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Geodesic Focusing Conditions in $f(Q)$ Gravity

Chunhui Liu

Hangzhou Institute for Advanced Study, UCAS

Collaborators: Chao-Qiang Geng, Ling-Wei Luo, and Jianhui Qiu

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2 Geometric Setup

3 Congruence Kinematics

4 Modified Raychaudhuri Equation and Focusing Condition

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Motivation and Central Questions

What does "focusing" mean beyond General Relativity?

For an initially converging timelike geodesic congruence, can its expansion θ be guaranteed to diverge to $-\infty$ in finite proper time?

1. How do metric geodesics differ from autoparallels in symmetric teleparallel geometry (STG)?
2. What is the correct covariant geodesic-deviation equation when the relative acceleration is defined with the STG connection?
3. Which assumptions lead to a matter-dependent Raychaudhuri equation and to a useful focusing condition in $f(Q)$ gravity?
4. What happens to this condition in a flat FLRW universe?

STG kinematics $\longrightarrow$ modified Raychaudhuri $\longrightarrow$ $T \leq T_{\mathrm{eff}}$ $\longrightarrow$ FLRW test

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Symmetric Teleparallel Geometry (STG)

Defining conditions

$$R^{\rho}{}_{\sigma\mu\nu}(\Gamma) = 0, \qquad T^{\rho}{}_{\mu\nu} = 2\Gamma^{\rho}{}_{[\mu\nu]} = 0.$$

Curvature and torsion of the independent connection vanish, but nonmetricity is allowed:

$$Q_{\rho\mu\nu} := \nabla_{\rho} g_{\mu\nu} \neq 0.$$

Connection decomposition

$$\Gamma^{\rho}{}_{\mu\nu} = \mathring{\Gamma}^{\rho}{}_{\mu\nu} + L^{\rho}{}_{\mu\nu},$$

$$L^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}(Q_{\alpha\mu\nu} - Q_{\mu\nu\alpha} - Q_{\nu\alpha\mu}).$$

- $\mathring{\Gamma}$ is the Levi–Civita connection; $L^{\rho}{}_{\mu\nu}$ is the *disformation*.
- In the coincident gauge, one can locally choose $\Gamma^{\rho}{}_{\mu\nu} = 0$; this does not imply $\mathring{\Gamma}^{\rho}{}_{\mu\nu} = 0$.
- Nonmetricity traces: $Q_{\rho} := Q_{\rho\mu\nu} g^{\mu\nu}$ and $\widetilde{Q}_{\rho} := Q_{\mu\nu\rho} g^{\mu\nu}$.

The $f(Q)$ Action and Field Equations

Nonmetricity-based gravitational dynamics

$$S = \frac{1}{2\kappa}\int \mathrm{d}^4x\,\sqrt{-g}\,f(Q) + S_{\mathrm{m}}, \qquad Q := Q_{\rho\mu\nu}P^{\rho\mu\nu}, \quad \kappa = 8\pi G.$$

Nonmetricity conjugate

$$P^{\rho\mu\nu} := -\frac{1}{4}Q^{\rho\mu\nu} + \frac{1}{2}Q^{(\mu\nu)\rho} + \frac{1}{4}\left(Q^{\rho} - \widetilde{Q}^{\rho}\right)g^{\mu\nu} - \frac{1}{4}g^{\rho(\mu}Q^{\nu)}.$$

The metric field equation (in the paper's conventions) is

$$f_Q\,\mathring{G}_{\mu\nu} + \frac{1}{2}(Qf_Q - f)g_{\mu\nu} + 2(\nabla_\rho f_Q)P^{\rho}{}_{\mu\nu} = \kappa T_{\mu\nu}, \quad f_Q := \frac{\mathrm{d}f}{\mathrm{d}Q}.$$

Trace equation: the bridge to matter-dependent focusing

$$\nabla_\rho\left[f_Q(Q^\rho - \widetilde{Q}^\rho)\right] + \frac{1}{2}f_Q Q_\rho(Q^\rho - \widetilde{Q}^\rho) - f_Q Q + 2(Qf_Q - f) = \kappa T, \qquad T := g^{\mu\nu}T_{\mu\nu}.$$

Key point: $f(Q) = Q$ gives STEGR (dynamically equivalent to GR); the paper takes $f_Q > 0$.

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Metric Geodesics vs. STG Autoparallels

Metric geodesic (extremizes proper time)

$$\dot{u}^\rho + \mathring{\Gamma}^\rho{}_{\mu\nu} u^\mu u^\nu = 0, \qquad g_{\mu\nu} u^\mu u^\nu = -1.$$

Writing this very same curve with the STG connection yields

$$\underbrace{u^\mu \nabla_\mu u^\rho}_{a^\rho} = \underbrace{L^\rho{}_{\mu\nu} u^\mu u^\nu}_{\text{disformation-induced acceleration}},$$

whereas an STG autoparallel obeys $u^\mu \nabla_\mu u^\rho = 0$.

A subtle identity

$$u^\mu \nabla_\mu u_\rho = a_\rho + Q_{\mu\nu\rho} u^\mu u^\nu.$$

- Raising/lowering indices does *not* commute with ∇.
- The physical curve and the choice of covariant description must be distinguished.

Geodesic Deviation: Curvature vs. Disformation

For neighboring metric geodesics $x^\rho(\tau)$ and $x^\rho(\tau) + n^\rho(\tau)$, compare the covariant formulations.

Levi–Civita description in GR

$$\frac{\mathring{D}^2 n^\rho}{\mathrm{d}\tau^2} + \mathring{R}_{\alpha\mu\nu}{}^\rho n^\alpha u^\mu u^\nu = 0.$$

STG description: the paper's main kinematic result

$$\frac{D^2 n^\rho}{\mathrm{d}\tau^2} - 2L^\rho{}_{\mu\alpha} u^\mu \frac{Dn^\alpha}{\mathrm{d}\tau} - n^\alpha (\nabla_\alpha L^\rho{}_{\mu\nu}) u^\mu u^\nu = 0.$$

- $L^\rho{}_{\mu\alpha}$ yields a velocity-dependent contribution; $\nabla_\alpha L^\rho{}_{\mu\nu}$ contributes to relative acceleration.
- The deviation equation is unchanged mathematically, however, its covariant expression physically depends on the connection used in $D/\mathrm{d}\tau$.

Congruence Expansion, Shear and Vorticity

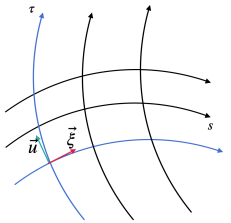
Let $x^\rho = x^\rho(\tau, s)$ parameterize the congruence.

$$u^\rho := \left.\frac{\partial x^\rho}{\partial \tau}\right|_s, \qquad \xi^\rho := \left.\frac{\partial x^\rho}{\partial s}\right|_\tau .$$

τ: proper time; s: geodesic label.

The Lie-dragging relation $[u, \xi]^\rho = 0$ gives

$$\frac{D\xi^\rho}{\mathrm{d}\tau} = \xi^\alpha \nabla_\alpha u^\rho, \qquad \bar{B}^\mu{}_\nu := \nabla_\nu u^\mu .$$



Tangent u^ρ and connecting vector ξ^ρ.

Spatial projection

$$h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu,$$
$$B^{\mu\nu} = h^\mu{}_\alpha h^\nu{}_\beta \bar{B}^{\alpha\beta},$$
$$B^{\mu\nu} = \sigma^{\mu\nu} + \omega^{\mu\nu} + \frac{1}{3}\theta h^{\mu\nu}.$$

Nonmetricity corrections

$$\theta = \bar{\theta} + a_\mu u^\mu,$$
$$\theta = \mathring{\theta} - \frac{1}{2} h^{\alpha\beta} Q_{\gamma\alpha\beta} u^\gamma .$$

- $u_\mu a^\mu$ need not vanish.
- Hypersurface orthogonality need not imply $\omega_{\mu\nu} = 0$.

Benchmark: Raychaudhuri Focusing in GR

For timelike metric geodesics, with the paper's scalar definitions $2\mathring{\sigma}^2 = \mathring{\sigma}_{\mu\nu}\mathring{\sigma}^{\mu\nu}$ and similarly for $\mathring{\omega}^2$,

$$\frac{\mathring{D}\mathring{\theta}}{\mathrm{d}\tau} = 2(\mathring{\omega}^2 - \mathring{\sigma}^2) - \frac{1}{3}\mathring{\theta}^2 - \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\,T\right)u^\mu u^\nu.$$

Focusing conditions in the familiar setting

- Hypersurface orthogonality in GR: $\mathring{\omega}_{\mu\nu} = 0$.
- Strong energy condition: $(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\,T)u^\mu u^\nu \geq 0$.

$$\boxed{\frac{\mathring{D}\mathring{\theta}}{\mathrm{d}\tau} \leq -\frac{1}{3}\mathring{\theta}^2.}$$

For $\mathring{\theta}(0) < 0$, this differential inequality gives finite-time focusing (no later than $3/|\mathring{\theta}(0)|$ under the hypotheses).

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A Geometric Raychaudhuri Equation in STG

The STG deviation equation, together with $D\xi^\rho/\mathrm{d}\tau = \xi^\alpha \bar{B}^\rho{}_\alpha$, gives

Deformation-tensor evolution

$$\frac{D\bar{B}^\rho{}_\alpha}{\mathrm{d}\tau} + \bar{B}^\rho{}_\mu \bar{B}^\mu{}_\alpha - \nabla_\alpha a^\rho = 0, \quad a^\rho = L^\rho{}_{\mu\nu} u^\mu u^\nu.$$

Taking a trace and using the projected kinematic decomposition,

$$\frac{D\bar{\theta}}{\mathrm{d}\tau} = 2(\omega^2 - \sigma^2) - \frac{1}{3}\theta^2 + (a_\alpha u^\alpha)^2 + 2u^\alpha a^\mu \nabla_\mu u_\alpha + \nabla_\rho a^\rho.$$

Key point: At this stage the relation is geometric; the matter trace T enters only after additional assumptions and field equations.

Essential Assumptions: Conversion + Weyl Type

Assumption 1: nonmetricity–acceleration conversion

$$A^\rho := g^{\rho\mu} u^\nu u^\sigma Q_{\sigma\mu\nu} + 2L^\rho{}_{\mu\nu} u^\mu u^\nu = 0.$$

Normalization establishes only $u_\rho A^\rho = 0$; $\boxed{A^\rho = 0}$ is an extra restriction.

Assumption 2: Weyl-type nonmetricity

$$Q_{\rho\mu\nu} = \varphi_\rho g_{\mu\nu} = \frac{1}{4} Q_\rho g_{\mu\nu}, \qquad \widetilde{Q}_\rho = \frac{1}{4} Q_\rho.$$

Together they imply

$$Q^\rho = -u^\rho (u^\alpha Q_\alpha), \quad Q = \frac{3}{32} Q_\rho Q^\rho, \quad \boxed{a^\rho = \frac{1}{8} Q^\rho}.$$

For hypersurface-orthogonal congruences, the Weyl ansatz also gives $\sigma_{\mu\nu} = \mathring{\sigma}_{\mu\nu}$ and $\omega_{\mu\nu} = 0$.

How Matter Enters the Evolution Equation

Under the two assumptions, the unprojected expansion equation reduces to

An intermediate Raychaudhuri equation

$$\frac{D\bar{\theta}}{\mathrm{d}\tau} = 2(\omega^2 - \sigma^2) - \frac{1}{3}\theta^2 + \frac{1}{6}Q + \frac{1}{8}\nabla_\rho Q^\rho.$$

The field-equation trace expresses the divergence of Q^ρ in terms of matter and gravitational variables:

$$\nabla_\rho Q^\rho = \frac{4\kappa T}{3f_Q} - Q^\rho \nabla_\rho \ln|f_Q| - 4Q - \frac{8}{3}\left(Q - \frac{f}{f_Q}\right).$$

- The $(4\kappa T)/(3f_Q)$ term supplies the explicit matter dependence.
- One must also convert from $\bar{\theta}$ to the projected expansion θ before formulating focusing.

Modified Raychaudhuri Equation in $f(Q)$ Gravity

Main result: projected expansion θ

$$\frac{D\theta}{\mathrm{d}\tau} = 2(\omega^2 - \sigma^2) - \frac{1}{3}\theta^2 + \frac{\kappa T}{6 f_Q} - \frac{1}{8} Q^\rho \nabla_\rho \ln |f_Q| - \frac{1}{2} Q - \frac{1}{3}\left(Q - \frac{f}{f_Q}\right) + \frac{1}{8} u^\nu u_\alpha \nabla_\nu Q^\alpha .$$

- The nonmetricity terms have no universal definite sign.
- The usual GR strong-energy-condition argument does not directly force $D\theta/\mathrm{d}\tau \leq -\theta^2/3$.
- The formula is derived under the Weyl-type and $A^\rho = 0$ restrictions; it is not claimed for arbitrary STG nonmetricity.

Effective Trace and the Focusing Criterion

With vanishing rotation, define F by

$$\frac{D\theta}{\mathrm{d}\tau} = F(Q, f_Q, T, \ldots) - \frac{1}{3}\theta^2.$$

A sufficient focusing condition

$$\boxed{F \le 0} \quad \Longleftrightarrow \quad \boxed{T \le T_{\mathrm{eff}}} \quad (f_Q > 0).$$

$$T_{\mathrm{eff}} = \frac{6 f_Q}{\kappa}\left[2\sigma^2 + \frac{1}{8}Q^\rho \nabla_\rho \ln|f_Q| + \frac{1}{2}Q + \frac{1}{3}\left(Q - \frac{f}{f_Q}\right) - \frac{1}{8}u^\nu u_\alpha \nabla_\nu Q^\alpha\right].$$

Key point: For $\theta(0) < 0$, the bound implies focusing in finite proper time; the condition is model/background dependent.

Flat FLRW: Which STG Connection Branch Survives?

For $\mathrm{d}s^2 = a^2(\eta)(-\mathrm{d}\eta^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j)$, set $\mathcal{H} = a'/a$ and $u^\mu = (1/a, 0, 0, 0)$.

Homogeneous and isotropic torsion-free connection (Hohmann 2021)

$$\Gamma^0{}_{00} = K_1, \qquad \Gamma^0{}_{ij} = K_2\delta_{ij}, \qquad \Gamma^i{}_{0j} = \Gamma^i{}_{j0} = K_3\delta^i{}_j.$$

Branch	K_1	K_2	K_3
1	γ_1	0	0
2	$\gamma_2'/\gamma_2 + \gamma_2$	0	γ_2
3	$-\gamma_3'/\gamma_3$	γ_3	0

The Weyl ansatz imposes $Q_{0ij} \propto g_{ij}$ and $Q_{i0j} = 0$; comparison with the FLRW $Q_{\rho\mu\nu}$ requires

$$\boxed{K_1 = K_2 = K_3} \implies \boxed{\text{Branch 1 with } \gamma_1 = 0} \quad \text{(coincident gauge)}.$$

FLRW Result: The Focusing Bound Is Saturated

In the coincident gauge, the two independent background equations are

$$3f_Q\mathcal{H}^2 + \frac{a^2}{2}(f - Qf_Q) = \kappa a^2\rho,$$
$$f_Q(2\mathcal{H}' + \mathcal{H}^2) + \frac{a^2}{2}(f - Qf_Q) + 2\mathcal{H}f'_Q = -\kappa a^2 p.$$

Effective trace evaluated using the Friedmann equations

$$T_{\rm eff} = -\frac{2}{\kappa a^2}\left[3f_Q(\mathcal{H}^2 + \mathcal{H}') + 3\mathcal{H}f'_Q + a^2(f - Qf_Q)\right]$$
$$= \boxed{-\rho + 3p = T}.$$

- $\boxed{T \leq T_{\rm eff}}$ is saturated: it places no new restriction on the perfect-fluid matter content.
- The result is independent of the functional form of $f(Q)$, *provided* $f_Q > 0$ and the preceding assumptions hold.
- **Scope:** flat FLRW, the compatible coincident-gauge branch, Weyl-type nonmetricity and the paper's conversion condition $A^\rho = 0$.

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Summary

- The acceleration is defined by the STG connection.
- We derive the Raychaudhuri equation in STG.
- The Weyl-type ansatz and $A^\rho = 0$ have been imposed in our discussion.
- In the flat FLRW universe:
  - only the coincident gauge (the branch of $K_1 = K_2 = K_3 = 0$) is compatible with the Weyl-type ansatz.
  - The focusing condition $T \leq T_{\text{eff}}$ is always satisfied.
    $\Longrightarrow$ The matter would not be constrained by focusing condition.