中国原子能科学研究院
China Institute of Atomic Energy
中核集团 CNNC

October 18–23, 2026, Nanjing, China

Study of Quantum Tunneling Behavior in Nuclear Reactions Based on the Coupled-Channels Model

Peiwei Wen

Collaborators: L. Y. Fu, C. J. Lin, O. Chuluunbaatar, S. Vinitsky, A. Gusev, H. M. Jia, H. Y. Li *et al.*

CONTENTS

OUTLINE

Introduction

Improvement of the coupled-channels model

Comparison of IWBC and OM methods

Summary and perspective

Contents

Introduction

Improvement of the coupled-channels model

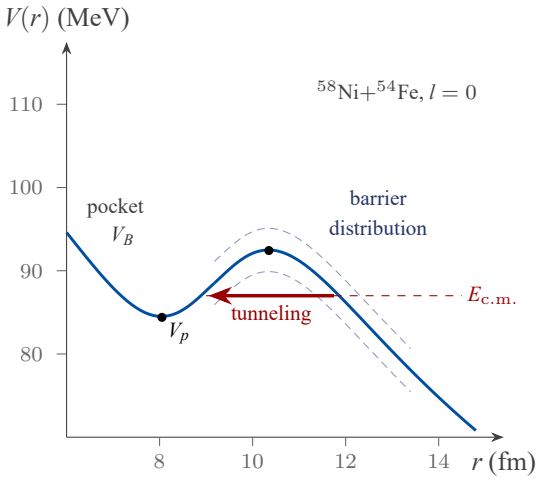
Comparison of IWBC and OM methods

Summary and perspective

Heavy-ion fusion: tunneling through the Coulomb barrier

- **tunneling** governs near-barrier fusion (capture for heavy reactions)
- **couplings** collective states ⇒ **barrier distribution** (dashed)
- **probes** superheavy synthesis; deep-sub-barrier **fusion hindrance**

B. B. Back *et al.*, Rev. Mod. Phys. **86**, 317 (2014)
Hagino and Takigawa, Prog. Theor. Phys. **128**, 1061 (2012)



Fusion barrier distribution (BD)

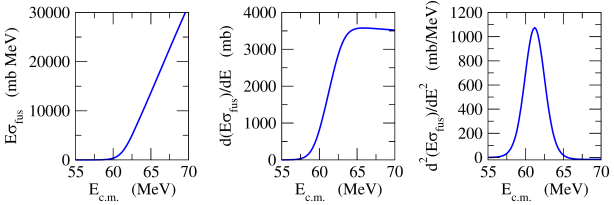
Volume 254, number 1,2 PHYSICS LETTERS B 17 January 1991

On the "distribution of barriers" interpretation of heavy-ion fusion

N. Rowley [a], G.R. Satchler [b] and P.H. Stelson [b]
[a] *SERC Daresbury Laboratory, Daresbury, Warrington WA4 4AD, UK*
[b] *Oak Ridge National Laboratory, Oak Ridge, TN 37831, USA* [1]

Received 8 August 1990; revised manuscript received 29 October 1990

The significance of the interpretation of heavy-ion fusion cross sections in terms of a distribution of potential barriers is discussed. The smoothing due to the quantal barrier penetration is shown to replace a set of discrete barriers by an effective continuous distribution. It is shown how this smoothed distribution may be obtained rather directly from the measured cross sections at near- and sub-barrier energies.



$$\sigma_{\rm fus}^{\rm cl}(E) = \pi R_b^2\,(1 - B/E)\,\theta(E - B)$$

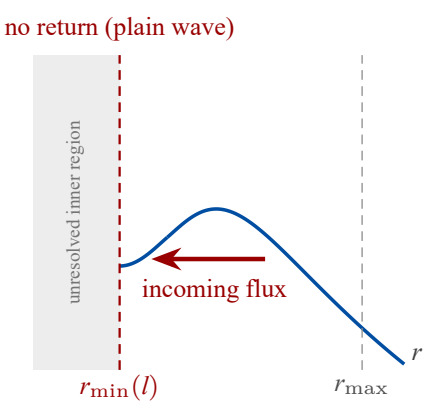
$$\frac{d^2}{dE^2}\left[E\sigma_{\rm fus}^{\rm cl}(E)\right] = \pi R_b^2\,\delta(E - B)$$

The **BD**:

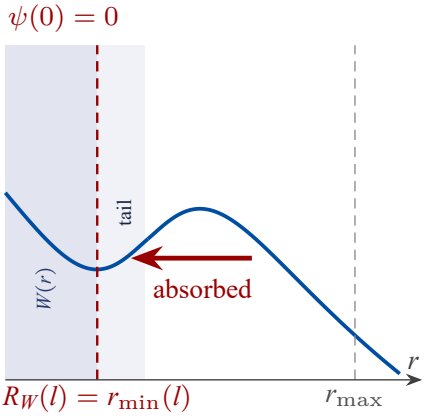
- **classical** **delta function**
- **realistic** **wide, from couplings**
- **role** sensitive probe of nuclear structure

Hagino and Takigawa, Prog. Theor. Phys. **128**, 1061 (2012)

Two prescriptions for the fusion (capture) flux: IWBC and OM



IWBC: only incoming waves at $r_{\min}(l)$



OM: $W(r)$ absorbs flux at $r_{\min}(l)$

Two prescriptions for the fusion (capture) flux

$\sigma_{\rm fus}$: flux removed from the elastic channel

	Optical model (OM)	**Incoming wave BC (IWBC)**
potential	$V(r) + \mathrm{i}W(r)$	$V(r)$ only
inner region	flux absorbed by W	**no-return** at $r_{\rm min}$
P_l	$1 - \sum_n \lvert S_{nn_0}\rvert^2$	inward flux at $r_{\rm min}$
parameters	W_0, R_W, a_W	none
typical use	light systems (FRESCO, CCFULL-sc)	heavy systems (CCFULL)

Open question

Common expectation: a sufficiently strong, short-ranged imaginary potential should reproduce IWBC. Is IWBC the $W_0 \to \infty$ limit of OM? — **never tested systematically in CC calculations**

Bethe, Phys. Rev. **57**, 1125 (1940)
G. H. Rawitscher, Nucl. Phys. 85, **337** (1966). (ingoing wave)
Hagino *et al.*, Comput. Phys. Commun. **123**, 143 (1999)

This work: when do OM and IWBC agree?

Why subtle?

- **reflection** absorbers also **reflect** — never perfectly black
- **status** known qualitatively; quantitative CC test missing
- **numerics** $P_l \sim 10^{-6}$–10^{-10} near the pocket

This work

- **system** ^{58}Ni+ ^{54}Fe (data: Stefanini *et al.*, 2010)
- **absorber** same real potential; $R_W(l) = r_{\min}(l)$ — **anchored where IWBC counts**
- **scope** single-channel + CC + 1D pure-imaginary
- **solver** KANTBP 3.1 finite elements; R-matrix

Goal: the criteria for $\sigma_{\rm fus}^{\rm OM} \approx \sigma_{\rm fus}^{\rm IWBC}$.

A. M. Stefanini *et al.*, Phys. Rev. C **81**, 037601 (2010)
L. F. Canto, Phys. Rev. C **98**, 044617 (2018)
D. Neuhauser and M. Baer, J. Chem. Phys. **90**, 4351 (1989)

Contents

Introduction

Improvement of the coupled-channels model

Comparison of IWBC and OM methods

Summary and perspective

Physical idea of the coupled-channels model

Two intrinsic levels, one coupling $F(r) = V_{01}(r)$:

$$\left[\frac{\hbar^2}{2\mu}\nabla^2 + V_l(r) + \begin{pmatrix} 0 & F(r) \\ F(r) & \epsilon_1 \end{pmatrix}\right]\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = E\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

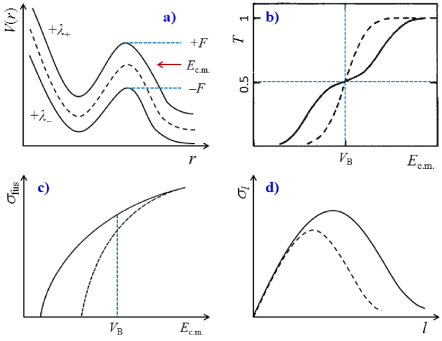
Local eigen-frequencies:

$$\lambda_\pm(r) = \left[\epsilon_1 \pm \sqrt{\epsilon_1^2 + 4F(r)^2}\right]/2$$

$$\epsilon_1 = 0: \quad \lambda_\pm(r) = \pm F(r)$$

Split barriers $\Rightarrow$ average penetration:

$$P_l(E) = \tfrac{1}{2}\left\{P_l^0[E; V_l{+}F] + P_l^0[E; V_l{-}F]\right\}$$



- **coupling** intrinsic motion mixes the channels:
  coupled eigenstates replace the bare levels

C. J. Lin, *Heavy-ion nuclear reactions* (2015)
P. Fröbrich and R. Lipperheide, *Theory of Nuclear Reactions* (1996)

Coupled-channels equations

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}+\frac{\hbar^2 l(l+1)}{2\mu r^2}+V_C(r)+V_N(r)+\mathrm{i}W^l(r)+\epsilon_n-E\right]\psi_{nn_0}(r)$$
$$+\sum_{n'=1}^{N}\left[V_{nn'}(r)+\mathrm{i}W^l_{nn'}(r)\right]\psi_{n'n_0}(r)=0,\qquad n=1,\ldots,N$$

- **channels** n: intrinsic; n_0: entrance (isocentrifugal)
- **matrices** $V_{nn'}, \mathrm{i}W_{nn'}$: coupling + absorption
- V_C uniform sphere, $R_C = r_C(A_P^{1/3} + A_T^{1/3})$, $r_C = 1.2$ fm

$$V_N(r) = -\frac{V_0}{1+e^{(r-R_V)/a_V}},$$
$$W^l(r) = -\frac{W_0}{1+e^{(r-R^l_W)/a_W}}$$

$$\sigma_{\mathrm{fus}}(E) = \frac{\pi\hbar^2}{2\mu E}\sum_{l=0}^{l_{\max}}(2l+1)\,P_l(E)$$

Coupled potential: collective vibration

Replace the potential radius by a dynamical operator, $R_0 \to R_0 + \hat{O}$, with $\hat{O}\,|\alpha\rangle = \lambda_\alpha\,|\alpha\rangle$:

$$\hat{O} = \frac{\beta_\lambda}{\sqrt{4\pi}} r_{\rm coup}\, A_T^{1/3} \left(a^\dagger_{\lambda 0} + a_{\lambda 0}\right),$$

on the potential,

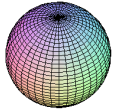
$$V'_N(r, \hat{O}) = -\frac{V_0}{1 + \exp\left[(r - R_0 - \hat{O})/a_0\right]}.$$

Full order: diagonalize the matrix $\hat{O}$,

$$O_{nm} = \frac{\beta_\lambda}{\sqrt{4\pi}} r_{\rm coup}\, A_T^{1/3} \left(\sqrt{m}\,\delta_{n,m-1} + \sqrt{n}\,\delta_{n,m+1}\right),$$

coupling matrix elements between phonons $|n\rangle$,$|m\rangle$:

$$\begin{aligned} V^{(N)}_{nm} &= \langle n|V'_N(r,\hat{O})|m\rangle - V^{(0)}_N \delta_{n,m} \\ &= \sum_\alpha \langle n|\alpha\rangle\langle\alpha|m\rangle V'_N(r,\lambda_\alpha) - V^{(0)}_N \delta_{n,m}. \end{aligned}$$



- **operator** surface displacement on the potential
- **couplings** all orders, by diagonalization

Bohr and Mottelson, *Nuclear Structure II* (1969)

Hagino *et al.*, Comput. Phys. Commun. **123**, 143 (1999)

The incoming wave boundary condition (IWBC)

The incoming wave boundary conditions (IWBC)

$$\psi_n(r) = \begin{cases} T_n \exp\left(-ik_n(r_{\min})r\right), & r \le r_{\min} \\ H_l^-(k_n r)\delta_{n,0} - R_n H_l^+(k_n r), & r \ge r_{\max} \end{cases}$$

Here $k_n = k_n(r \to +\infty)$, and $k_n(r)$ is the local wave number for n-th channel

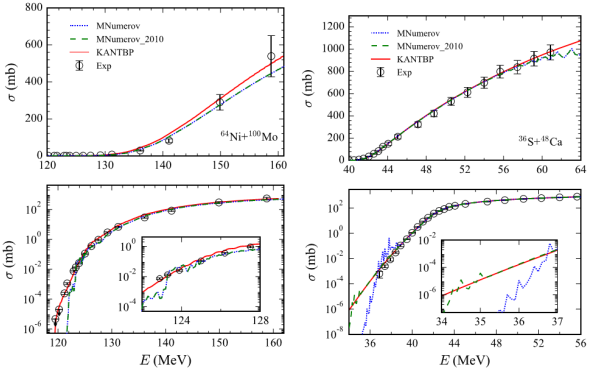
$$k_n(r) = \sqrt{\frac{2\mu}{\hbar^2}\left(E - \epsilon_n - \frac{l(l+1)\hbar^2}{2\mu r^2} - V_N^{(0)}(r) - \frac{Z_P Z_T e^2}{r} - V_{nn}(r)\right)}$$

Problems in IWBC.

- **Problem** At $r_{\min}$, $k_n(r)$ involves only the diagonal part V_{nn}.
- **Result** Fluctuation happens by neglecting off-diagonal matrix elements.
- **Solution** Linear transformation is needed at $r_{\min}$.

V.V. Samarin, V.I. Zagrebaev, 2004 *NPA* **734** E9;
V.I. Zagrebaev, V.V. Samarin, 2004 *Phys. Atom. Nucl.* **67** 1462;

High accuracy solutions FEM (KANTBP)



(M) Numerov (CCFULL-type)	**FEM** (this work)
sub-barrier **fluctuations**	smooth; agrees with R-matrix
—	unitarity $\lesssim 10^{-10}$

► **Boundary condition**

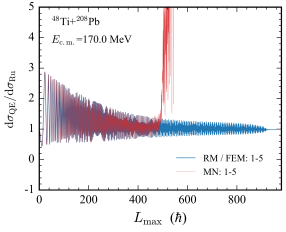
off-diagonal couplings are considered

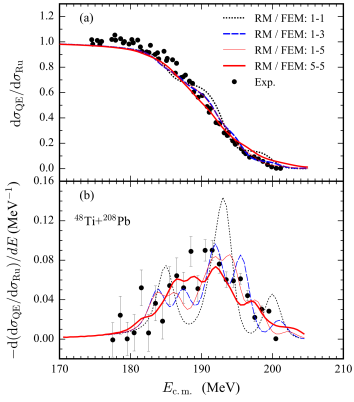
Chuluunbaatar *et al.*, Comput. Phys. Commun. **278**, 108397 (2022)
P. W. Wen, C. J. Lin, et al, Phys Rev C **103**, 054601 (2021).
P. W. Wen, O. Chuluunbaatar, et al, Phys Rev C **101**, 014618 (2020).

Quasi-elastic scattering: vibration

TABLE II. The properties of the single-phonon states included in the present coupled-channels calculations. $\hbar\omega$ and β are the excitation energy and the dynamical deformation parameter, respectively.

Nucleus	I^π	$\hbar\omega$ (MeV)	β
^{208}Pb	3^-	2.614	0.110[a]
^{48}Ti	2^+	0.983	0.269[b]
^{54}Cr	2^+	0.834	0.250[b]
^{56}Fe	2^+	0.846	0.239[b]
^{64}Ni	2^+	1.346	0.179[b]
^{70}Zn	2^+	0.884	0.228[b]

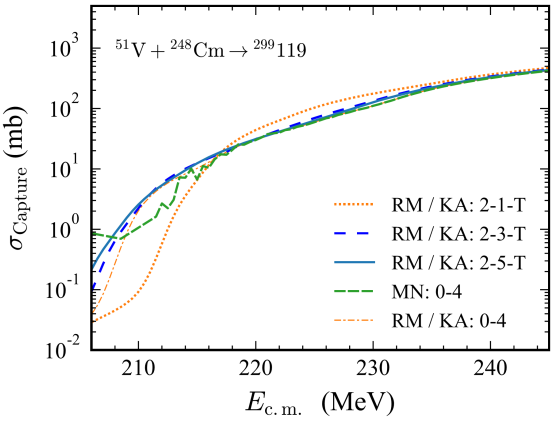




- **QE** counterpart of tunneling
- **numerics** R-matrix + FEM: cures the Numerov instability
- **physics** **higher-order vibrations and high-spin state** shape the QE barrier distribution

Wen *et al.*, Phys. Lett. B **863**, 139383 (2025)
Descouvemont, Comput. Phys. Commun. **200**, 199 (2016)
Zamrun *et al.*, Phys. Rev. C **77**, 034604 (2008)

Capture cross sections constrained by QE



- **input** QE observables **constrain** the capture cross section
- **output** input for fusion and multinucleon-transfer models for superheavy nuclei

Product

a consistent capture cross section where direct capture data are unreachable

Wen *et al.*, Phys. Lett. B **863**, 139383 (2025)
Chuluunbaatar *et al.*, Phys. Part. Nucl. **56**, 1407 (2025)

Contents

Introduction

Improvement of the coupled-channels model

Comparison of IWBC and OM methods

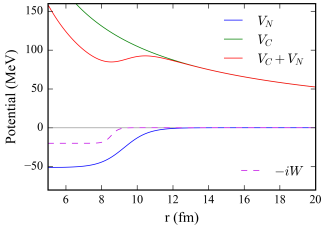
Summary and perspective

Optical potential and absorption

§9.8 光学模型

1. 光学模型的基本思想

前已指出，光学模型认为，原子核好比一个半透明的玻璃球，入射粒子与靶核的作用如同光波射在玻璃球上，一部分透射或反射，一部分被吸收。透射或反射的光波相当于粒子被散射。被吸收的光波相当于粒子进入靶核，引起核反应。所以，光学模型认为靶核对于入射粒子



最简单的势阱形式是早期提出的复数直角势阱，即

$$V(r) = \begin{cases} -(V_0 + iW) & r \leqslant R \\ 0 & r > R \end{cases} \tag{9.8-1}$$

其中 V_0 和 W 是正实数，R 是道半径。

首先讨论一下复势(9.8-1)式具有什么效应。入射粒子在复势中的一维薛定谔方程是

$$\frac{d^2\psi}{dx^2} + \frac{2\mu}{\hbar^2}(E' + V_0 + iW)\psi = 0 \tag{9.8-2}$$

式中 μ 是入射粒子与靶核的折合质量，E' 是入射粒子的相对运动动能。(9.8-2)式的解具有平面波的形式

$$\psi = e^{\pm iK_c x} \tag{9.8-3}$$

这里波数 K_c 是复数，

$$K_c = \frac{1}{\hbar}\sqrt{2\mu(E' + V_0 + iW)} \tag{9.8-4}$$

令

$$K_c = K_1 + iK_2 \tag{9.8-5}$$

则当 $W \ll E' + V_0$ 时，将(9.8-4)式中的根式展开可得

$$K_1 \approx \frac{1}{\hbar}\sqrt{2\mu(E' + V_0)} \tag{9.8-6}$$

$$K_2 \approx \frac{WK_1}{2(E' + V_0)} \tag{9.8-7}$$

把(9.8-5)式代入(9.8-3)式，并取正号以表示 ψ 是向 x 正向传播的平面波，则有

$$\psi = e^{iK_1 x} e^{-K_2 x} \tag{9.8-8}$$

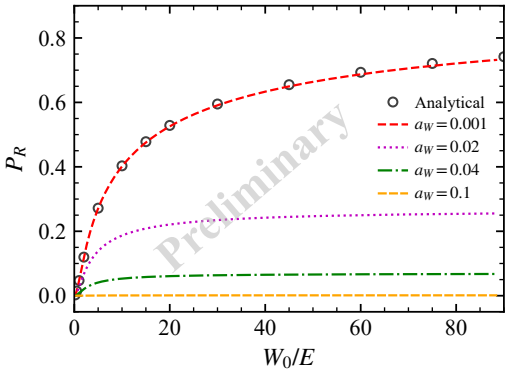
因 $|\psi|^2 = e^{-2K_2 x}$，所以(9.8-8)式表明 ψ 是指数衰减波，衰减系数 $2K_2 \propto W$，复势的虚数部分 W 表示对入射粒子的吸收，W 越大吸收越强烈。

Textbook statement — and question

the imaginary part of a complex potential absorbs flux: the larger W, the stronger — **under which conditions?**

X.-T. Lu (*Nuclear Physics*, textbook, in Chinese)

Quantum reflection from an imaginary step



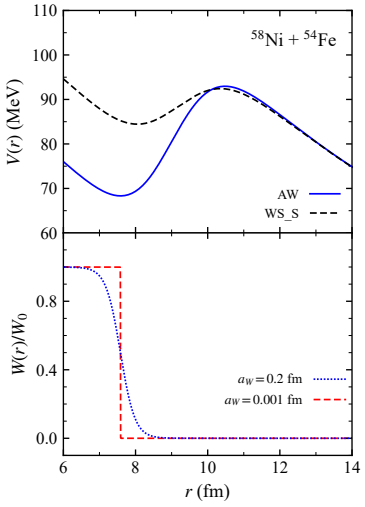
- **model** 1D step: $V(r) = -\mathrm{i}W_0$ for $r \geq 0$ — analytic

$$P_R = \left| \frac{1 - \sqrt{1 + \mathrm{i}W_0/E}}{1 + \sqrt{1 + \mathrm{i}W_0/E}} \right|^2$$

- **limits** $W_0/E \to \infty$: $P_R \to 1$ — **a strong sharp absorber is a mirror**
- **check** $a_W = 0.001$ fm tracks the step; 0.02–0.04 fm: visible

D. Neuhauser and M. Baer, J. Chem. Phys. **90**, 4351 (1989) (TDSE)

Setup: ^{58}Ni+ ^{54}Fe, absorber at the pocket minimum

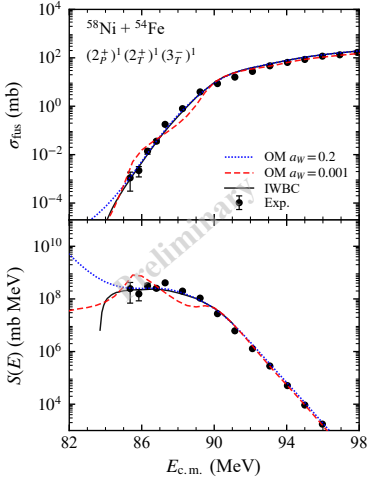


WS_S: a fitted shallow Woods-Saxon potential.

	AW	WS_S
V_0 (MeV)	71.734	53.137
r_V (fm)	1.1765	1.1944
a_V (fm)	0.6692	0.7545
$r_{\min}$, $l = 0$ (fm)	7.58	8.05
$a_W = 0.2$ fm and 0.001 fm (**sharp**)		

- **barriers** similar; **different pockets**
- **sharp edge** limitation test

Coupled channels: comparison

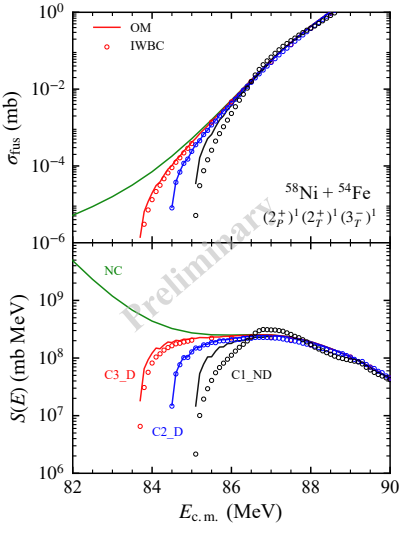


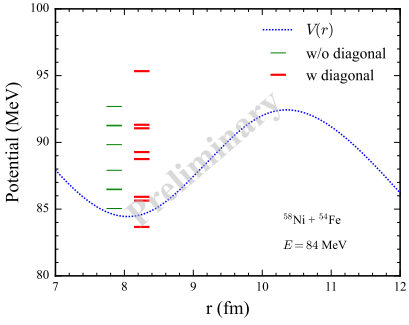
	state	E_x (MeV)	β
^{58}Ni	2^+	1.454	0.183
^{54}Fe	2^+	1.408	0.195
^{54}Fe	3^-	4.782	0.115

- **couplings** sub-barrier enhancement ⇒ near the data
- **differences** sub-barrier energy region
- $S(E)$ separates the deep sub-barrier trends

A. M. Stefanini *et al.*, Phys. Rev. C **81**, 037601 (2010)
A. M. Stefanini *et al.*, Phys. Rev. C **82**, 014614 (2010)

Criterion: the same minimum cutoff energy





- **IWBC_D** off-diagonal elements considered
- **IWBC_ND** only diagonal elements
- **cutoff** C1, C2, C3: $V(r_{\min}(l)) > E$
  C3 (red) is recommend.

Contents

Introduction

Improvement of the coupled-channels model

Comparison of IWBC and OM methods

Summary and perspective

Summary and perspective

- **CC model** improved boundary condition and numerical solution with **R-matrix + finite elements** (KANTBP)
- **QE capture** capture cross sections **constrained by QE**
- **OM–IWBC** IWBC is **not** the $W_0 \to \infty$ limit;
  quantum reflection from imaginary potential plays a critical role

Three criteria for close OM–IWBC results (preliminary)		
1	OM	$a_W > 0.1$ fm: reflection negligible
2	IWBC	eigenchannel diagonalization at $r_{\min}$
3	both	the same minimum cutoff energy

Perspective: modified potential pocket minimum; driving potential for DNS model.

中国原子能科学研究院
China Institute of Atomic Energy
中核集团
CNNC

Thank you for attention!

The new method KANTBP

The coupled-channels Schrödinger equation

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}+\frac{l(l+1)\hbar^2}{2\mu r^2}+V_N^{(0)}(r)+\frac{Z_P Z_T e^2}{r}+\epsilon_n-E\right]\psi_{nn_o}+\sum_{n'=1}^{N}V_{nn'}(r)\psi_{n'n_o}(r)=0, \quad (1)$$

with

- n_o is a number of the open entrance channel with a positive relative energy $E_{n_o}=E-\epsilon_{n_o}>0$, $n_o=1,...,N_o\le N$.
- $\{\psi_{nn_o}(r)\}_{n=1}^{N}$ are components of a desirable matrix solution.

Let **W** is the symmetric matrix of dimension $N\times N$

$$W_{nm}=W_{mn}=\frac{2\mu}{\hbar^2}\left[\left(\frac{l(l+1)\hbar^2}{2\mu r^2}+V_N^{(0)}(r)+\frac{Z_P Z_T e^2}{r}+\epsilon_n\right)\delta_{nm}+V_{nm}(r)\right]. \quad (2)$$

Then the equation can be expressed as

$$-\psi''_{nm}(r)+\sum_{m'}W_{nm'}\psi_{m'm}(r)=\frac{2\mu E}{\hbar^2}\psi_{nm}(r), \quad (3)$$

The new method KANTBP

Diagonalize the matrix at $r = r_{\min}$

$$\mathbf{WA} = \mathbf{A}\tilde{\mathbf{W}}, \quad \{\tilde{\mathbf{W}}\}_{nm} = \delta_{nm}\tilde{W}_{mm}, \quad \tilde{W}_{11} \le \tilde{W}_{22} \ldots \le \tilde{W}_{NN}. \tag{4}$$

The functions $y_m(r)$ are solutions of the uncoupled equations

$$y_m''(r) + K_m^2 y_m(r) = 0, \quad K_m^2 = \frac{2\mu E}{\hbar^2} - \tilde{W}_{mm}. \tag{5}$$

In open channels at $K_m^2 > 0$, $m = 1, ..., M_o \le N$ the solutions $y_m(r)$ have the form:

$$y_m(r) = \frac{\exp(-\imath K_m r)}{\sqrt{K_m}}. \tag{6}$$

In this case $\psi_{nn_o}(r)$ expressed by the linear combinations of the linear independent solutions

$$\psi_{nn_o}(r) = \sum_{m=1}^{M_o} A_{nm} y_m(r) \hat{T}_{mn_o}, \quad r = r_{\min}. \tag{7}$$

In this way, the off-diagonal matrix elements have been considered in the calculation.

The new method KANTBP

Summary of the boundary conditions for open channels

$$\psi^{as}_{nn_o}(r) = \begin{cases} \sum_{m=1}^{M_o} A_{nm} \frac{\exp(-\imath K_m r)}{\sqrt{K_m}} \hat{T}_{mn_o}, & r = r_{\min}, \\ \hat{H}_l^-(k_n r)\delta_{n,n_o} - \hat{H}_l^+(k_n r)\hat{R}_{nn_o}, & r = r_{\max}. \end{cases} \tag{8}$$

In this case, the partial tunneling probability from the ground state ($n_o = 1$) is

$$P_l(E) \equiv T^{(l)}_{n_o n_o}(E). \tag{9}$$

At fixed orbital momentum l, it is given by summation over all possible intrinsic states:

$$T^{(l)}_{n_o n_o}(E) = \sum_{m=1}^{M_o} \left|\hat{T}_{mn_o}\right|^2, \quad R^{(l)}_{n_o n_o}(E) = \sum_{n=1}^{N_o} \left|\hat{R}_{nn_o}\right|^2, \quad T^{(l)}_{n_o n_o}(E) = 1 - R^{(l)}_{n_o n_o}(E) \tag{10}$$

The condition $T^{(l)}_{n_o n_o}(E) + R^{(l)}_{n_o n_o}(E) - 1 = 0$ fulfills with ten significant digits by the element method KANTBP.

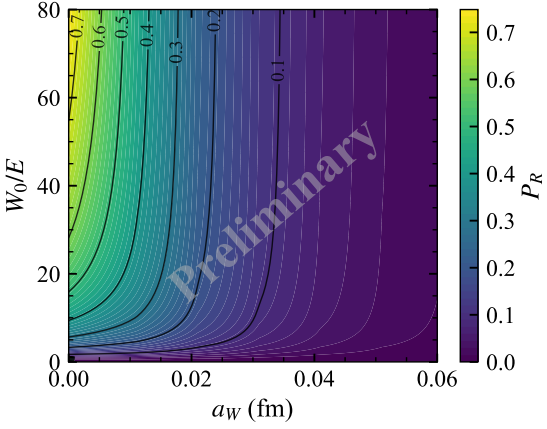
O. Chuluunbaatar, A. A. Gusev, A.G. Abrashkevich *et al*, CPC. 177, 649 (2007)

A. A. Gusev, O. Chuluunbaatar, S. I. Vinitsky *et al*, CPC 185, 3341 (2014)

A. A. Gusev, O. Chuluunbaatar, S. I. Vinitsky *et al*, Math. Mod. Geom. 3, 2 22 (2015)

V. I. Zagrebaev, Phys. Rev. C 78 047602 (2008)

Reflection map of the pure imaginary Woods--Saxon



- **map** P_R over $W_0/E \in [0, 80]$, $a_W \in [0, 0.06]$ fm (1D, pure imaginary)
- **band** sizable reflection: **narrow band** along the sharp-edge axis
- $a_W = 0.2$ fm: $P_R \lesssim 10^{-6}$ over the whole depth range

Criterion 1 (OM)

$a_W \gtrsim 0.1$ fm: reflection negligible — tail still inside the pocket