

How to Bootstrap a Dark Photon

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Outline

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- 05 Conclusions

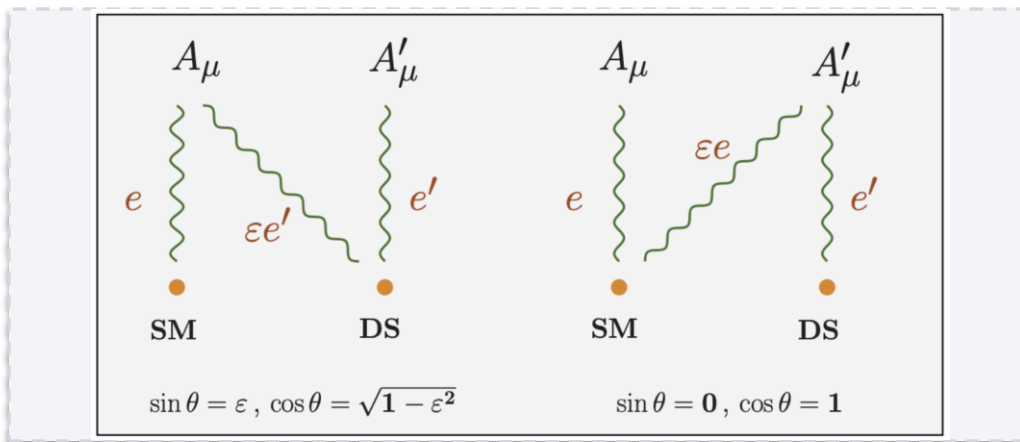
Introduction

The Dark Photon and bootstrapping

01

The Dark Photon

Kinetic mixing as a portal to dark sectors



Mixing of standard and dark photon · arXiv:2005.01515

- ✓ Dark sectors can be connected with the Standard Model through portals.
- ✓ An extra dark U(1) gauge field mixes Kinetically with the Standard Model photon.
- ✓ The dark photon acts as a portal to dark matter or as dark matter itself.

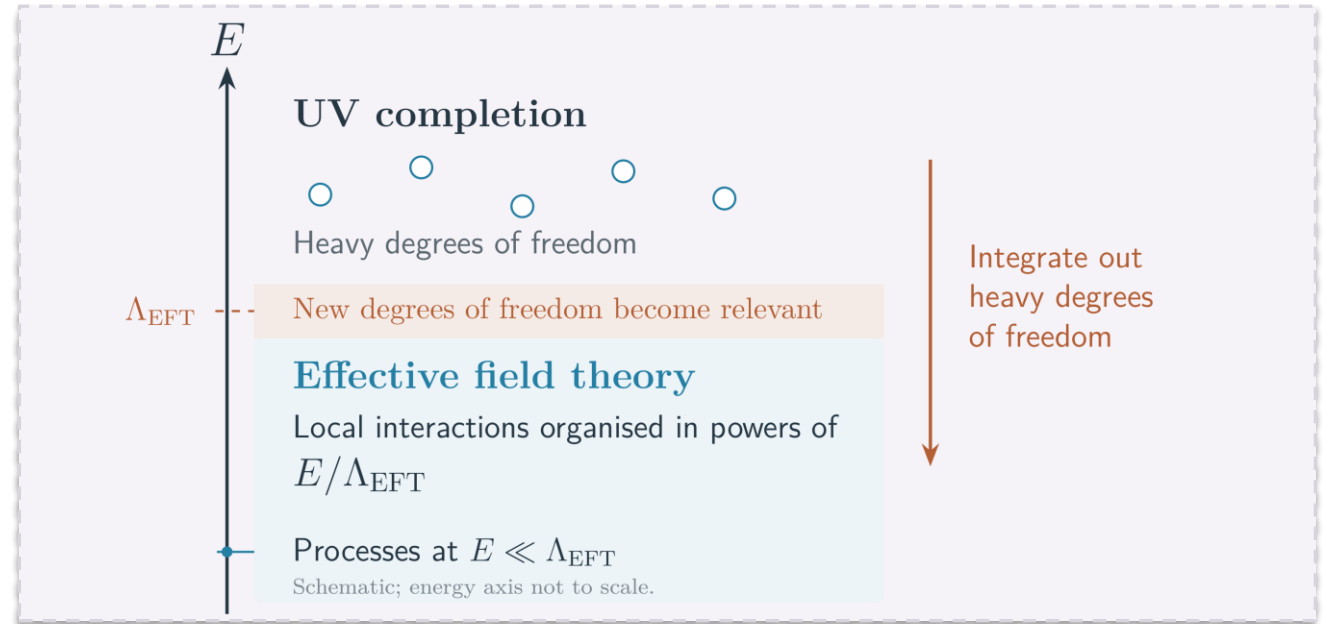
$$\mathcal{L}_{\text{mix}} = -\frac{\epsilon}{2} F_{\mu\nu} F'^{\mu\nu}.$$

Kinetic mixing allows for dark photon theories to be probed by non-gravitational sources.

Effective Field Theories

EFTs and UV completions

- ✓ EFTs lose predictive control at energies near the cutoff.
- ✓ UV-completing an EFT imposes restrictions on its parameters.



Schematic energy scale

Bootstrapping consists of using S-matrix properties of a UV theory to constrain the underlying EFT.

Positivity bounds

For scalar theories

Scalar scattering amplitude in the forward limit.

$$\mathcal{M}_{\phi\phi}(s, 0) = \mathcal{M}_{\text{light poles}}(s, 0) + B_{\phi}(s, 0),$$

$$B_{\phi}(s, 0) = b_0 + b_2(s - 2m_{\phi}^2)^2 + \mathcal{O}\left((s - 2m_{\phi}^2)^4\right),$$

Positivity bounds from twice subtracted dispersion relations.

$$b_2 \equiv \frac{1}{2} \partial_s^2 B_{\phi}(s, 0) \Big|_{s=2m_{\phi}^2} = \frac{2}{\pi} \int_{4m_{\phi}^2}^{\infty} \frac{d\sigma \operatorname{Im} \mathcal{M}_{\phi\phi}(\sigma, 0)}{(\sigma - 2m_{\phi}^2)^3} \geq 0.$$

- ✓ Dispersion relations relate low energy WC to absorptive scattering data of the UV theory.
- ✓ The optical theorem enforces a non-negative absorptive part for the amplitude.
- ✓ The argument can be extended to particles with spin.

The S-matrix properties of the UV theory restrict the sign of certain low-energy observables.

arXiv:hep-th/0602178

State-of-the-art

Current bounds known from bootstrapping dark photons

02

Higgsed dark photon model

Dark sector low-energy model

Dark photon Lagrangian

$$\mathcal{L}_D = -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} - (\mathcal{D}_\mu\Phi_D)^*\mathcal{D}^\mu\Phi_D - \lambda_D\left(|\Phi_D|^2 - \frac{v_D^2}{2}\right)^2, \quad \mathcal{D}_\mu = \partial_\mu - ig_DA'_\mu$$

Spontaneous Symmetry Breaking

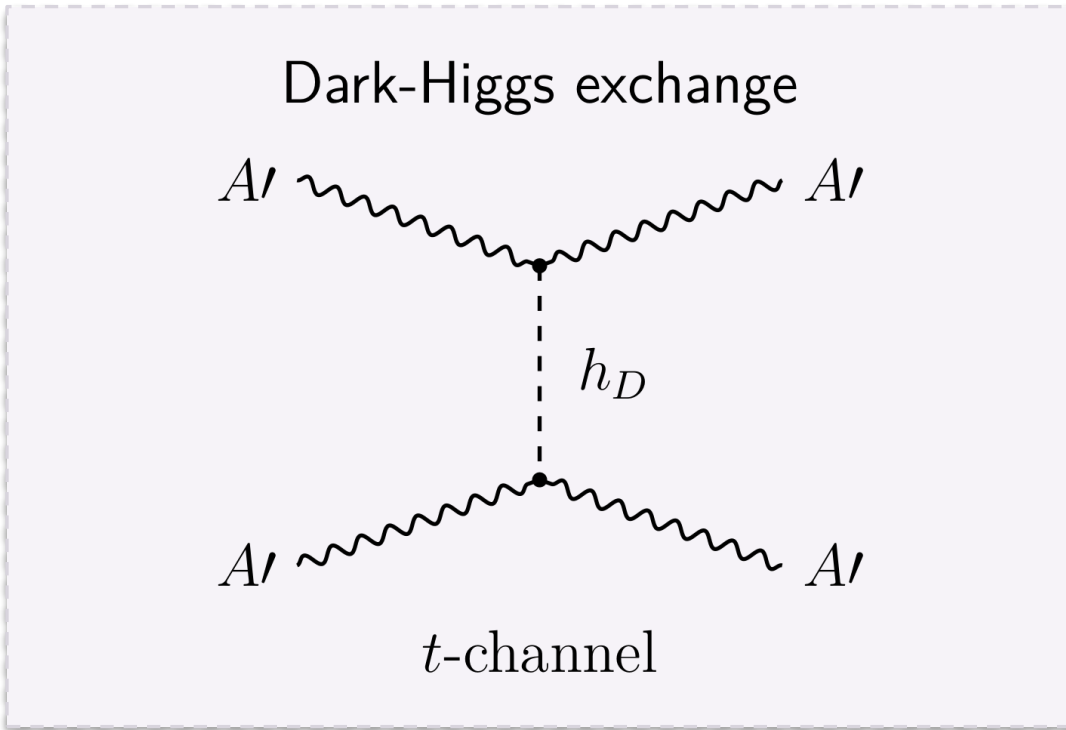
$$\Phi_D = \frac{v_D + h_D}{\sqrt{2}} \quad (\text{unitary gauge}), \quad m = g_D v_D, \quad m_h^2 = 2\lambda_D v_D^2$$

- ✓ The Dark Photon acquires mass via Spontaneous Symmetry Breaking.
- ✓ The Dark Higgs can be integrated out in certain energy regimes.

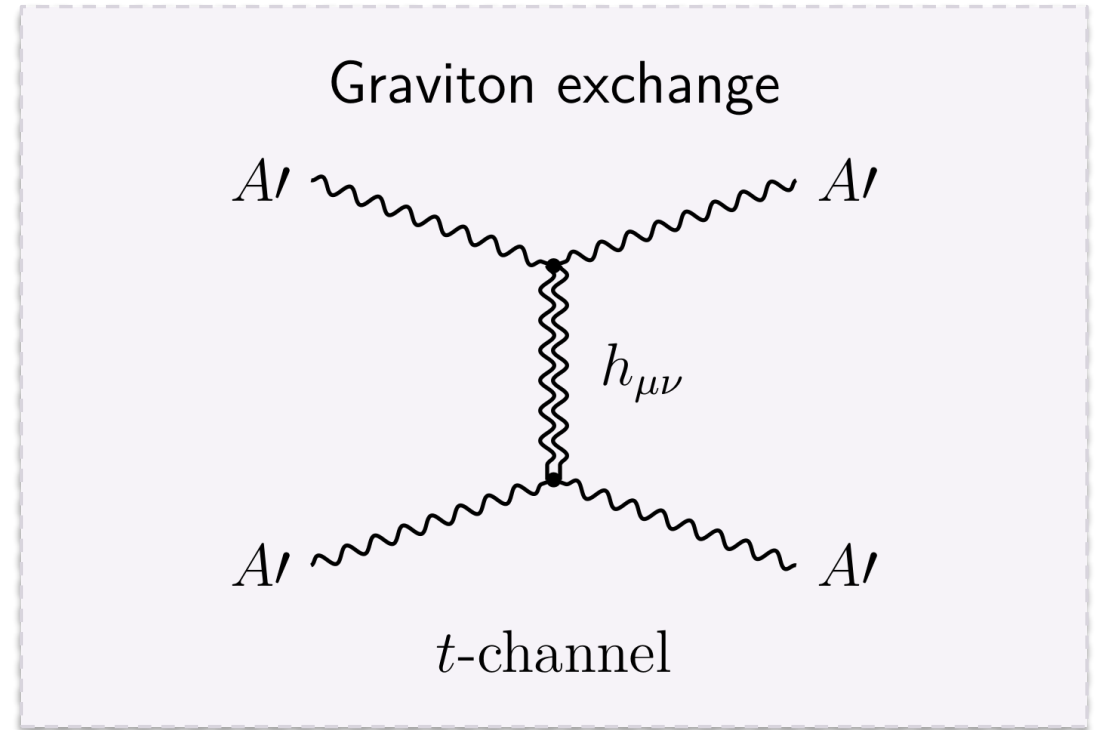
$$\sqrt{|s|}, \sqrt{|t|}, \sqrt{|u|} \ll m_h \implies \text{local expansion of Higgs exchange.}$$

The goal is to bootstrap the parameters of this model with minimal UV assumptions.

Dark photon scattering setup



+ crossed channels



+ crossed channels

We consider tree-level scattering of identical particles to obtain positivity bounds.

The graviton pole

Graviton pole in the forward limit

Graviton pole exchange term is divergent in the forward limit

$$|\mathcal{M}_{\text{grav}}(s, t)| \sim \frac{s^2}{M_{\text{Pl}}^2 |t|}, \quad t \longrightarrow 0^-.$$

Forward-limit positivity bounds do not apply here

$$b_2 \equiv \frac{1}{2} \partial_s^2 B_\phi(s, 0) \Big|_{s=2m_\phi^2} = \frac{2}{\pi} \int_{4m_\phi^2}^{\infty} \frac{d\sigma \operatorname{Im} \mathcal{M}_{\phi\phi}(\sigma, 0)}{(\sigma - 2m_\phi^2)^3} \geq 0.$$

- ✓ We cannot apply the previous positivity bounds when there is graviton exchange.
- ✓ Unitarity alone cannot establish non-negativity of the corresponding low-energy coefficient.
- ✓ The alternatives are either removing the pole or finding non-forward methods.

Therefore: positivity bounds with gravity need a careful treatment.

Pole subtraction

Full pole subtraction

Pole subtraction positivity is conditional on analyticity, boundedness and positive absorptive data for the remainder.

$$\mathcal{M}_{\text{rem}}(s, t) \simeq \mathcal{M}(s, t) - \frac{s^2}{M_{\text{Pl}}^2 |t|}, \quad t < 0$$

$$b_2 = \frac{2}{\pi} \int_{4m_\phi^2}^{\infty} \frac{d\sigma \operatorname{Im} \mathcal{M}_{\text{rem}}(\sigma, 0)}{(\sigma - 2m_\phi^2)^3} \geq 0$$

Thrice subtracted dispersion relations

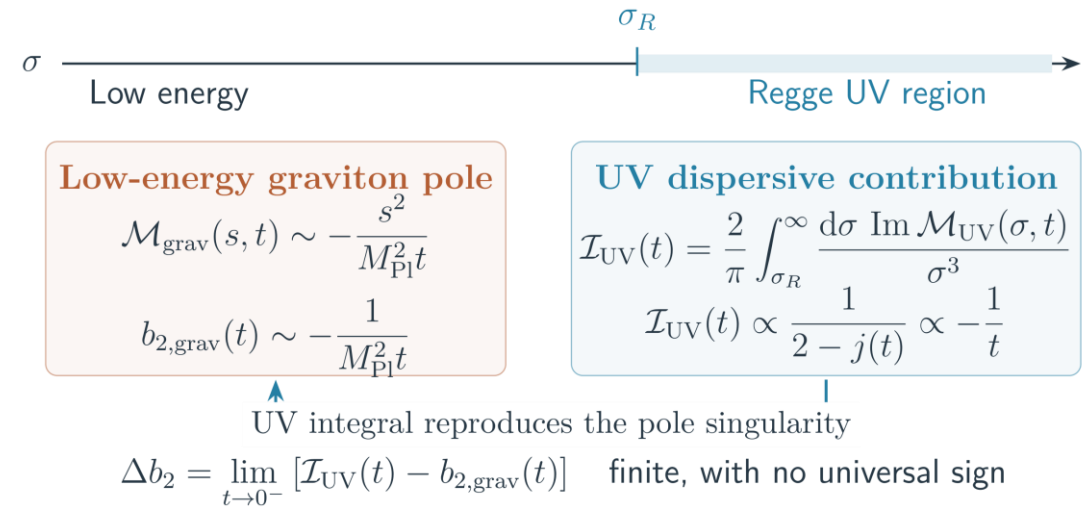
s^2 coefficient information is lost. Higher-derivative constraints are possible.

$$B_\phi(s, t) = c_0(t) + c_1(t)s + c_2(t)s^2 + \frac{1}{\pi} \int_{4m_\phi^2}^{\infty} \frac{d\sigma \operatorname{Im} \mathcal{M}_{\phi\phi}(\sigma, t)}{\sigma^3} \left[\frac{s^3}{\sigma - s} + \frac{u^3}{\sigma - u} \right],$$
$$u = 4m_\phi^2 - s - t$$

These approaches are not very suitable to obtain general optimal bounds.

Reggeisation

Extra UV assumptions



$$\mathcal{M}_{\text{UV}}(s, t) \propto \beta(t) \left(\frac{s}{s_0} \right)^{j(t)}, \quad j(t) = 2 + \alpha' t + \dots, \quad \alpha' \sim M_R^{-2}.$$

- ✓ Assuming Regge behaviour, the UV integral tames the IR graviton pole, leaving a finite remainder.

Under a specified Regge ansatz, the t-pole cancels in 4D spacetime.

Bounds on the kinetic mixing

Assuming Reggeisation, the following bounds for kinetic mixing were found using photon and dark photon scattering. arXiv:2205.12835.

Dark photon polarisation	Lower bound on kinetic mixing
Transverse	$\varepsilon_{\min}^{(T)} = \sqrt{\frac{11}{5760\pi\alpha}} \frac{m_W \Lambda_{\text{NST}}}{m_e M_{\text{Pl}}} \simeq 1.9 \times 10^{-11} \left(\frac{\Lambda_{\text{NST}}}{1 \text{ TeV}} \right),$
Longitudinal	$\varepsilon_{\min}^{(L)} = \sqrt{\frac{11}{1440\pi\alpha}} \frac{m_W^2 \Lambda_{\text{NST}}}{m m_e M_{\text{Pl}}} \simeq 3.0 \times 10^{-3} \left(\frac{\Lambda_{\text{NST}}}{1 \text{ TeV}} \right) \left(\frac{1 \text{ keV}}{m} \right).$

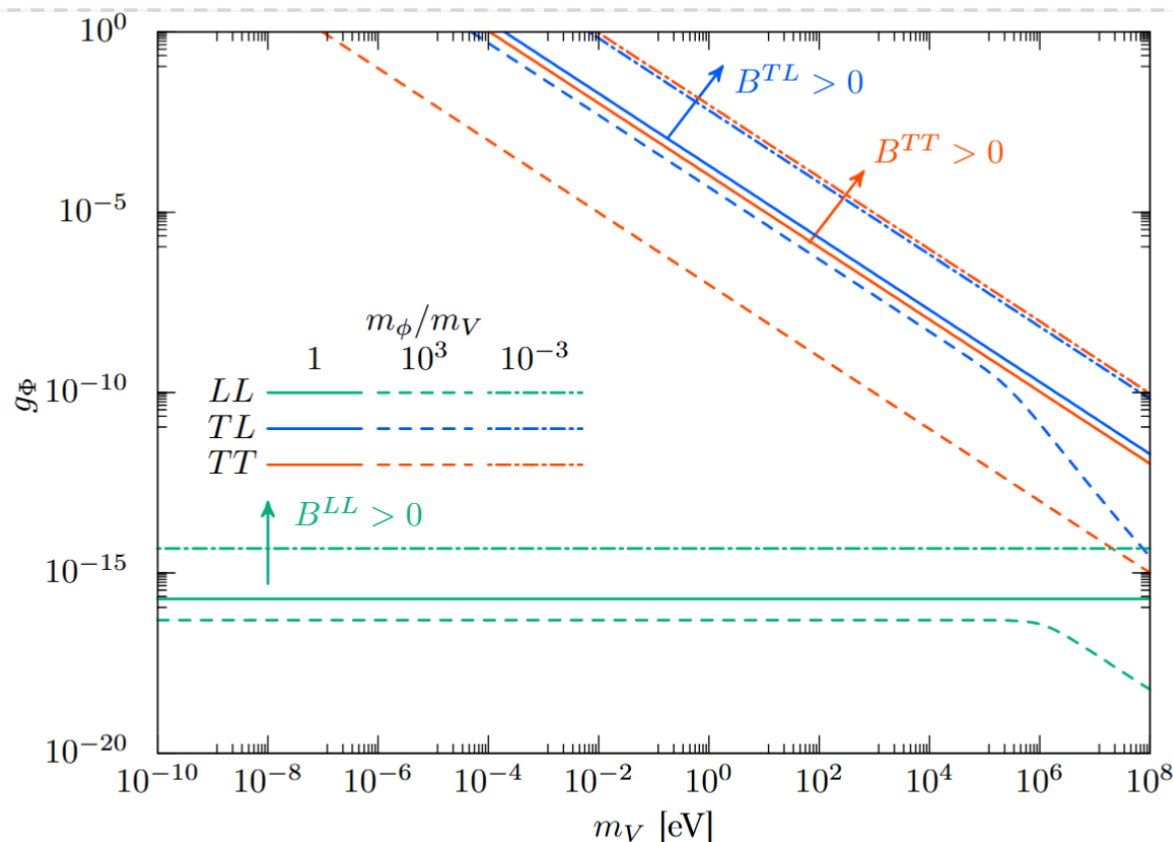
$$\mathcal{L}_{\text{mix}} = -\frac{\varepsilon}{2} F_{\mu\nu} F'^{\mu\nu} \quad \varepsilon \geq \varepsilon_{\min}$$

- ✓ This analysis considers Standard Model loops but neglects QCD interactions.

The kinetic mixing of the Dark Photon was constrained assuming Reggeisation in a Higgsless model.

Bounds on the dark-photon coupling

Assuming Reggeisation, the following for the dark photon coupling were found using pure dark-photon scattering. arXiv:2305.10058



Lower bounds on the dark-photon-Higgs coupling for 1 TeV cutoff. arXiv:2305.10058

Reggeisation also allows to constrain the Higgsed dark-photon model. We want to loosen the UV assumptions.

Formalism

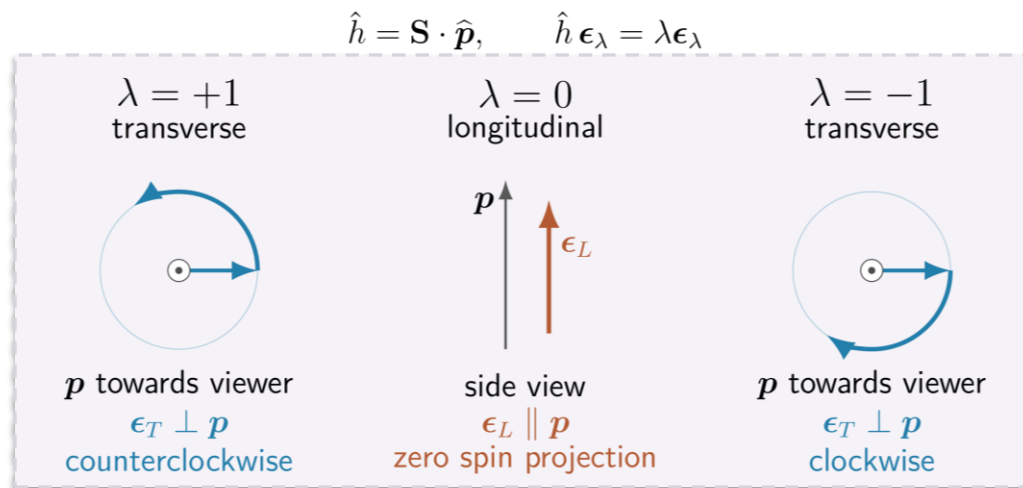
Bootstrap formalism for massive spin-1 particles

03

Helicity properties

Identical-massive-vector scattering amplitudes

- ✓ A generic $2 \rightarrow 2$ amplitude could have 81 different polarisation configurations.
- ✓ Parity, Time-reversal and Bose Exchange relate most configurations.
- ✓ It is sufficient to consider 17 helicity configurations. The rest are redundant.



Helicity and polarisation. Arrows represent time evolution.

$$\lambda_i \in \{-1, 0, +1\}, \quad 3^4 = 81 \xrightarrow{P, T, \text{Bose}} 17.$$

We work with a list of 17 canonical amplitudes.

Structures and functions

Amplitude characterization

Mass-dimension-zero structure

$$e_1 = (\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_4),$$

Mass-dimension-two structure

$$e_4 = (\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot p_4)(\epsilon_4 \cdot p_3) \\ + (\epsilon_3 \cdot \epsilon_4)(\epsilon_1 \cdot p_2)(\epsilon_2 \cdot p_1)$$

- ✓ Structures are a basis of kinematic invariants for a generic $2 \rightarrow 2$ gauge boson process.
- ✓ Functions encode the dynamical information. They obey crossing relations. $\{f, g, \tilde{g}, h, \tilde{h}\}$
- ✓ Decomposing amplitudes like this is useful to track symmetry effects.

Five generating functions. ←

We fix the following amplitude representation:

$$\mathcal{M}^I(s, t, u) = \sum_{r=1}^{17} E_r^I(s, t, u) F_r(s, t, u),$$

arXiv:2402.13327

Kinematic branch cuts

Reduced amplitude

- ✓ Structures are functions of s, t, u after choosing a frame.
- ✓ Structures insert square root factors into the amplitude.
- ✓ Rescaling the amplitude removes unphysical cuts.

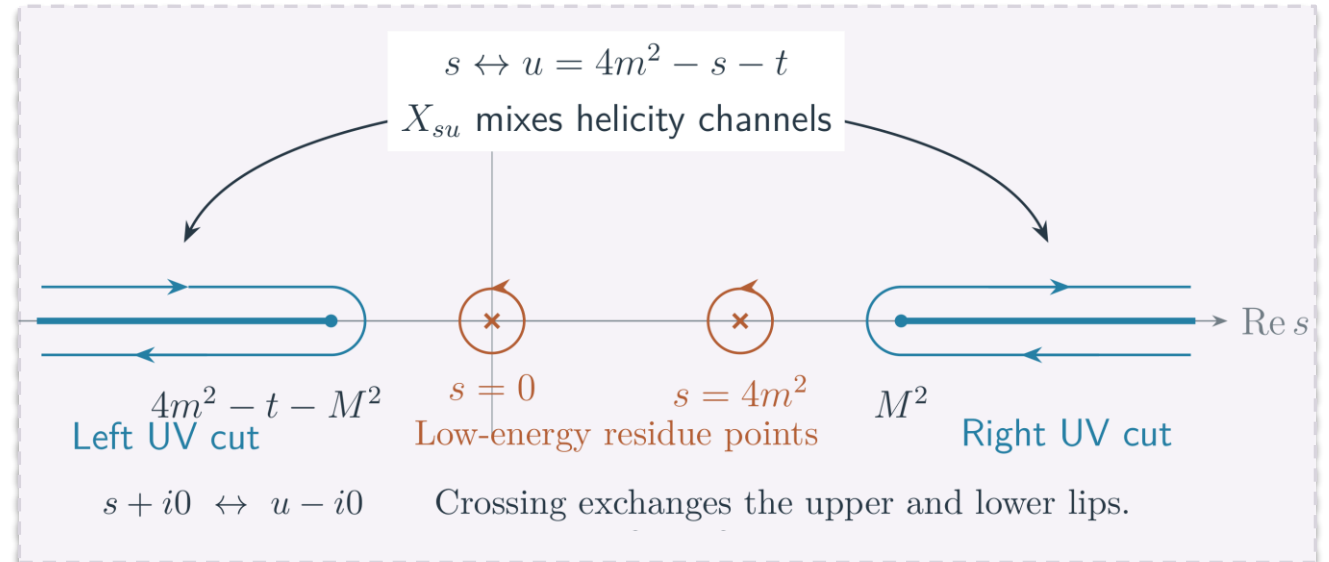
$$\widetilde{\mathcal{M}}^I(s, t) = \begin{cases} \frac{ms}{\sqrt{-ts(s+t-4m^2)}} \mathcal{M}^I(s, t), & I = 2, 4, 9, 10, 11, \\ \mathcal{M}^I(s, t), & \text{otherwise.} \end{cases}$$

Unphysical branch cuts appear after fixing a reference frame. They must be removed.

arXiv:2402.13327

Crossing relations

Connecting the left and right cuts



- ✓ Functions obey crossing relations: $\mathbf{F}(u, t, s) = C_{su} \mathbf{F}(s, t, u)$.
- ✓ Using the amplitude characterization, we can deduce the amplitude crossing relations.

$$X_{su}(s, t, u) = \tilde{E}(u, t, s) C_{su} \tilde{E}(s, t, u)^{-1}, \quad \tilde{E} = RE$$

The crossed amplitude is: $\widetilde{\mathcal{M}}(u, t, s) = X_{su}(s, t, u) \widetilde{\mathcal{M}}(s, t, u),$

arXiv:2402.13327

Unitarity

Partial-wave positivity

Partial-wave decomposition of the high-energy amplitude

$$\mathcal{M}^\lambda(s, t) = 16\pi \sqrt{\frac{s}{s - 4m^2}} \sum_J (2J + 1) \times d_{\lambda_1 - \lambda_2, \lambda_4 - \lambda_3}^J(\theta) \mathcal{M}_J^\lambda(s)$$

Parameter definitions

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \quad \cos \theta = 1 + \frac{2t}{s - 4m^2}.$$

- ✓ We define the spectral density $\rho_J^\lambda(\sigma) := \text{Disc } \mathcal{M}_J^\lambda(\sigma) = \frac{\mathcal{M}_J^\lambda(\sigma + i0) - \mathcal{M}_J^\lambda(\sigma - i0)}{2i}$
- ✓ Unitarity of the partial waves implies $\rho_{J,ab}(\sigma) = \sum_X B_{JaX}(\sigma) B_{JbX}^*(\sigma), \quad \rho_J(\sigma) \succeq 0$

Unitarity gives positive semidefinite spectral matrices without taking the forward limit.

Arcs

The low-energy side of the dispersion relations

- ✓ Arcs encode the low-energy information of dispersion relations.
- ✓ They are observables related to Wilson Coefficients. Many definitions exist.
- ✓ Usual definitions involve the forward limit, but this can be changed.

$$\langle V_i \rangle := \sum_J \int_{M^2}^{\infty} d\sigma \operatorname{Tr}[\rho_J(\sigma) V_{i,J}(\sigma)] .$$

$$V_{i,J}(\sigma) = \partial_t^n \left[K_{I,k,J}^{\text{right}}(\sigma, t) + K_{I,k,J}^{\text{crossed}}(\sigma, t) \right] \Big|_{t=0} ,$$
$$i = (I, k, n)$$

Spectral kinematic forward matrices

$$a_{k,n}^I = \partial_t^n \left(\operatorname{Res}_{s=0} + \operatorname{Res}_{s=4m^2} \right) \frac{\widetilde{\mathcal{M}}_{\text{low}}^I(s, t)}{s^{k+1}} \Big|_{t=0} ,$$

Dispersion relations are now compactly written.

$$a_i = \langle V_i \rangle ,$$

Null constraints

The low-energy side of the dispersion relations

- ✓ Dispersion relations relate low-energy and high-energy information.
- ✓ Null constraints correlate different contributions to UV spectral data.
- ✓ There is an infinite number of them, and they form a vector space.

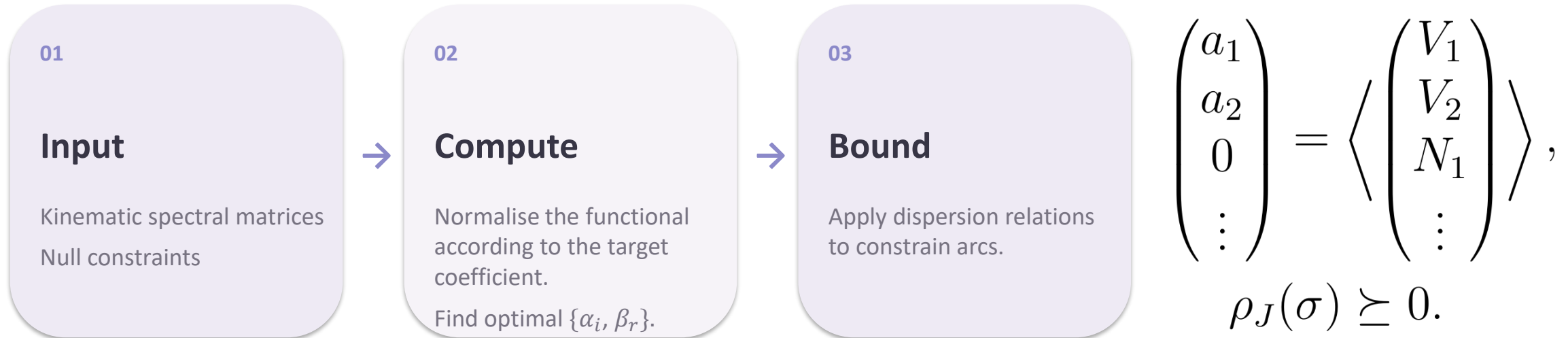
- ✓ Null constraints are obtained from dispersion relations with the same low-energy contribution.

$$a_i = \langle V_i^{(1)} \rangle = \langle V_i^{(2)} \rangle \implies \langle N_i \rangle = 0, \quad N_i := V_i^{(1)} - V_i^{(2)}.$$

Null relations contain useful and necessary information for the optimisation problem.

The optimisation problem

Find minimal or maximal solutions for upper and lower bounds, respectively.



Optimise α_i, β_r such that $\sum_i \alpha_i V_{i,J}(\sigma) + \sum_r \beta_r N_{r,J}(\sigma) \succeq 0$ for every allowed J , $\sigma \geq M^2$.

Non-forward bootstrap

Treating the graviton t-pole

04

Finite momentum transfer

Series expansion for small t

Schematic tree-level arc definition

$$t = -q^2 < 0, \quad a_{\text{lead}}(t) = \frac{r_g}{t} + b_0 + b_1 t + \dots$$

Loop non-analytic terms

$$\delta \mathcal{M}_{\text{loop}}(s, t) \propto s^2 \ln\left(\frac{-s - i0}{\mu^2}\right) \ln\left(\frac{-t}{\mu^2}\right), \quad t < 0$$

- ✓ We can try to define the dispersion relations away from the forward limit.
- ✓ At loop order, non-analytic terms (like logarithms of t) will appear in the series.
- ✓ The UV partial-wave expansion is not guaranteed to be convergent for all $t \neq 0$.

Finite- t dispersion relations need careful treatment of analyticity and convergence.

arXiv: 2407.02346

Smearing

Beyond point-wise convergence

Smeared kinematic matrices

$$K_{W,J}(\sigma) := \int_0^M dq W(q) V_J(\sigma, -q^2),$$

Smeared arcs

$$\mathcal{S}_W := \int_0^M dq W(q) a(-q^2).$$

- ✓ Integrate the dispersion relations over momentum transfer with a weight function.
- ✓ The smearing function is subject to positivity constraints, but suitable weights can yield convergent smeared relations.

$$K_{W,J}(\sigma) \succeq 0 \quad (\text{all allowed } J, \sigma \geq M^2) \\ \implies \mathcal{S}_W \geq 0$$

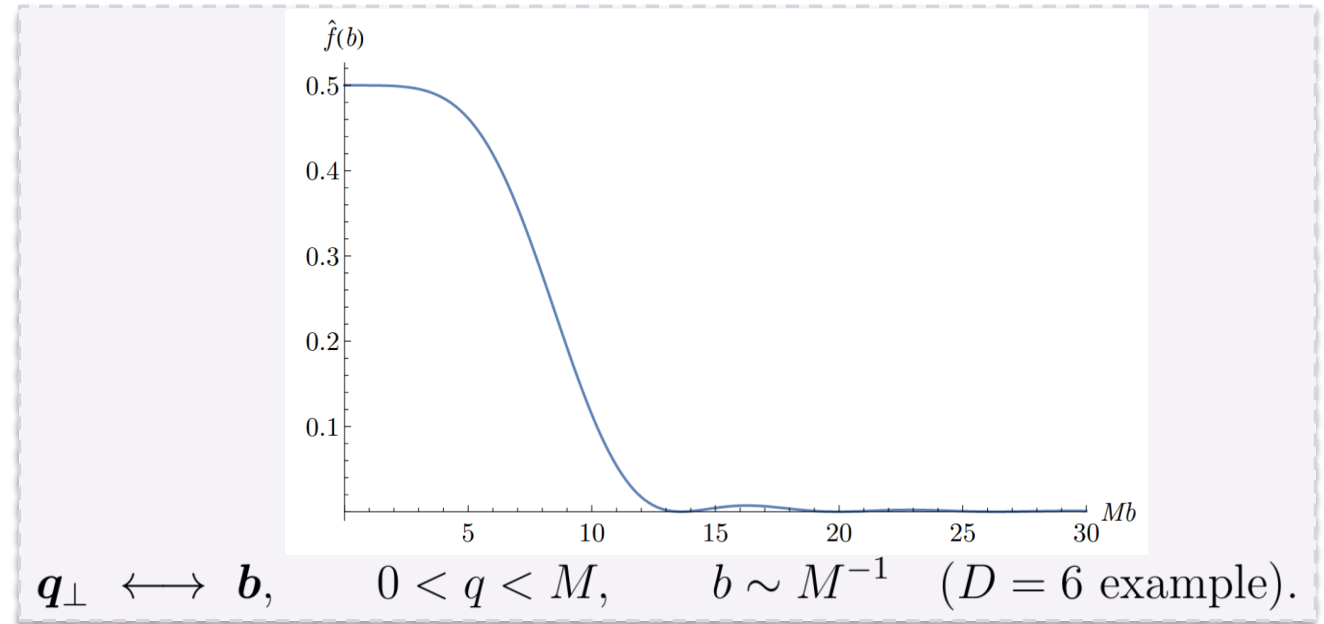
Smearing can realise non-forward positivity.

arXiv: 2102.08951, 2407.02346

Smearing function

Impact-parameter interpretation

- ✓ Momentum transfer is conjugate to impact parameter.
- ✓ The smearing function $\hat{f}(b)$ is positive and radially symmetric in the $\mathbb{R}^{D-2}$ impact-parameter space.

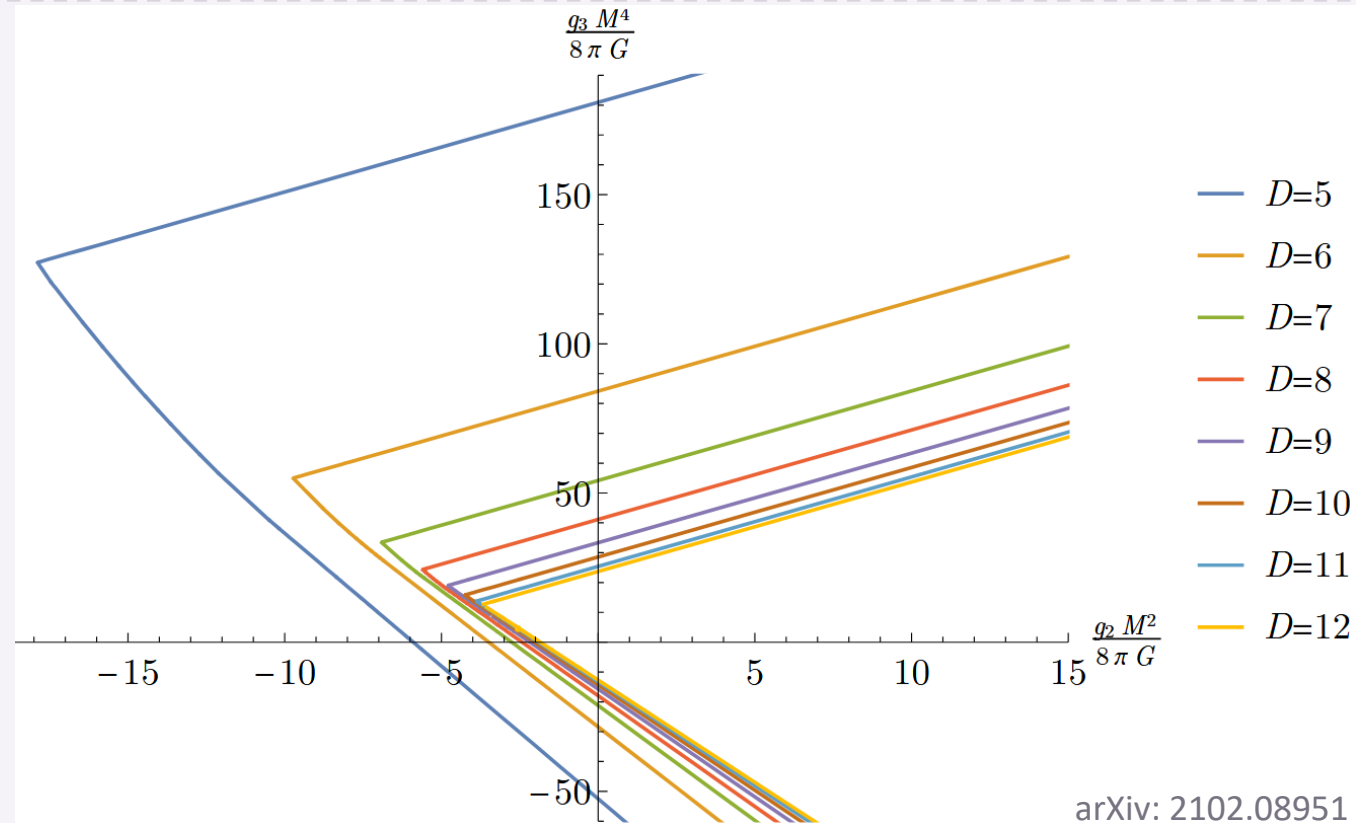


Example of a Fourier-transformed smearing function in D=6. arXiv: 2102.08951

The momentum-space smearing function $f(q = \sqrt{-t})$ can have support for $-M^2 < t < 0$.

Bounds from smeared dispersion relations

Bounds for massless scalar EFT coefficients with gravity in different spacetime dimensions.



The region to the left of the graphs is forbidden and the region to the right is allowed. M is the EFT cutoff.

Smearing yields quantitative bounds for a massless scalar coupled to gravity.

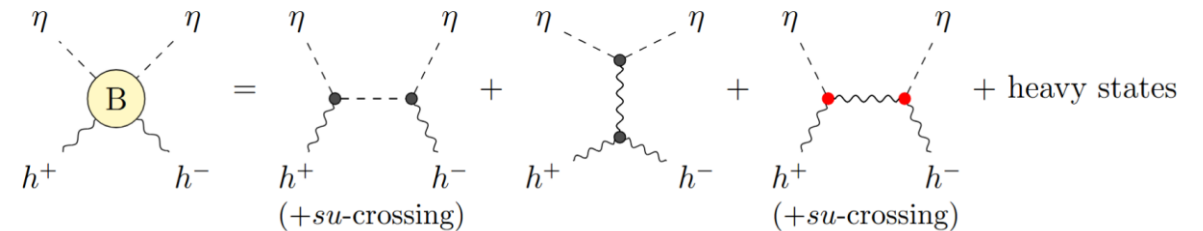
Smearing in 4D

Pseudoscalar and graviton scattering

$$\frac{M_2^2 \kappa_g}{M_{\text{Pl}} f_\eta} \lesssim 30 \sqrt{\ln \left(\frac{M_2}{M_{\text{IR}}} \right)}.$$

arXiv:2411.14422

- ✓ The smeared t-pole is log-divergent in 4D and requires an IR regulator M_{IR} .
- ✓ M_2 is the mass of the lightest $J \geq 2$ states mediating the scattering.
- ✓ κ_g is the anomalous ηhh coupling and f_η is the pseudoscalar decay constant.



Smearing in 4D is feasible with an IR regulator.

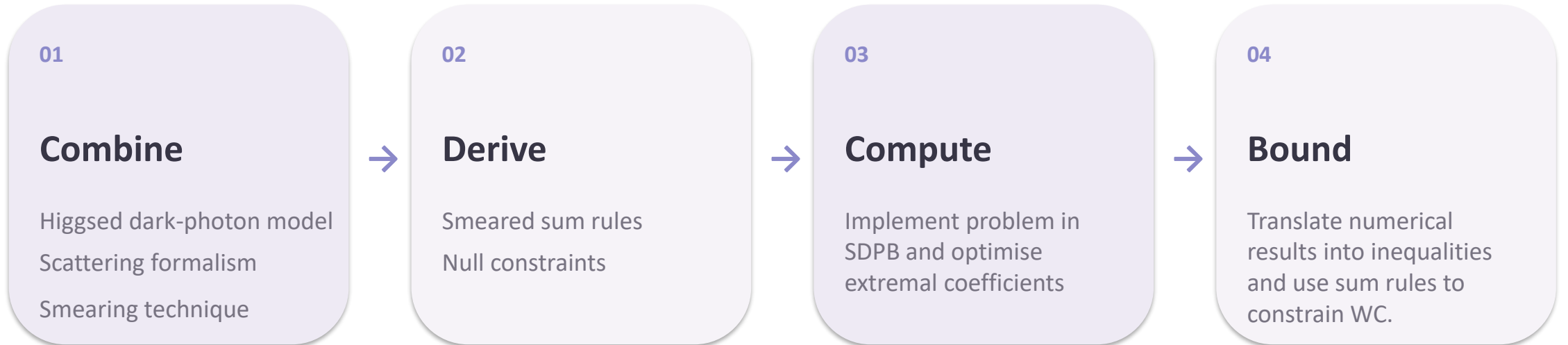
Conclusions

How to obtain bound for a dark photon

05

Remaining steps

Dark-photon bootstrapping roadmap



We must combine smearing and scattering formalisms, then implement them numerically in SDPB.

Conclusions

Main issues with dark photon bootstrap and how to address them.

- 01 Gravity obstructs forward positivity.
- 02 Spin complicates crossing relations.
- 03 Positivity creates an optimisation problem.

SOLUTIONS DEVELOPED SEPARATELY

The next step is to construct smeared matrix sum rules and solve the optimisation problem.

What constraints follow after matching these sum rules to a Higgsed dark-photon model?

Thank you

Questions?

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Backup

Canonical polarisation configuration and structures

There are 17 independent polarisation configurations and 17 independent structures. We can choose a basis for reference.

Raw helicity amplitudes	81
Canonical representatives	17
Canonical labels	++++ , +++0 , +++- , 000+ , 0000 , ++00 , +0+0 , 0++0 , ++0- , +0+- , 0++- , ++-- , +-+- , -++- , +00- , +0-0 , +-00

Polarisation-matrix benchmark

Bertucci et al., structure basis

Reference expression	Notebook output
$e^1_{++++} = 1$ $e^2_{++++} = \frac{t^2}{(s-4m^2)^2}$	$e^1_{++++} = 1$ $e^2_{++++} = \frac{t^2}{(s-4m^2)^2}$

Comparison: exact symbolic match

We successfully reproduce the massive-vector scattering formalism from arXiv:2402.13327.

Functions and generating functions

The 17 structures are accompanied by 17 functions, but they are not independent. They can be written from the crossing of 5 generating functions.

Five generating functions -> 17 crossing-related functions

Bertucci et al., Eq. (2.8)

Function	Bertucci et al.	Notebook FunctionVector	Match
F_1	$fBench(s, t, u)$	$fBench(s, t, u)$	✓
F_2	$fBench(t, s, u)$	$fBench(t, s, u)$	✓
F_3	$fBench(u, s, t)$	$fBench(u, s, t)$	✓
F_4	$gBench(s, t, u)$	$gBench(s, t, u)$	✓
F_5	$gBench(t, s, u)$	$gBench(t, s, u)$	✓
F_6	$gBench(u, s, t)$	$gBench(u, s, t)$	✓
F_7	$gtBench(s, t, u)$	$gtBench(s, t, u)$	✓
F_8	$gtBench(s, u, t)$	$gtBench(s, u, t)$	✓
F_9	$gtBench(t, s, u)$	$gtBench(t, s, u)$	✓
F_{10}	$gtBench(t, u, s)$	$gtBench(t, u, s)$	✓
F_{11}	$gtBench(u, s, t)$	$gtBench(u, s, t)$	✓
F_{12}	$gtBench(u, t, s)$	$gtBench(u, t, s)$	✓
F_{13}	$hBench(s, t, u) + htBench(s, t, u) + htBench(s, u, t)$	$hBench(s, t, u) + htBench(s, t, u) + htBench(s, u, t)$	✓
F_{14}	$hBench(t, u, s) + htBench(s, u, t)$	$hBench(t, u, s) + htBench(s, u, t)$	✓
F_{15}	$hBench(u, s, t) + htBench(s, t, u)$	$hBench(u, s, t) + htBench(s, t, u)$	✓
F_{16}	$2 \text{ htBench}(t, s, u)$	$2 \text{ htBench}(t, s, u)$	✓
F_{17}	$2 \text{ htBench}(u, s, t)$	$2 \text{ htBench}(u, s, t)$	✓

Overall comparison: all 17 components match exactly

We successfully reproduce the massive-vector scattering formalism from arXiv:2402.13327.

Regulating factor for the reduced amplitudes

Structures introduce kinematic branch cuts. They are removed by the reduced factor. Only 5 polarisation configurations need this.

Regulated Wigner kernel

Bertucci et al., Eq. (C.11)

I	Helicity	Bertucci et al.	Notebook	Match
2	+++0	$\frac{ms}{\sqrt{-st(-4m^2+s+t)}}$	$\frac{ms}{\sqrt{stu}}$	✓
4	000+	$\frac{ms}{\sqrt{-st(-4m^2+s+t)}}$	$\frac{ms}{\sqrt{stu}}$	✓
9	++0-	$\frac{ms}{\sqrt{-st(-4m^2+s+t)}}$	$\frac{ms}{\sqrt{stu}}$	✓
10	+0+-	$\frac{ms}{\sqrt{-st(-4m^2+s+t)}}$	$\frac{ms}{\sqrt{stu}}$	✓
11	0++-	$\frac{ms}{\sqrt{-st(-4m^2+s+t)}}$	$\frac{ms}{\sqrt{stu}}$	✓
otherwise	remaining 12 helicities	1	1	✓

The notebook uses $4m^2 - s - t$, so $st(4m^2 - s - t)$.

Overall comparison: Eq. (C.11) reproduced exactly

We successfully reproduce the massive-vector scattering formalism from arXiv:2402.13327.

Spectral kinematic matrix (right cut)

This is one specific explicit expression for one of the $V_I(s, t)$ matrices in the forward limit. The right (s-channel) cut is shown.

Master spectral decomposition: right and left cuts

Bertucci et al., Eq. (C.10); row-12 forward benchmark: Eqs. (C.16) and (C.26)

Observable row $I = 12$, arc order $k = 2$, t -order $= 0$

Right/direct-cut contribution

Sector	ℓ	Reference decomposition	Notebook	Match
ee	2	$\begin{pmatrix} \frac{1}{s^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{s^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	✓
oe	2	$\begin{pmatrix} \frac{1}{s^2} & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{s^2} & 0 \\ 0 & 0 \end{pmatrix}$	✓
eo	3	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	✓
oo	3	(0)	(0)	✓

We successfully reproduce the massive-vector scattering formalism from arXiv:2402.13327.

Spectral kinematic matrix (left cut)

This is one specific explicit expression for one of the $V_J(s, t)$ matrices in the forward limit. The left (crossed u-channel) cut is shown.

Left/crossed-cut contribution

Sector	ℓ	Reference decomposition	Notebook	Match
ee	2	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{s}{(-4m^2+s)^3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{s}{(-4m^2+s)^3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	✓
oe	2	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	✓
eo	3	$\begin{pmatrix} \frac{s}{(-4m^2+s)^3} & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{s}{(-4m^2+s)^3} & 0 \\ 0 & 0 \end{pmatrix}$	✓
oo	3	(0)	(0)	✓

Right cut: exact match Left cut: exact match

We successfully reproduce the massive-vector scattering formalism from arXiv:2402.13327.

Spectral kinematic matrix (total)

This is one specific explicit expression for a different $V_I(s, t)$ matrix in the forward limit. The sum of left and right cuts is shown.

Explicit spectral V-matrix

Bertucci et al., Eq. (C.25)

Reference expression	Notebook output
$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2 s^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{s}{(4 m^2-s)^3} & 0 \\ \frac{1}{2 s^2} & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2 s^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{s}{(4 m^2-s)^3} & 0 \\ \frac{1}{2 s^2} & 0 & 0 & 0 \end{pmatrix}$

Comparison: exact symbolic match

We successfully reproduce the massive-vector scattering formalism from arXiv:2402.13327.

Analytical full constraint

In order to find a null constraint, we look for combinations of dispersion relations that cancel its IR side.

Published F17 contour null

Bertucci et al., Eqs. (3.23)–(3.26)

$$\text{ContourIntegral} \left[\frac{\frac{F_{17}(s,t)}{t^2} - \frac{F_{17}(t,s)}{s^2}}{s t} \right] = 0$$

Canonical contour term	Eq. (3.26)	Notebook	Match
$F_{16} \quad \{2, 0\}$	1	1	✓
$F_{17} \quad \{0, 2\}$	1	1	✓
$F_{17} \quad \{2, 0\}$	−1	−1	✓

Full contour-vector length	612
Non-zero paper terms	3
Canonical null-space dimension	279
Non-zero canonical coordinates	1
Embedded in canonical null space	✓

We successfully reproduce the massive-vector scattering formalism from arXiv:2402.13327.

Spectral null constraint

Spectral matrix decomposition of a null constraint. This is necessary for a consistent implementation together with the spectral kinematic matrices.

Forward spectral null constraint

Bertucci et al., Eq. (C.28), second relation

Reference kernel: $\frac{s}{(s-4m^2)^3} - \frac{1}{s^2}$

The notebook defines the same homogeneous null with the opposite overall sign; therefore $-g2NullV$ is displayed in the paper convention.

Sector	ℓ	Bertucci et al., Eq. (C.28)	$-g2NullV$	Match
ee	2	$-\frac{1}{s^2} + \frac{s}{(-4m^2+s)^3}$	$-\frac{1}{s^2} + \frac{s}{(-4m^2+s)^3}$	✓
		0	0	
		$\frac{1}{s^2} - \frac{s}{(-4m^2+s)^3}$	$\frac{1}{s^2} - \frac{s}{(-4m^2+s)^3}$	
		0	0	
oe	2	0	0	✓
		0	0	
		0	0	
		0	0	
oe	2	$\begin{pmatrix} -\frac{1}{s^2} + \frac{s}{(-4m^2+s)^3} & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{s^2} + \frac{s}{(-4m^2+s)^3} & 0 \\ 0 & 0 \end{pmatrix}$	✓

Sector	ℓ	Bertucci et al., Eq. (C.28)	$-g2NullV$	Match
eo	3	$\begin{pmatrix} \frac{1}{s^2} - \frac{s}{(-4m^2+s)^3} & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{s^2} - \frac{s}{(-4m^2+s)^3} & 0 \\ 0 & 0 \end{pmatrix}$	✓
oo	3	(0)	(0)	✓

Orientation-independent check: $V_{\text{paper}} = -V_{g2NullV}$ ✓

We successfully reproduce the massive-vector scattering formalism from arXiv:2402.13327.

Spectral null constraint crosscheck

Spectral null constraints can be expanded as weighted spectral densities. We check several components of the previous null constraint explicitly.

Spectral null generated from the F17 contour relation

Bertucci et al., Eq. (3.27)

Representative components at $\ell = 7$

Density	Bertucci et al., Eq. (3.27)	Generated by notebook	Match
ρ^1_7	$-\frac{448 m^4 (54 m^2 - s)}{s^2 (4 m^2 - s)^5}$	$-\frac{448 (\text{delta } m^4 - 50 m^6)}{\text{delta}^5 (\text{delta} + 4 m^2)^2}$	✓
ρ^2_7	$-\frac{16 \sqrt{7} m^3 (6696 m^2 - 174 s)}{3 s^{3/2} (s - 4 m^2)^5}$	$\frac{32 (29 \sqrt{7} \text{delta } m^3 - 1000 \sqrt{7} m^5)}{\text{delta}^5 (\text{delta} + 4 m^2)^{3/2}}$	✓
ρ^{17}_7	$-\frac{64 \sqrt{21} m^4 (212 m^2 - 3 s)}{s^2 (4 m^2 - s)^5}$	$-\frac{64 \sqrt{21} (3 \text{delta } m^4 - 200 m^6)}{\text{delta}^5 (\text{delta} + 4 m^2)^2}$	✓

Complete symbolic comparison: 17 / 17 spectral weights ✓

We successfully reproduce the massive-vector scattering formalism from arXiv:2402.13327.

Optimisation problem: technical summary

For the numerical solution of the optimisation problem, we need to truncate angular momentum and number of nulls. They directly affect convergence.

From analytic nulls to the finite SDP basis

Bertucci et al., Sec. 3.2.2 and Table 2

Stage	Notebook baseline	Paper production
Contour coefficient space	612	—
Canonical null space	279	—
Numerical selection	$\langle \text{SPoleCap} \rightarrow 3, \text{DeltaPoleCap} \rightarrow 3 \rangle$	$a + b < 11$
Filter / production spins	{7, 8}	$\ell_{\text{max}} = 202 + \text{higher-spin samples}$
Retained nulls	16	127
Null-space working precision	400	—
Null-space tolerance	1.00×10^{-200}	—

The 16-null notebook baseline is a diagnostic custom truncation; the 127-null column records the published production target.

We successfully reproduce the massive-vector scattering formalism from arXiv:2402.13327.