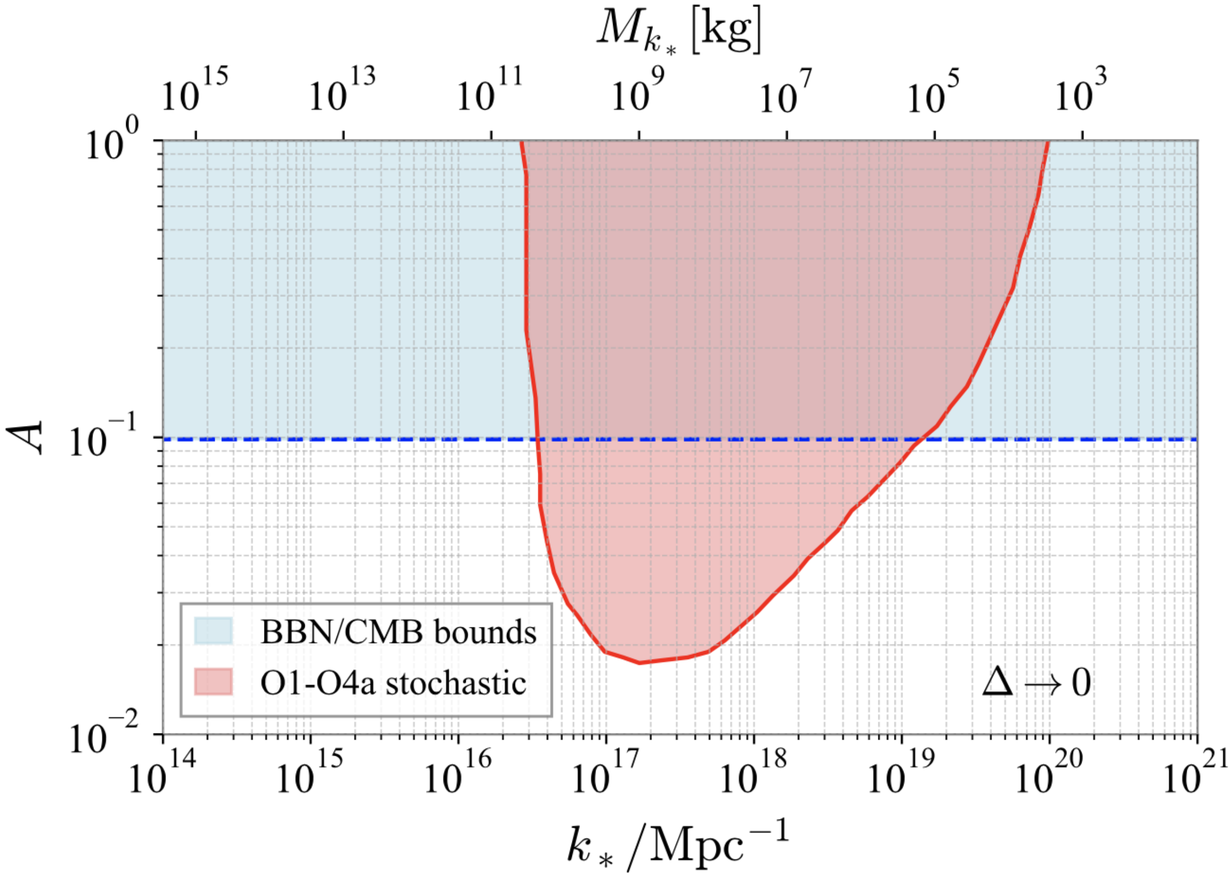
$\Omega_{\rm GW}(f) \leq 4.8 \times 10^{-8}$ @ 25Hz

$\Omega_{\rm GW}(f) \leq 2.0 \times 10^{-9}$ @ 25Hz

Mairi's Talk

LIGO constraints





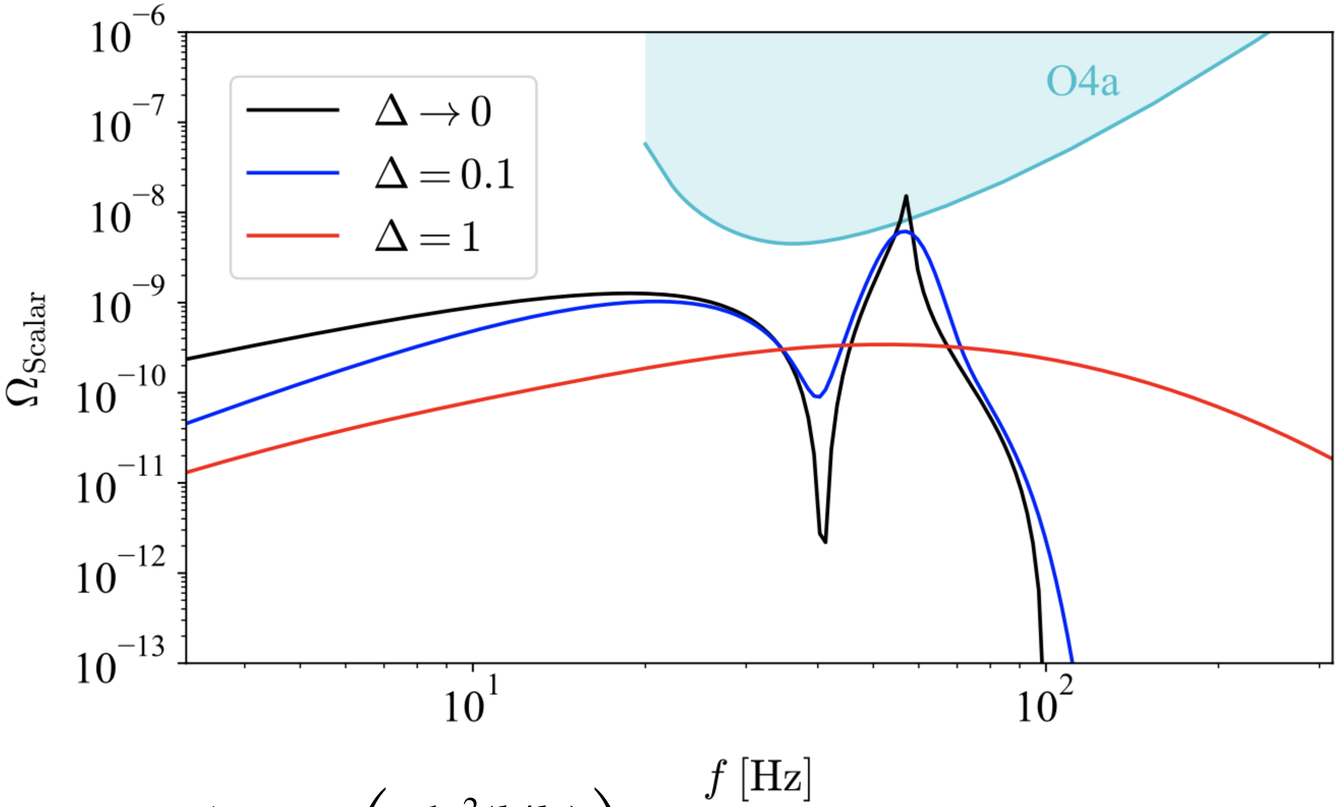
$$\mathcal{P}_\zeta = \frac{A}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\ln^2(k/k_*)}{2\sigma^2}\right)$$

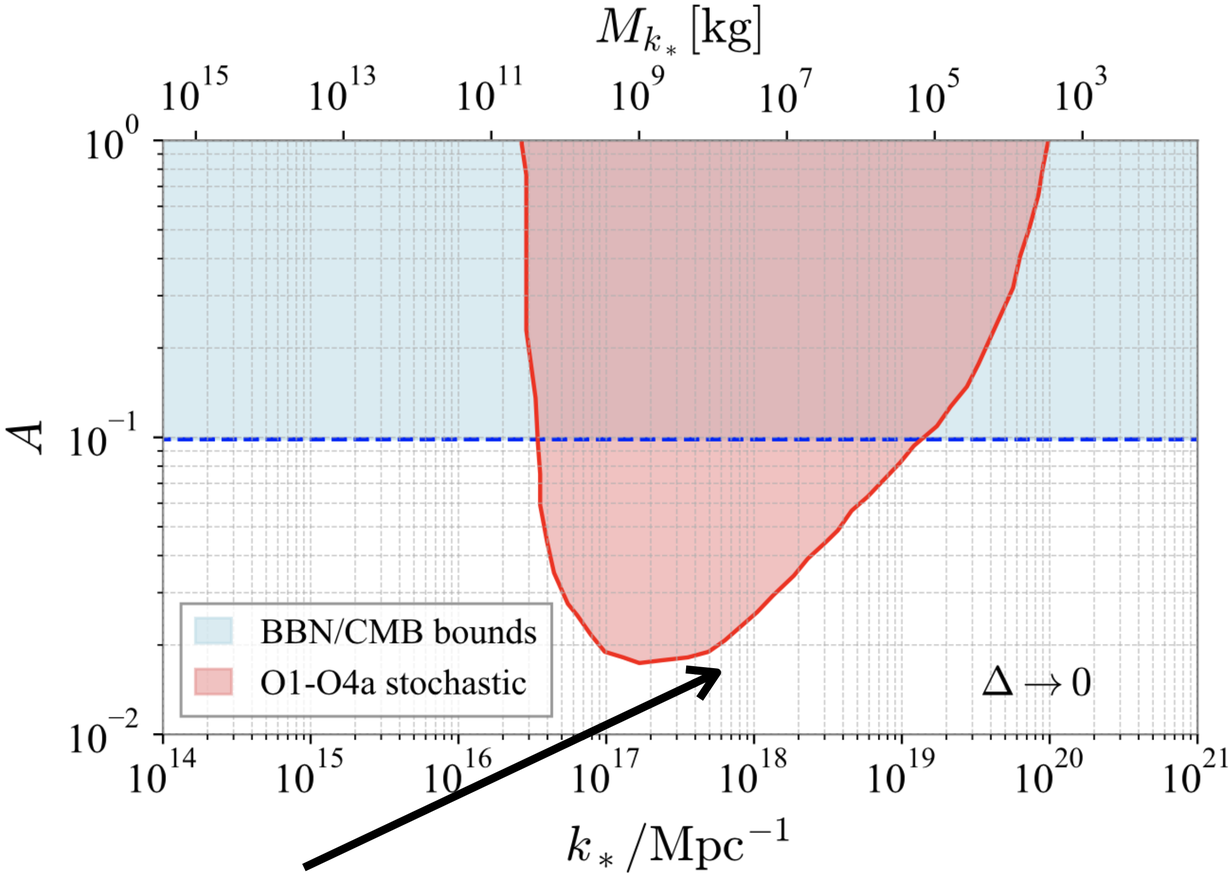
Lognormal ansatz:
SP and Sasaki, 2005.12306

Mairi's Talk

LVK, 2510.26848

LIGO constraints





$$\mathcal{P}_\zeta = \frac{A}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\ln^2(k/k_*)}{2\sigma^2}\right)$$

Lognormal ansatz:
SP and Sasaki, 2005.12306

$$f_{\rm PBH} \sim 4\times 10^8 \left(\frac{M}{M_\odot}\right)^{-1/2} {\rm erfc}\frac{\zeta_c}{\sqrt{2A}} \sim 10^9$$

$f_{\rm PBH} < 1$ implies $A \lesssim 9\times 10^{-3}$

Mairi's Talk

LVK, 2510.26848

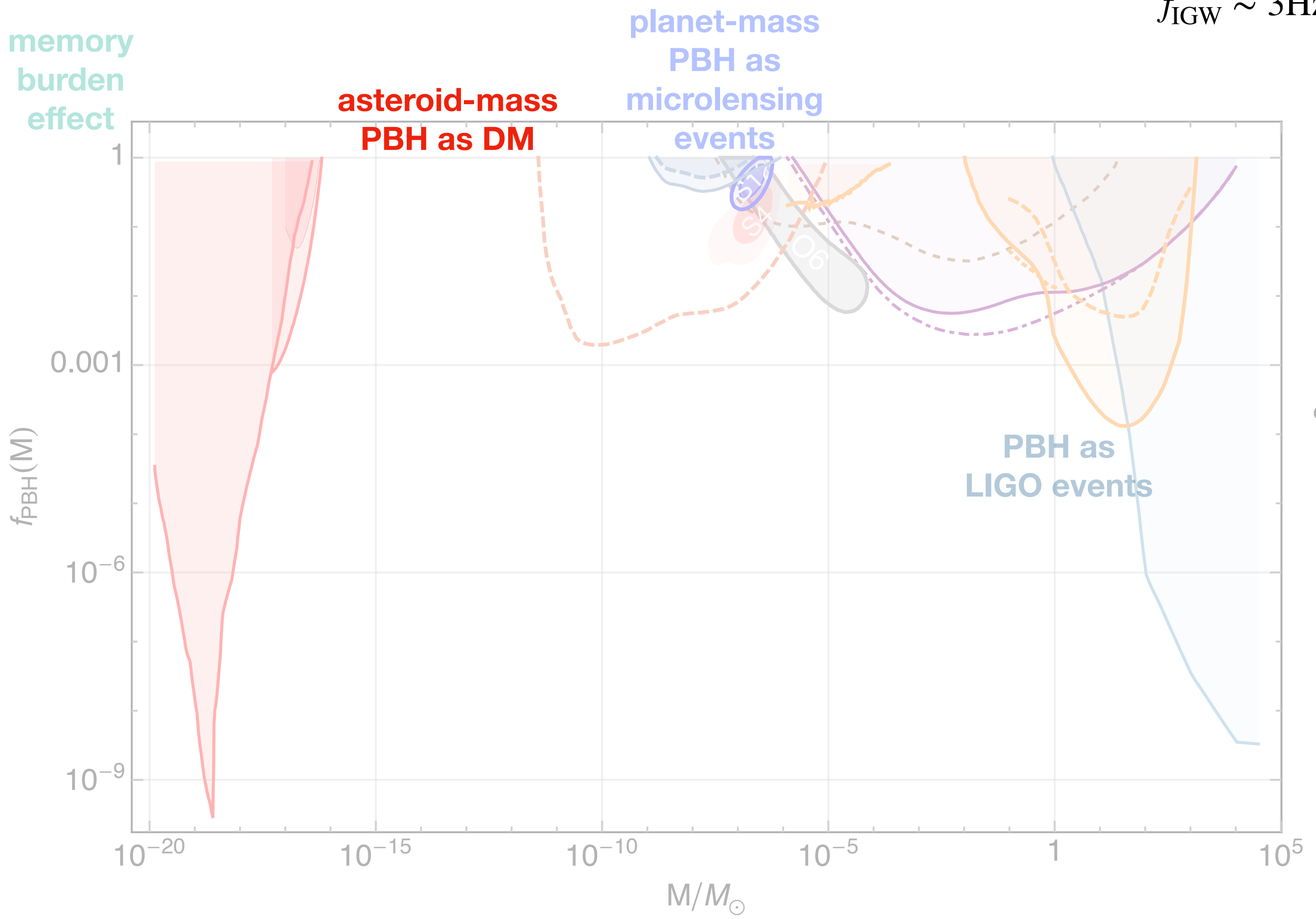
PBH-IGW crosscheck

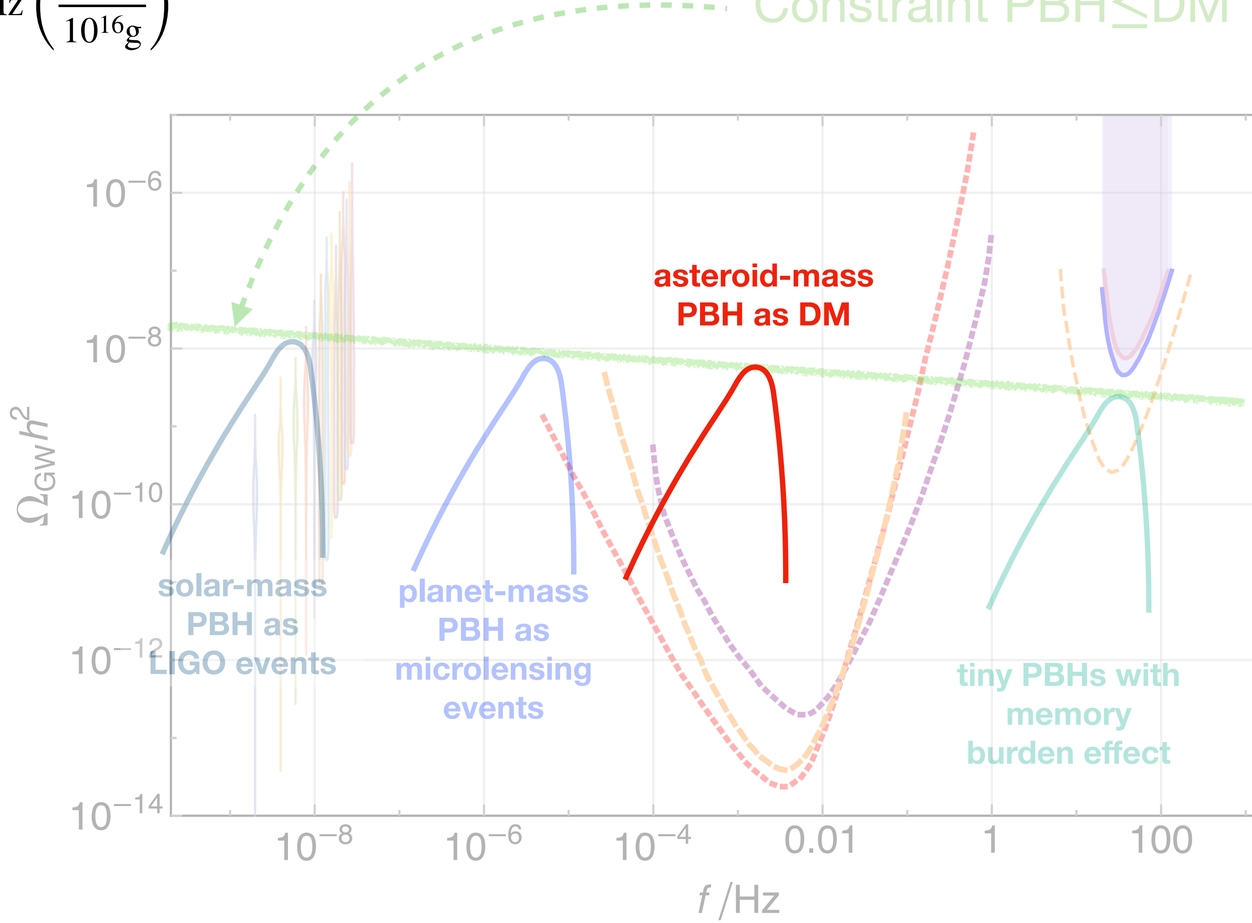
PBH constraints

crosscheck

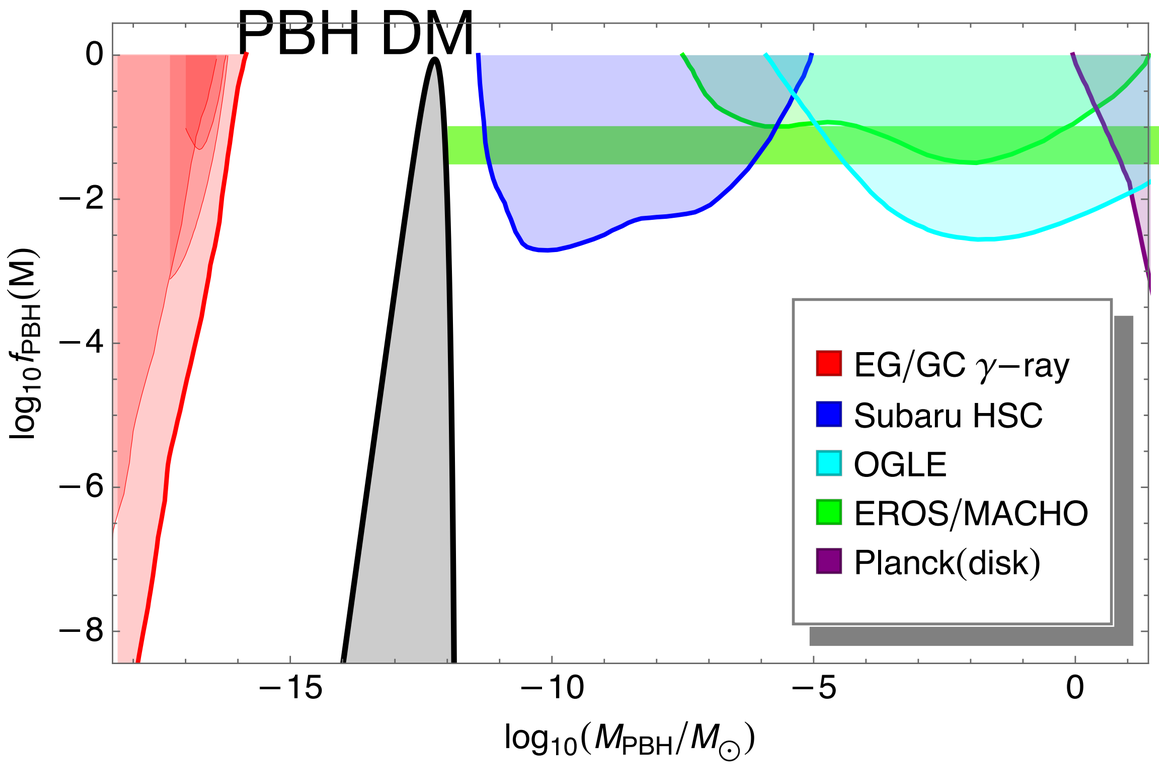
IGW constraints

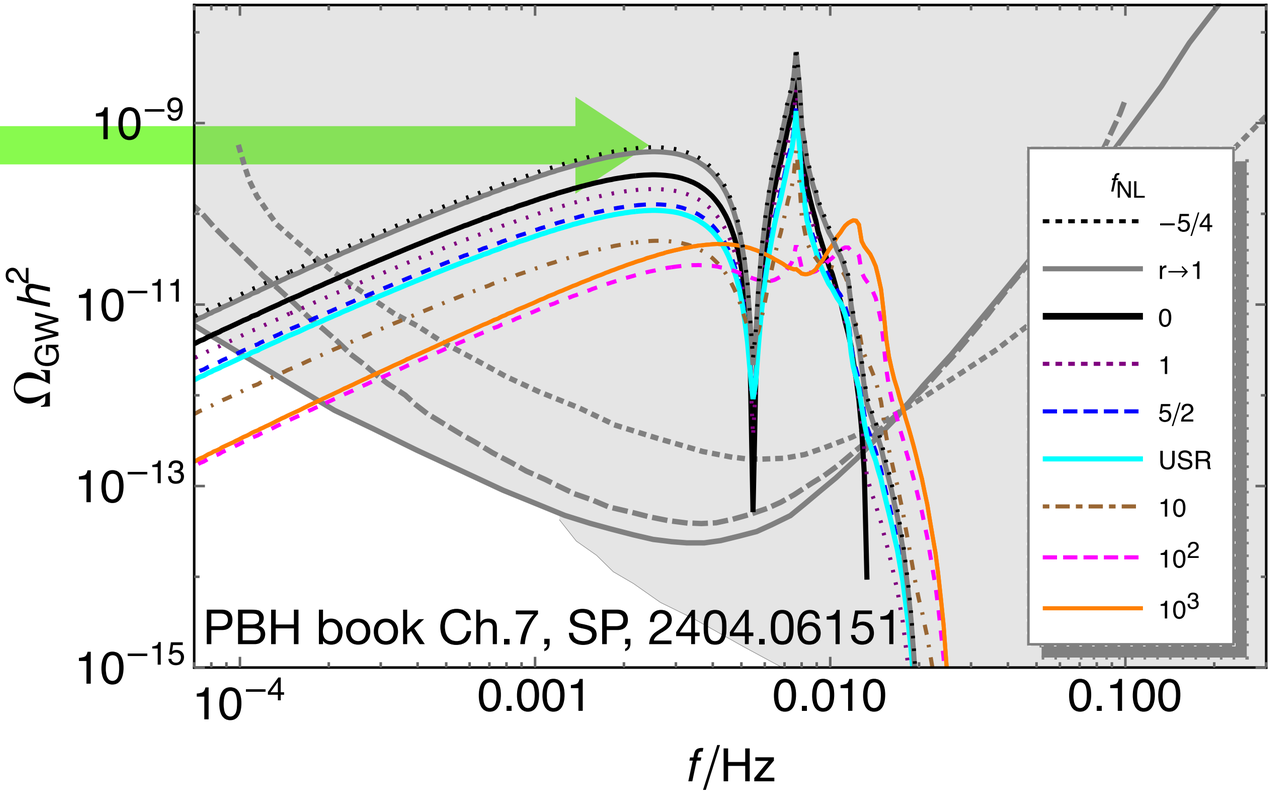
$$f_{\rm IGW} \sim 3{\rm Hz}\left(\frac{M_{\rm PBH}}{10^{16}{\rm g}}\right)^{-\frac{1}{2}}$$





Probing PBH dark matter by LISA/Taiji/TianQin

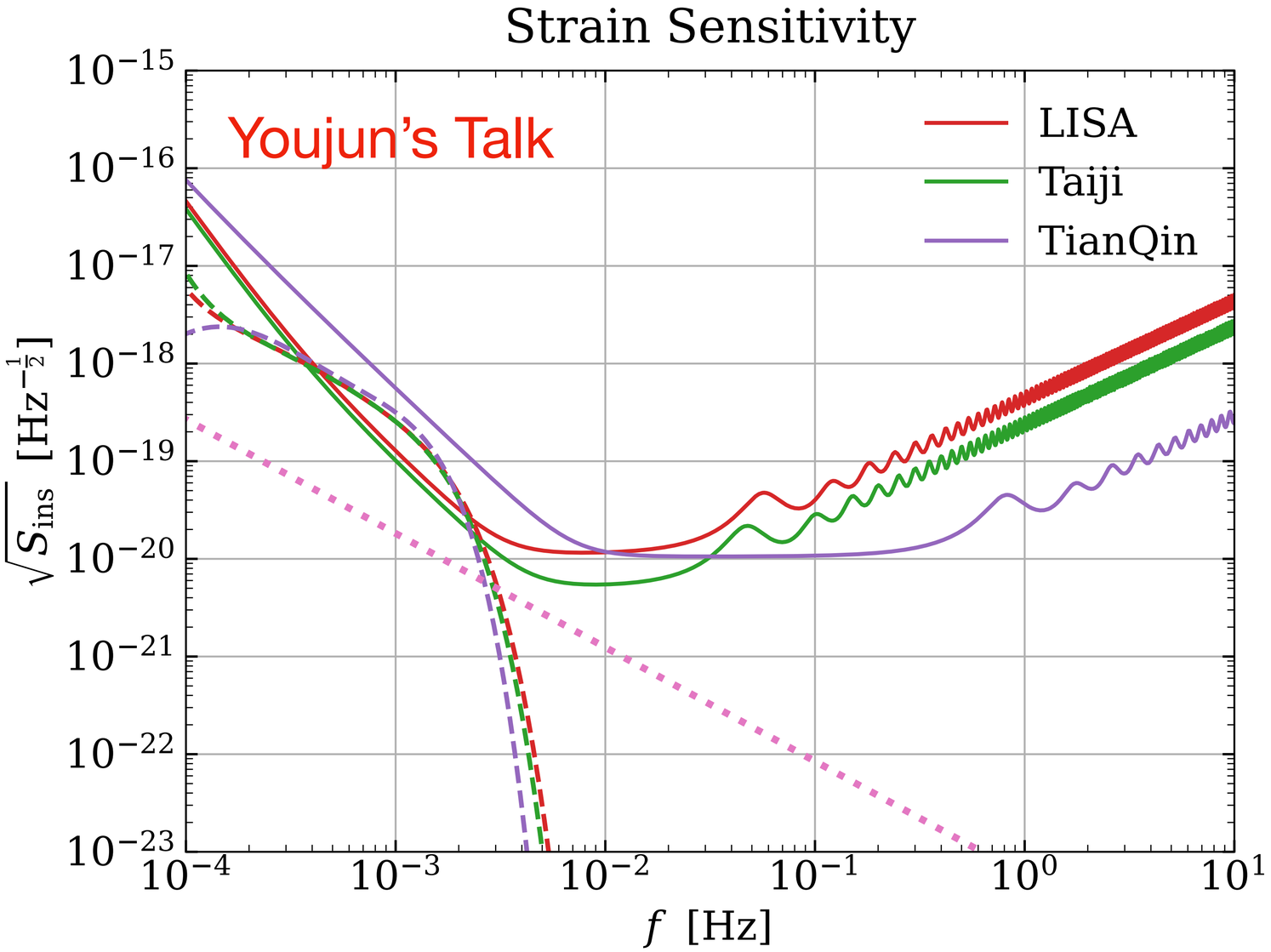




- IGW is rubust against non-Gaussianity
- The only PBH DM window can be probed by space-based inteferometers LISA/Taiji/TianQin

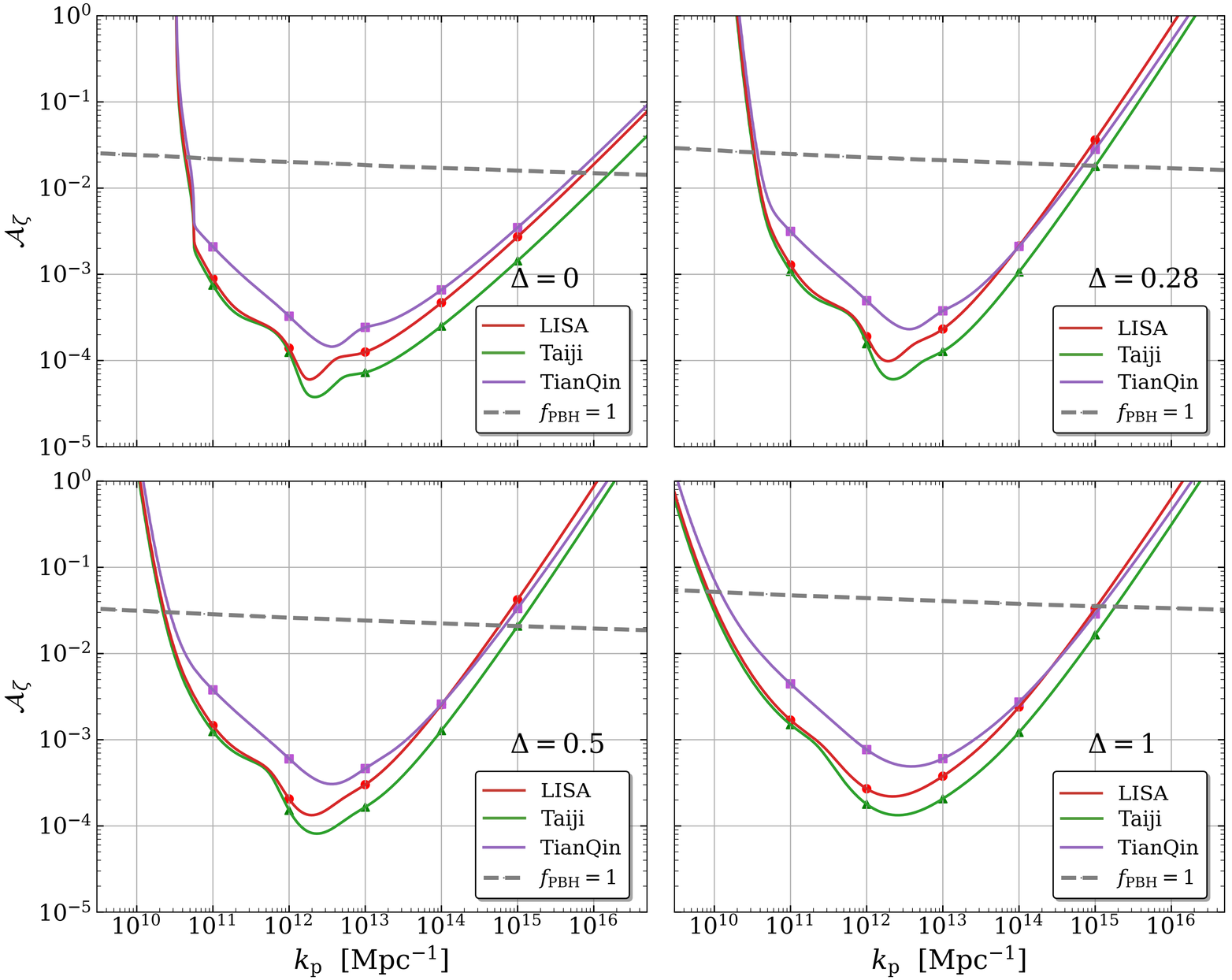
Probing PBH dark matter by induced GW
(even when there is non-Gaussianity)
Cai, SP, Sasaki, 1810.11000, PRL 122, 201101

Probing PBH dark matter by LISA/Taiji/TianQin



- Use the sensitivity curves to predict the detectable parameter space
- $h_c \rightarrow \Omega_{\rm IGW} \rightarrow A_\zeta \rightarrow f_{\rm PBH}$
- The entire asteroid-mass window can be covered
- SNR: $10^3 \sim 10^4$ for PBH dark matter

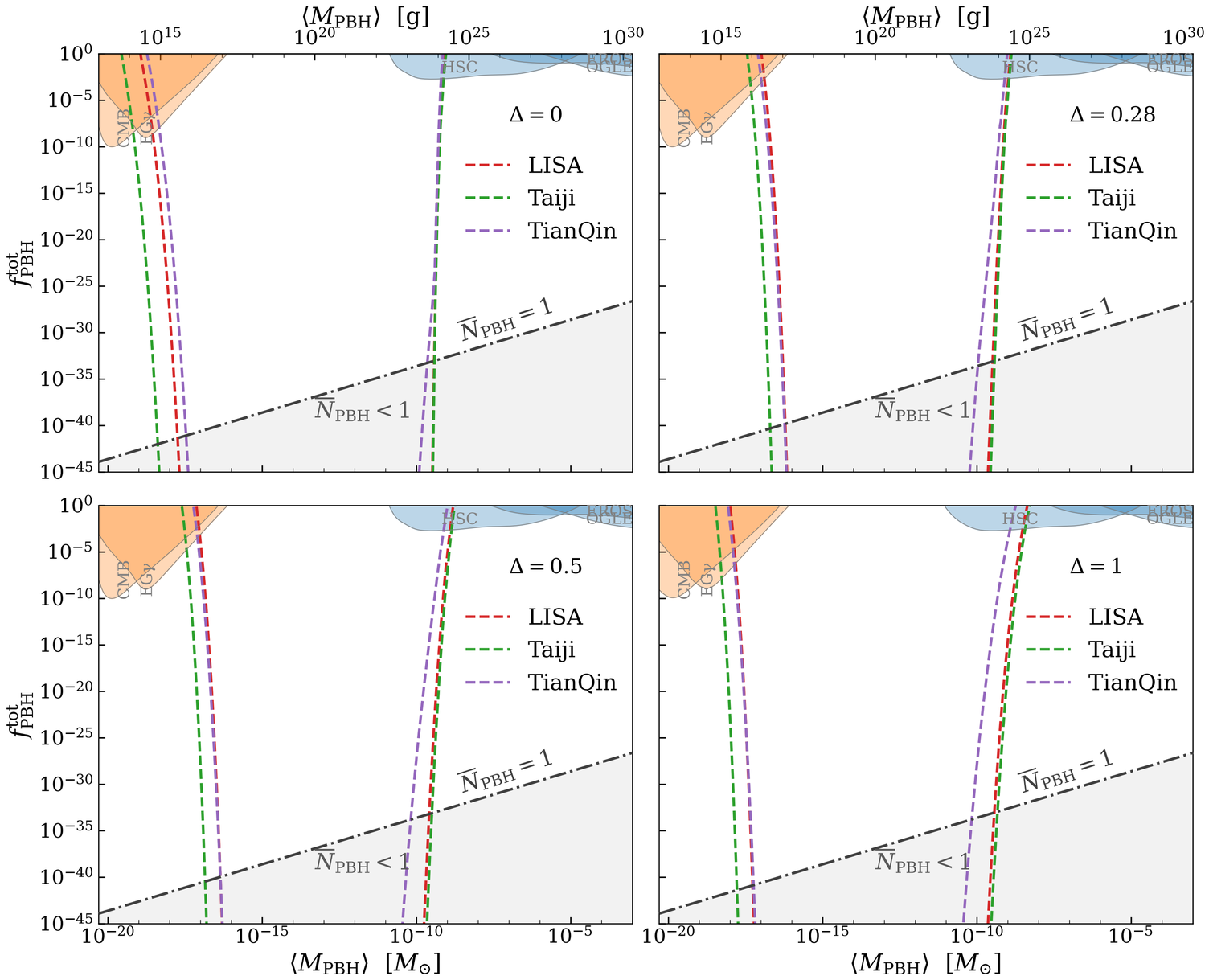
TianQin Whitebook, 2502.20138
Hong, SP, Wang, Zhang, 2601.05069
Taiji White book, to appear
c.f. Iovino et al. 2512.13648 (for LISA)



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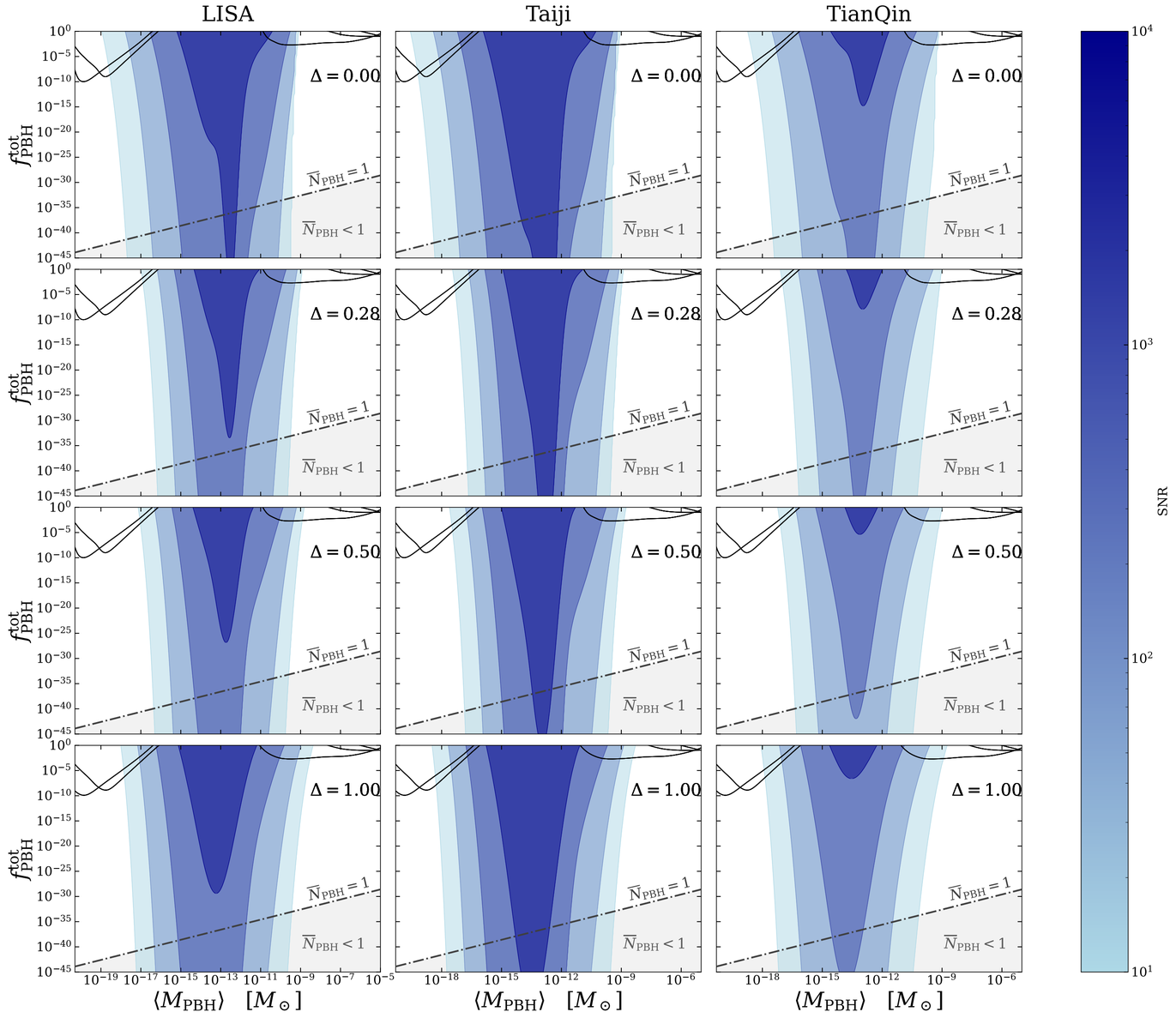
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TianQin Whitebook, 2502.20138
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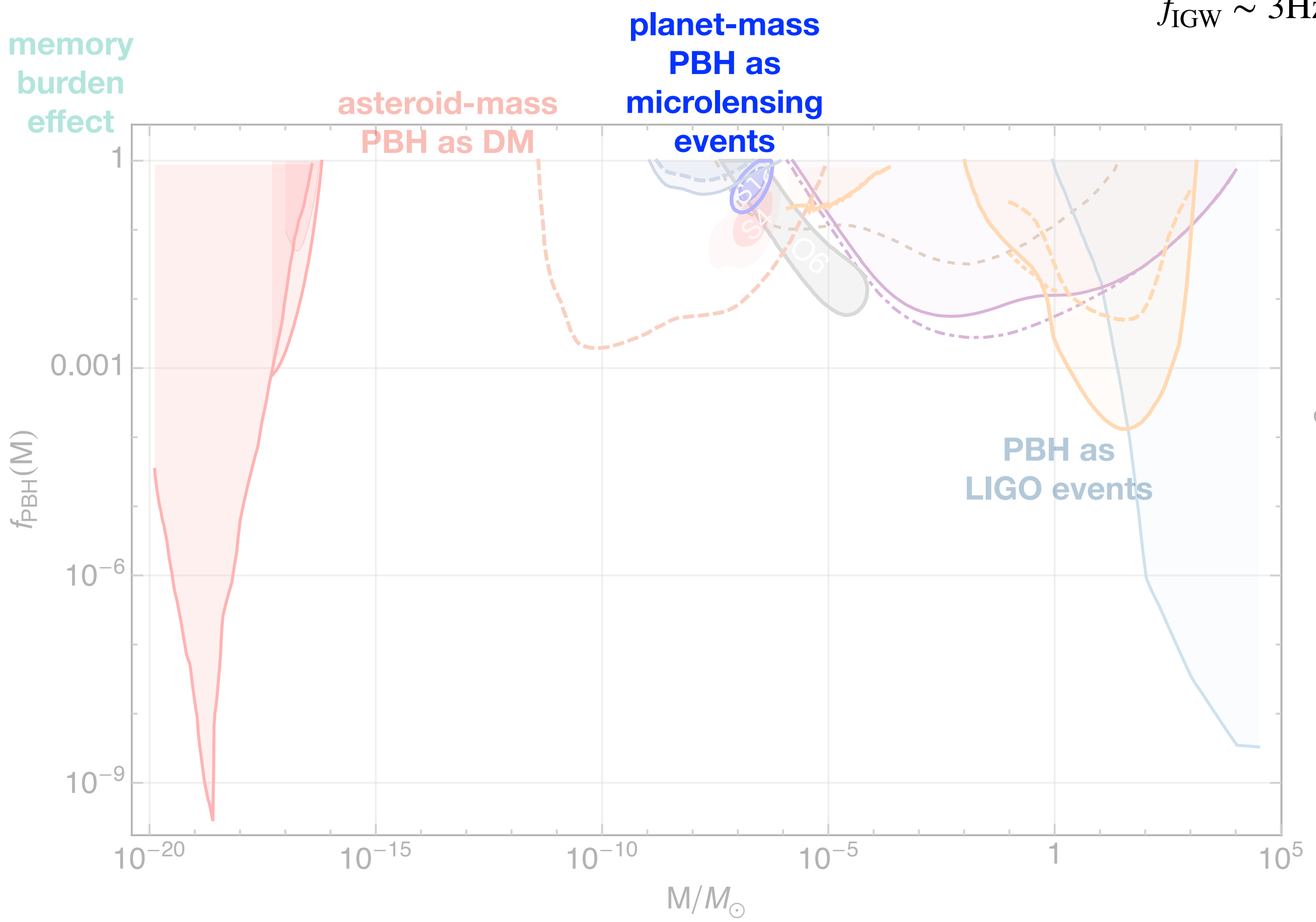
PBH-IGW crosscheck

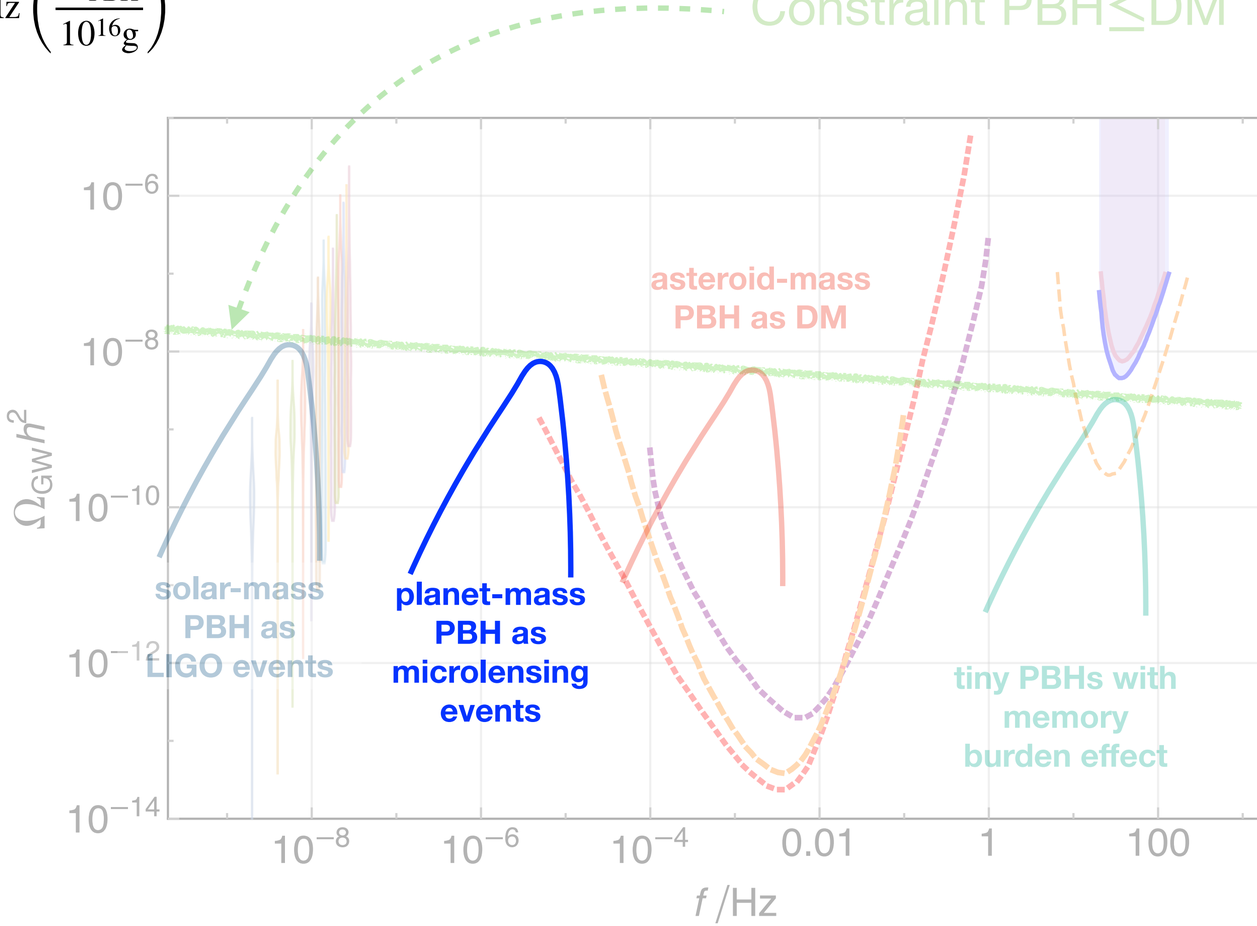
PBH constraints

crosscheck

IGW constraints

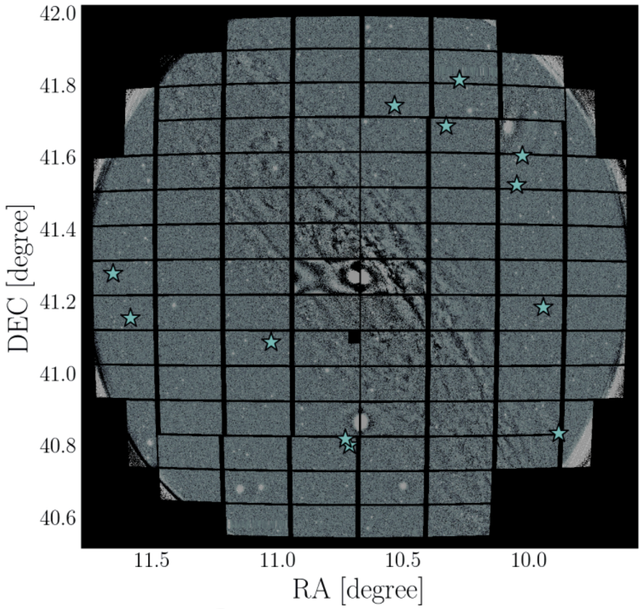
$$f_{\rm IGW} \sim 3{\rm Hz}\left(\frac{M_{\rm PBH}}{10^{16}{\rm g}}\right)^{-\frac{1}{2}}$$

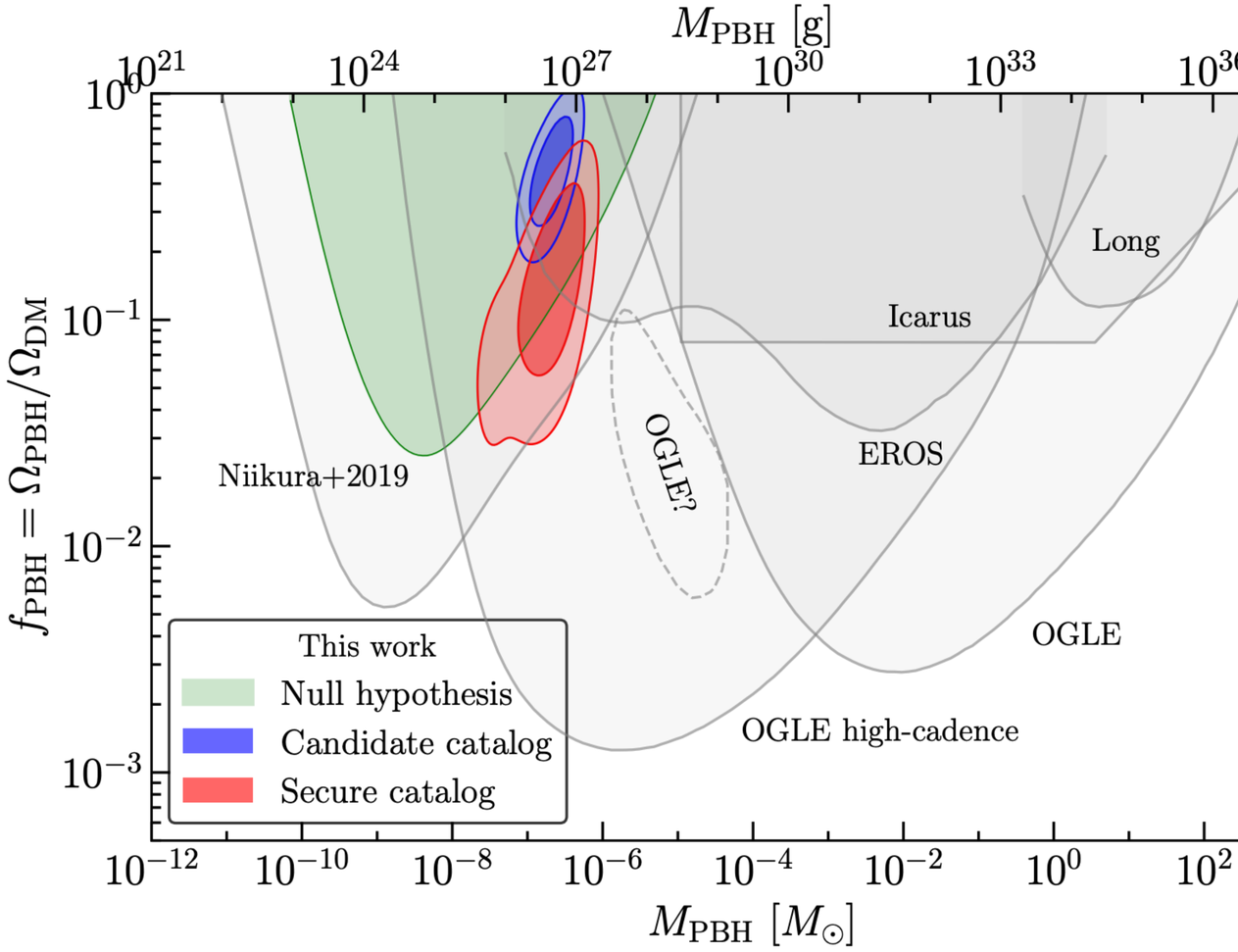




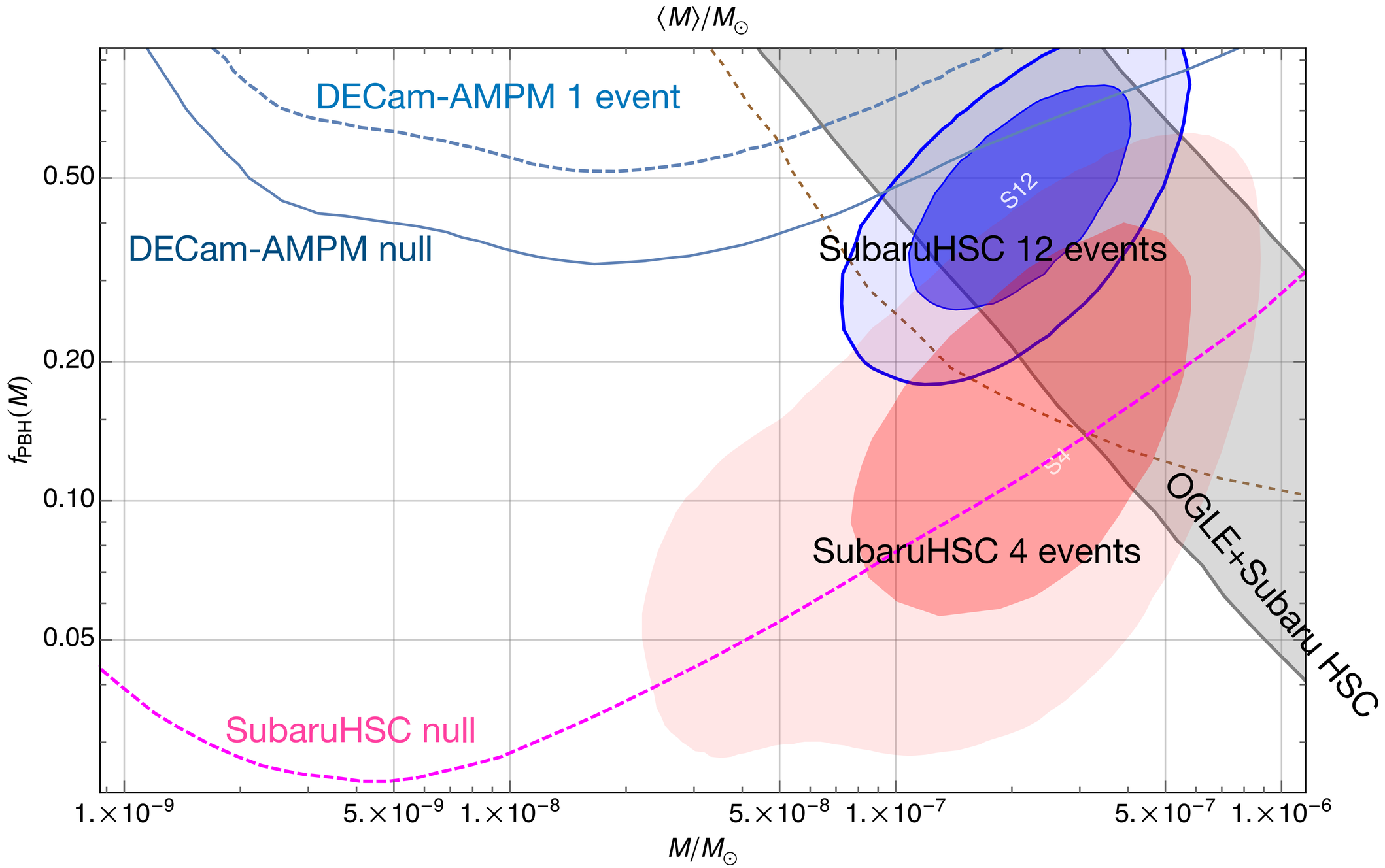
Planet-mass PBH

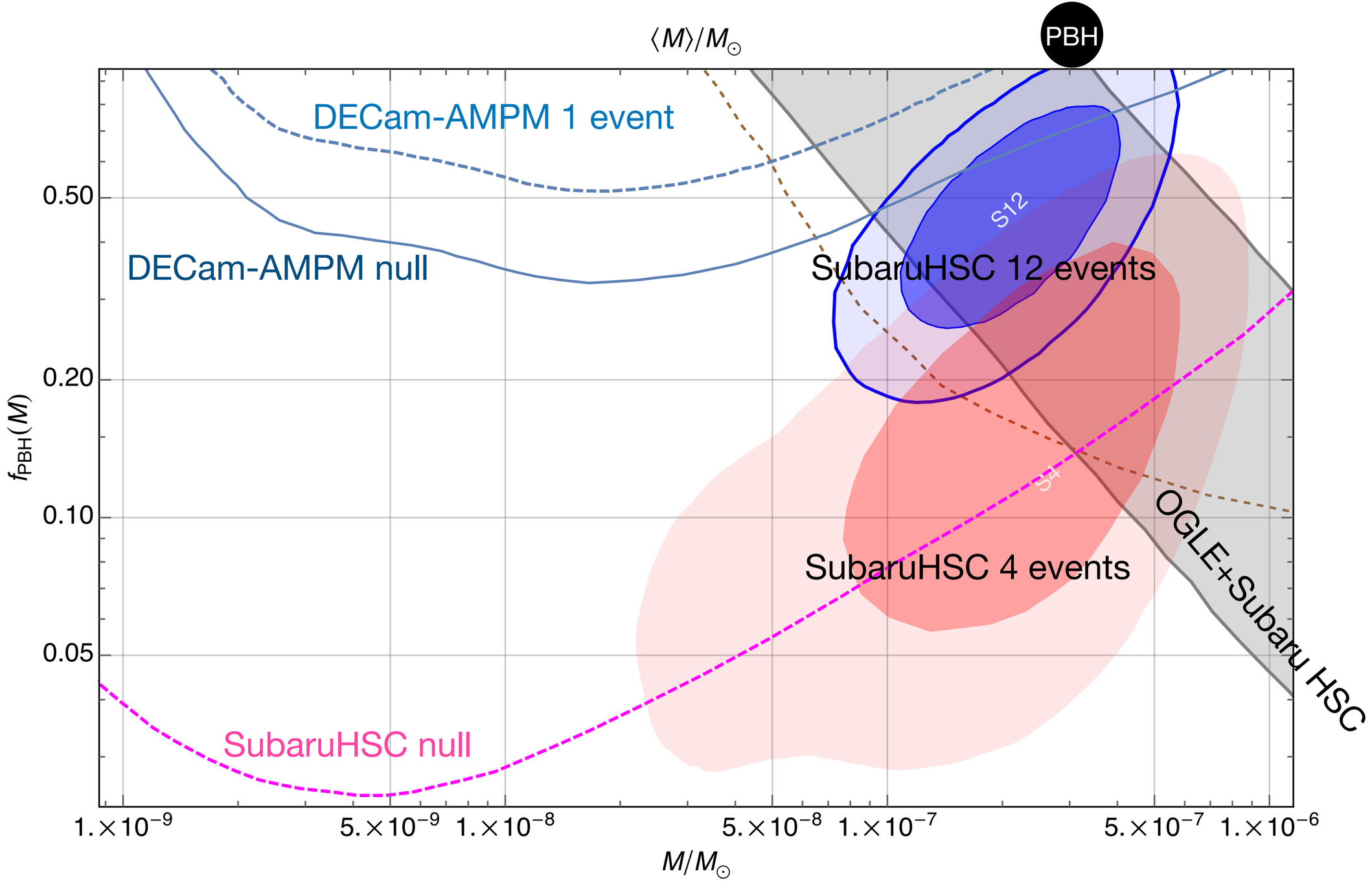
Subaru-HSC

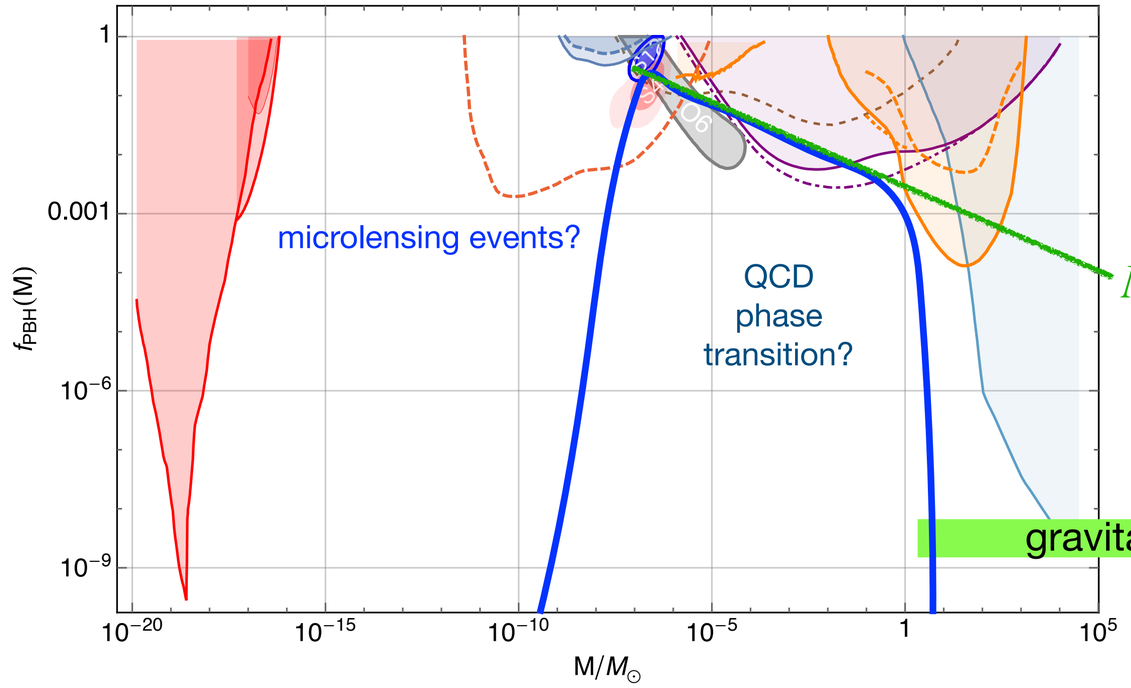




Sugiyama et al 2602.05840

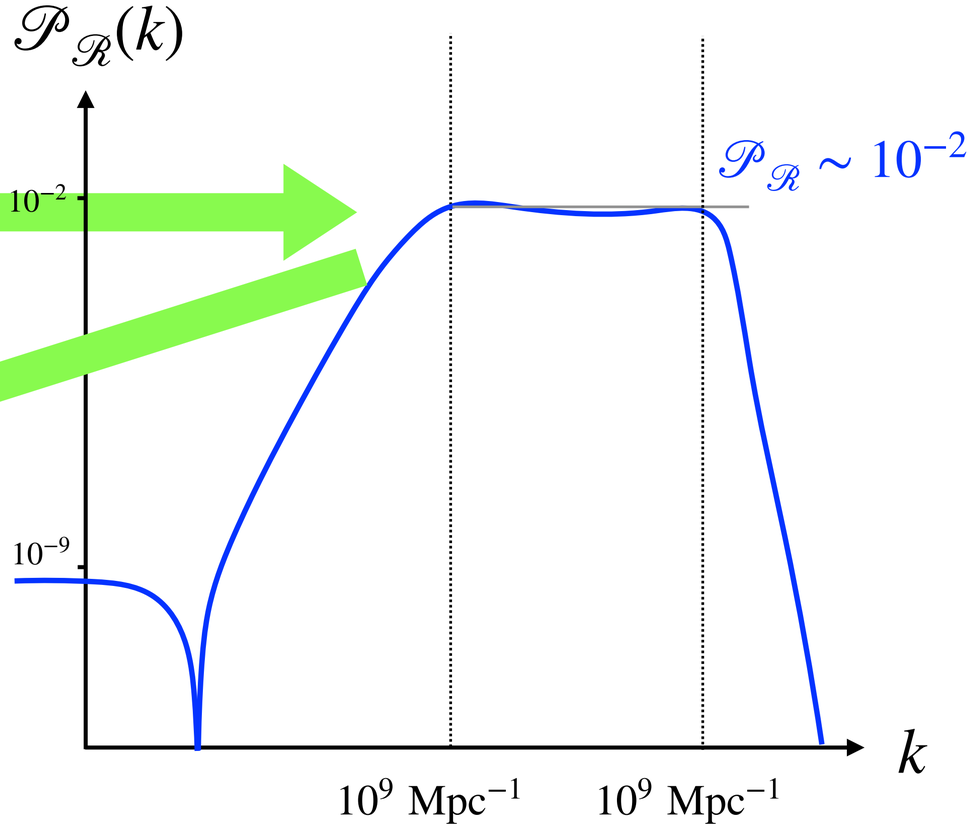

⟨M⟩/M⊙
DECam-AMPM 1 event
DECam-AMPM null
SubaruHSC 12 events
S12
S4
SubaruHSC 4 events
OGLE+Subaru HSC
SubaruHSC null
0.50
0.20
0.10
0.05
f_PBH(M)
1.×10^−9
5.×10^−9
1.×10^−8
5.×10^−8
1.×10^−7
5.×10^−7
1.×10^−6
M/M⊙


⟨M⟩/M⊙
PBH
DECam-AMPM 1 event
DECam-AMPM null
SubaruHSC 12 events
S12
S4
SubaruHSC 4 events
OGLE+Subaru HSC
SubaruHSC null
0.50
0.20
0.10
0.05
f_PBH(M)
1.×10^−9
5.×10^−9
1.×10^−8
5.×10^−8
1.×10^−7
5.×10^−7
1.×10^−6
M/M⊙

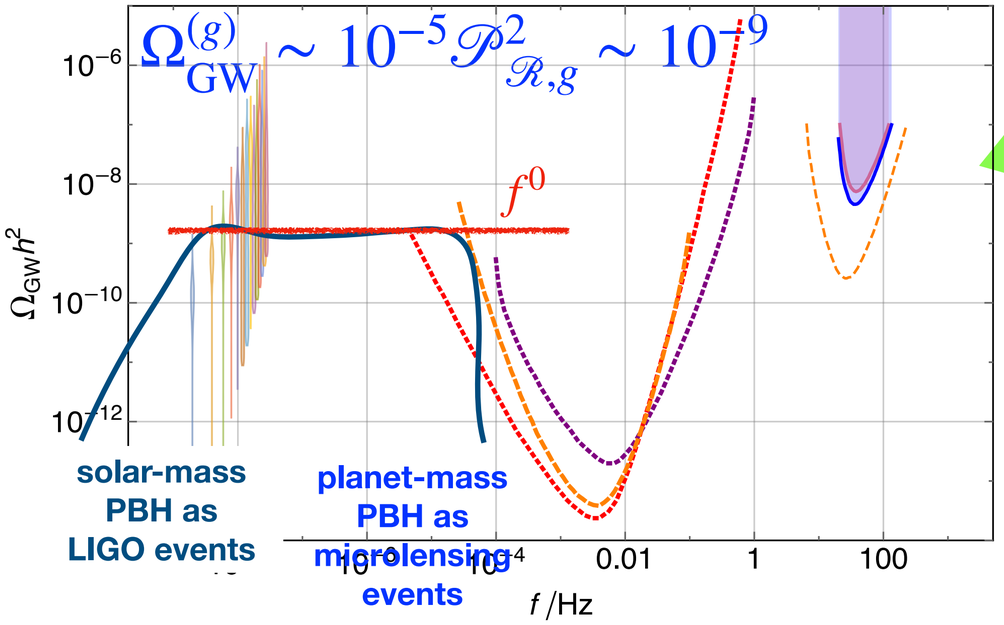


$$f_{\rm IGW} \sim 3{\rm Hz}\left(\frac{M_{\rm PBH}}{10^{16}{\rm g}}\right)^{-\frac{1}{2}}$$

gravitational collapse

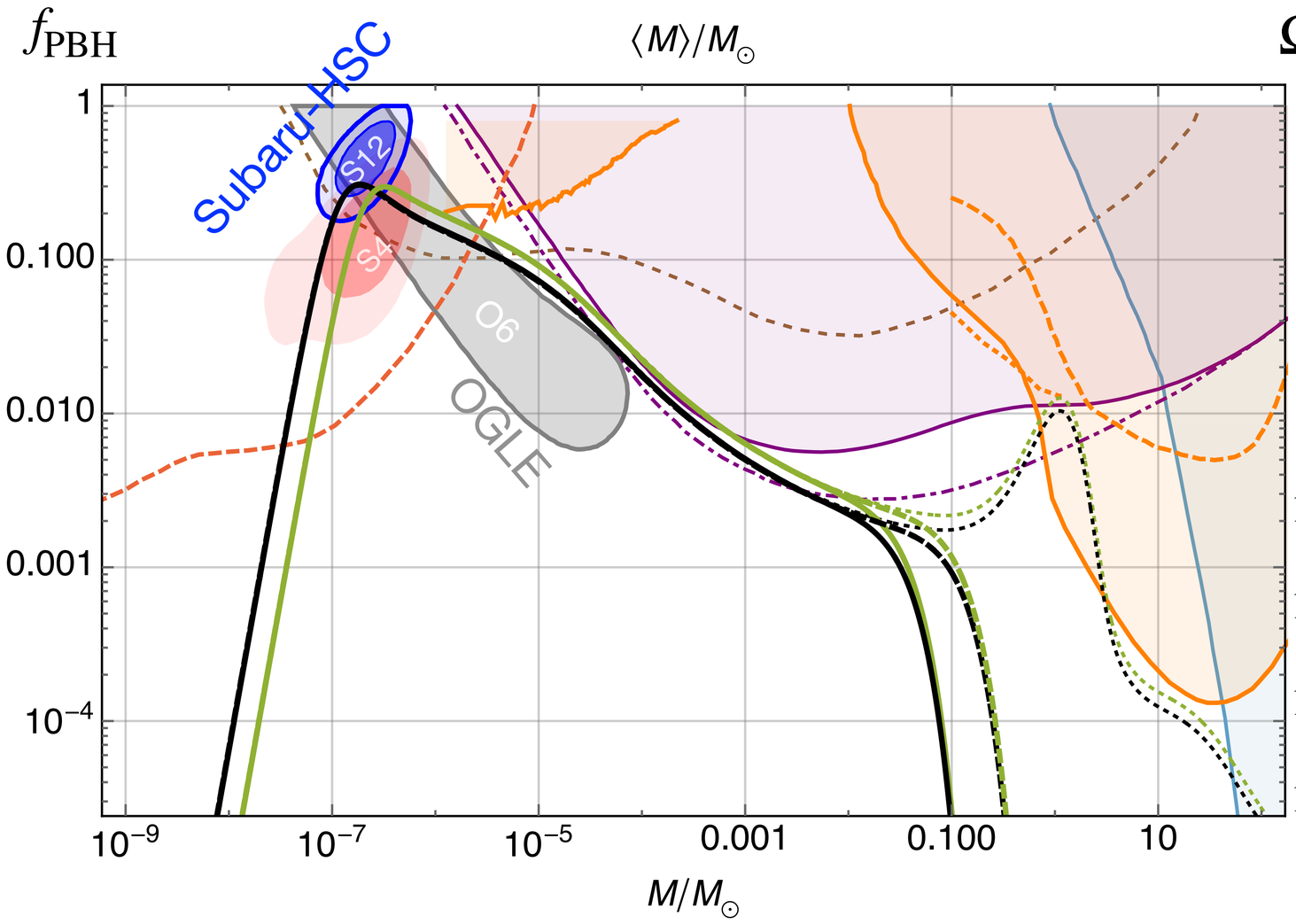


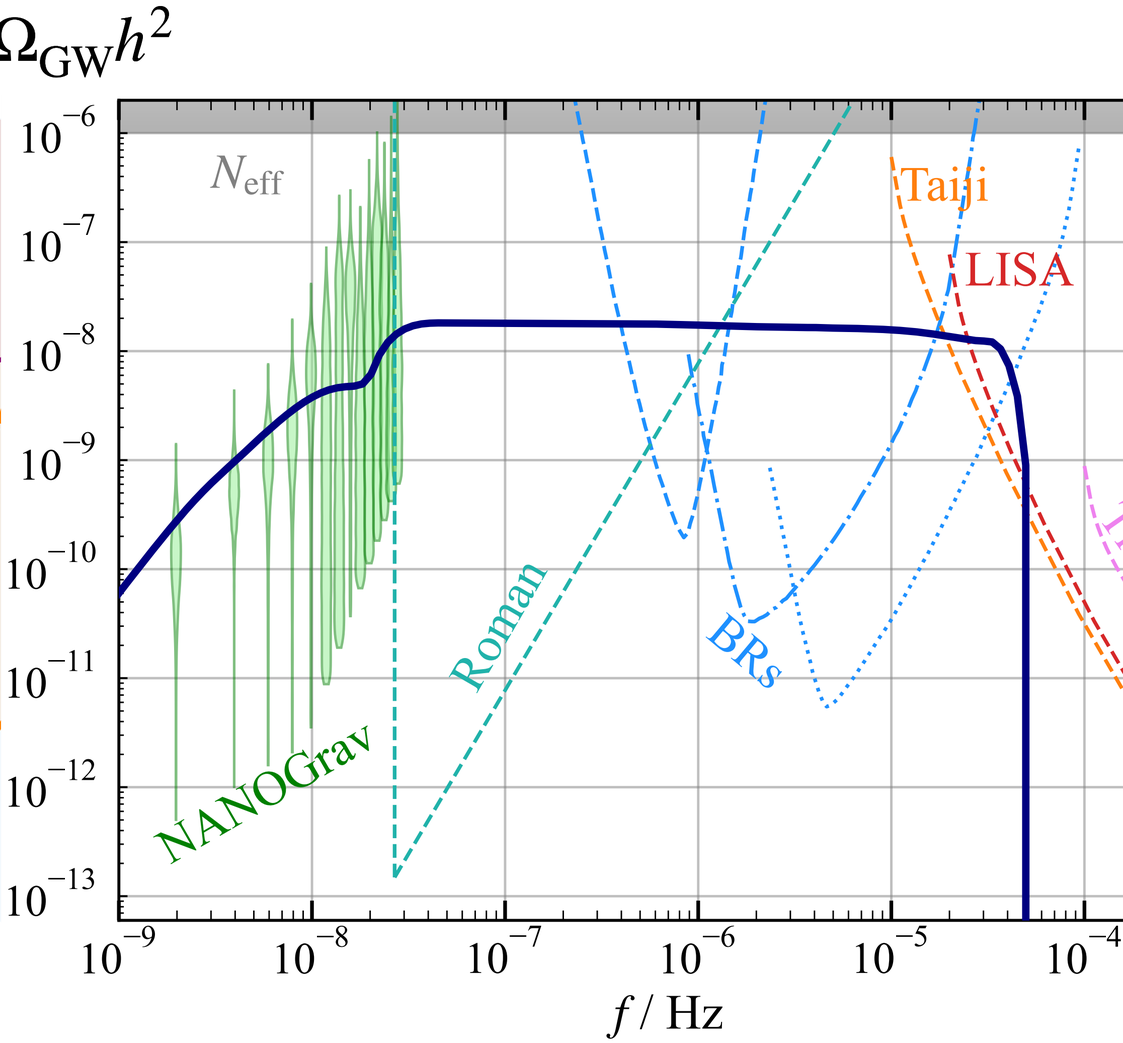
nonlinear coupling $h\mathscr{R}\mathscr{R}$



Saito & Yokoyama 0812.4339
Cai, SP, Sasaki, 1810.11000

Planet-mass PBH DM

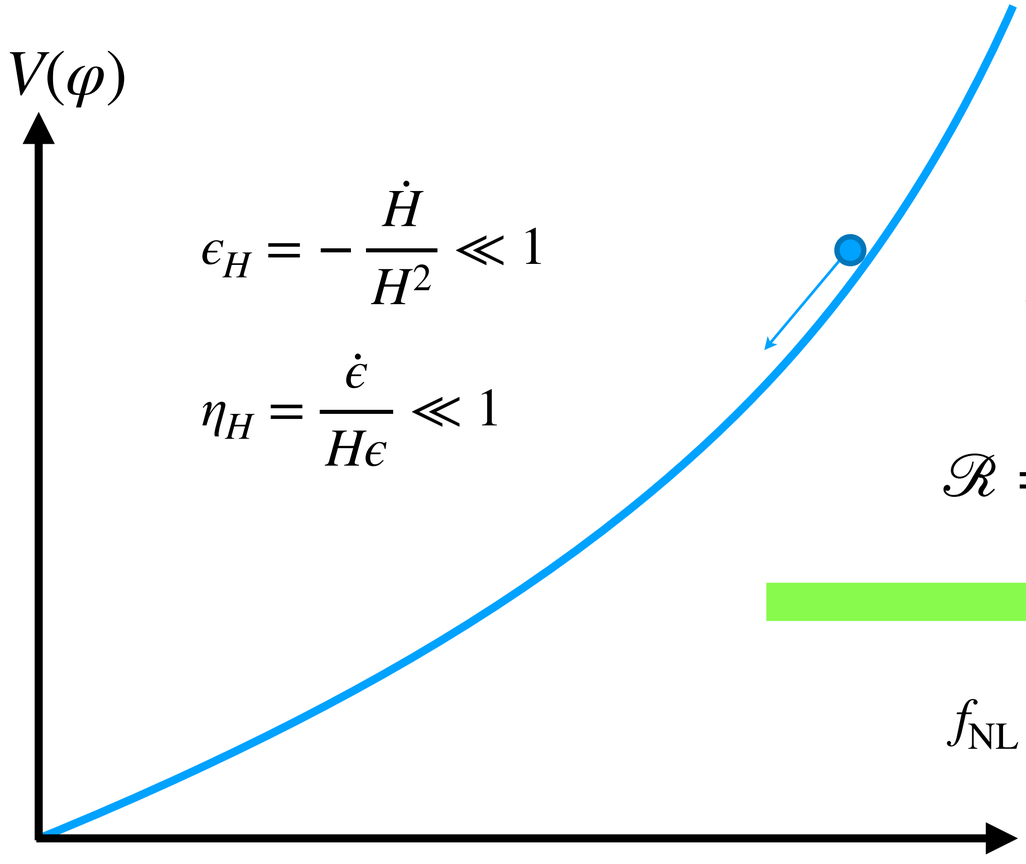




Domenech, SP, Wang, 2602.24061

Non-Gaussianity

Slow-roll inflation



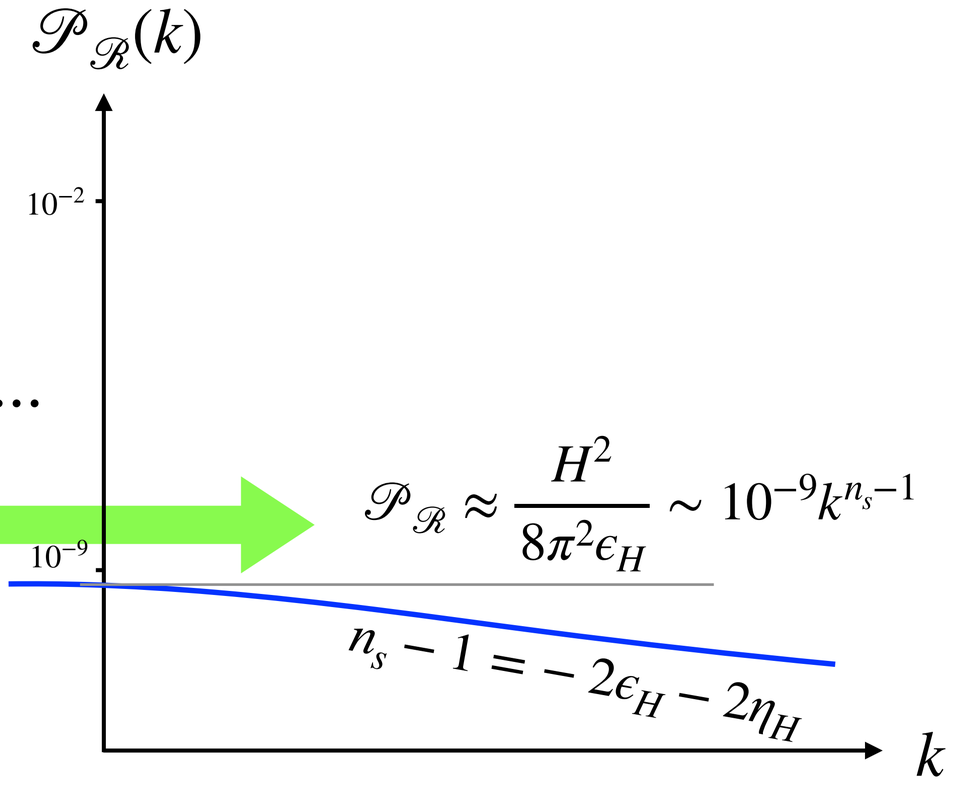
$$\epsilon_H = -\frac{\dot{H}}{H^2} \ll 1$$

$$\eta_H = \frac{\dot{\epsilon}}{H\epsilon} \ll 1$$

$$\mathscr{R}_g = -\frac{H}{\dot{\varphi}}\delta\varphi$$

$$\mathscr{R} = \mathscr{R}_g + \frac{3}{5} f_{\rm NL} \mathscr{R}_g^2 + \cdots$$

$$f_{\rm NL} = -\frac{5}{12}(n_s - 1)$$

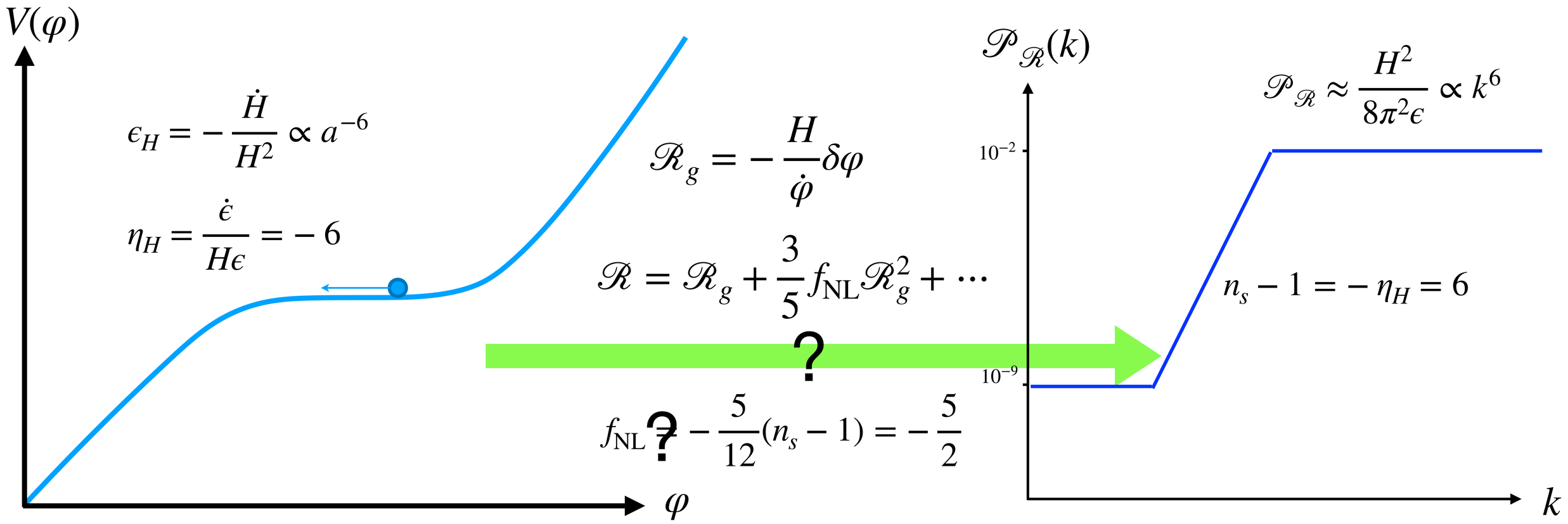


$$\mathscr{P}_{\mathscr{R}} \approx \frac{H^2}{8\pi^2\epsilon_H} \sim 10^{-9} k^{n_s - 1}$$

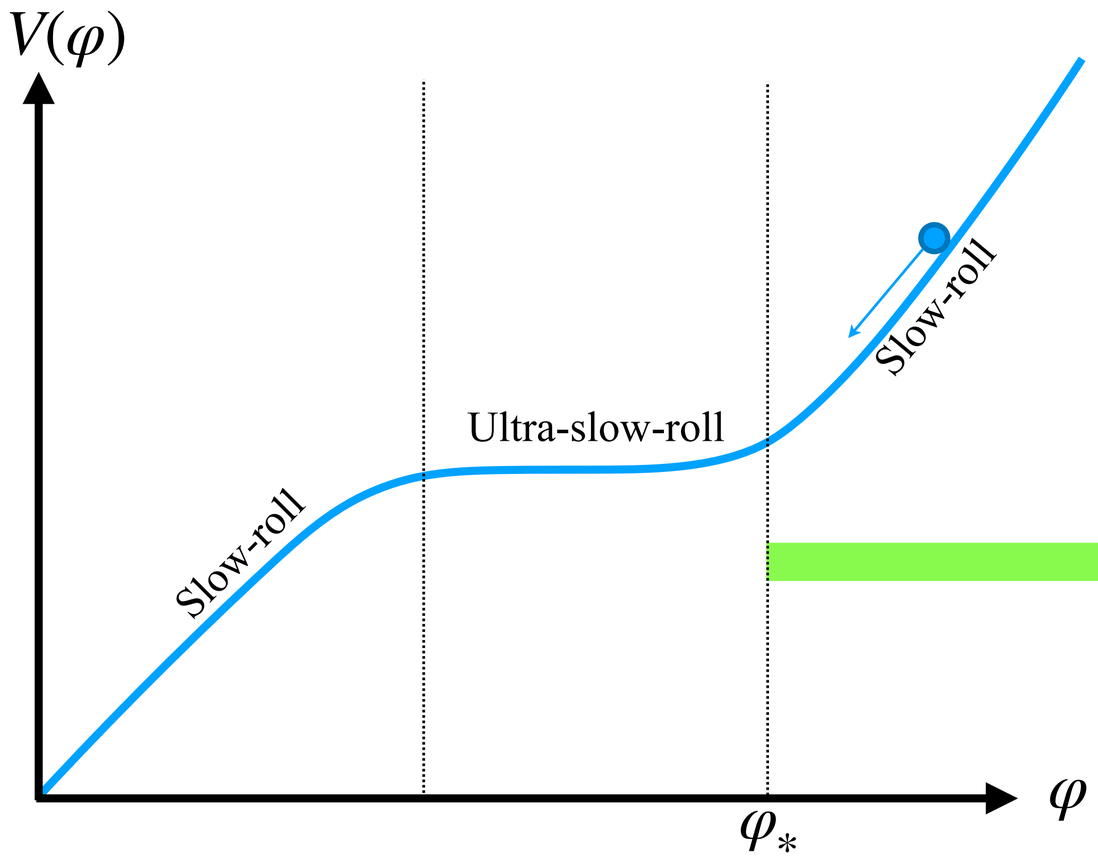
$$n_s - 1 = -2\epsilon_H - 2\eta_H$$

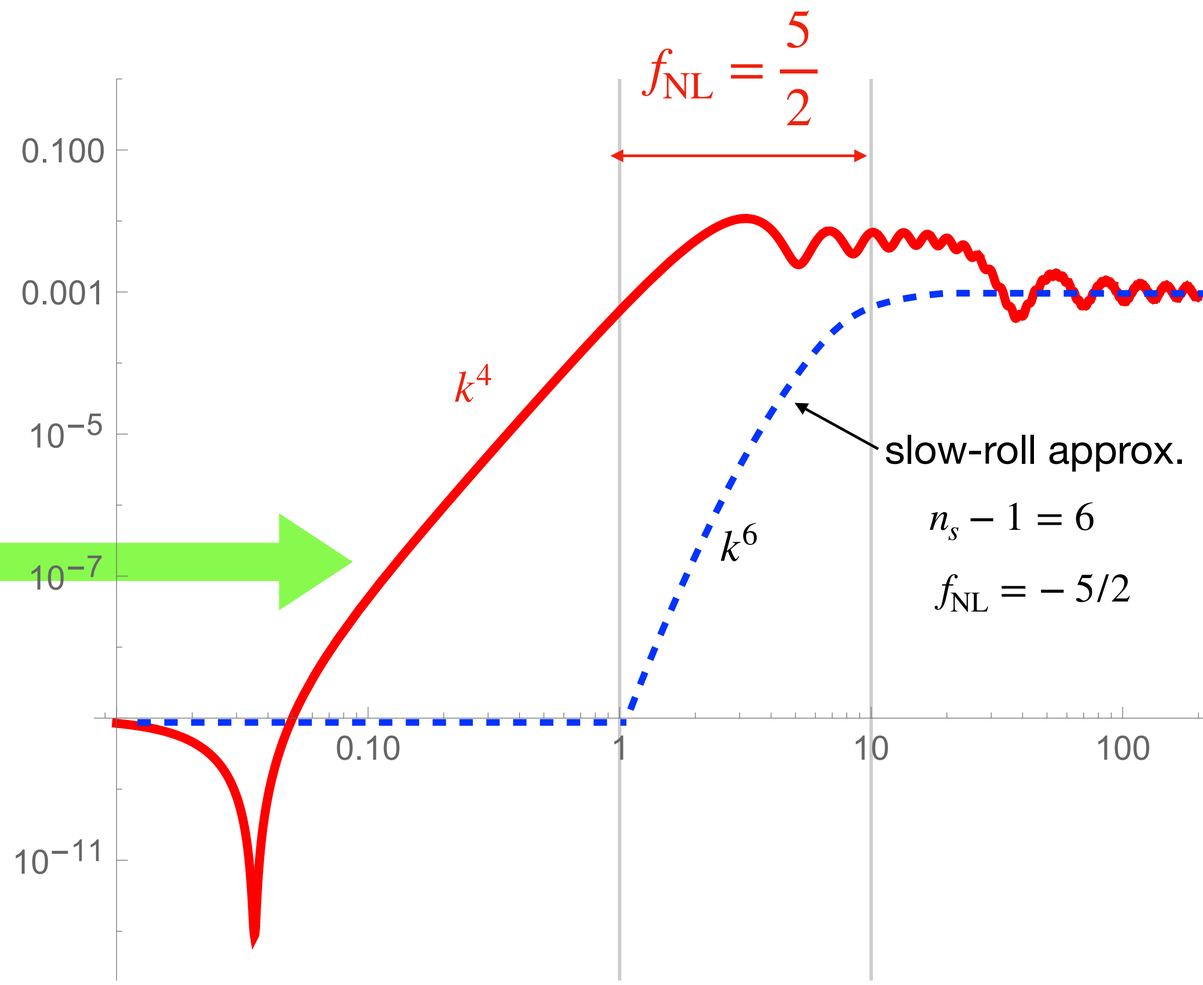
Stewart and Sasaki, astro-ph/9507001
Lyth and Roquigez, astro-ph/0504045
Maldadena, astro-ph/0210603

Ultra-slow-roll inflation



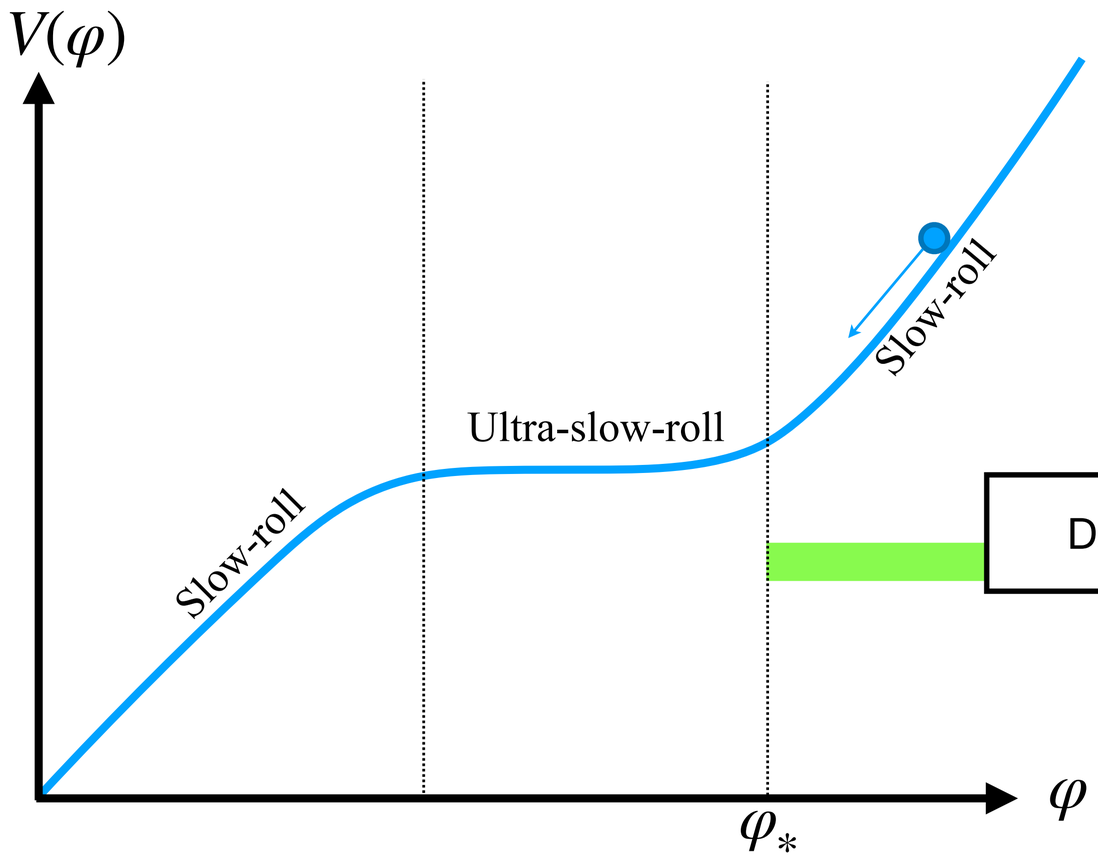
Ultra-slow-roll inflation



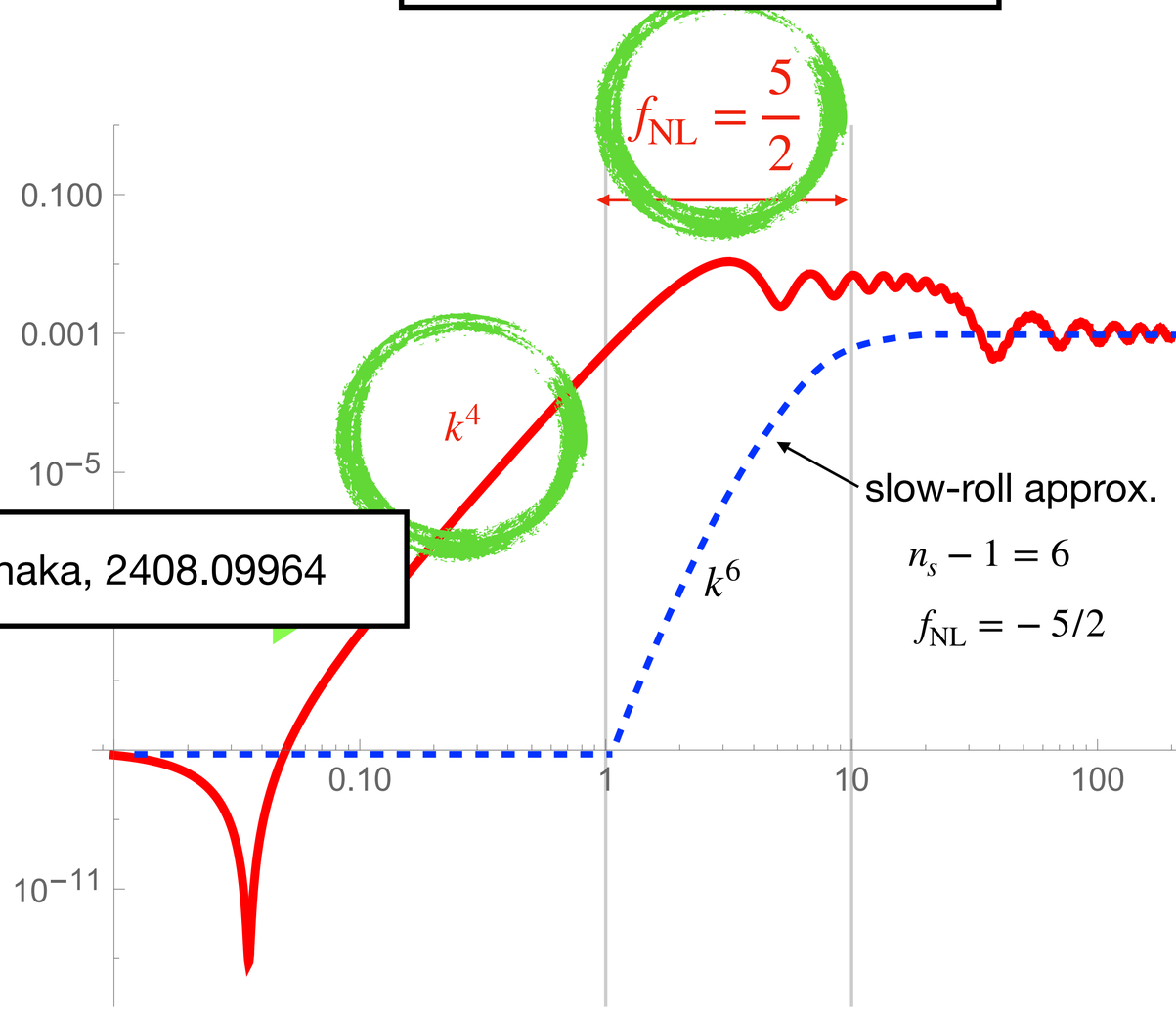


Ultra-slow-roll inflation

SP, Sasaki, 2211.13932

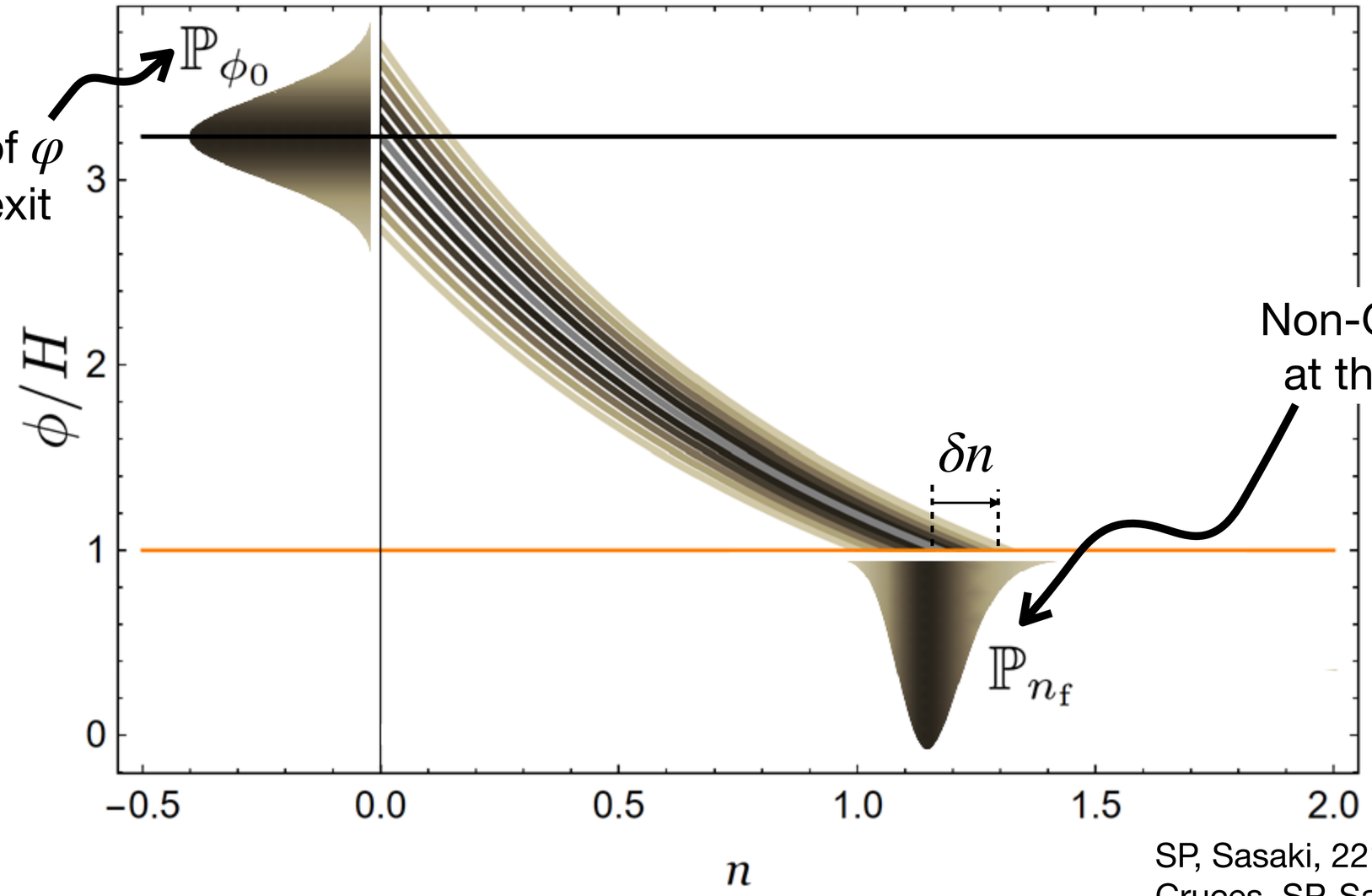


Danilo, SP, Tanaka, 2408.09964



δN formalism

Gaussian PDF of φ at the horizon exit



Non-Gaussian PDF of ζ at the end of inflation

SP, Sasaki, 2211.13932, PRL 131, 011002
Cruces, SP, Sasaki, 2505.24590

Logarithmic Duality

SP, Sasaki, 2211.13932, PRL 131, 011002
SP, 2404.06151 (PBH book ch.7)

$$\mathscr{R}(\delta\varphi, \delta\pi)$$

$$\mathscr{R} = \frac{1}{\lambda_\pm} \ln\left(1 + \frac{\delta\pi + \lambda_\mp \delta\varphi}{\pi + \lambda_\mp \varphi}\right) - \frac{1}{\lambda_\pm} \ln\left(1 + \frac{\delta\pi_*}{\pi_* + \lambda_\mp \varphi_*}\right) + \cdots$$

Logarithmic Duality

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c.f. consistency relation

$$f_{\rm NL} = \frac{5}{2}\epsilon_V - \frac{5}{6}\eta_V$$

$$\left(f_{NL} = -\frac{5}{6}\lambda_- \approx -\frac{5}{6}\eta\right)$$

$\lambda_- \ll 1$

$\lambda_+ \approx 3$

$$\mathscr{R} = -H\frac{\delta\varphi}{\dot\varphi} + \frac{3}{5} f_{\rm NL}\left(-H\frac{\delta\varphi}{\dot\varphi}\right)^2$$

Slow-roll inflation
Stewart and Sasaki, 1995
Lyth and Roquigez, 2005

Logarithmic Duality

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Slow-roll inflation
Stewart and Sasaki, 1995
Lyth and Roquigez, 2005

$\lambda_- = -1/\mu$

$\lambda_+ = 3 - 1/\mu$

$$\mathscr{R} = -\mu\ln\left(1-\frac{\mathscr{R}_g}{\mu}\right)$$

Constant-roll
Atal, Garriga, Marcos-Caballero, 1905.13202
Atal, Cid, Escrivà, Garriga, 1908.11357
Escrivà, Atal, Garriga, 2306.09990
Inui, Motohashi, SP, et al, 2409.13500

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Atal, Garriga, Marcos-Caballero, 1905.13202
Atal, Cid, Escrivà, Garriga, 1908.11357
Escrivà, Atal, Garriga, 2306.09990
Inui, Motohashi, SP, et al, 2409.13500

$\lambda_- = 0$

$\lambda_+ = 3$

$$\mathscr{R} = -\frac{1}{3}\ln\left(1+\frac{\delta\pi_*}{\pi_*}\right)$$

Ultra-slow-roll
Namjoo, Firouzjahi, Sasaki, 1210.3692
Cai, Chen, et al 1712.09998
Biagetti et al 1804.07124
Passaglia et al 1812.08243

Logarithmic Duality

SP, Sasaki, 2211.13932, PRL 131, 011002
SP, 2404.06151 (PBH book ch.7)

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Atal, Garriga, Marcos-Caballero, 1905.13202
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Escrivà, Atal, Garriga, 2306.09990
Inui, Motohashi, SP, et al, 2409.13500

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Ultra-slow-roll
Namjoo, Firouzjahi, Sasaki, 1210.3692
Cai, Chen, et al 1712.09998
Biagetti et al 1804.07124
Passaglia et al 1812.08243

$\lambda_- = -3/2$
$\lambda_- = 9/2$

$$\mathscr{R} = \frac{2}{3}\ln(1+\delta)$$

Curvaton scenario,
SP and Sasaki, 2112.12680
Ferrante et al, 2211.01728
Hooper et al. 2308.00756

Logarithmic Duality

SP, Sasaki, 2211.13932, PRL 131, 011002
SP, 2404.06151 (PBH book ch.7)

$$\mathscr{R}(\delta\varphi, \delta\pi)$$

$$\mathscr{R} = \frac{1}{\lambda_\pm}\ln\left(1+\frac{\delta\pi+\lambda_\mp\delta\varphi}{\pi+\lambda_\mp\varphi}\right) - \frac{1}{\lambda_\pm}\ln\left(1+\frac{\delta\pi_*}{\pi_*+\lambda_\mp\varphi_*}\right)+\cdots$$

c.f. consistency relation

$$f_{\rm NL} = \frac{5}{2}\epsilon_V - \frac{5}{6}\eta_V$$

$$\left(f_{NL} = -\frac{5}{6}\lambda_- \approx -\frac{5}{6}\eta\right)$$

$\lambda_- \ll 1$
$\lambda_+ \approx 3$

$$\mathscr{R} = -H\frac{\delta\varphi}{\dot\varphi} + \frac{3}{5}f_{\rm NL}\left(-H\frac{\delta\varphi}{\dot\varphi}\right)^2$$

Slow-roll inflation
Stewart and Sasaki, 1995
Lyth and Roquigez, 2005

$\lambda_- = -1/\mu$
$\lambda_+ = 3 - 1/\mu$

$$\mathscr{R} = -\mu\ln\left(1-\frac{\mathscr{R}_g}{\mu}\right)$$

Constant-roll
Atal, Garriga, Marcos-Caballero, 1905.13202
Atal, Cid, Escrivà, Garriga, 1908.11357
Escrivà, Atal, Garriga, 2306.09990
Inui, Motohashi, SP, et al, 2409.13500

$\lambda_- = 0$
$\lambda_+ = 3$

$$\mathscr{R} = -\frac{1}{3}\ln\left(1+\frac{\delta\pi_*}{\pi_*}\right)$$

Ultra-slow-roll
Namjoo, Firouzjahi, Sasaki, 1210.3692
Cai, Chen, et al 1712.09998
Biagetti et al 1804.07124
Passaglia et al 1812.08243

$\lambda_- = -3/2$
$\lambda_- = 9/2$

$$\mathscr{R} = \frac{2}{3}\ln(1+\delta)$$

Curvaton scenario,
SP and Sasaki, 2112.12680
Ferrante et al, 2211.01728
Hooper et al. 2308.00756

$$\mathscr{R} = -\frac{1}{\lambda}\ln\left(f(\mathscr{R}_{\rm G})\right)$$

Extensions,
Celoria et al, 2103.09244
Kawaguchi et al, 2305.18140

Probability Distribution Function

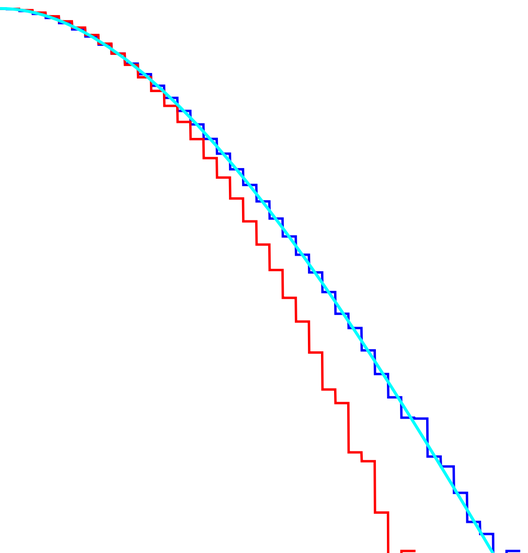
SP, Sasaki, 2211.13932, PRL 131, 011002
SP, 2404.06151 (PBH book ch.7)

For the simplest single-logarithm case: $\mathscr{R} \equiv \delta N = \frac{1}{\lambda_-} \ln\left(1 + \frac{\delta\pi + \lambda_+ \delta\varphi}{\pi + \lambda_+ \varphi}\right)$

$$\Downarrow \quad P(\mathscr{R})d\mathscr{R} = P(\delta\varphi)d\delta\varphi$$

Gaussian PDF with variance $\sigma^2_{\delta\varphi}$

$$P(\mathscr{R}) = \frac{e^{\lambda_- \mathscr{R}}}{\sqrt{2\pi}\sigma_{\delta\varphi}} \left|\lambda_-\right| \varphi \exp\left[-\frac{\varphi^2}{2\sigma^2_{\delta\varphi}}\left(e^{\lambda_- \mathscr{R}} - 1\right)^2\right]$$



Probability Distribution Function

SP, Sasaki, 2211.13932, PRL 131, 011002
SP, 2404.06151 (PBH book ch.7)

For the simplest single-logarithm case: $\mathscr{R} \equiv \delta N = \frac{1}{\lambda_-} \ln\left(1 + \frac{\delta\pi + \lambda_+ \delta\varphi}{\pi + \lambda_+ \varphi}\right)$

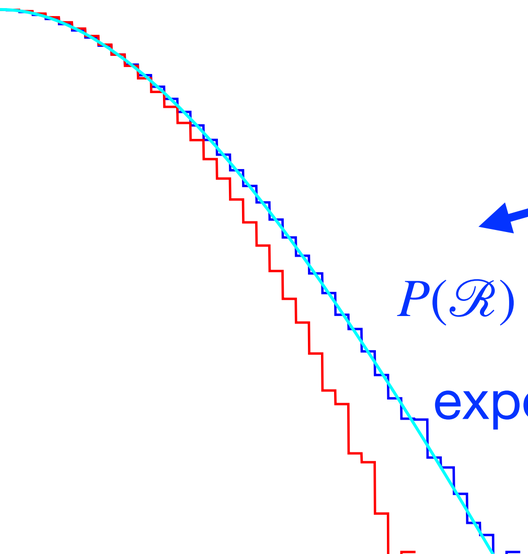
$\Downarrow$ $P(\mathscr{R})d\mathscr{R} = P(\delta\varphi)d\delta\varphi$ → Gaussian PDF with variance $\sigma^2_{\delta\varphi}$

$$P(\mathscr{R}) = \frac{e^{\lambda_- \mathscr{R}}}{\sqrt{2\pi}\sigma_{\delta\varphi}} \left|\lambda_-\right| \varphi \exp\left[-\frac{\varphi^2}{2\sigma^2_{\delta\varphi}}\left(e^{\lambda_- \mathscr{R}} - 1\right)^2\right]$$

$\lambda_- < 0$

$P(\mathscr{R}) \sim e^{\lambda_- \mathscr{R}}$

exponential tail



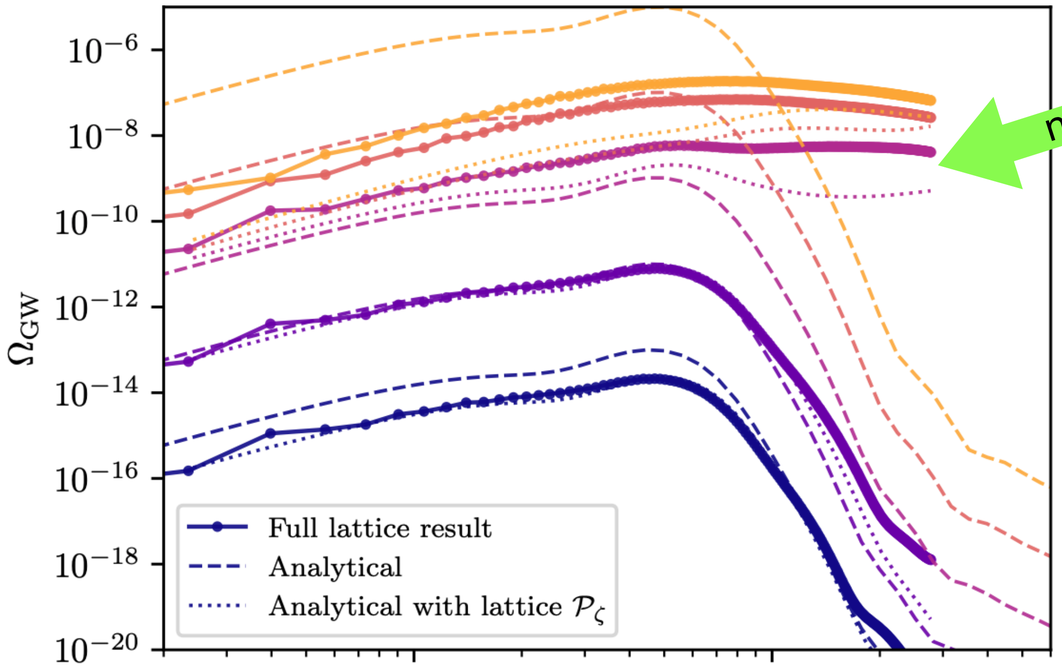
$\lambda_- > 0$

$P(\mathscr{R}) \sim \exp\left(-c^2 e^{2\lambda_- \mathscr{R}}\right)$

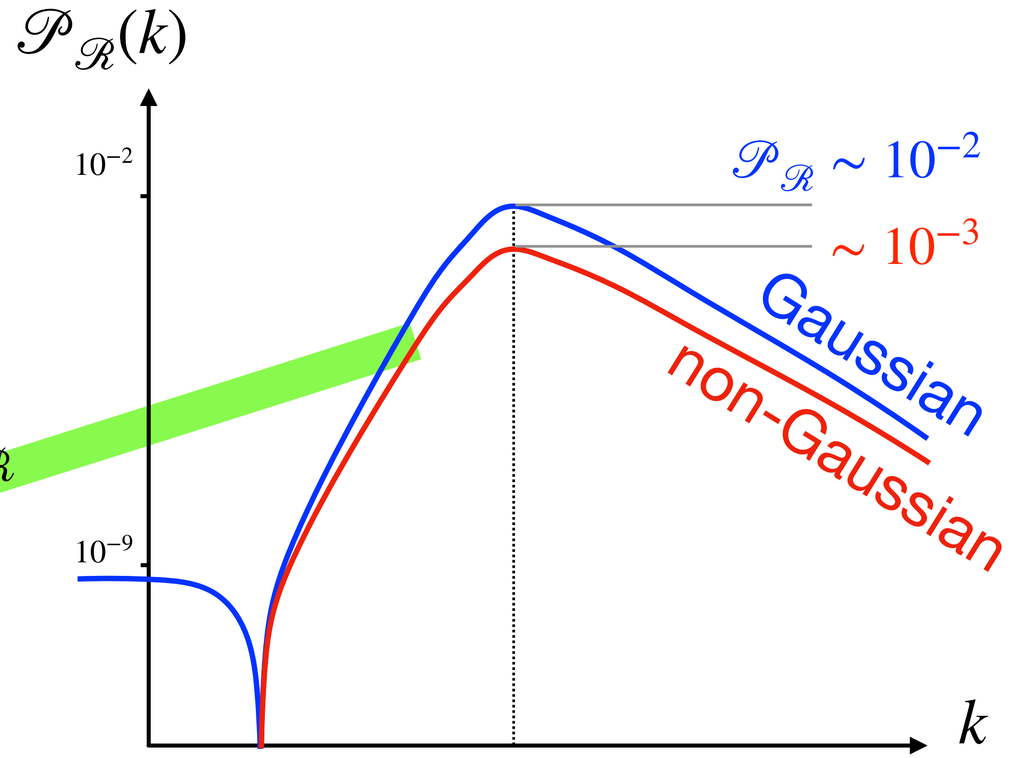
Gumbel-distribution-like tail

PBH Dark Matter

- Quadratic expansion: Cai, SP, Sasaki 1810.11000
- Higher orders: Adshead+ 2105.01659. Abe+ 2209.13891. Garcia-Saenz+ 2207.14267. Yuan+, 2308.07155; Li+, 2309.07792. Perna+, 2403.06962
- Lattice simulation marks the breakdown of perturbativity: Zeng+ 2508.10812, Caravano+ 2604.03628
- IGW is robust against non-Gaussianity when PBH amount is fixed (?)



nonlinear coupling $h\mathcal{R}\mathcal{R}$
with non-Gaussianity



Zeng, Ning, Cai, Wang 2508.10812
Caravano, Franciolini, Renaux-Petel 2604.03628

Conclusion

- We resolved the gauge issue of secondary GWs, by establishing the connection between strain and observed time shift.

- PBHs are viable dark matter candidates, which can only be probed by induced GW. The associated IGW is an important target for next-generation GW detectors.

- Non-Gaussianity is essential in establishing the PBH-IGW cross check; but large NGs can only be studied case by case.