Nuclear α-cluster structures from valence-space microscopic cluster model

Zhen Wang[✉†] (王震), Dong Bai[‡] (柏栋), Zhongzhou Ren[†] (任中洲)

✉ wang_zhen@tongji.edu.cn

[†] Tongji University, Shanghai 200092, China

[‡] Hohai University, Nanjing 211100, China

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4. Summary and Prospect

Alpha clustering in nuclei

Alpha clustering exists across the nuclide chart and plays a crucial role in understanding nuclear structures.

- **Light nuclei**: ^{8}**Be**, $N\alpha$ condensate state in ^{12}**C**, ^{16}**O**, ^{20}**Ne**, etc.
- **Heavy nuclei**: α decay (*e.g.*, ^{212}**Po**, ^{104}**Te**, etc.).

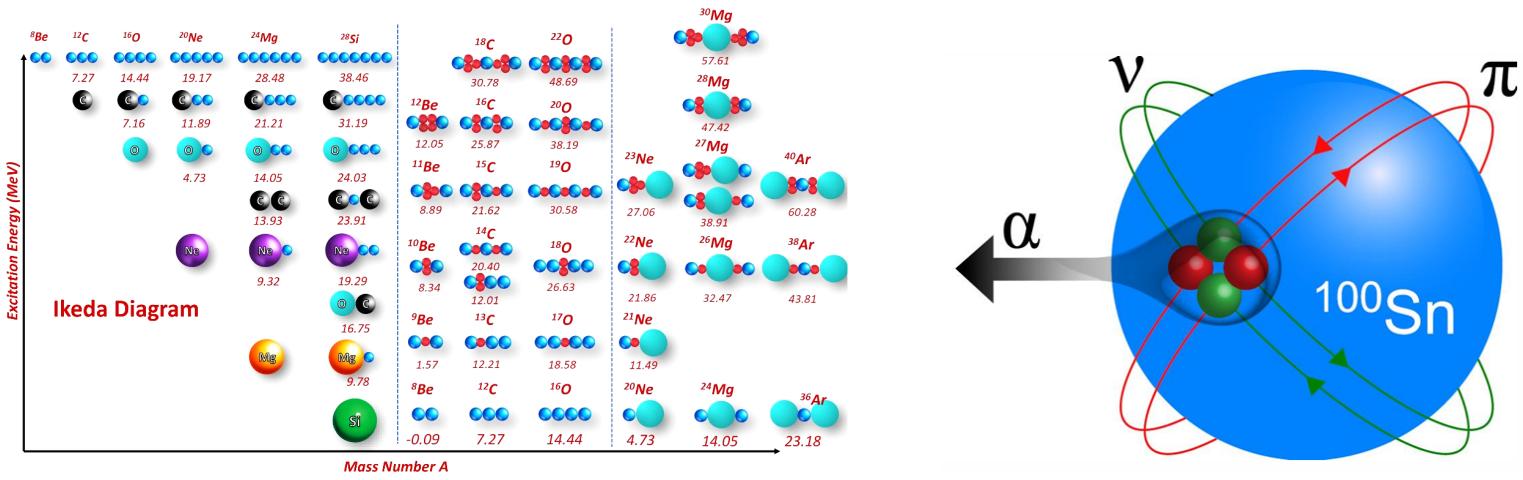


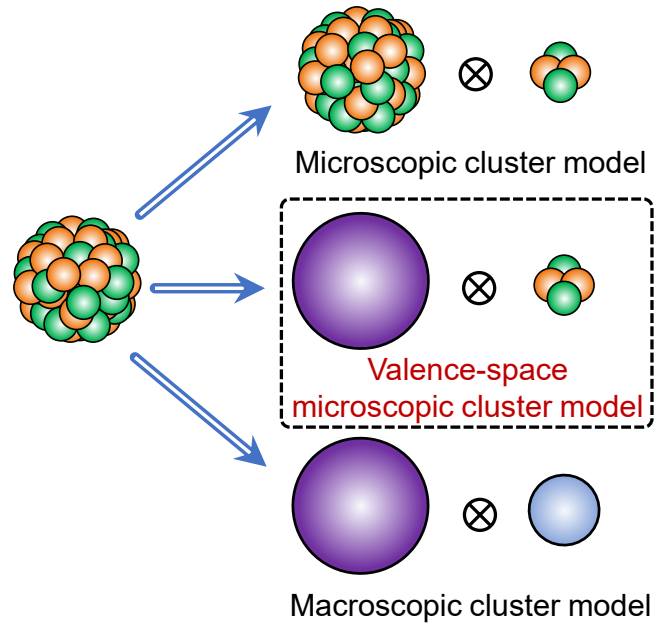
Fig. 1: The Ikeda threshold diagram (Left), and the α decay of ^{104}Te (Right).

[*Prog. Theo. Phys. Supp.* 68, 464 (1968); *Phys. Rept.* 432, 43 (2006).]

Theoretical description of α clustering

Theoretically, the **microscopic** and **macroscopic** cluster models are widely used to study α-cluster structures.

1. Microscopic cluster model:
   - Nucleon degrees of freedom;
   - Rigorous antisymmetrization;
   - Computationally expensive;
   - *e.g.,* GCM, THSR, AMD, etc.
2. Macroscopic cluster model:
   - Cluster degrees of freedom;
   - Lower computational cost;
   - Lose critical structural details.



Motivation: Try to extend microscopic cluster models to heavy nuclei while retaining critical structural details.

— Valence-space microscopic cluster model (VS-MCM).

Theoretical framework of VS-MCM

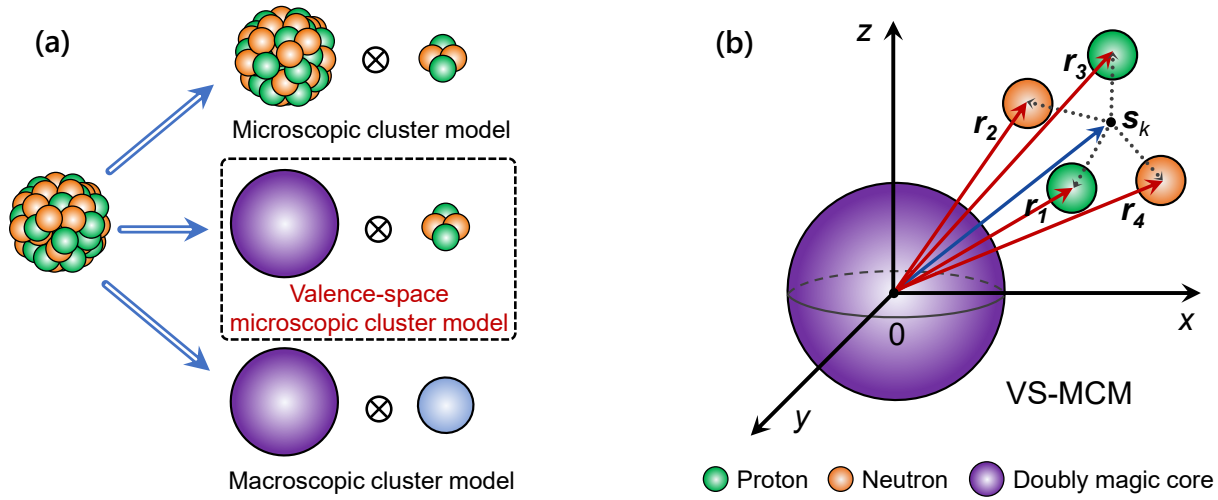
VS-MCM takes the **valence nucleons** outside the doubly magic core as active degrees of freedom. The core nucleons are treated as **frozen**.

- The Hamiltonian for the valence nucleons $\{p\uparrow, p\downarrow, n\uparrow, n\downarrow\}$ is given as

$$\hat{H} = \sum_{i=1}^{4} \frac{\hat{\boldsymbol{p}}_i^2}{2m_i} + \sum_{i=1}^{4} \hat{U}_i + \sum_{i<j}^{4} \hat{V}_{ij} + \frac{1}{2m_c} \left(\sum_{i=1}^{4} \hat{\boldsymbol{p}}_i \right)^2 , \tag{1}$$

$$= \sum_{i=1}^{4} \left(\frac{\hat{\boldsymbol{p}}_i^2}{2\mu_i} + \hat{U}_i \right) + \sum_{i<j}^{4} \left(\hat{V}_{ij} + \frac{\hat{\boldsymbol{p}}_i \cdot \hat{\boldsymbol{p}}_j}{m_c} \right) . \tag{2}$$

- The last term in Eq. (1) is sourced from the recoil kinetic energy of the core.



Theoretical framework of VS-MCM

The nucleon-nucleon interaction V_{ij} includes the nuclear part V_{ij}^{N} and Coulomb one V_{ij}^{C} as well, which takes the form as

$$V_{ij}^{N}=\sum_{k=1}^{N_g}V_k\exp(-\frac{\boldsymbol{r}_{ij}^2}{a_k^2})(W_k-M_kP_{ij}^{\sigma}P_{ij}^{\tau}+B_kP_{ij}^{\sigma}-H_kP_{ij}^{\tau}), \tag{3}$$

and

$$V_{ij}^{C}=\left(m_{\tau_i}+\frac{1}{2}\right)\left(m_{\tau_j}+\frac{1}{2}\right)\frac{e^2}{\boldsymbol{r}_{ij}}. \tag{4}$$

The effective core-nucleon potential U_i contains the nuclear part and the Coulomb one for the protons:

$$U_N(r)=\frac{V_0}{1+\exp[(r-R)/a]}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\frac{V_{LS}\cdot(\boldsymbol{l}\cdot\boldsymbol{\sigma})}{1+\exp[(r-R)/a]}, \tag{5}$$

$$U_C(r)=\left(m_\tau+\frac{1}{2}\right)Z_ce^2\cdot\begin{cases}\frac{3-(r/R)^2}{2R}, & r\le R,\\ \frac{1}{r}, & r>R.\end{cases} \tag{6}$$

Modified Brink wave function

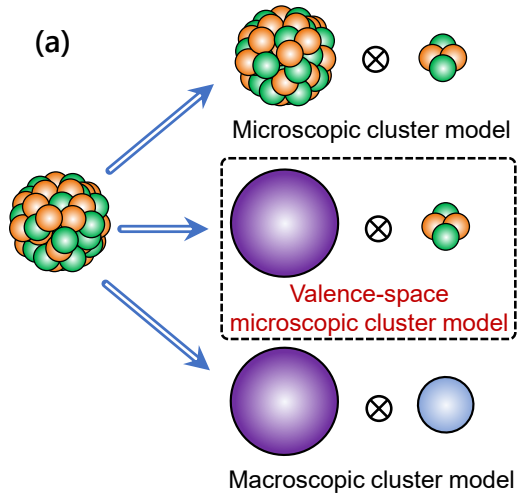
The valence α-cluster is given by a modified Brink wave function $\Phi_{\boldsymbol{s}_k}$ as

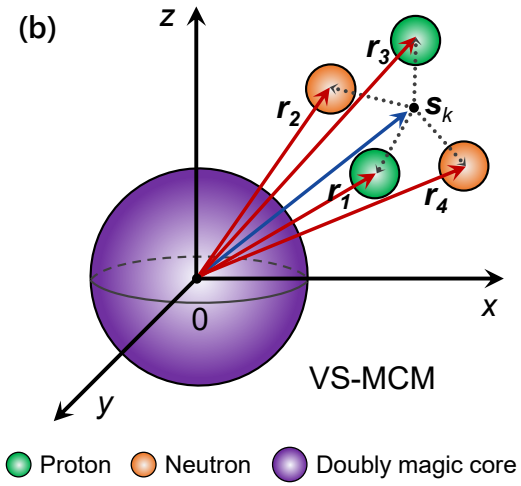
$$\Phi_{\boldsymbol{s}_k} = \frac{1}{\sqrt{4!}} \det\{\hat{\psi}_{1\boldsymbol{s}_k}(\boldsymbol{r}_1)\chi_{\sigma_1\tau_1} \cdots \hat{\psi}_{4\boldsymbol{s}_k}(\boldsymbol{r}_4)\chi_{\sigma_4\tau_4}\}. \tag{7}$$

The core orbits are removed from the model space by applying the Pauli projection operator $\hat{\mathscr{P}}$ in the single-particle wave function $\hat{\psi}_{i\boldsymbol{s}_k}(\boldsymbol{r}_i)$ as

$$\hat{\psi}_{i\boldsymbol{s}_k}(\boldsymbol{r}_i) = \hat{\mathscr{P}}\psi_{i\boldsymbol{s}_k}(\boldsymbol{r}_i) = \left[1 - \sum_{\gamma\in\text{core}} |\varphi_\gamma\rangle\langle\varphi_\gamma|\right]\psi_{i\boldsymbol{s}_k}(\boldsymbol{r}_i), \tag{8}$$

$\psi_{i\boldsymbol{s}_k}(\boldsymbol{r}_i)$ denotes the single-particle orbit in the α particle in free space. $\varphi_\gamma(\boldsymbol{r}_i)$ is the single-particle orbit occupied by the core nucleus.





Hill-Wheeler equation

The wave function of α-cluster state with the definite spin j is given by superposing the projected wave functions as

$$\Phi_j = \sum_k f_j^k \cdot \Phi_j^k, \quad \Phi_j^k = \frac{2j+1}{8\pi^2} \int \mathscr{D}^{j*}_{mm'}(\Omega) \hat{\mathscr{R}}(\Omega) \Phi_{\boldsymbol{s}_k} \mathrm{d}\Omega, \tag{9}$$

The weight coefficient f_j^k and the energy E_j of the α-cluster state Φ_j can be determined by solving the following Hill-Wheeler equation:

$$\sum_k \left[\left\langle \Phi_j^{k'} \middle| \hat{H} \middle| \Phi_j^k \right\rangle - E_j \left\langle \Phi_j^{k'} \middle| \Phi_j^k \right\rangle \right] \cdot f_j^k = 0. \tag{10}$$

Taking the generator coordinate $\boldsymbol{s}_{k'}$ along z-axis and $\boldsymbol{s}_k$ in the xz plane with a relative angle ξ, one can obtain

$$\left\langle \Phi_j^{k'} \middle| \hat{H} \middle| \Phi_j^k \right\rangle = \frac{2j+1}{2} \int \left\langle \Phi_{\boldsymbol{s}_{k'}} \middle| \hat{H} \hat{\mathscr{R}}_y(\xi) \middle| \Phi_{\boldsymbol{s}_k} \right\rangle P_j(\cos\xi) \sin\xi \, \mathrm{d}\xi, \tag{11}$$

$$\left\langle \Phi_j^{k'} \middle| \Phi_j^k \right\rangle = \frac{2j+1}{2} \int \left\langle \Phi_{\boldsymbol{s}_{k'}} \middle| \hat{\mathscr{R}}_y(\xi) \middle| \Phi_{\boldsymbol{s}_k} \right\rangle P_j(\cos\xi) \sin\xi \, \mathrm{d}\xi. \tag{12}$$

Proof of principle: ^{20}Ne and ^{44}Ti

We introduce the Pauli index $\mathscr{P}_\alpha = \left|\left\langle \psi_{\boldsymbol{s}_k}(\boldsymbol{r}) \middle| \hat{\psi}_{\boldsymbol{s}_k}(\boldsymbol{r}) \right\rangle\right|^2$ to show the degree of Pauli blocking effects.

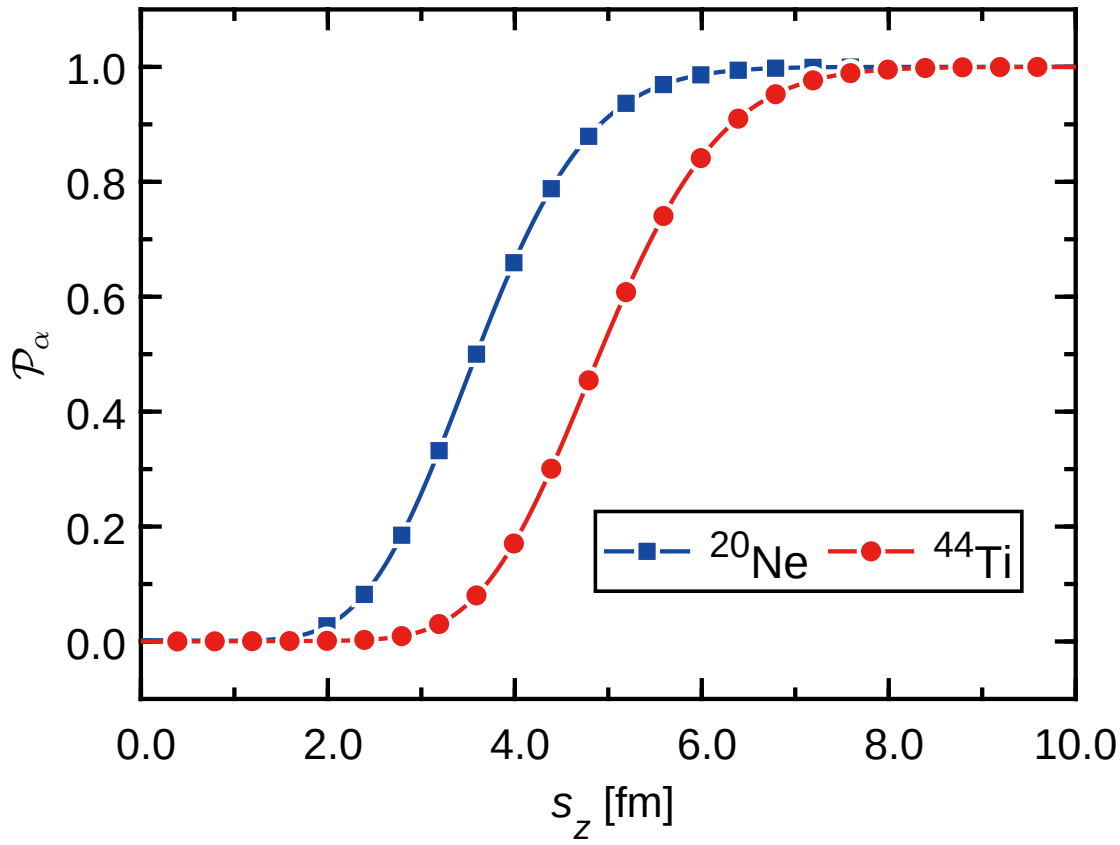


Fig. 2: The dependence of the Pauli index $\mathscr{P}_\alpha = \left|\left\langle \psi_{\boldsymbol{s}_k}(\boldsymbol{r}) \middle| \hat{\psi}_{\boldsymbol{s}_k}(\boldsymbol{r}) \right\rangle\right|^2$ on the intercluster separation s_z.

Proof of principle: ^{20}Ne and ^{44}Ti

The energy levels of the inversion doublet $K^\pi = 0_1^\pm$ bands in ^{20}Ne.

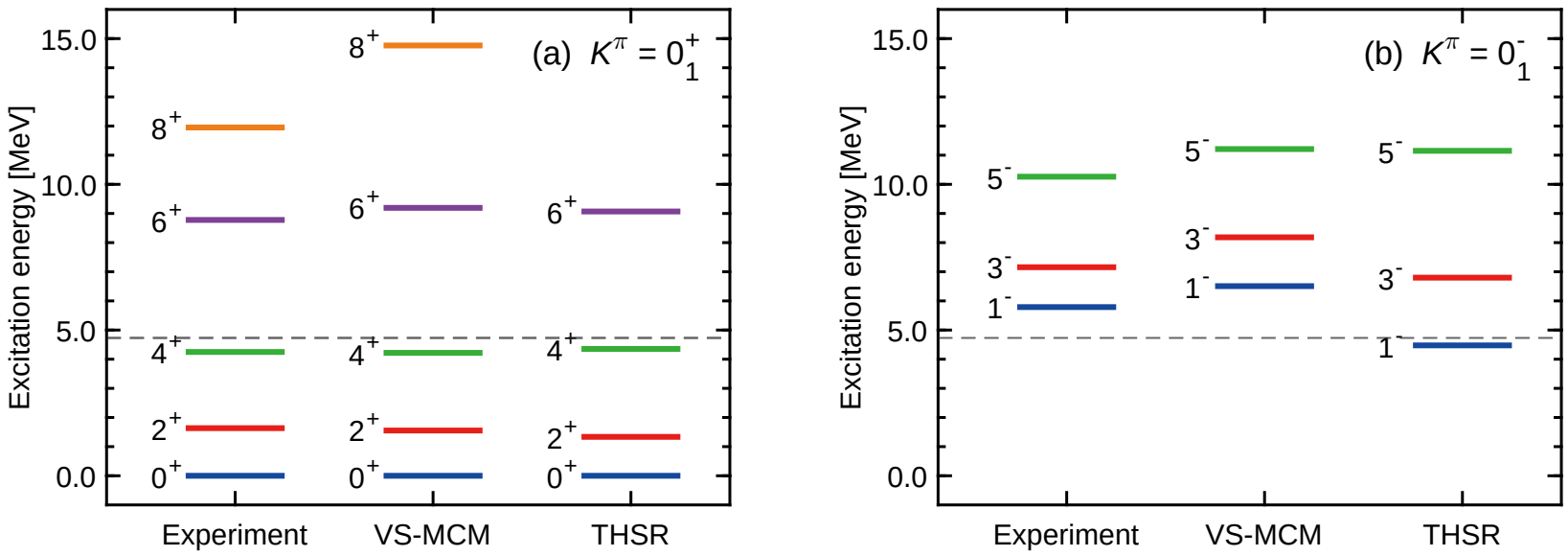


Fig. 3: The energy levels of the inversion doublet $K^\pi = 0_1^\pm$ bands in ^{20}Ne reproduced by the VS-MCM in this work, compared with the theoretical ones obtained by THSR wave function as well as the experimental data.

[*Phys. Rev. Lett.* 110 , 262501 (2013).]

Proof of principle: ^{20}Ne and ^{44}Ti

The energy levels of the inversion doublet $K^\pi = 0^\pm_1$ bands in ^{44}Ti.

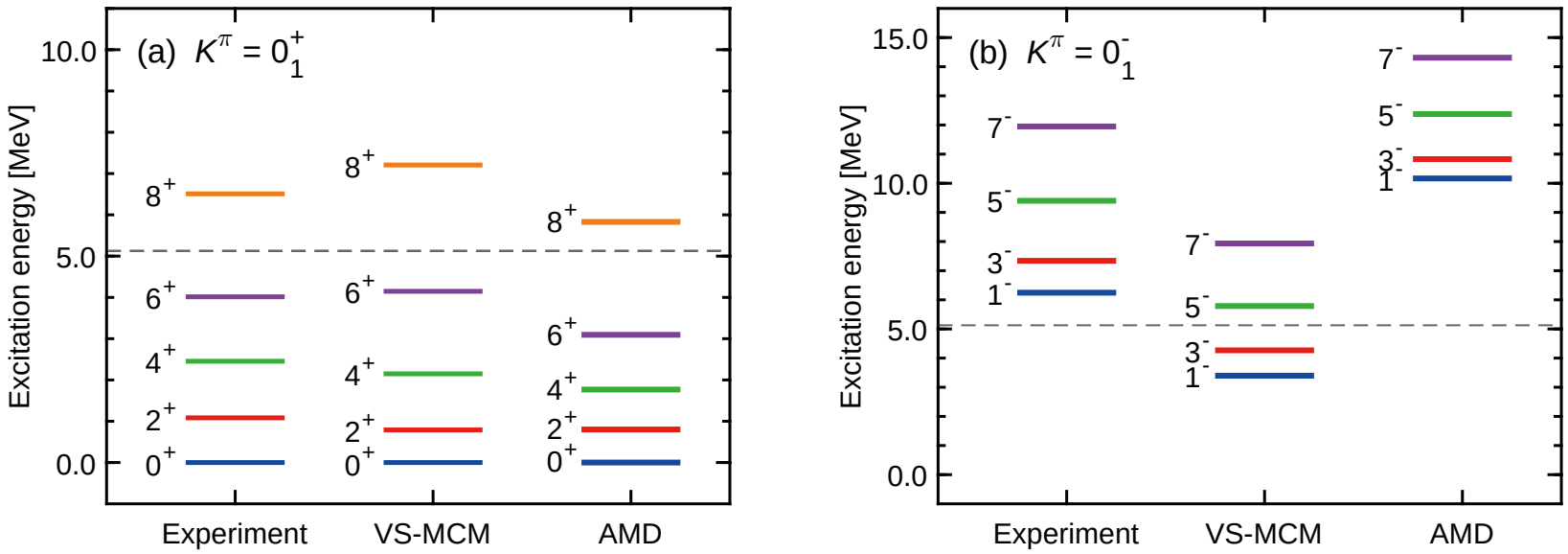


Fig. 4: The energy levels of the inversion doublet $K^\pi = 0^\pm_1$ bands in ^{44}Ti reproduced by the VS-MCM in this work, compared with the theoretical ones obtained by AMD as well as the experimental data.

[*Nucl. Phys. A* 767, 58−80 (2006).]

The deviations in negative band: Coupling to the core excited state?

Summary and Prospect

1. We propose the VS-MCM as a novel framework to investigate the nuclear α-cluster structures.
2. The energy levels of $K^\pi = 0_1^\pm$ bands in ^{20}Ne and ^{44}Ti are studied by VS-MCM as a proof of principle.
3. **Prospect**:
   - Extend to study the α-cluster structure above heavier doubly magic core, *e.g.*, $^{104}\mathrm{Te} = {}^{100}\mathrm{Sn} + \alpha$, etc.
   - Considering the system where more α clusters and valence nucleons can exist outside the heavy doubly magic core;
   - Modified Brink wave function $\rightarrow$ AMD basis?
   - …

Thanks for your attention !