Moment of inertia of strongly deformed configuration

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Introduction

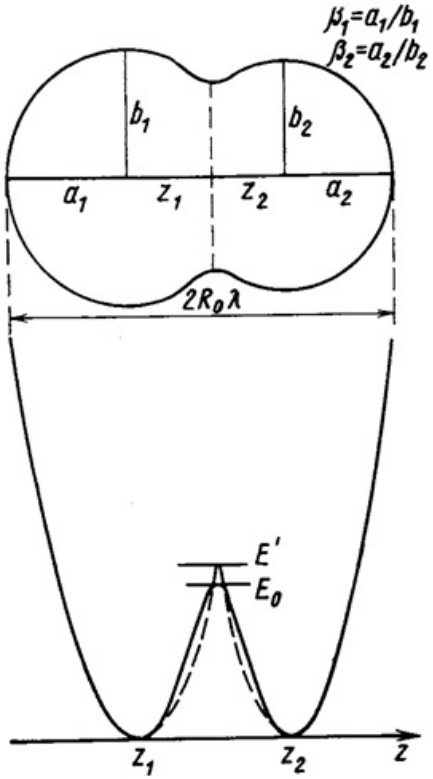
The moment of inertia is used for:

- the description of strongly deformed nuclear states: superdeformed and hyperdeformed nuclei;
- the study of dinuclear systems;
- the study of induced fission;
- the analysis of α-decays to rotational states in superheavy nuclei (prediction of 2_1^+ states);
- the generation of angular momentum for fragments of spontaneous fission.

Adiabatic two-center model of the nucleus (TCSM[1])

Hamiltonian of the system

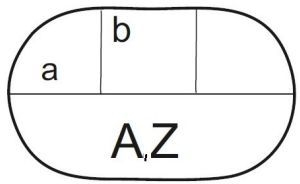
$$H = (-\hbar^2/2m)\nabla^2 + V(\rho, z) + V_{ls}(\vec{r}, \vec{p}, \vec{s}) + V_{l^2}(\vec{r}, \vec{l})$$



$$V_{ls} = -\frac{2\hbar\kappa}{m\omega_0'}(\nabla V \times \mathbf{p})\,\mathbf{s}$$

$$V_{l^2} = -\kappa\mu\hbar\omega_0' l^2 + \kappa\mu\hbar\omega_0'\frac{N(N+3)}{2}\delta_{if}$$

- $\lambda = L/2R_0$
- $\eta_A = \dfrac{A_1 - A_2}{A_1 + A_2}$
- $\varepsilon = E_0/E'$
- $\beta_i = a_i/b_i$ $(i = 1, 2)$, $\beta_i = \beta_j$



[1] J. Maruhn, W. Greiner, Z. Physik. **251**, 431 (1972)

Basis wave functions

Wave functions

$$\varphi_{n_z n_\rho m s_z}(z, \rho, \varphi, s_z) = \frac{e^{im\varphi}}{\sqrt{2\pi}} \varphi_{n_z}(z) \chi_{n_\rho}^{|m|}(\rho) \chi_{\frac{1}{2} s_z}$$

- $\{z, \rho, \varphi\}$ – cylindrical coordinates of the nucleon,
- $\chi_n^{|m|}(\rho) = \sqrt{\frac{2n!}{(n+|m|)!}} k^{\frac{|m|+1}{2}} e^{-\frac{1}{2}k\rho^2} \rho^{|m|} L_n^{|m|}(k\rho^2)$ – radial part of the wave function,
- $\varphi_{n_z}(z)$ – longitudinal part of the wave function,
- $\chi_{\frac{1}{2} s_z}$ – spinor,
- s_z – projection of the nucleon spin onto the z-axis,
- $L_n^{|m|}(k\rho^2)$ – Laguerre polynomials.

Moment of inertia

Total angular momentum

$$\hat{J}_x = \hat{L}_x + \hat{S}_x + a\hat{P}_y$$

$$\hat{L}_x = \sum_{i=1}^{A} \hat{l}_x, \quad \hat{S}_x = \sum_{i=1}^{A} \hat{s}_x, \quad \hat{P}_y = \sum_{i=1}^{A} \hat{p}_y$$

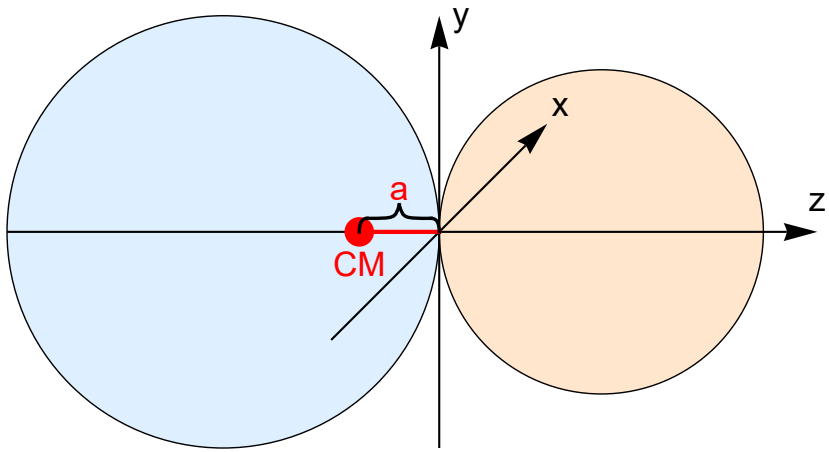


Figure: Schematic DNS

The nuclear moment of inertia in Belyaev's approach is calculated as follows:

$$\mathcal{J} = \sum_{i,j} \frac{1}{E_i + E_j} \Big(|\langle j| \hat{l}_x + \hat{s}_x |i\rangle| (u_i v_j - v_i u_j) + |\langle j| a\hat{p}_y |i\rangle| (u_i v_j + v_i u_j) \Big)^2$$

Single-particle operators:

$\hat{l}_x$ – orbital momentum

$\hat{s}_x$ – spin momentum

$\hat{p}_y$ – momentum

Inglis moment of inertia and the sticking limit

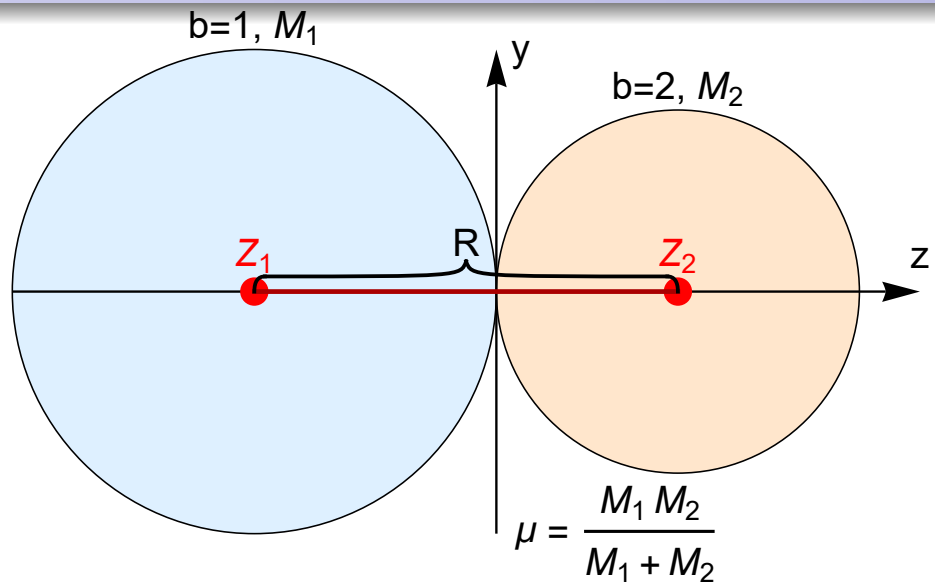
TCSM wave function

$$|\phi_b\rangle = \mathbf{B}\mathbf{c}_b$$

$\mathbf{B} = (\mathbf{B}_1\,\mathbf{B}_2)$ – basis states of each fragment,

$\mathbf{c}$ – coefficients,

$\hat{T}_b$ – translation operator.



Overlap matrix

$$\mathbf{S} = \mathbf{B}^\dagger\mathbf{B} = \begin{pmatrix} \mathbf{1}_1 & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{1}_2 \end{pmatrix}$$

Asymptotics

$$\mathbf{S}_{12} = \mathbf{S}_{21}^\dagger \to 0$$

$$\mathbf{j}_{21}^\dagger = \mathbf{j}_{12} = \mathbf{S}_{12}\mathbf{F}_2 \to 0$$

Translated $\hat{j}_x$

$$\hat{T}_b^\dagger \hat{j}_x \hat{T}_b = \hat{j}_x - Z_b\hat{p}_y,$$

$$\mathbf{j}_x = \begin{pmatrix} \hat{j}_{1,x} - Z_1\hat{p}_{1,y} & \hat{j}_{12,x} \\ \hat{j}_{21,x} & \hat{j}_{2,x} - Z_2\hat{p}_{2,y} \end{pmatrix}$$

Separated fragments

$$\mathcal{J}_x^I(R) \xrightarrow[\text{separated fragments}]{} \mathcal{J}_{1,x}^I + \mathcal{J}_{2,x}^I + \mu R^2$$

Dependence on distance

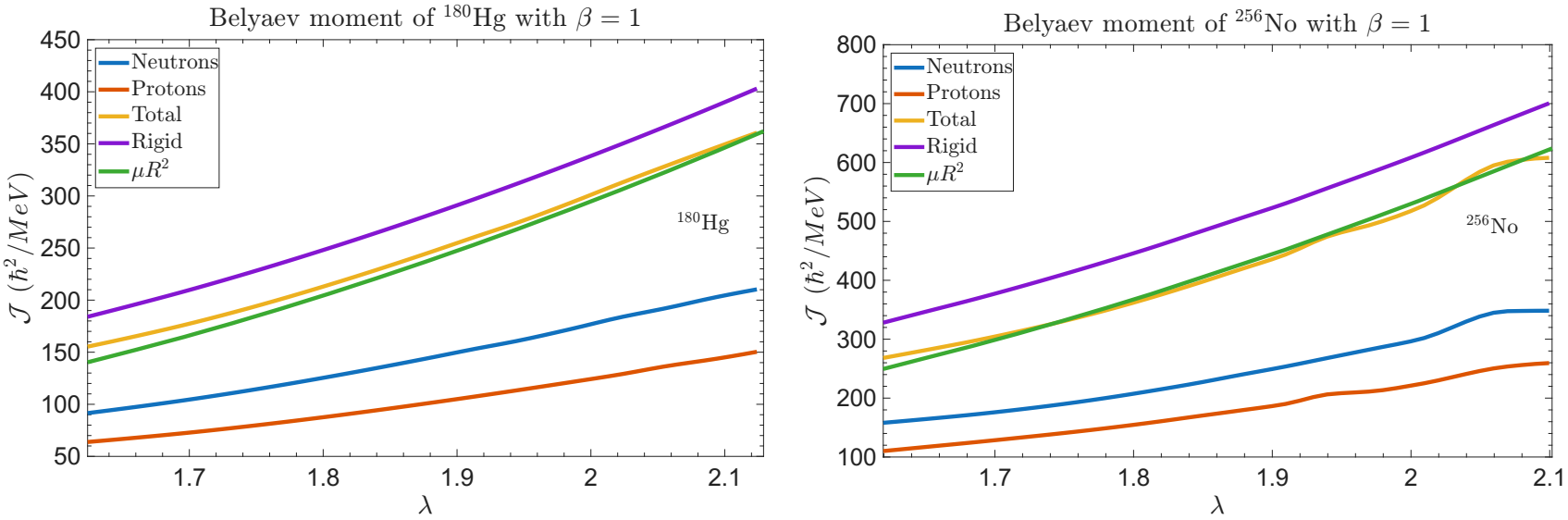
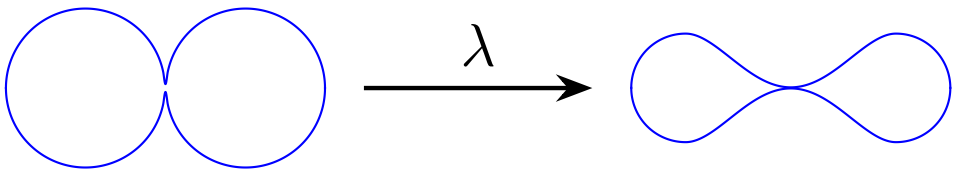


Figure: Depending undeformed ($\beta = 1$) ^{180}Hg and ^{256}No on λ



As the distance between parts increases, the moment of inertia **tends to the** μR^2.

Dependence on deformation

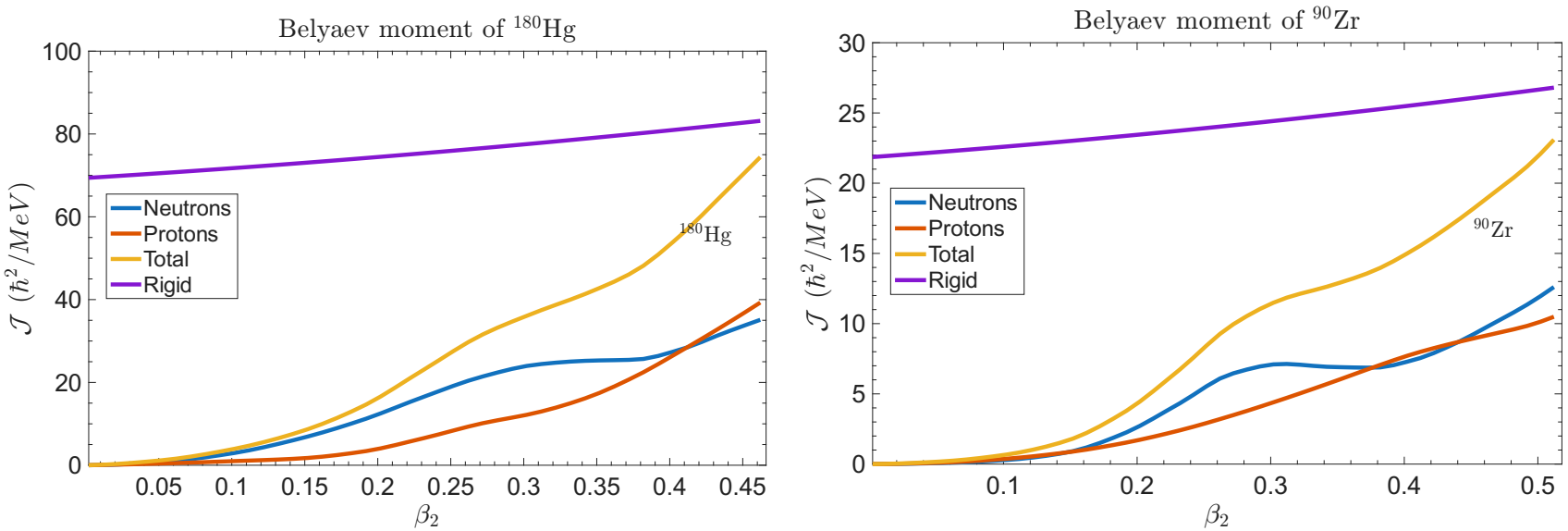
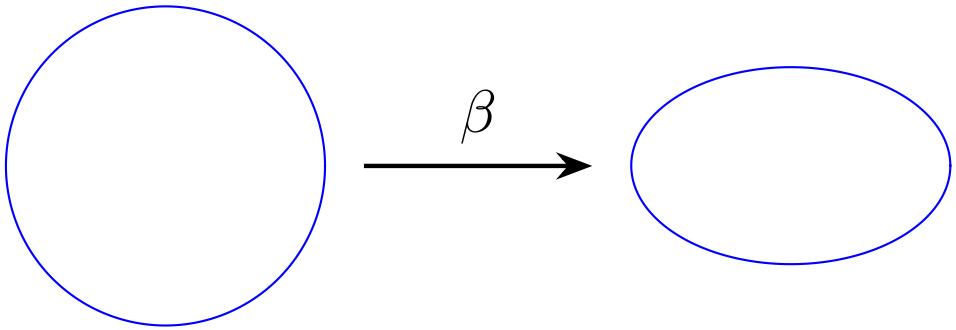


Figure: Depending ^{180}Hg and ^{90}Zr on β_2



As the nuclear deformation increases, the moment of inertia **tends to the rigid-body value**.

Dependence on neck parameter

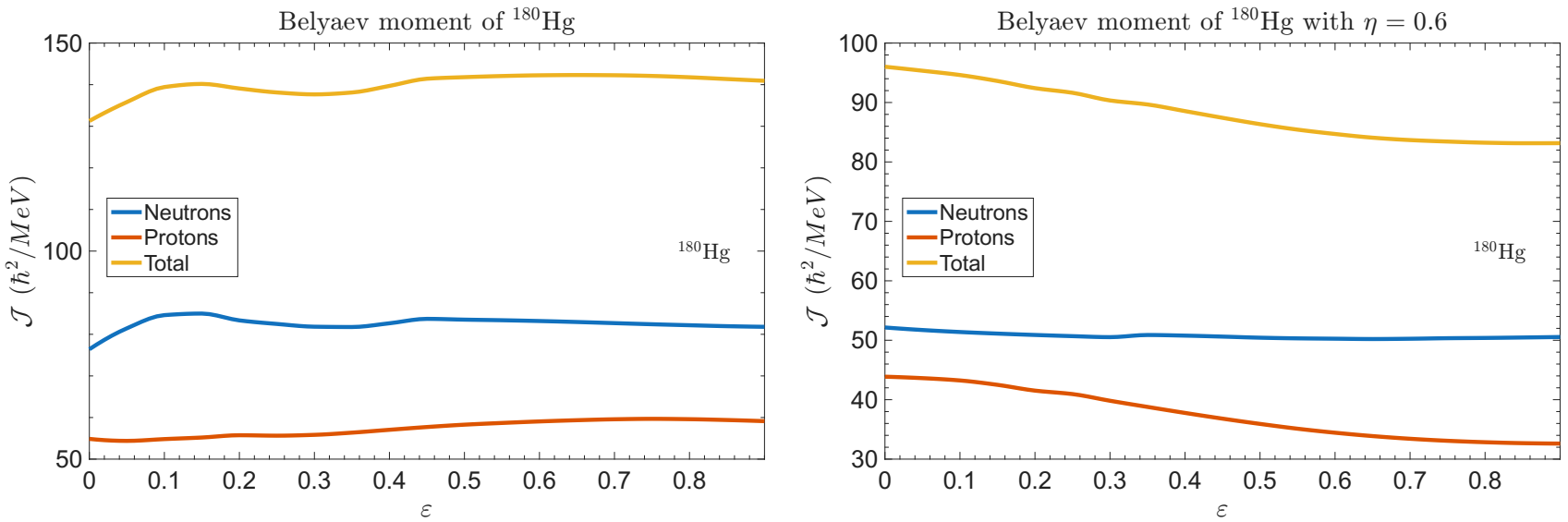


Figure: Depending ^{180}Hg on ε with different mass asymmetry (symmetric $\eta = 0$ and asymmetric $\eta = 0.6$)



Dependence on mass asymmetry

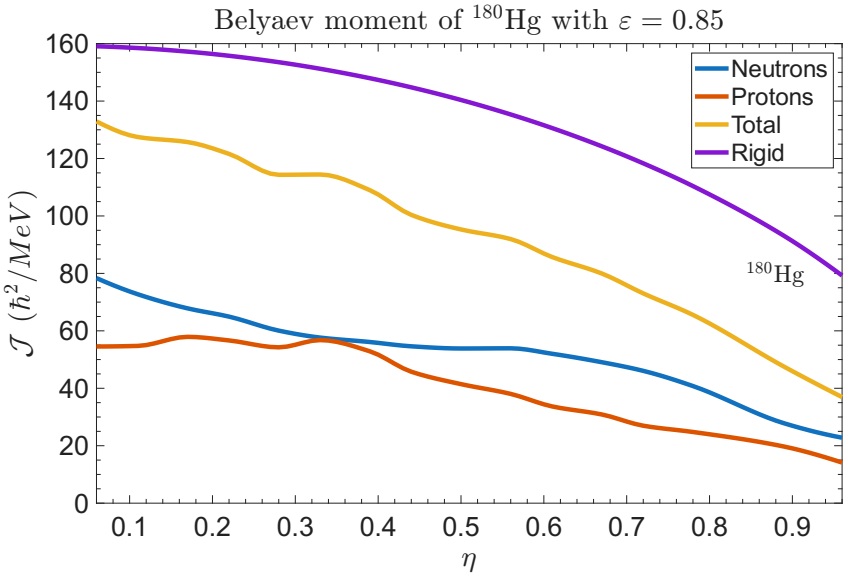
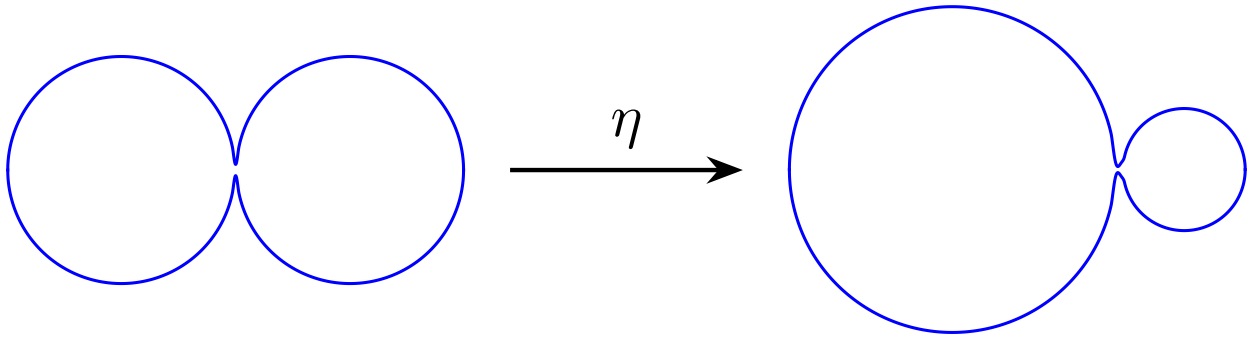


Figure: Depending ^{180}Hg on η with $\varepsilon = 0.85$

The ratio of the moment of inertia to the rigid-body moment of inertia varies in the range from 0.46 to 0.85, i.e.

$$0.46 \leq \frac{\mathcal{J}}{\mathcal{J}^{\text{rig}}} \leq 0.85$$



Changing moment of inertia from g.s. to s.p.

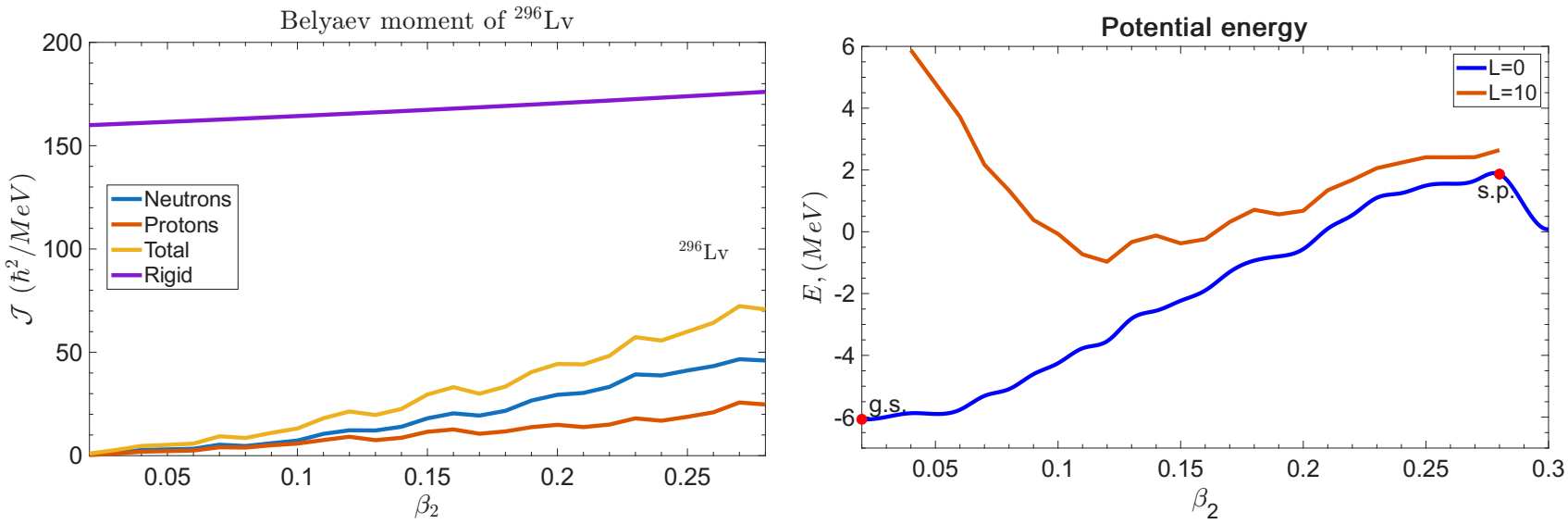


Figure: Changing ^{296}Lv moment of inertia and potential from g.s. to s.p.

The difference $\frac{1}{\mathcal{J}_{\text{g.s.}}} - \frac{1}{\mathcal{J}_{\text{s.p.}}}$ for total moment ≈ 0.9696 and ≈ 0.0006 - for rigid body moment of inertia

Comparison with experiment [1]

	$\mathcal{J}_n$ $(\frac{\hbar^2}{MeV})$	$\mathcal{J}_p$ $(\frac{\hbar^2}{MeV})$	$\mathcal{J}$ $(\frac{\hbar^2}{MeV})$	$E_{2_1^+}$ (keV)	$E_{2_1^+}^{\exp}$ (keV)
$^{236}\mathrm{U}$	35	14	50	61	45.24
$^{254}\mathrm{No}$	41	23	64	47	44.2
$^{220}\mathrm{Ra}$	1.1	0.4	1.5	2041	178.37
$^{252}\mathrm{Cf}$	34	24	60	50	45.72

Table: Moment of inertia for ground states and $E_{2_1^+}$ energy (with experimental data)

[1] NNDC, Evaluated Nuclear Structure Data File (ENSDF), Brookhaven National Laboratory.

$E(2_1^+)$ prediction from Grodzins relations [1]

Grodzins relation

$$E(2_1^+) \times B(E2; 0_1^+ \to 2_1^+) = \hbar^2 q^2 \langle 0_1^+ | (B^{-1})_{00} | 0_1^+ \rangle$$

- $(B^{-1})_{\mu\mu'}$ - inverted inertia tensor,
- $q = 3/4\pi eZR_0^2$,
- $R_0 = r_0 A^{1/3}$

For small deformation $E(2_1^+)$ can be underestimate.

$$E(2_1^+) = \frac{1}{2\beta_2^2 R_0^2}\left(\frac{2}{5}\frac{1}{\Sigma_{rot}} + \frac{2}{5}\frac{1}{\Sigma_\gamma} + \frac{1}{5}\frac{1}{\Sigma_\beta}\right)$$

Σ_γ, Σ_{rot} and Σ_β - some sums depending on single particle, quasiparticle, Fermi energy and energy gap

[1] N. Yu. Shirikova, A. V. Sushkov, L. A. Malov, E. A. Kolganova, and R. V. Jolos, Phys. Rev. C **105**, 024309 (2022). doi:10.1103/PhysRevC.105.024309

Comparison with the predictions obtained from Grodzins relations

	$\mathcal{J}_n$ $(\frac{\hbar^2}{MeV})$	$\mathcal{J}_p$ $(\frac{\hbar^2}{MeV})$	$\mathcal{J}$ $(\frac{\hbar^2}{MeV})$	$E_{2_1^+}$ (keV)	$E_{2_1^+}$ (keV)[1]
^{254}No	41	23	64	47	^{260}No: 42
^{292}Lv	11	3	14	217	^{288}Lv: 202
^{296}Lv	2	1	3	1109	
^{294}Og	20	12	33	91	^{292}Og: 532
^{298}120	12	7	19	157	^{296}120: 176
^{272}Hs	44	27	70	43	75
^{276}Ds	35	21	56	53	89
^{282}Cn	29	19	48	62	^{280}Cn: 86

Table: Moment of inertia for ground states and $E_{2_1^+}$ energy (with predictions)

[1] N. Yu. Shirikova, A. V. Sushkov, L. A. Malov, E. A. Kolganova, and R. V. Jolos, Phys. Rev. C **105**, 024309 (2022). doi:10.1103/PhysRevC.105.024309

Conclusion

- Systematic calculation of moment of inertia.
- $E_{2_1^+}$ are in agreement with available experimental data.
- For strongly deformed nuclei, the relation $\frac{\mathcal{J}}{\mathcal{J}^{\text{rig}}} \approx 0.8$.
- In DNS, $\mathcal{J}$ decreases to ≈ 0.46 with increasing η.
- $\frac{\mathcal{J}_{\text{s.p.}}}{\mathcal{J}_{\text{g.s.}}} \approx 70$ for nearly spherical ground states.

Matrix elements

In cylindrical coordinates, matrix elements have the form:

$$\langle n_{z_f}\, n_{\rho_f}\, m_f\, s_{zf}|\, \hat{I}_x\, |n_{z_i}\, n_{\rho_i}\, m_i\, s_{zi}\rangle = -\frac{1}{2}\hbar\, \delta_{s_{z_f}\, s_{z_i}}$$

$$\Big(\big(\sqrt{n_{\rho_i}+m_i+1}\,\delta_{n_{\rho_f}\, n_{\rho_i}}\delta_{m_f\, m_i+1} + \sqrt{n_{\rho_i}+1}\,\delta_{n_{\rho_f}\, n_{\rho_i}+1}\delta_{m_f\, m_i-1}\big)\cdot$$

$$\cdot \langle n_{z_f}|\, \frac{1}{\sqrt{k}}\frac{\partial}{\partial z} + \sqrt{k}\, z\, |n_{z_i}\rangle - \langle n_{z_f}|\, \frac{1}{\sqrt{k}}\frac{\partial}{\partial z} - \sqrt{k}\, z\, |n_{z_i}\rangle \cdot$$

$$\big(\sqrt{n_{\rho_i}}\,\delta_{n_{\rho_f}\, n_{\rho_i}-1}\delta_{m_f\, m_i+1} + \sqrt{n_{\rho_i}+m_i}\,\delta_{n_{\rho_f}\, n_{\rho_i}}\delta_{m_f\, m_i-1}\big)\Big)$$

$$\langle n_{z_f}\, n_{\rho_f}\, m_f\, s_{zf}|\, \hat{p}_y\, |n_{z_i}\, n_{\rho_i}\, m_i\, s_{zi}\rangle = \delta_{s_{zf}\, s_{zi}}\delta_{n_{z_f}\, n_{z_i}}\frac{i\sqrt{k}}{2}\times$$

$$\times\Big[\big(\sqrt{n_{\rho_i}+m_i}\,\delta_{n_{\rho_f}\, n_{\rho_i}} + \sqrt{n_{\rho_i}+1}\,\delta_{n_{\rho_f}\, n_{\rho_i}+1}\big)\delta_{m_f\, m_i-1}$$

$$+\big(\sqrt{n_{\rho_i}+m_i+1}\,\delta_{n_{\rho_f}\, n_{\rho_i}} + \sqrt{n_{\rho_i}}\,\delta_{n_{\rho_f}\, n_{\rho_i}-1}\big)\delta_{m_f\, m_i+1}\Big]$$

$$\langle n_{z_f}\, n_{\rho_f}\, m_f\, s_{zf}|\, \hat{s}_x\, |n_{z_i}\, n_{\rho_i}\, m_i\, s_{zi}\rangle = \frac{1}{2}\delta_{n_{z_f}\, n_{z_i}}\delta_{n_{\rho_f}\, n_{\rho_i}}\delta_{m_f\, m_i}\big(\delta_{s_{zf}\, s_{zi}+1} + \delta_{s_{zf}\, s_{zi}-1}\big)$$