

Violation and Restoration of Probability Positivity in Perturbative Quantum Field Theory

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Outline

- 1 Introduction
- 2 The Amplitude-Level Prescription
- 3 Numerical Results
- 4 Summary

From Quantum Mechanics to Quantum Field Theory

- Quantum Mechanics: according to Born's statistical interpretation (1926), first solve the wave function ψ ; then its modulus square $|\psi|^2$ is the probability of an event
— positivity is self-evident [1]
- Quantum field theory: works in a similar way

wave function ψ $\leftrightarrow$ **matrix element** $\mathcal{M}$ (computed from Feynman diagrams)

The probability of a process — its cross section or decay width — should in principle be the modulus square of the matrix element,

$$\Gamma \propto \int d\Phi \sum_{\text{pol}} |\mathcal{M}|^2,$$

— positivity preserved by construction

- In practice, we expand the amplitude $\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1 + \mathcal{M}_2 + \dots$, square it, and keep only the terms up to a fixed order
— traditional practice of perturbative QFT

Why truncate the amplitude square

- Early Era: the founding papers of renormalized QED — Tomonaga (1946), Schwinger (1948), Feynman (1949), Dyson (1949) — all computed cross sections as $|\mathcal{M}|^2$ expanded in powers of α , discarding the terms beyond *without comment* [2, 3, 4, 5, 6, 7, 8]
- Cancellation of IR divergence in inclusive processes
 - infrared divergences of virtual corrections cancel against real emission only when the squared amplitude is truncated at a fixed order $\Rightarrow$ *the truncation was forced, not optional*
 - Bloch–Nordsieck mechanism; Kinoshita–Lee–Nauenberg (KLN) theorem [9, 10, 11]
 - *positivity is no longer guaranteed as the unavoidable price of infrared finiteness*
- No problem found in QED
 - although not guaranteed, negative probability never arose ($\alpha \sim 1/137$).
- However in QCD, where α_s is large, the discarded terms can be as large as the kept ones $\Rightarrow$ *negative probability might appear*

The Sign Crisis in Exclusive QCD Processes

- $J/\psi \rightarrow 3\gamma$:
 - NLO computed nearly five decades ago [12, 13]:

$$\Gamma = \Gamma_{\text{LO}} \left(1 - 12.6 \frac{\alpha_s}{\pi} \right) \sim [-0.04, -0.56] \Gamma_{\text{LO}} \quad \text{as } \alpha_s \in [0.259, 0.388]$$

- The traditional perturbative expansion for $J/\psi \rightarrow 3\gamma$ is pathological order by order:

$$\Gamma = \Gamma_{\text{LO}} \left(1 - 12.6 \frac{\alpha_s}{\pi} - 5.32 \langle v^2 \rangle + 68.9 \frac{\alpha_s}{\pi} \langle v^2 \rangle - 28.7 \frac{\alpha_s^2}{\pi^2} \right)$$

- $\mathcal{O}(v^2)$: become totally negative; $\mathcal{O}(\alpha_s v^2)$: accidentally “rescues” ; NNLO: the rescue fails
- $\Upsilon \rightarrow 3\gamma$: also becomes negative at NNLO [14]
- $e^+e^- \rightarrow J/\psi + J/\psi$: significantly decreased at NLO; become negative at NNLO [15, 16, 17]
- even at NNLO, the crisis persists! (seems unable to resolved by including higher order terms)
- A 40-year puzzle — signals a **fundamental issue** in the organization of the perturbative series

Thinking over ...

- Recall when we **lose** the positivity of amplitude square:
 - the cancellation of IR divergence forced us to truncate **the amplitude square** in a fixed order
 - positivity is thus no longer guaranteed
- Why no problem in QED
 - everything is fine when the coupling is really small
 - But in QCD, the coupling constant can be large (no so small as you think)
- Is there a way to **restore** the positivity beside praying for a good convergence
 - inclusive processes: unknown, current truncation is forced by KLN theorem
 - exclusive processes: YES
- for exclusive processes, the answer is simple and natural: truncate the amplitude
no need for cancellation of IR divergence = no need to lose the positivity at very beginning

Exclusive Processes: The Amplitude-Level Prescription

- Exclusive processes: all external particles are colorless (QCD) or electrically neutral (QED)
⇒ no real emission at any order
- IR divergences cancel **amplitude by amplitude**: each renormalized amplitude $\mathcal{A}^{(l)}$ is finite
⇒ the KLN constraint is absent
- The complete modulus square of the truncated amplitude is the proper probability, positive at every order M :

$$\Gamma_M^{\text{amp}} = \int d\phi \sum_{\text{pol}} \left| \sum_{l=0}^M \mathcal{A}^{(l)} \right|^2$$

- Every amplitude-level truncated observable admits a compact effective representation

$$\Gamma^{\text{amp}} = \Gamma_{\text{LO}} \left| 1 + \frac{\alpha_s}{\pi} (c_1 + ic_2) + \frac{\alpha_s^2}{\pi^2} (c_3 + ic_4) \right|^2$$

c_i : effective real numbers from the one- and two-loop amplitude ratios

- The traditional approach expands the square and truncates again
— discarding extra terms and lose positivity

Result I: $J/\psi \rightarrow 3\gamma$ — first result@QCD NNLO

- First complete NNLO QCD correction to $J/\psi \rightarrow 3\gamma$ within NRQCD [18, 14], five decades after the NLO
- **Effective** representation:

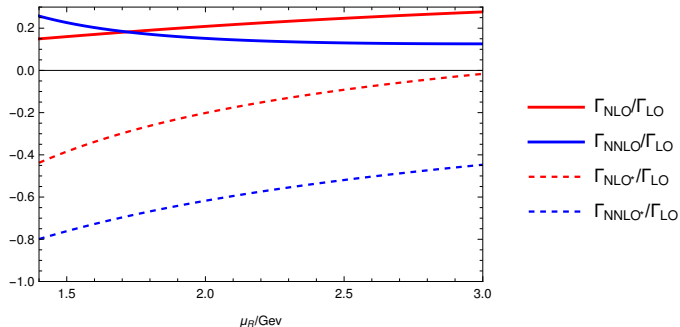
$$\Gamma = \Gamma_{\text{LO}} \left| 1 + (-0.51 + i 0.19) + (-0.36 + i 0.14) \right|^2$$

- The series inside the modulus: 1, 0.54, 0.39 — converges geometrically
- Widths:
 - Amplitude level: $\Gamma_{\text{NLO}}/\Gamma_{\text{LO}} = 0.276$, $\Gamma_{\text{NNLO}}/\Gamma_{\text{LO}} = 0.126$ — **positive and well ordered**
 - Traditional: -0.02 , -0.44 — **negative at NLO and NNLO**
- Including the known relativistic correction up to $\mathcal{O}(\alpha_s v^2)$ as a rough estimation: [19]

$$\Gamma(J/\psi \rightarrow 3\gamma) = 0.96^{+4.32}_{-0.13} \text{ eV} \quad \text{vs. BESIII: } 1.046 \pm 0.167 \pm 0.185 \text{ eV}$$

- Markedly improved agreement with the high-precision BESIII measurement

$J/\psi \rightarrow 3\gamma$: Renormalization-Scale Dependence



- Solid lines (amplitude expansion): **positive**, much weaker scale dependence
- Dashed lines (traditional expansion): (almost) always negative, hard to see convergence
- The NNLO band overlaps with NLO — perturbative stability

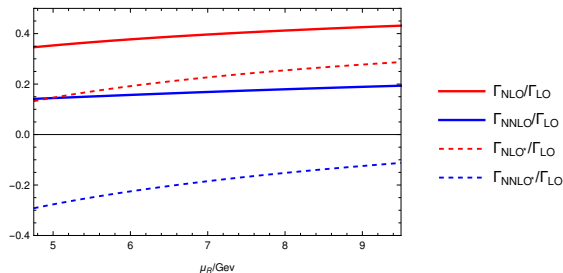
Result II: $\Upsilon \rightarrow 3\gamma$ — First NNLO result

- $\alpha_s(2m_b) \approx 0.18 < \alpha_s(2m_c) \approx 0.25$:
better expansion is expected for Υ than for J/ψ
- **Yet** the traditional NNLO result is **negative over the entire scale range**
- Amplitude level [14]:

$$\Gamma = \Gamma_{\text{LO}} \left| 1 + (-0.36 + i 0.13) + (-0.27 + i 0.10) \right|^2$$

- Series inside the modulus: 1, 0.38, 0.29;
 $\Gamma_{\text{NLO}}/\Gamma_{\text{LO}} = 0.426$, $\Gamma_{\text{NNLO}}/\Gamma_{\text{LO}} = 0.190$
- Traditional: 0.28, -0.11

after including the relativistic correction: $\Gamma(\Upsilon \rightarrow 3\gamma) = 0.0086^{+0.0028}_{-0.0006} \text{ eV}$
— much smaller uncertainties than J/ψ , better convergence as expected



Result III: $e^+e^- \rightarrow J/\psi + J/\psi$ Production (revisit)

- NLO correction [15]: negative and substantial — the sign crisis first appeared in production
- NNLO corrections [16, 17]: the traditional truncation gives **negative result** again
- Amplitude level [17] (columns renormalized to a common wave-function-at-origin value):

$$\Gamma = \Gamma_{\text{LO}} \left| 1 + (-0.37 + i 0.02) + (-0.39 - i 0.003) \right|^2$$

- Series inside the modulus: 1, 0.37, 0.39 vs. traditional series: 1, -0.74, -0.64
- $\Gamma_{\text{NLO}}/\Gamma_{\text{LO}} = 0.40$, $\Gamma_{\text{NNLO}}/\Gamma_{\text{LO}} = 0.058$; traditional: 0.26, -0.38

Result IV: Eight Exclusive Channels at B Factories (revisit)

Process	2nd term	3rd term	$\Gamma_{\text{NLO}}^{\text{amp}}/\Gamma_{\text{LO}}$	$\Gamma_{\text{NNLO}}^{\text{amp}}/\Gamma_{\text{LO}}$	$\Gamma_{\text{NLO}}^{\text{sq}}/\Gamma_{\text{LO}}$	$\Gamma_{\text{NNLO}}^{\text{sq}}/\Gamma_{\text{LO}}$	series inside $ \cdot ^2$	traditional series
$J/\psi \rightarrow 3\gamma$	$-0.51 + i 0.19$	$-0.36 + i 0.14$	0.276	0.126	-0.02	-0.44	1, 0.54, 0.39	1, 1.02, 0.42
$\Upsilon \rightarrow 3\gamma$	$-0.36 + i 0.13$	$-0.27 + i 0.10$	0.426	0.190	0.28	-0.11	1, 0.38, 0.29	1, 0.72, 0.39
$J/\psi + J/\psi$	$-0.37 + i 0.02$	$-0.39 - i 0.003$	0.40	0.058	0.26	-0.38	1, 0.37, 0.39	1, 0.74, 0.64
$J/\psi + \eta_c$	$0.55 + i 0.38$	$0.22 + i 0.09$	2.55	3.35	2.10	2.99	1, 0.67, 0.24	1, 1.10, 0.89
$J/\psi + \chi_{c0}$	$0.39 + i 0.05$	$0.07 + i 0.06$	1.93	2.14	1.78	2.07	1, 0.39, 0.09	1, 0.78, 0.29
$J/\psi + \chi_{c1}$	$0.12 + i 0.37$	$-0.14 + i 0.22$	1.39	1.31	1.24	1.11	1, 0.39, 0.26	1, 0.24, 0.13
$J/\psi + \chi_{c2}$	$0.07 + i 0.16$	$-0.20 + i 0.08$	1.17	0.81	1.14	0.77	1, 0.17, 0.22	1, 0.14, 0.37
$\eta_c + \gamma$	$-0.09 + i 0.09$	$-0.17 + i 0.01$	0.84	0.56	0.82	0.50	1, 0.13, 0.17	1, 0.18, 0.32
$\chi_{c0} + \gamma$	$0.07 + i 0.12$	$0.01 - i 0.01$	1.16	1.18	1.14	1.18	1, 0.14, 0.01	1, 0.14, 0.04
$\chi_{c1} + \gamma$	$-0.11 + i 0.09$	$-0.10 + i 0.03$	0.80	0.64	0.78	0.60	1, 0.14, 0.10	1, 0.22, 0.18
$\chi_{c2} + \gamma$	$-0.30 + i 0.10$	$-0.19 - i 0.02$	0.50	0.27	0.40	0.12	1, 0.32, 0.19	1, 0.60, 0.28

Amplitude level: **positive and better convergence for all 11 processes**; the traditional truncation turns negative for the decays and double- J/ψ . Data: (decays)[14], (double J/ψ)[17], (8 channels, three-loop α_s at $\mu_R = \sqrt{s}/2$)[20].

More results: application in exclusive $J/\psi + \gamma$ in UPC

- Colleagues have applied our amplitude-level scheme to exclusive $J/\psi + \gamma$ production in ultraperipheral collisions (UPC)
- In the traditional perturbative treatment the large- p_T differential cross section comes out **negative**; at the amplitude level the same observable is **positive**
- Reference:

X.-J. Zhan, T.-F. Feng, X.-G. Wu,

“Amplitude-level NLO predictions for exclusive $J/\psi + \gamma$ production in ultraperipheral collisions”,
arXiv:2610.04293 [hep-ph]

- Actually, there is **no need** to emphasis the effectiveness of amplitude-level prescription in the sign problem, since **it preserves positivity all the time**

Summary and Discussions

- QM $\rightarrow$ QFT: the modulus square of the matrix element *is* the probability
— the amplitude-level prescription is a natural choice that **preserves the positivity of probability**
- The traditional truncation of amplitude square **violates the positivity**
 - ① **Inclusive** processes: forced by the KLN theorem, **any other choices?**
 - ② **QED**: harmless, since $\alpha \simeq 1/137$
 - ③ **Exclusive QCD**: *not acceptable* — five decades of negative rates are the consequence
- Amplitude-level prescription ~~restores~~ **preserves** the positivity:

$$\Gamma^{\text{amp}} = \Gamma_{\text{LO}} \left| 1 + \frac{\alpha_s}{\pi} (c_1 + ic_2) + \frac{\alpha_s^2}{\pi^2} (c_3 + ic_4) \right|^2$$

- application: better convergence found in 11 exclusive processes
first complete NNLO for $J/\psi \rightarrow 3\gamma$ gives $\Gamma = 0.96_{-0.13}^{+4.32}$ eV (BESIII: $1.046 \pm 0.167 \pm 0.185$ eV)
- also applied by others in exclusive $J/\psi + \gamma$ production in UPC[21]
- More examples besides Quarkonium physics?

Thank you for your attention!

Input Parameters

Parameter	Central Value
m_c	1.5 GeV
$M_{J/\psi}$	$2m_c$
$ R_{J/\psi}(0) ^2$	0.934 GeV ³
$\langle v^2 \rangle_{J/\psi}$	0.225
μ_R	$2m_c$
μ_Λ	m_c
$\alpha_s^{(3)}(\mu_R)$	0.25406 (3-loop)

Parameter	Central Value
m_b	4.75 GeV
M_Υ	9.46 GeV
$ R_\Upsilon(0) ^2$	6.477 GeV ³
μ_R	$2m_b$
$\alpha_s^{(4)}(\mu_R)$	0.18055 (3-loop)

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