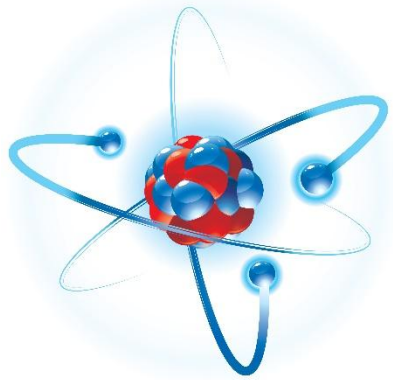
Microscopic Approaches to the Nuclear Level Density in Relativistic Density Functional Theory

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Outline

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- **Nuclear level density**
- **Combinatorial method based on relativistic density functional**
- **Microscopic statistical method based on relativistic density functional**
- **Summary and prospective**

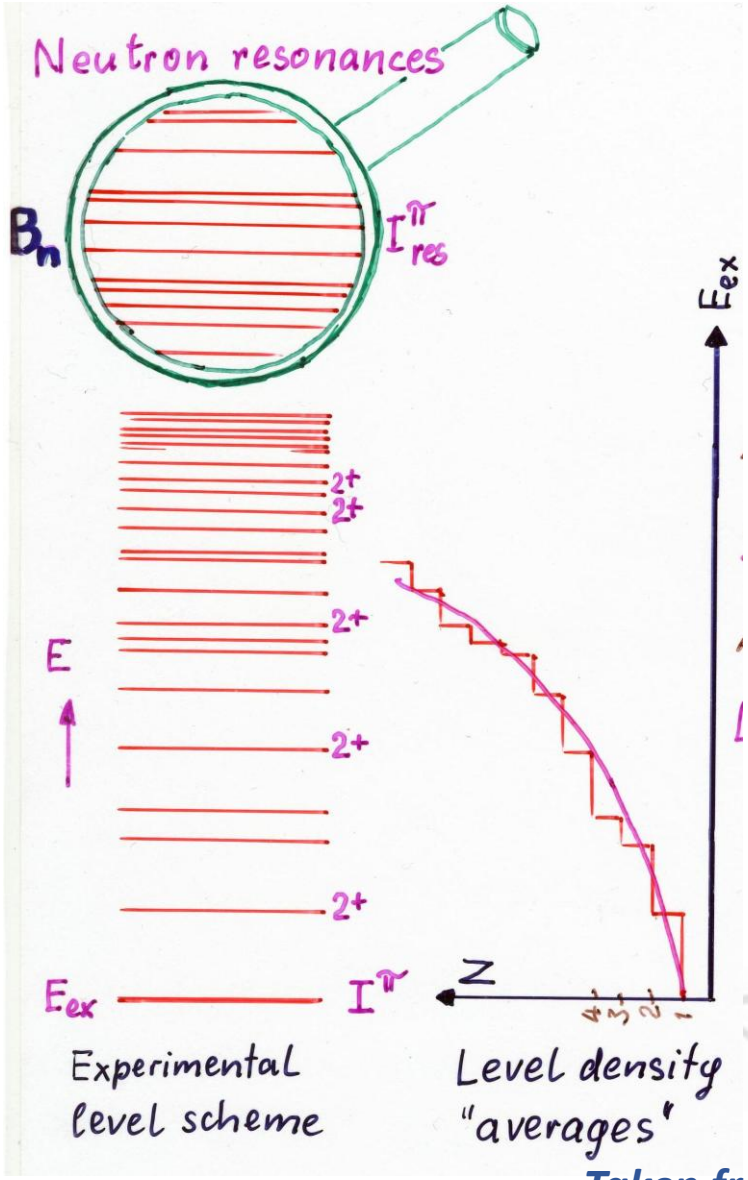
Nuclear level density

Nuclear level density (NLD): the average number of energy levels per unit energy interval (at given spin **J**, parity **P**)

Important in nuclear physics:

- ✓ Provides nuclear structure information at high excitation states and the behavior of nucleons within the nucleus
- ✓ Essential for statistical theory models of nuclear reactions:
  - various nuclear reaction rates;
  - low-energy neutron capture;
  - thermonuclear rates for nucleosynthesis;
  - Nuclear reactor design (fission and fusion);
  - ……

Directly governs the accuracy of nuclear reaction calculations

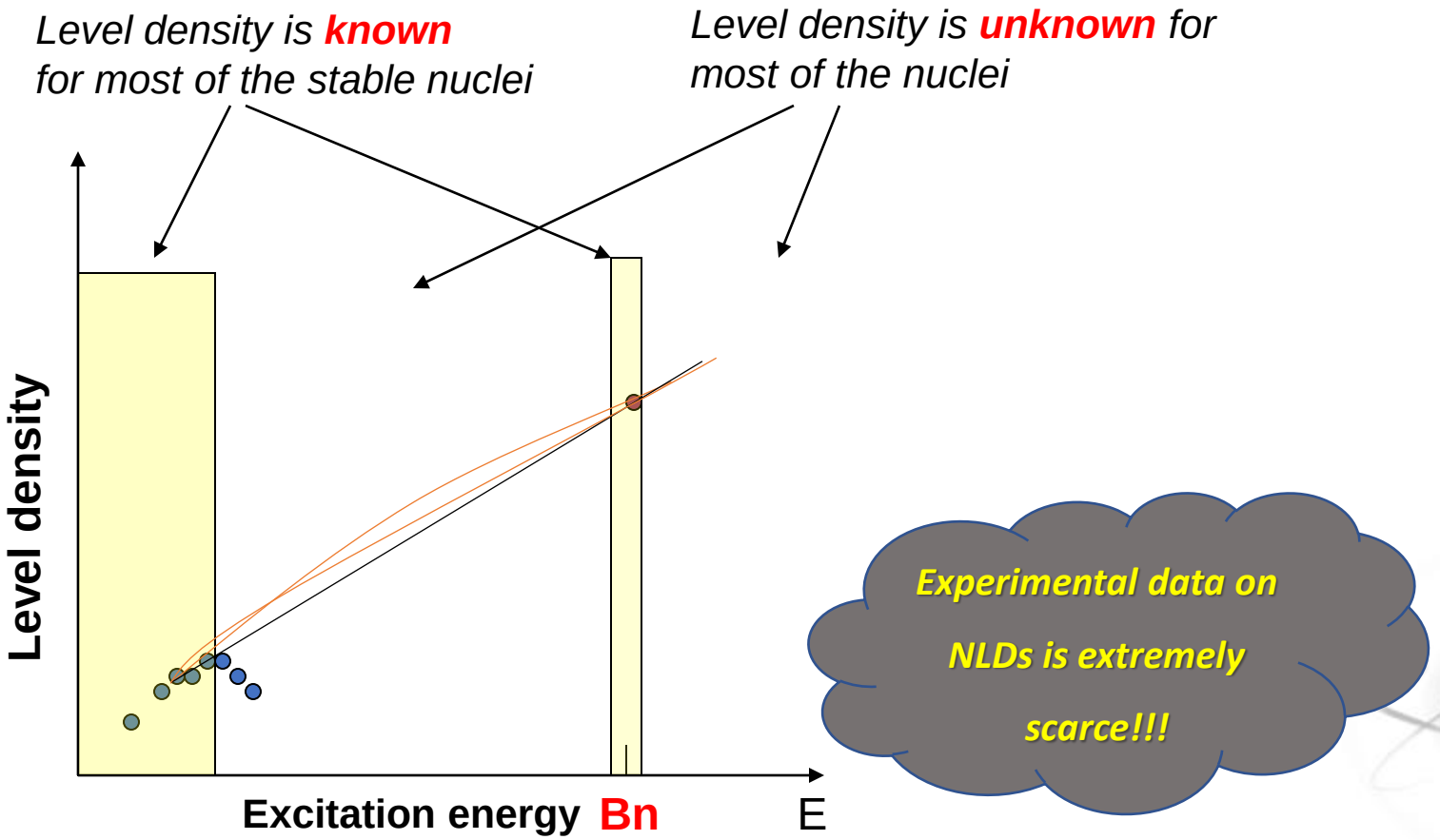


Taken from Egidy

Experimental level density

How to obtain：

- ✓ **原子核低激发态能级累计数(cumulative number of low-lying discrete levels)**
- ✓ **中子结合能处测量的平均能级间距(neutron resonance spacing at the neutron binding energy)**
- ✓ **复合核反应(compound-nucleus reactions 出射粒子谱分析获得剩余核能级密度信息)**
- ✓ **Oslo method**



Theoretical calculation： Fermi gas model

Fermi gas model: non-interacting fermions、statistical physics methods *Bethe, Phys. Rev. 1936*

$$\rho(E) = \frac{\sqrt{\pi}}{12}\frac{\exp[2\sqrt{aE}]}{a^{1/4}E^{5/4}}$$

Further improvements： shell、pairing and collective effects

- **Constant Temperature Model 常温模型** *Gilbert, Can. J. Phys., 1965*
- **Back-shifted(返移)费米气体模型** *Wilg, Nucl. Phys.A, 1973*
- **Generalized Superfluid Model广义超流模型** *Ignatyuk, Sov. J. Nucl. A, 1973*
- **……**

✓ **Most widely used methods for calculating level density: simple and convenient (formulas with a limited number of parameters), well describing experimental results (reaction).**

Calculation methods for level density: limitations and improvements

Phenomenological methods

Fermi gas model

- ✓ **Most widely used，simple and convenient、good description**
- × **limited scope of application, uncertain extrapolation: relies on experimental data to adjust parameters, while experimental data is extremely scarce.**

Microscopic methods

- ✓ **Based on well-established nuclear structure models: independent of parameters, providing a microscopic, unified, and self-consistent description of nuclear properties;**
- ✓ **incorporates the true physical picture of level densities.**

Microscopic approaches for level density

✓ **Shell-model Monte Carlo / Shell model / Projected shell model （Shell-model-based methods)**

Y. Alhassid, S. Liu, H. Nakada. Phys. Rev. Lett., 1999, 83:4265.
W. E. Ormand. Phys. Rev. C, 1997, 56:R1678.
J. A. White, S. E. Koonin, D. J. Dean. Phys. Rev. C, 2000, 61:034303.
J. Chen, M. Liu, C. Yuan, et al. Phys. Rev. C, 2023, 107:054306.
Wang J.-Q., Dutta S., Wang L.-J., et al., Phys. Rev. C 108(3) (2023) 034309.

✓ **Microscopic statistical method**

J. B. French, K. F. Ratcliff. Phys. Rev. C, 1971, 3:94.
B. Strohmaier, S. M. Grimes, H. Satyanarayana. Phys. Rev. C, 1987, 36:1604.
S. M. Grimes, T. N. Massey. Phys. Rev. C, 1995, 51:606.
J. Zhao, T. Nikšic, and D. Vretenar, Phys. Rev. C 102, 054606 (2020).
W. Zhang, W. Gao, G.-T. Zhang, and Z.-Y. Li, Nucl. Sci. Tech. 34, 124 (2023)
Zi-Cheng Wang, Peng-Xiang Du, Jian Li, et al. Phys. Rev. C, 2026, 114:024345.

✓ **Combinatorial methods based on density functional theory**

S. Hilaire, J. Delaroche. A. Koning, Nucl. Phys. A, 1998. 632:417.
S. Hilaire, J. Delaroche, M. Girod. Eur. Phys. J. A, 2001. 12:169.
S. Hilaire, S. Goriely. Nucl. Phys. A, 2006. 779:63.
S. Goriely, S. Hilaire, A. J. Koning. Phys. Rev. C, 2008. 78:064307.
J. Berger, M. Martinot. Nucl. Phys. A, 1974. 226:391.
K.-P. Geng, P.-X. Du, J. Li, and D.-L. Fang, Nucl. Sci. Tech. 34, 141 (2023)
X. F. Jiang, X. H. Wu, P. W. Zhao, and J. Meng, Phys. Lett. B 849, 138448 (2024).
S. Goriely, W. Ryssens, S. Hilaire, and A.J. Koning, Phys. Rev. C 113(2026), 014320
P. Du and J. Li, arXiv:2605.04862.

✓ **The QRPA plus boson expansion**

Hilaire S., Goriely S., Péru S., et al., Phys. Lett. B 843 (2023) 137989

Outline

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- Nuclear level density
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Combinatorial methods

Single-particle levels (ε_i)

From relativistic density functional theory

Combinatorial method

$$\mathcal{Z}(x_1, x_2, x_3, x_4, y, t) = \prod_{k=1}^{4} \prod_{i=1}^{I_k} \left(1 + x_k p_i^k y^{\varepsilon_i^k} t^{m_i^k}\right)$$

$$C_N(U, M, P) \prod_{k=1}^{4} x_k^{N_k} p y^U t^M$$

$$\rho_I(U, M, P) = \frac{1}{\varepsilon_0} C_N(U, M, P)$$

Incoherent particle-hole state density ρ_I

$$\mathcal{Z}_{vib} = \prod_{\lambda} \prod_{\mu} \sum_{N_{ph}} [x y^{\varepsilon} t^{\mu} p_{\lambda}]^{N_{ph}}$$

Collective vibrational excitation

$$\rho_{def}(U, J, P) = 1/2[\sum_{K=-J, K\neq 0}^{J} \rho_I\left(U - E_{rot}^{J,K}, K, P\right)]$$

$$+ \delta_{(J_{even})} \delta_{(P=+)} \rho_I\left(U - E_{rot}^{J,0}, 0, P\right) + \delta_{(J_{odd})} \delta_{(P=-)} \rho_I\left(U - E_{rot}^{J,0}, 0, P\right)$$

Collective rotational excitation

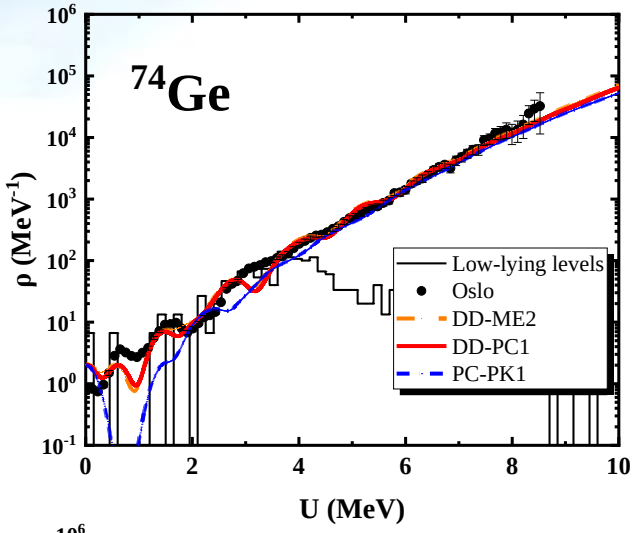
$$\rho(U, J, P) = [1 - f_{dam}(U)] \rho_{sph}(U, J, P)$$

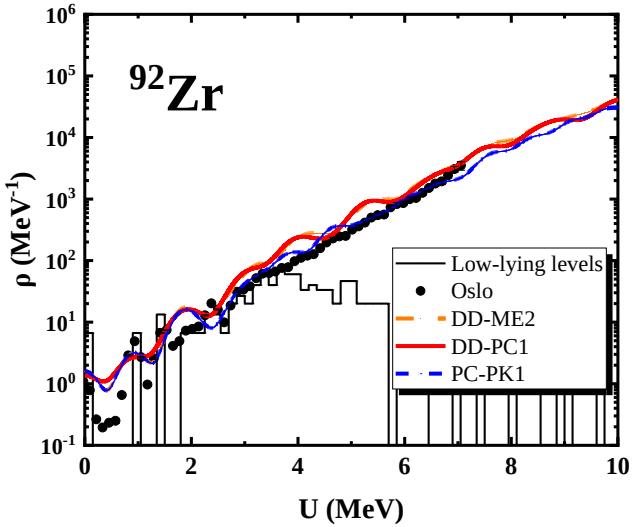
$$+ f_{dam}(U) \rho_{def}(U, J, P)$$

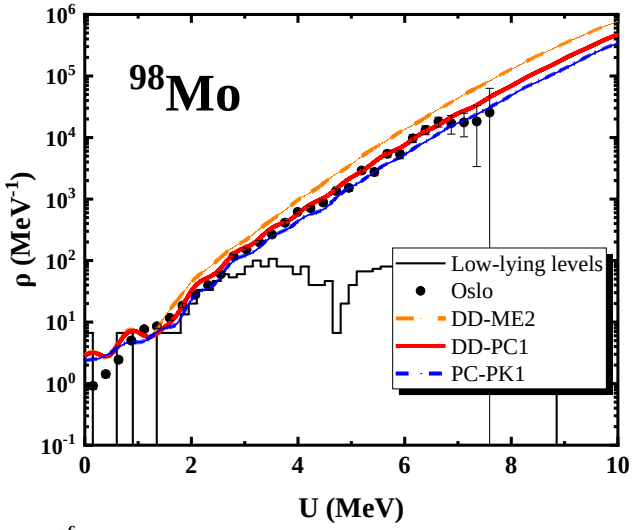
Energy-spin-parity dependent level density

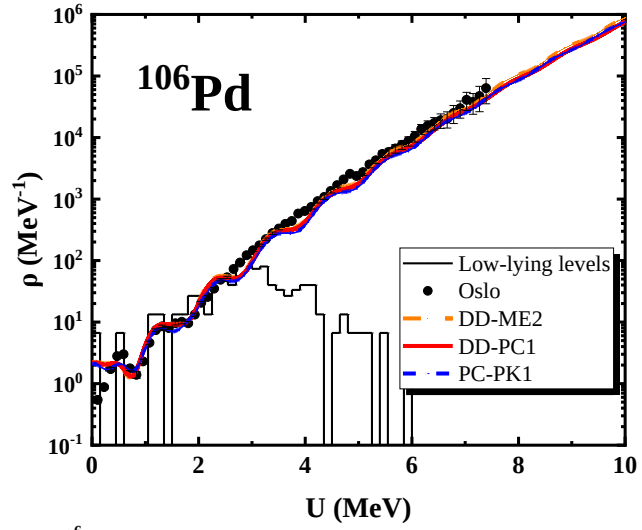
$$\rho_{sph}(U, J, P) = \rho_I(U, M = J, P) - \rho_I(U, M = J + 1, P)$$

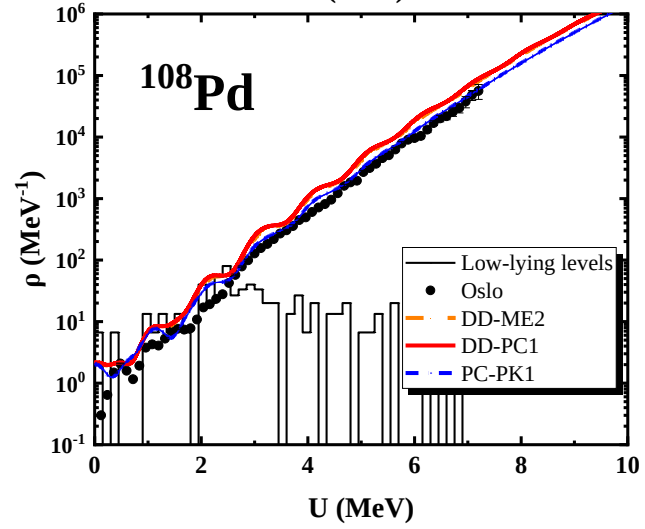
Results of level-density calculations

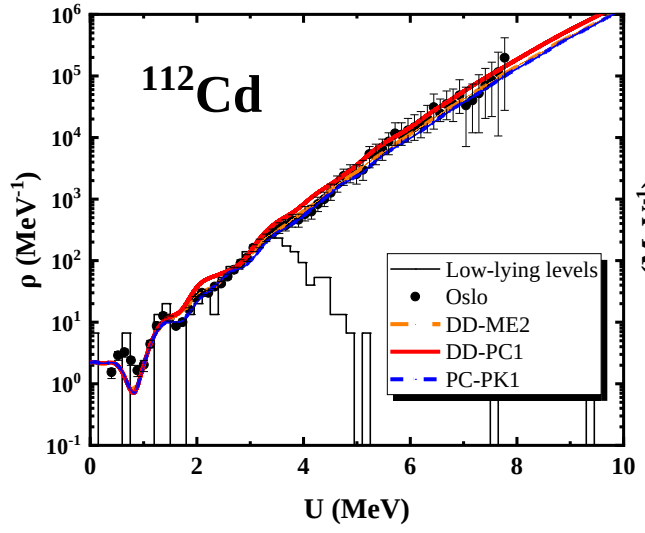


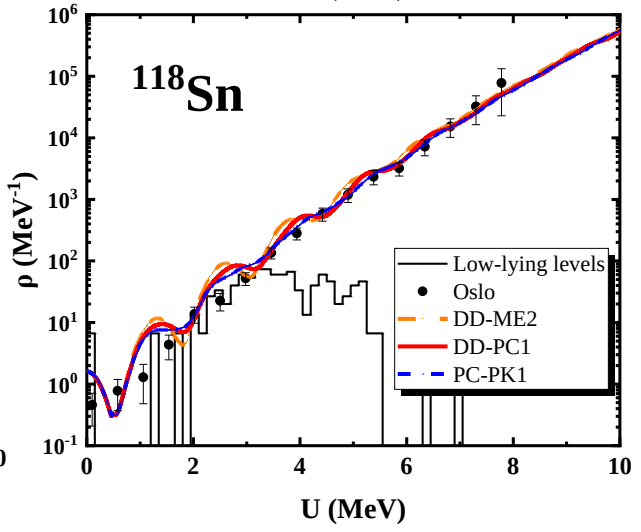


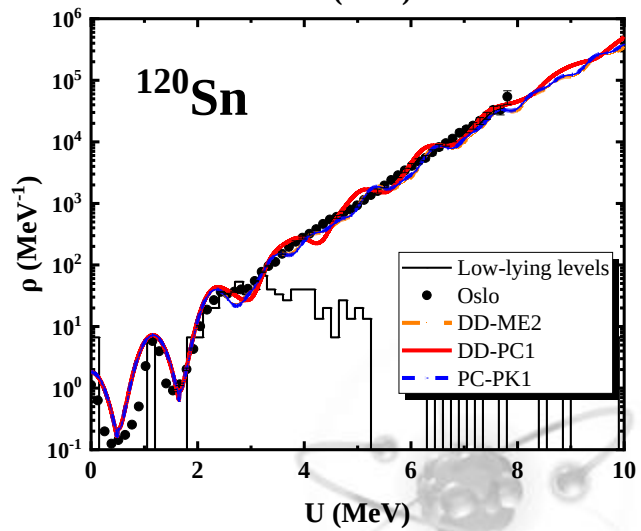






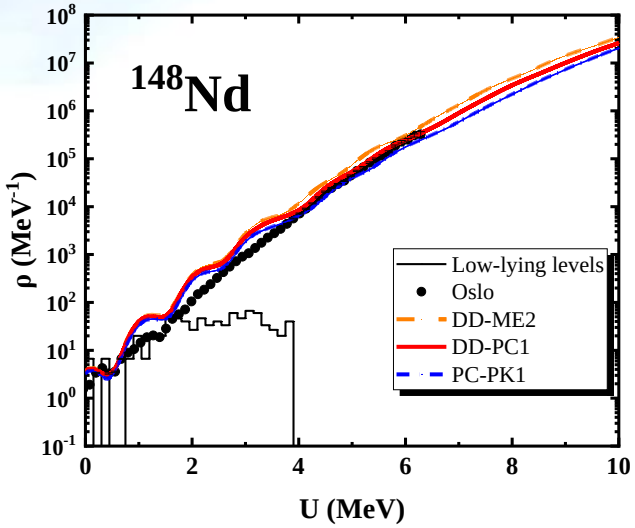


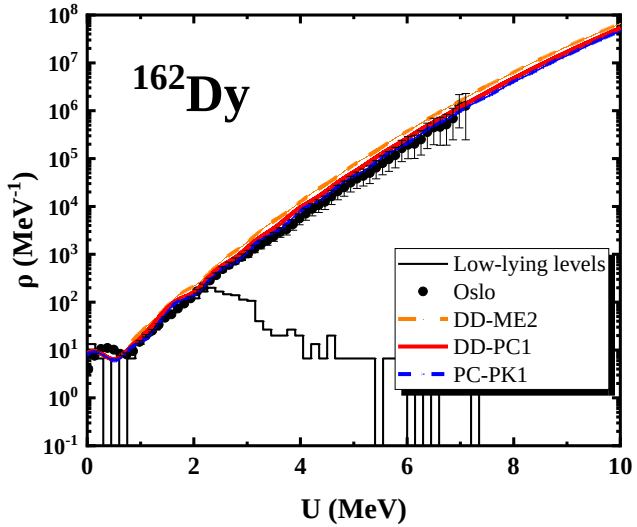


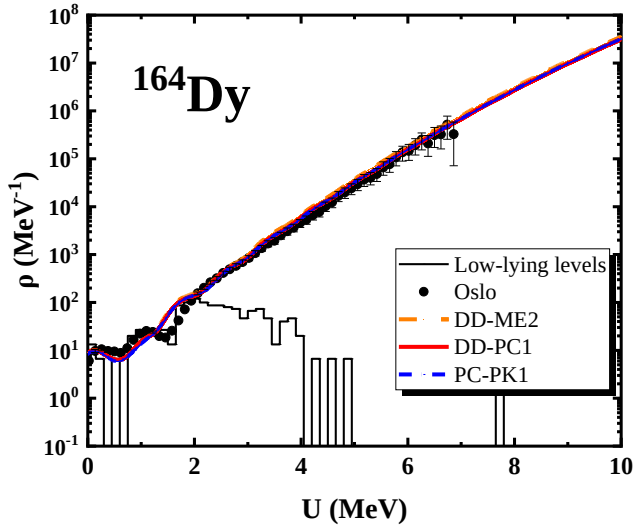


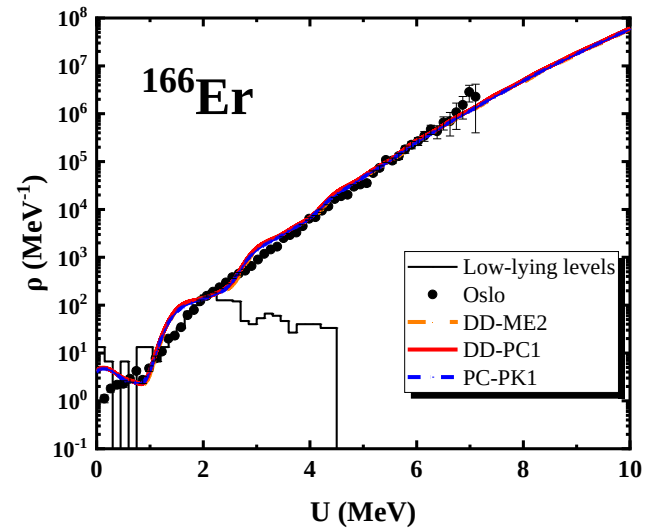
- ✓ Predictions from three different relativistic functionals are in good agreement with each other and the Oslo data.
- ✓ The non-statistical, step-like shell effects of the discrete levels are naturally reproduced at the low-energy area.

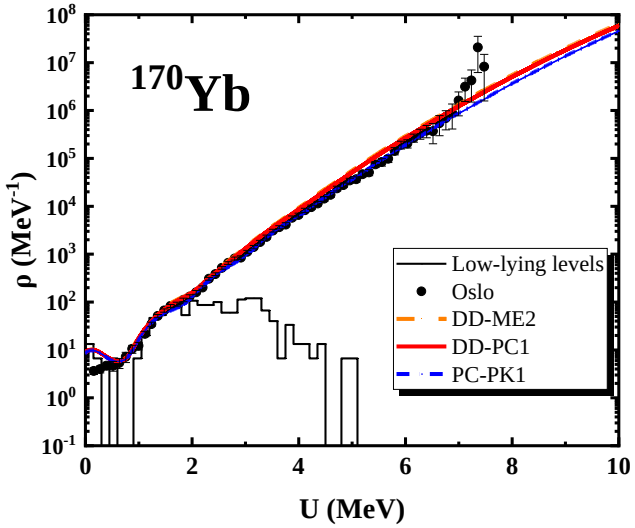
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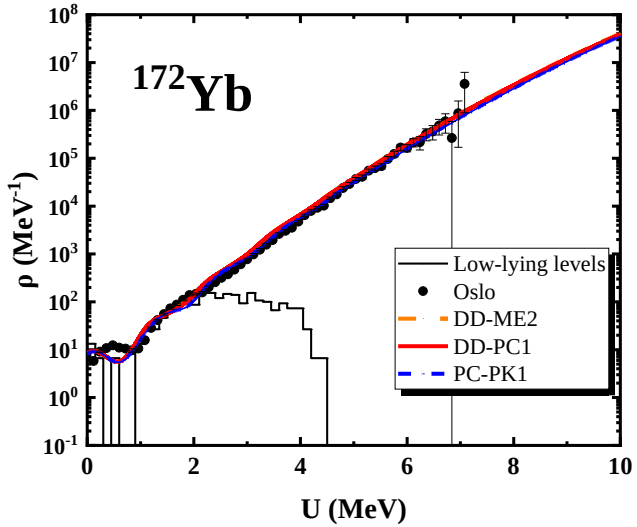


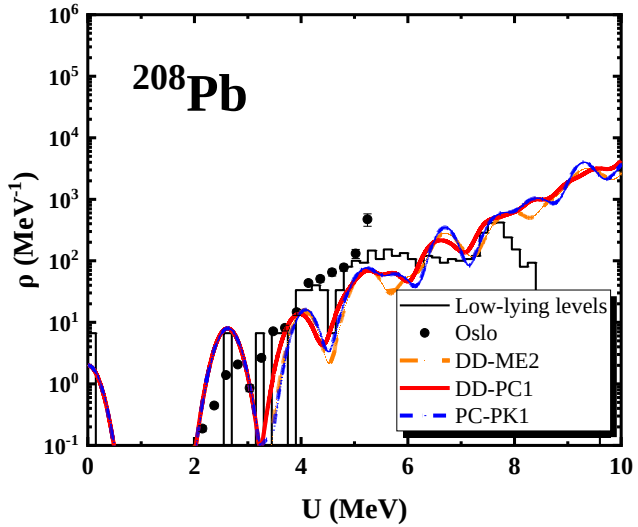


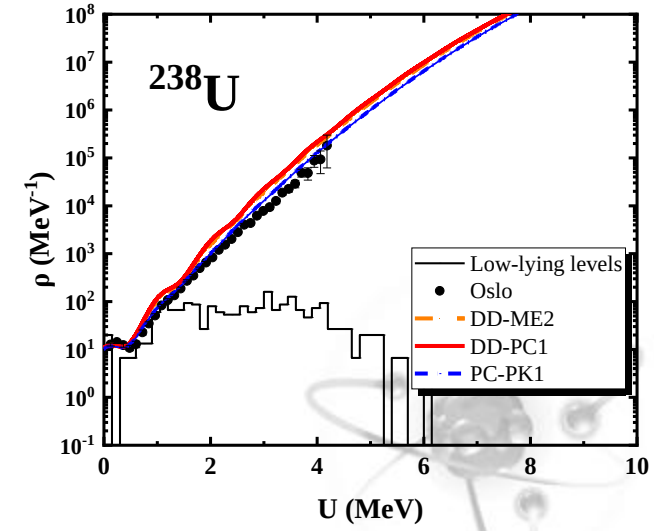






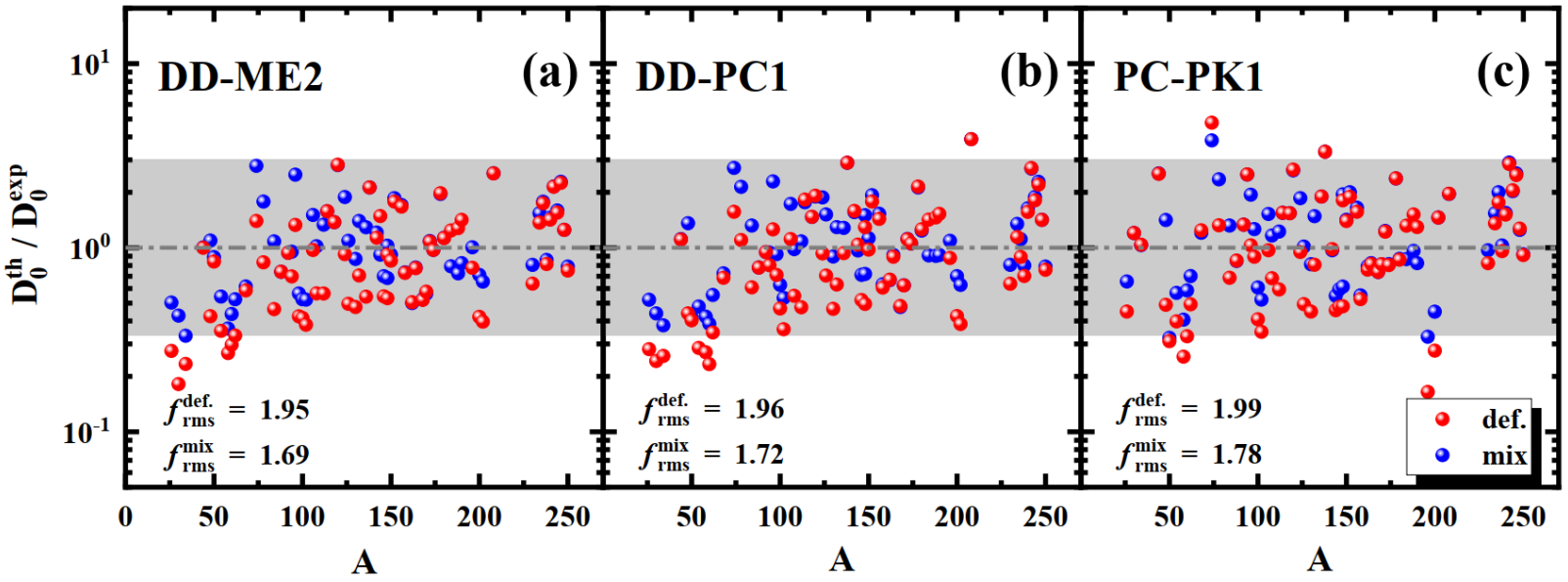






P. Du and J. Li, Nuclear level densities in the relativistic Hartree-Bogoliubov plus combinatorial Framework, arXiv:2605.04862v1.

s-wave neutron resonance spacing of 67 even-even nuclei



- ✓ The most reliable experimental level-density data are the neutron resonance spacings at the neutron separation energy $\boldsymbol{S_n}$ (s-wave D_0 and p-wave D_1):

$$f_{rms} = \exp\left[\frac{1}{N}\sum_{i=1}^{N}\left(\ln\frac{D_{k,i}^{th.}}{D_{k,i}^{exp.}}\right)^2\right]^{1/2} \quad (k = 0,1)$$

- ✓ DD-ME2 yielded the best results, f_{rms} is 1.69 for 67 even-even nuclei, achieving the same accuracy as the phenomenological model (CTM, BFM).
- ✓ The results obtained with DD-ME2, DD-PC1, and PC-PK1 are in close agreement, f_{rms} is all around 1.7.

P. Du and J. Li, Nuclear level densities in the relativistic Hartree-Bogoliubov plus combinatorial Framework, arXiv:2605.04862v1.

Outline

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- Nuclear level density
- Combinatorial method based on relativistic density functional
- **Microscopic statistical method based on relativistic density functional**
- Summary and prospective

Microscopic statistical method

Solve BCS equations at different excitation energies (temperatures $T = \frac{1}{\beta}$)

Single-particle levels, total energy, etc.

From relativistic density functional theory

λ_k, Δ_k, S(U) , etc. at different excitation energies

$$N_k = \sum_\nu \left(1 - \frac{\varepsilon_{k\nu} - \lambda_k}{E_{k\nu}} tanh\left[\frac{1}{2}\beta E_{k\nu}\right]\right) \qquad \frac{2}{G_k} = \sum_\nu \frac{tanh[\beta E_{k\nu}/2]}{E_{k\nu}} \qquad (k = n, p)$$

$$E(T) = \sum_{k=n,p}\{\sum_\nu \varepsilon_{k\nu}\left(1 - \frac{\varepsilon_{k\nu} - \lambda_k}{E_{k\nu}} tanh\left[\frac{1}{2}\beta E_{k\nu}\right]\right) - \frac{\Delta_k^2}{G_k}\} \qquad U = E(T) - E(0)$$

$$\omega(U) = \frac{exp[S(U)]}{(2\pi)^{3/2}\sqrt{D}}$$

Energy-spin dependent level density ρ

Intrinsic level density ω

$$\rho_{def}(U,J) = \frac{1}{2}\sum_{K=-J}^{J} \frac{1}{\sqrt{2\pi\sigma^2}} exp\left[\left(-\frac{J(J+1)}{2\sigma_\perp^2}\right) - K^2\left(\frac{1}{\sigma^2} - \frac{1}{\sigma_\perp^2}\right)/2\right]\omega_{def}(U)$$

Damping effect

$$\rho(U,J) = f_{dam}(U)K_{vib}\rho_{def}(U,J) + [1 - f_{dam}(U)]K_{vib}\rho_{sph}(U,J)$$

Rotational enhancement

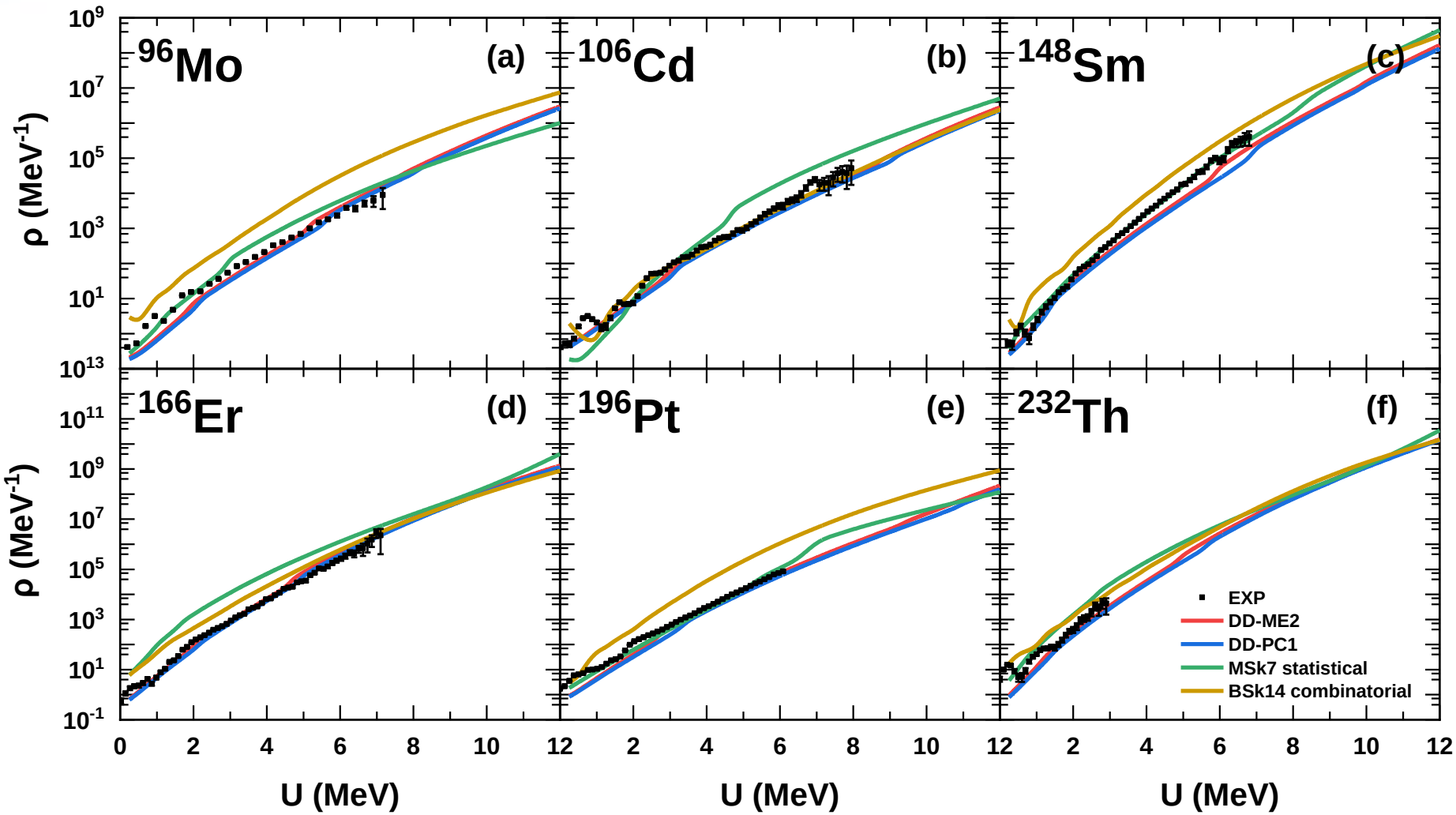
$$\rho_{sph}(U,J) = \frac{2J+1}{2\sqrt{2\pi}\sigma^3} exp\left[-\frac{J(J+1)}{2\sigma^2}\right]\omega_{sph}(U)$$

Collective excitations of the intrinsic level density

Vibrational enhancement

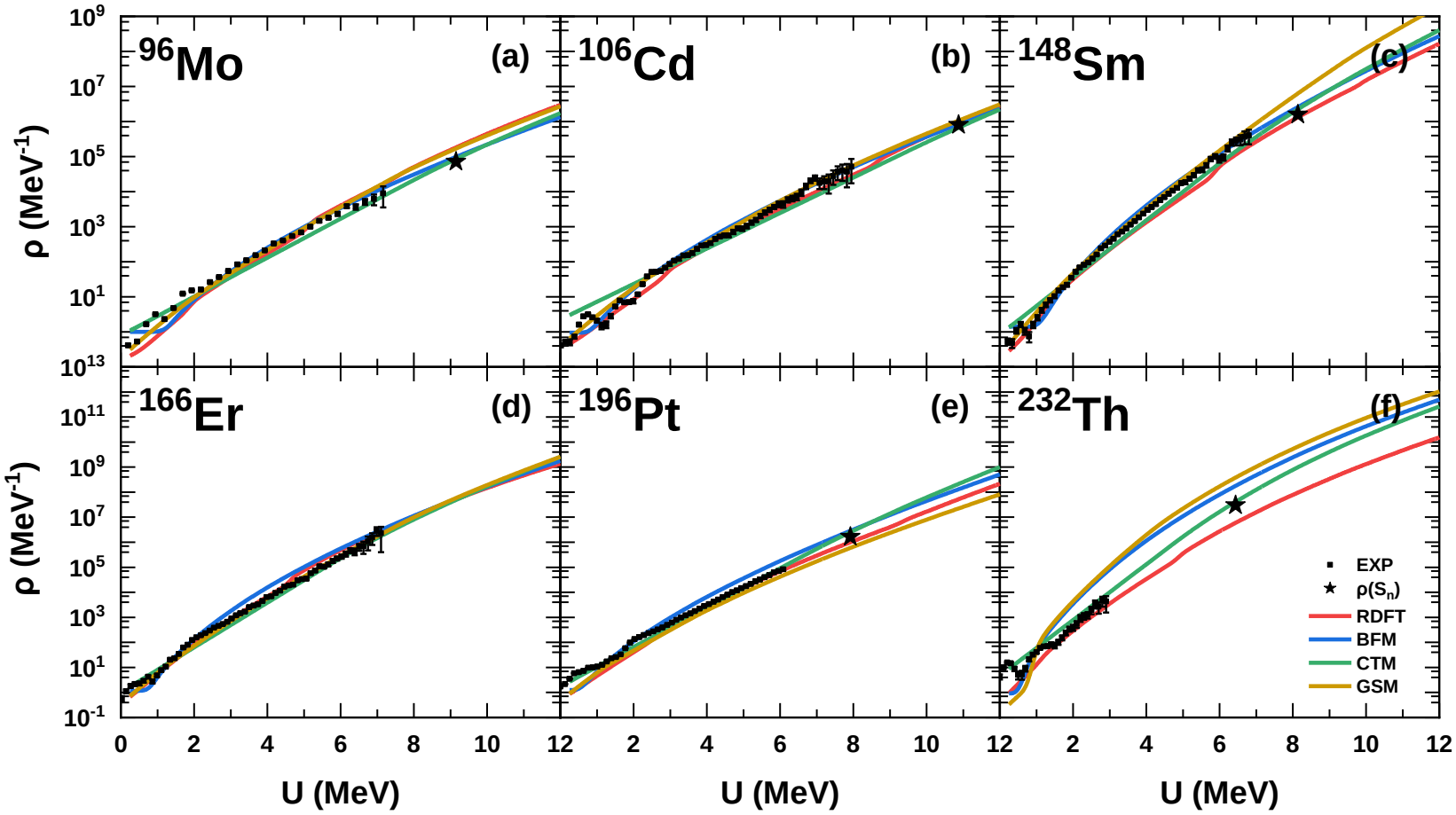
$$K_{vib} = exp[\delta S - (\delta U/T)]$$

Level density: comparison with microscopic calculations



- ✓ RDFT interactions: DD-ME2 and DD-PC1 give similar level densities
- ✓ Overall agreement: RDFT level densities are comparable to MSk7 (statistical) and BSk14 (combinatorial) results.

Level density: comparison with phenomenological calculations



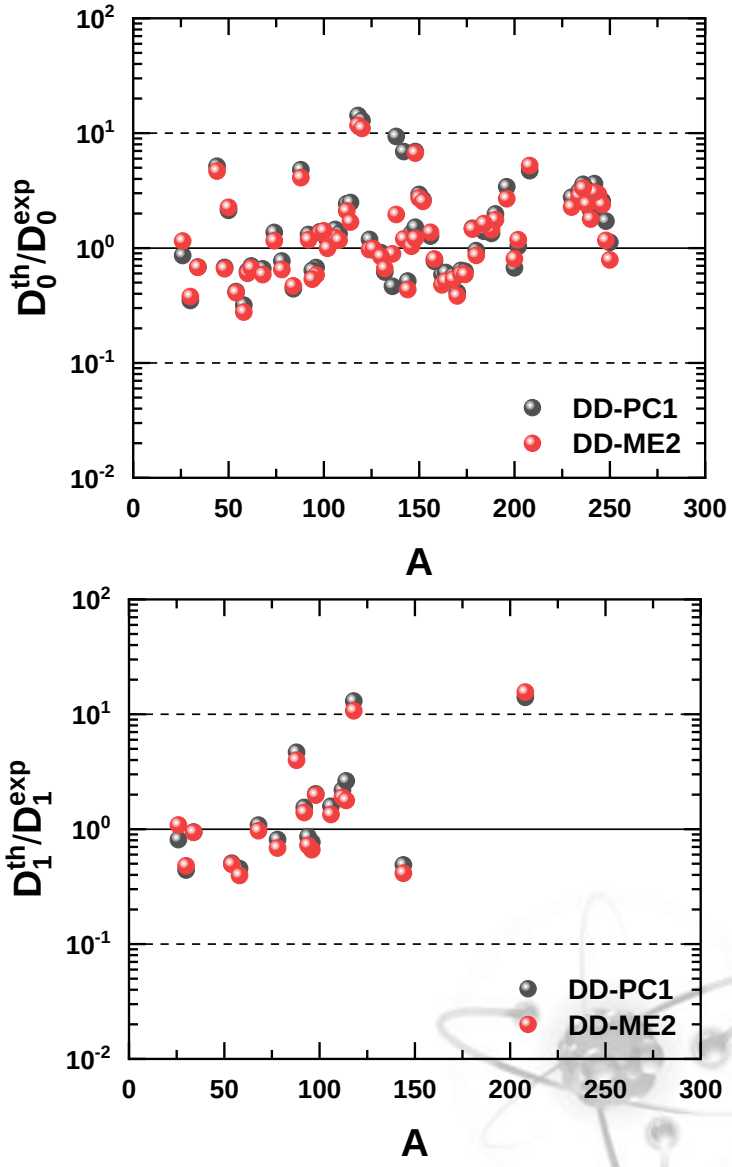
Statistical level densities from DD-ME2 vs. other phenomenological models: despite some differences, all show the same trend; agreement varies by model, with small deviations below the neutron separation energy.

Neutron resonance spacing of 67 even-even nuclei

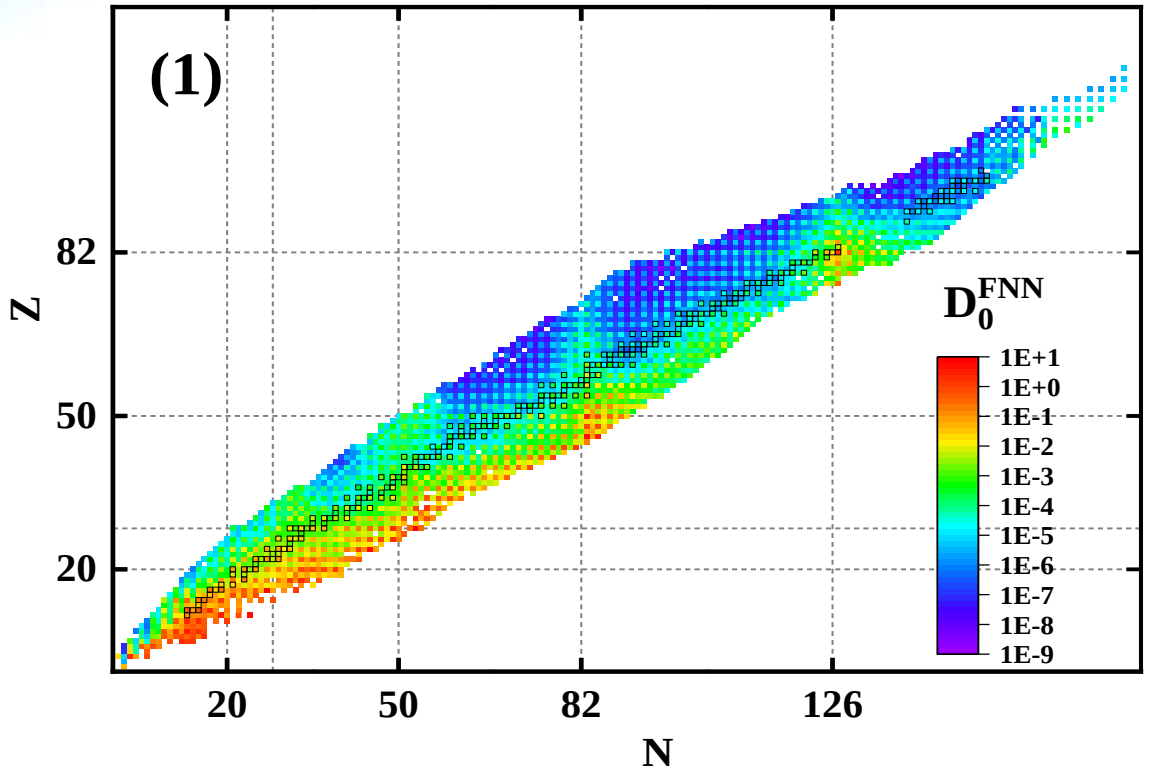
RMS deviations (f_{rms}) of neutron resonance spacings for 67 even-even nuclei:

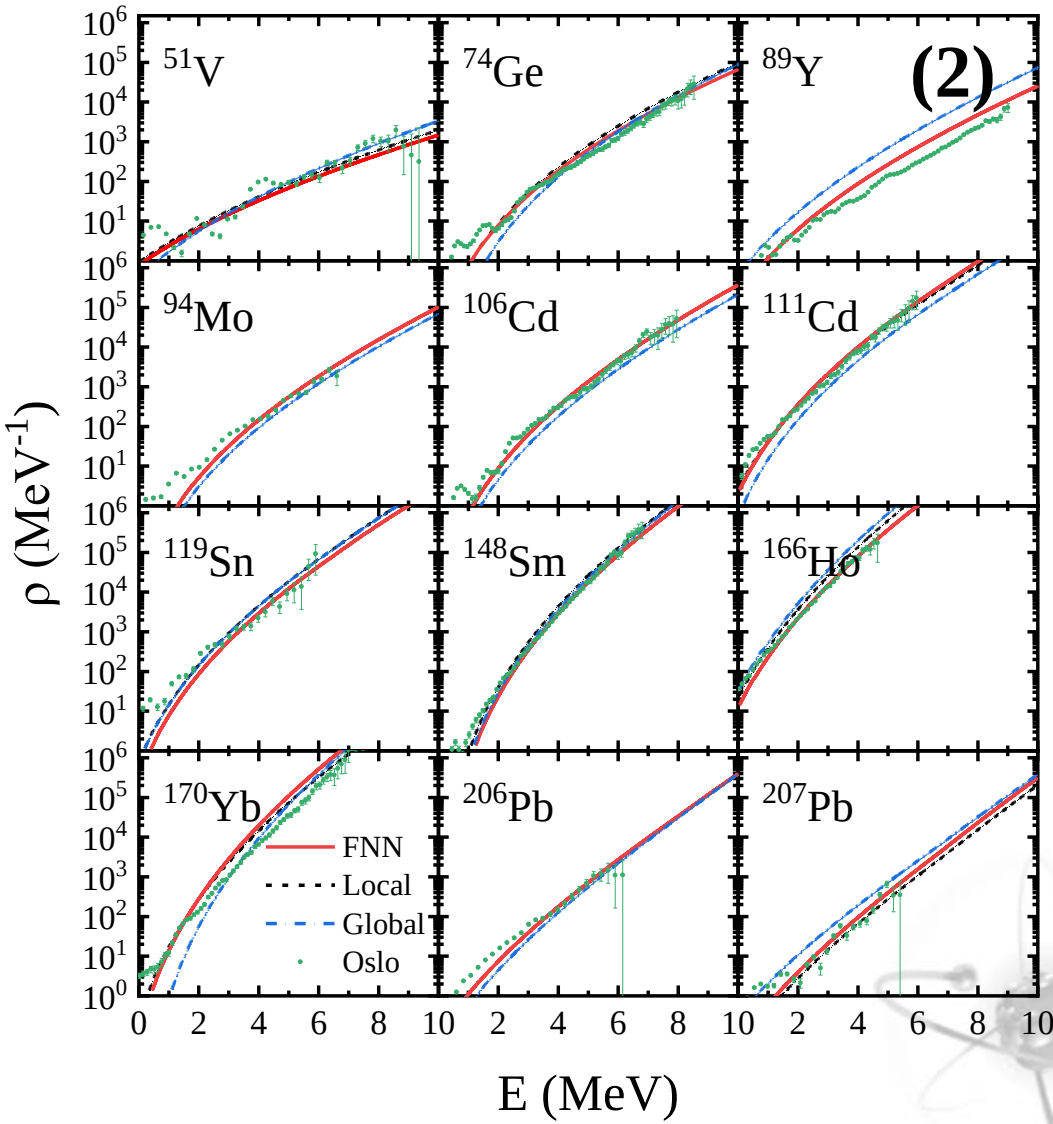
$$f_{rms} = \exp\left[\frac{1}{N}\sum_{i=1}^{N}\left(\ln\frac{D_{k,i}^{th.}}{D_{k,i}^{exp.}}\right)^2\right]^{1/2} \quad (k = 0,1)$$

- ✓ 67 D_0 data points: f_{rms}= 2.29 for DD-ME2 and 2.56 for DD-PC1: Relativistic combinatorial method without pairing correlations, f_{rms}=3.62 (66 nuclei); nonrelativistic microscopic statistical method, f_{rms}=2.14 (278 nuclei); BFM, f_{rms}=1.78 (295 nuclei); QRPA-based boson expansion method, f_{rms}=1.65 (48 nuclei).
- ✓ 18 D_1 data points: RMS deviation = 2.84 for DD-ME2 and 2.93 for DD-PC1
- ✓ For the combined 67 D_0 and 18 D_1 data points: f_{rms} = 2.41 for DD-ME2 and 2.64 for DD-PC1: nonrelativistic combinatorial method with pairing correlations, 2.3 (289 nuclei).



Determining level density parameters using a feedforward neural network (FNN)



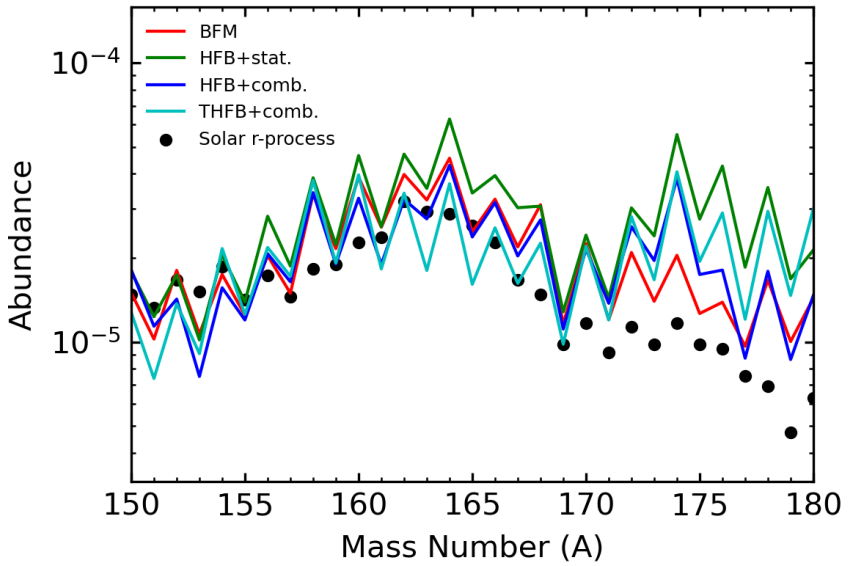


- ✓ The back-shifted Fermi gas model (BSFM) parameters for nearly 3,000 nuclei on the nuclide chart were predicted using FNN, and D_0 was calculated.
- ✓ The level density derived from the predicted parameters reproduces the experimental data from Oslo method.

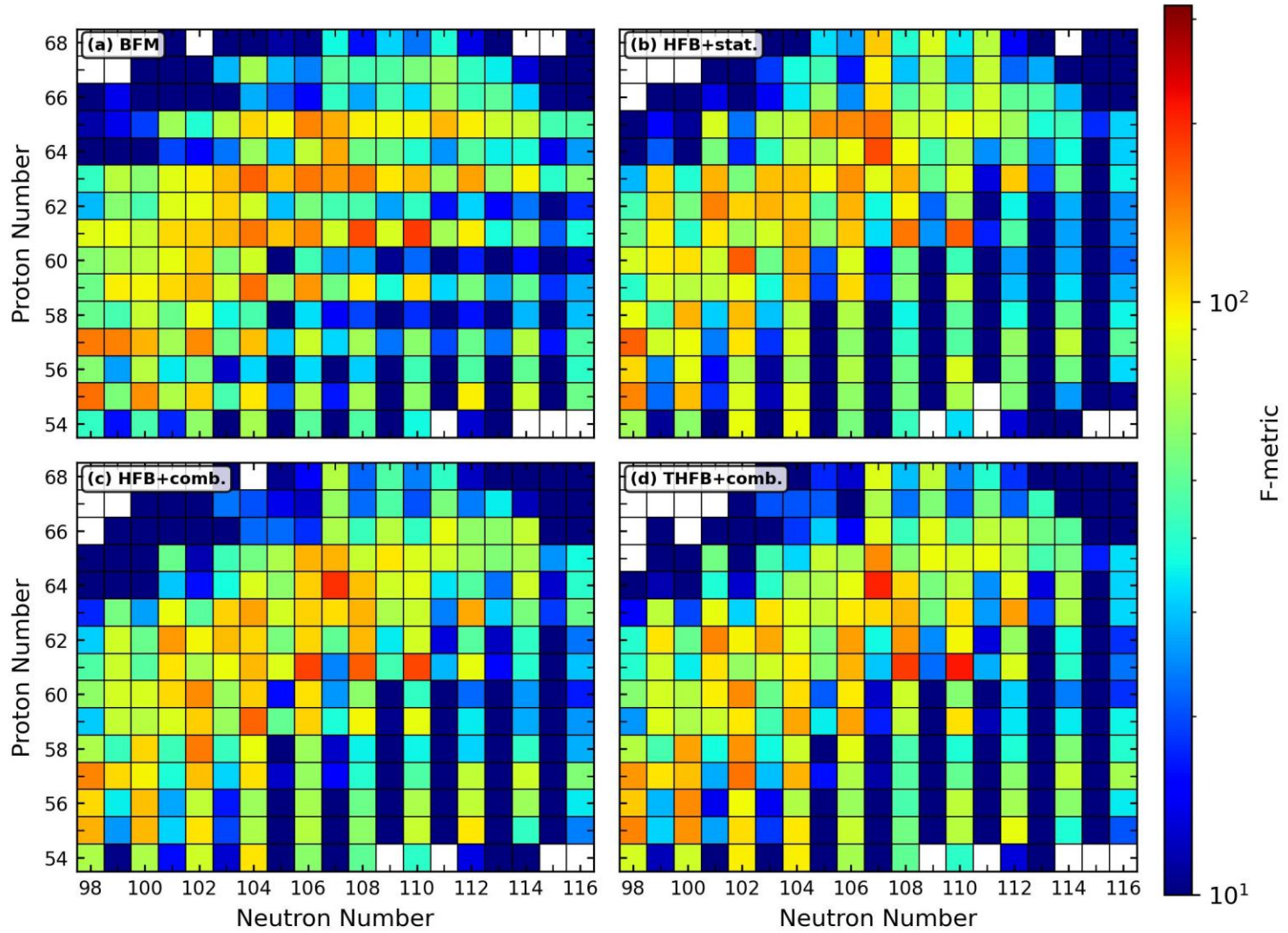
P. X. Du, T. S. Shang, K. P. Geng, J. Li, & D. L. Fang, Inference of parameters for the back-shifted Fermi gas model using a feedforward neural network, Phys. Rev. C, 2024 ,109(4), 044325.

Nuclear Level Density Effects on the r-Process Rare-Earth Peak

Rare-earth peak: a nuclear-structure-sensitive feature of the r-process abundance pattern at A ≈ 164.



- **Phenomenological models**: Data-calibrated Fermi-gas models with limited treatment of fine nuclear structure, restricting extrapolation far from stability.
- **Microscopic models**: Self-consistent calculations based on established nuclear models, incorporating deformation, pairing correlations, and other structure effects.
- Nuclear level density connects microscopic nuclear structure to macroscopic rare-earth abundance patterns via the r-process rare-earth peak.



Rare-earth abundance sensitivities to individual neutron-capture rates in the cold r-process for four NLD models: (a) a phenomenological model; (b)–(d) microscopic models.

Hang Xu, Peng-Xiang Du, Jian Li, and Dong-Liang Fang, Impact of Nuclear Level Density on r-process Rare-Earth Peak Nucleosynthesis, The Astrophysical Journal, 1009, 61 (2026)

Summary and prospective

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- **Combinatorial method based on relativistic density functional**
  - ✓ **Relativistic mean field theory**
  - ✓ **Relativistic Hartree Bogoliubov theory**
  - ➢ ***Nuclear level density in fissioning Nuclei*** ⟹ **further refinement**
- **Predictions in machine learning**
  - ✓ **Inference of parameters for the back-shifted Fermi gas model using a feedforward neural network**
  - ➢ ***Fissioning nuclei*** ⟹ Construct an NLD parameter data table (database), integrated into nuclear reaction code, etc.
- **Applications of NLD: neutron capture reaction ⇒ fission reaction, superheavy nucleus synthesis**

Joint Undergraduate Program in Physics · Jilin University – Tomsk Polytechnic University

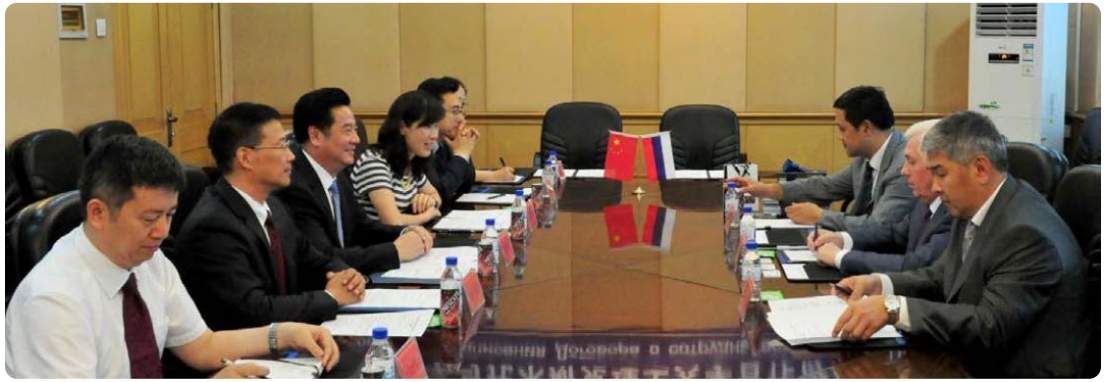
- ✓ **China–Russia Cooperative Undergraduate Education Program · Approved by the Chinese Ministry of Education;**
- ✓ **Started in 2002 · More than 690 graduates to date;**
- ✓ **In recent years, 60–70 graduates per year**

PROGRAM MODEL

Years 1–3
at Jilin University
→
Year 4
at Tomsk Polytechnic University
→
Dual Degrees
JLU + TPU diplomas

· Physics courses progressively taught **in Russian**, with zero-start Russian training

· **Mutual credit recognition**; students registered at both universities



JLU–TPU cooperation agreement signing ceremony

STRENGTHS OF BOTH PARTNERS

Jilin University · College of Physics

· **Double First-Class** physics discipline — one of the first 7 in China

· 2 academicians of the Chinese Academy of Sciences

· 148 full professors among 198 faculty members

Tomsk Polytechnic University

· Founded in **1896**; National Research University of Russia

· Physics ranked **Top 10** among Russian universities

· Home to Russia's only university research nuclear reactor

228	72.8%	115	51 (abroad)
graduates in 2023–2025	pursued graduate study	to Double First-Class universities & CAS institutes	Moscow State University; Tomsk Polytechnic University UTokyo · UZH · NUS …

Acknowledgments

Peng-Xiang Du Zi-Cheng Wang Hang Xu Kun-Peng Geng Tian-Shuai Shang	Jilin University
Prof. Dong-Liang Fang	Institute of modern physics, CAS
Prof. Shan-Gui Zhou	Institute of theoretical physics, CAS
Prof. Timur M. Shneidman	Bogoliubov laboratory of theoretical physics, JINR

Thank you!

Welcome to Changchun: discuss the physics of strong interacting



Changchun Ice and Snow New World

长春冰雪新天地





冰天雪地也是金山银山 Ice and snow are also invaluable assets

















Jilin Rime



Traditional Winter Fishing



Heaven Lake on Mount Changbai



