NON-RENORMALIZABLE THEORIES: FROM WEAK INTERACTIONS TO COSMOLOGY

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The foundation:

- Quantum field theory is formulated for all types of interactions independently on renormalizability
- The Standard Model of particle interactions is based on renormalizable theories. The principle of renormalizability served as a guiding idea in construction of the SM
- The problems of NR interactions are related to the growth of the amplitudes with energy violating unitarity and uncontrollable UV divergences
- However, the NR interactions are not totally obscure and allow for proper treatment that may lead to solving of these problems
- The key ingredient is the R-operation that removes the UV divergences which equally works for NR theories as well as for the renormalizable ones

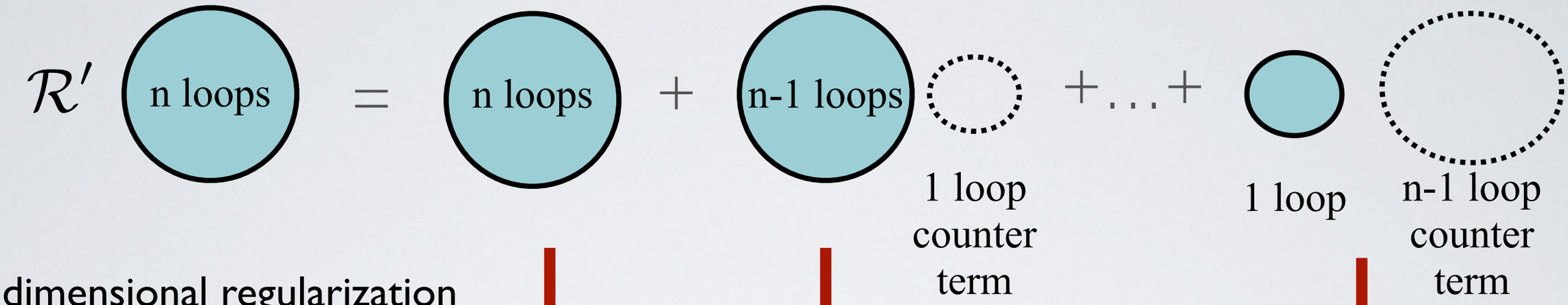
The foundation:

- Local Quantum Field Theory obeying the requirements of causality, unitarity and analyticity has a remarkable property: after applying the R-operation all UV divergent structures are local in coordinate space or are at most polynomial in momentum space (Bogoliubov-Parasiuk theorem).

- This is true for any local QFT irrespectively of its renormalizability or non-renormalizability

- All these statements lead to relations between the subsequent orders of PT resulting in the RG equations which aimed on the summation of infinite series of PT for the asymptotics of the Green functions, amplitudes, potential, etc

BPHZ R-operation



- dimensional regularization

$$d = 4 - 2\epsilon$$

$$\mathcal{R}'G_n = \frac{A_n^{(n)}(\mu^2)^{n\epsilon}}{\epsilon^n} + \frac{A_{n-1}^{(n)}(\mu^2)^{(n-1)\epsilon}}{\epsilon^n} + \ldots + \frac{A_1^{(n)}(\mu^2)^{\epsilon}}{\epsilon^n}$$

$$+ \quad \frac{B_n^{(n)}(\mu^2)^{n\epsilon}}{\epsilon^{n-1}} + \frac{B_{n-1}^{(n)}(\mu^2)^{(n-1)\epsilon}}{\epsilon^{n-1}} + \ldots + \frac{B_1^{(n)}(\mu^2)^{\epsilon}}{\epsilon^{n-1}}$$

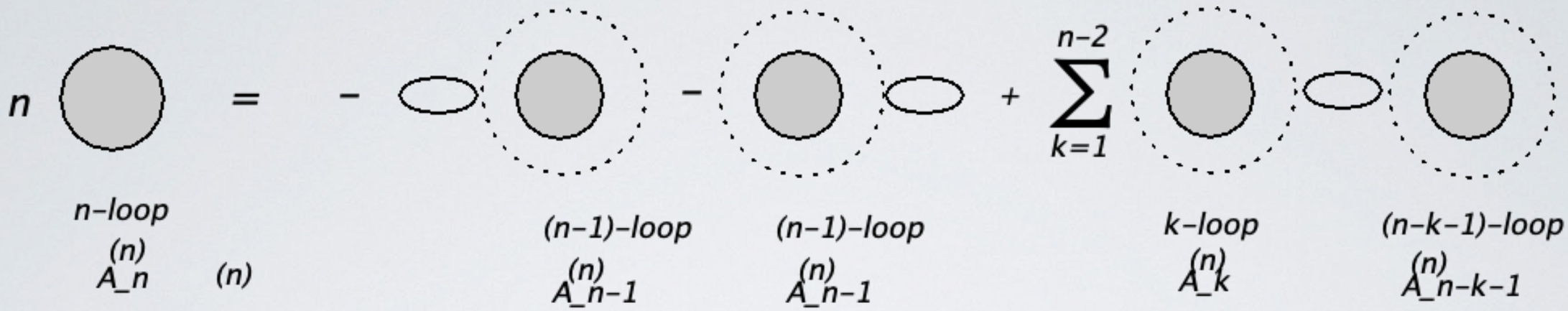
$$+ \quad \frac{C_n^{(n)}(\mu^2)^{n\epsilon}}{\epsilon^{n-2}} + \frac{C_{n-1}^{(n)}(\mu^2)^{(n-1)\epsilon}}{\epsilon^{n-2}} + \ldots + \frac{C_1^{(n)}(\mu^2)^{\epsilon}}{\epsilon^{n-2}}$$

$$+ \quad \text{lower pole terms,}$$

$A_k^{(n)}$ $B_k^{(n)}$ $C_k^{(n)}$ $(\mu^2)^{k\epsilon}$ terms appear after subtraction of (n-k) loop counter terms

The requirement of locality imposes relations to these divergent contributions relating them to the lower order ones

From Recurrence Relation to the RG Equation



- This is the general recurrence relation that reflects the locality of counterterms in any theory
- In renormalizable theories A_n is a constant and this relation is reduced to the algebraic one
- In non-renormalizable theories for the scattering amplitudes A_n depends on kinematics and one has to integrate through the one-loop diagrams
- For the effective potential A_n depends on the fields and one has to differentiate

Taking the sum $\sum_n A_n(-z)^n = A(z)$ one can transform the recurrence relation into RG equation

$$\frac{dA(z)}{dz} = -2 \oint A(z) + \oint A(z) \otimes A(z)$$

$$\frac{d}{dz} = \frac{d}{d\log\mu^2}$$

This is the generalized RG equation valid in any (even non-renormalizable) theory!

Demonstration

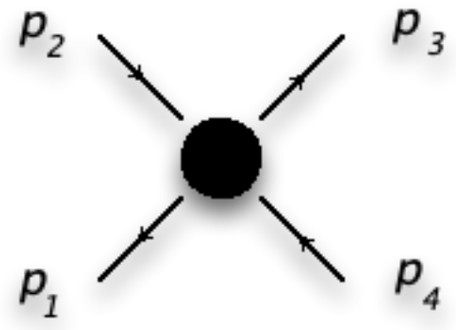
V-A four fermion Interaction

The interaction Lagrangian

$$\mathcal{L}_{int} = -\frac{G_F}{4}\bar{\psi}\frac{\gamma^\mu(1-\gamma^5)}{2}\psi\bar{\psi}\frac{\gamma^\nu(1-\gamma^5)}{2}\psi = -\frac{G_F}{4}\bar{\psi}\gamma^\mu P_L\psi\bar{\psi}\gamma^\nu P_L\psi$$

The tree level

$$A_4^{(0)} = \langle 13\rangle[42].$$



The one-loop amplitude

$$A_4^{(1)} = S_1 + T_1 + U_1 = \left(-\frac{16}{3}\frac{s}{\epsilon} - \frac{16}{3}\frac{t}{\epsilon}\right)\langle 13\rangle[42].$$

Recurrence Relations and RG Equation for the Scattering Amplitude

$$A_n(s,t,u) = S_n(s,t,u) + T_n(s,t,u) + U_n(s,t,u)$$

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$$nS_n(s,t,u) = -4s\int_0^1 dx \sum_{k=0}^{n-1}\sum_{p=0}^{k} \frac{[s(-s-u)]^p[x(1-x)]^{p+1}}{p!(p+1)!(p+2)^{-1}} \times$$

$$-\frac{dA(s,t,u)}{dz} = \times \frac{d^pA_k(s,-s-u',u')}{du'^p}\frac{d^pA_{n-1-k}(s,-s-u',u')}{du'^p}\Big|_{u'\to -sx}$$

$$= -4s\int_0^1 dx \sum_{p=0}^{\infty} \frac{[s(-s-u)]^p[x(1-x)]^{p+1}}{p!(p+1)!(p+2)^{-1}} \left(\frac{d^pA(s,-s-u',u')}{du'^p}\right)^2$$

$$-4t\int_0^1 dx \sum_{p=0}^{\infty} \frac{[t(-t-u)]^p[x(1-x)]^{p+1}}{p!(p+1)!(p+2)^{-1}} \left(\frac{d^pA(-t-u',t,u')}{du'^p}\right)^2$$

$$+4u\int_0^1 dx \sum_{p=0}^{\infty} \frac{[u(-s-u)]^p[x(1-x)]^{p+1}}{p!(p+1)!(p+3)^{-1}} \left(\frac{d^pA(s',-s'-u,u)}{ds'^p}\right)^2$$

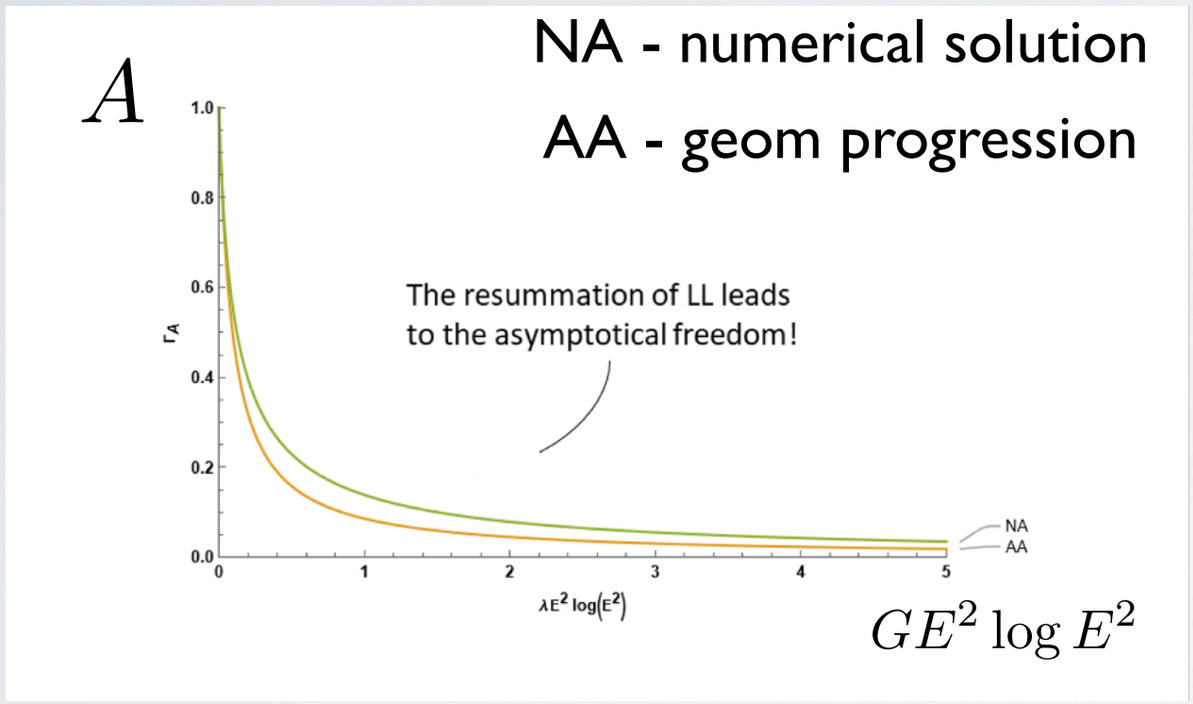
$$+4u\int_0^1 dx \sum_{p=0}^{\infty} \frac{[u(-t-u)]^p[x(1-x)]^{p+1}}{p!(p+1)!(p+3)^{-1}} \left(\frac{d^pA(-t'-u,t',u)}{dt'^p}\right)^2$$

- With the help of this equation one can find the asymptotic behaviour of the amplitude in the high energy regime

Numerical Solution of RG Equation for the Scattering Amplitude

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$$s = 4E^2,\ t = u = -2E^2,\ E \to \infty \qquad \frac{Amplitude}{A_{Tree}} = A(E^2, G\log E^2/\mu^2)$$

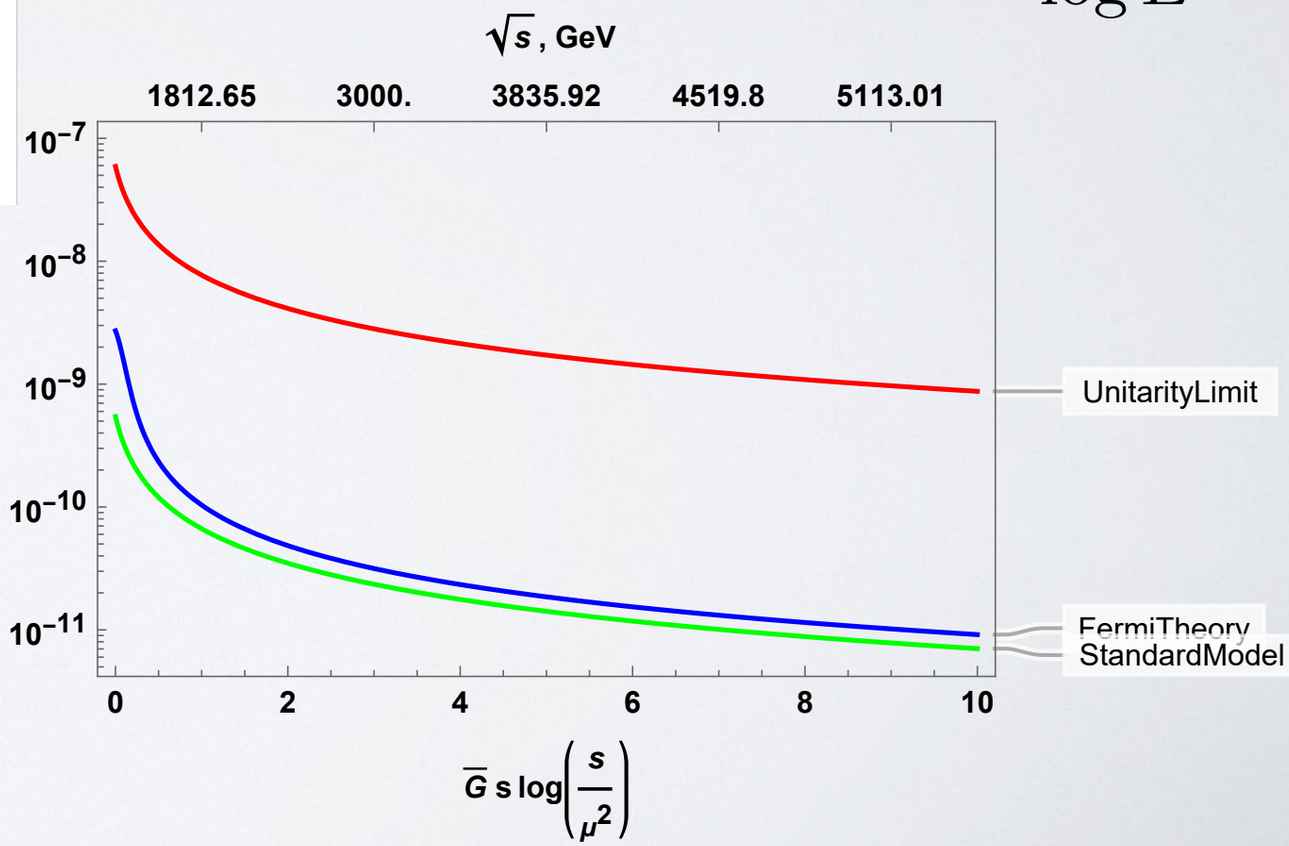


$$A_{Tree} \sim GE^2$$

$$A \sim \frac{1}{GE^2 \log E^2}$$

$$Amp = A_{Tree}A \sim \frac{1}{\log E^2}$$

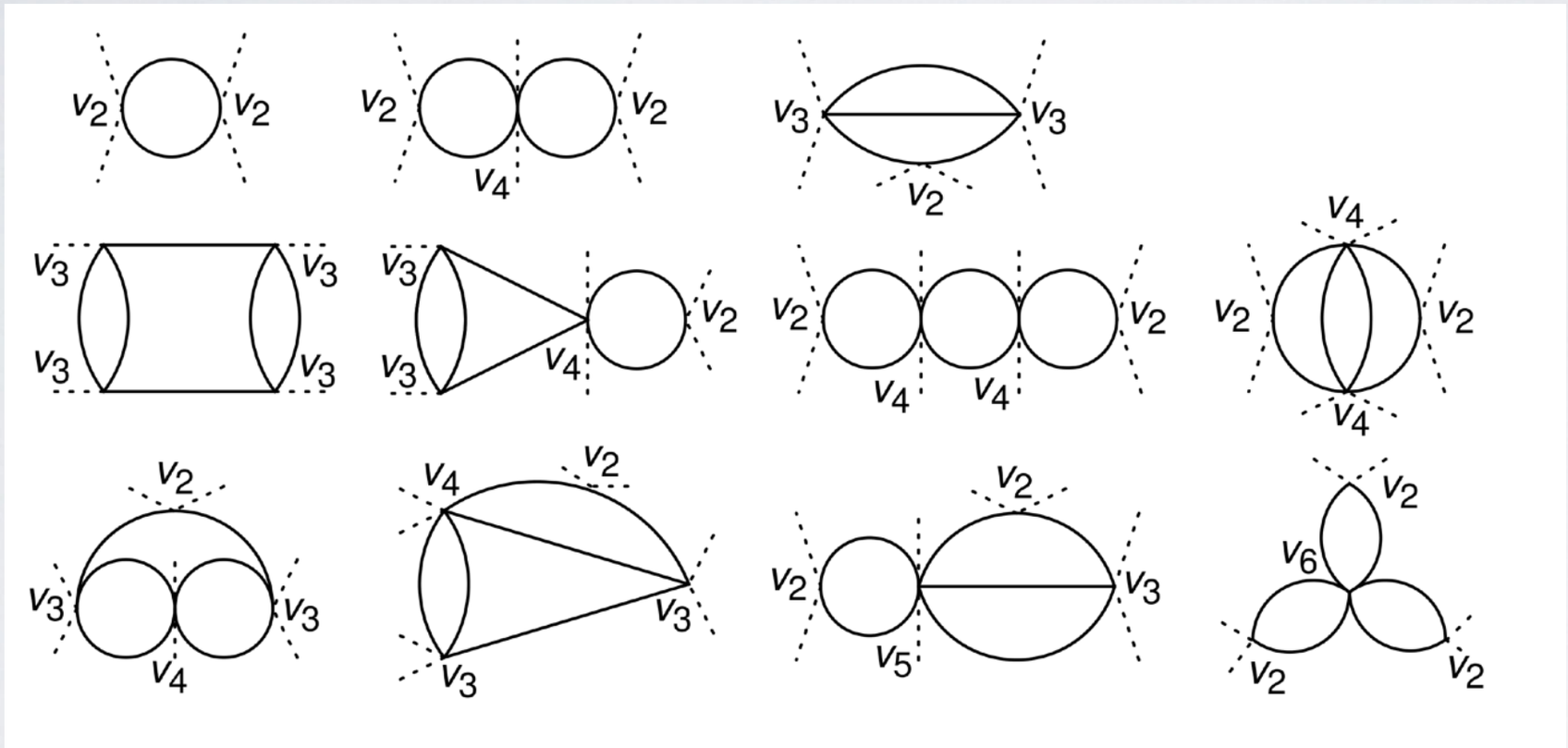
Asymptotic freedom !!
Just like in the SM



Effective Potential in Scalar Theory

V_{eff} Is the sum of all vacuum 1PI diagrams

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - gV_0(\phi)$$



$$v_2(\phi) \equiv \frac{d^2 V_0(\phi)}{d\phi^2}$$

$$v_n \equiv d^n V_0 / d\phi^n$$

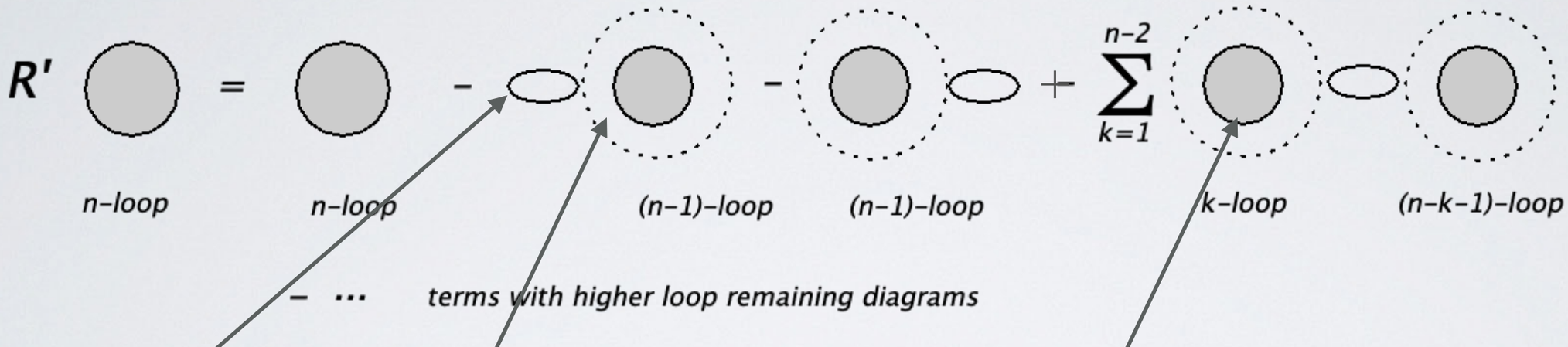
Shown are UV divergent vacuum diagrams in arbitrary scalar theory up to three loops

$$V_{eff} = g \sum_{n=0}^{\infty} (-g)^n V_n .$$

Recurrence relations for the leading poles for effective potential

Kazakov, Iakhibbaev, Tolkachev 22

Action of R'-operation on divergent diagram



v_2 $D_2\Delta V_{n-1}$ $D_2\Delta V_k$ $D_2 = \dfrac{d^2}{d\phi^2}$

$$n\Delta V_n = \frac{1}{2}v_2 D_2\Delta V_{n-1} + \frac{1}{4}\sum_{k=1}^{n-2} D_2\Delta V_k D_2\Delta V_{n-1-k}, \quad n \geq 2 \qquad \Delta V_1 = \frac{1}{4}v_2^2$$

$$n\Delta V_n = \frac{1}{4}\sum_{k=0}^{n-1} D_2\Delta V_k D_2\Delta V_{n-1-k}, \quad n \geq 1, \quad \Delta V_0 = V_0$$

RG pole equation for arbitrary potential

$$\Sigma(z,\phi) = \sum_{n=0}^{\infty} (-z)^n \Delta V_n(\phi) \qquad z = \frac{g}{\epsilon}$$

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RG pole equation

$$\frac{d\Sigma}{dz} = -\frac{1}{4}(D_2\Sigma)^2 \qquad \Sigma(0,\phi) = V_0(\phi)$$

This a non-linear partial differential equation!

Effective potential

$$V_{eff}(g,\phi) = g\Sigma(z,\phi)\big|_{z\to -\frac{g}{16\pi^2}\log g v_2/\mu^2}. \qquad v_2(\phi) \equiv \frac{d^2V_0(\phi)}{d\phi^2}$$

Demonstration

α -attractor Inflaton Potential

Kallosh, Linde 13

Inflaton action with hyperbolic geometry

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R(g) + \frac{1}{2}\frac{\partial_\mu\phi\partial^\mu\phi}{1-\frac{\phi^2}{6\alpha}} - V(\phi)\right]$$

Transition to the standard kinetic term

$$\partial\phi/\sqrt{1-\frac{\phi^2}{1-6\alpha}} = \partial\varphi \qquad\qquad \phi = \sqrt{6\alpha}\tanh\left(\frac{\varphi}{\sqrt{6\alpha}}\right)$$

Inflaton action of α -attractor model

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R(g) + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V\left(\sqrt{6\alpha}\tanh\left(\frac{\varphi}{\sqrt{6\alpha}}\right)\right)\right].$$

T- model

n=2 T_2 - model

$$gV_T(\varphi) = g\tanh^n\left(\frac{\varphi}{\sqrt{6\alpha}M_{Pl}}\right)$$

Numerical Solution for T_2 - model

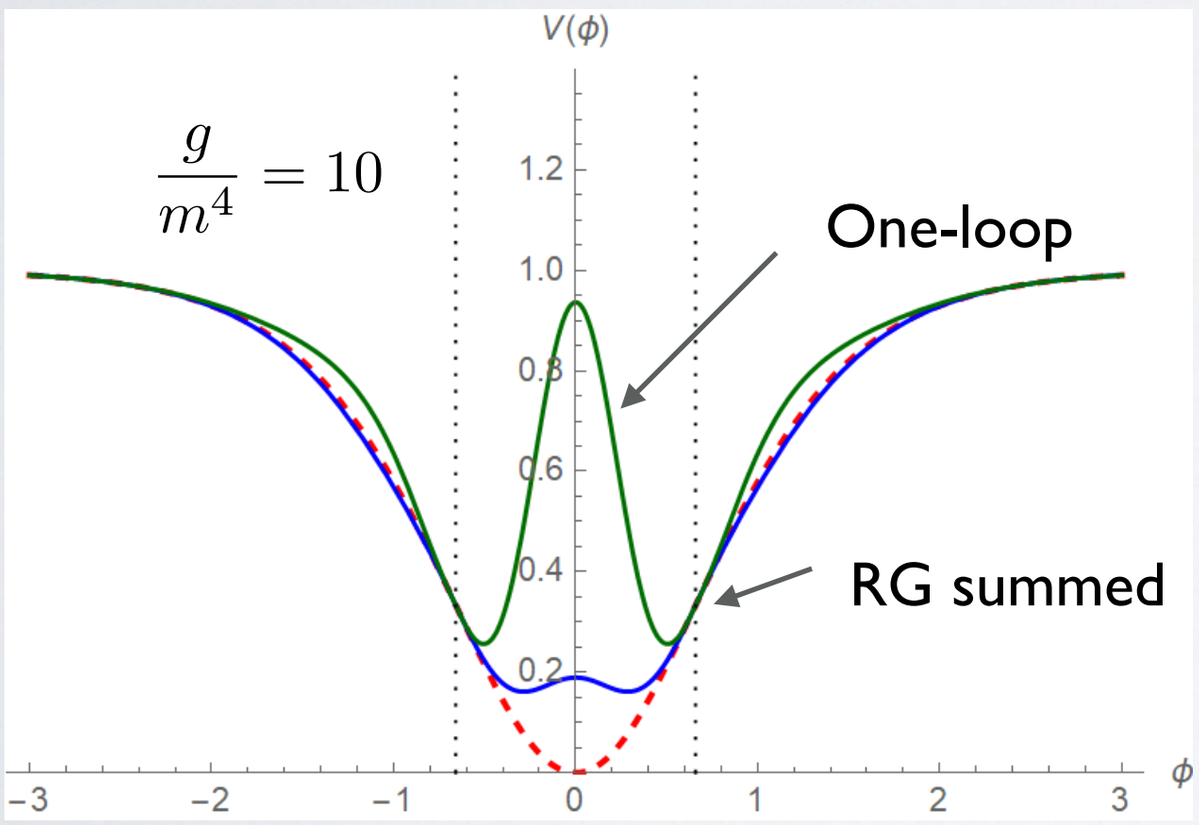
Kazakov, Iakhibbaev, Tolkachev '23

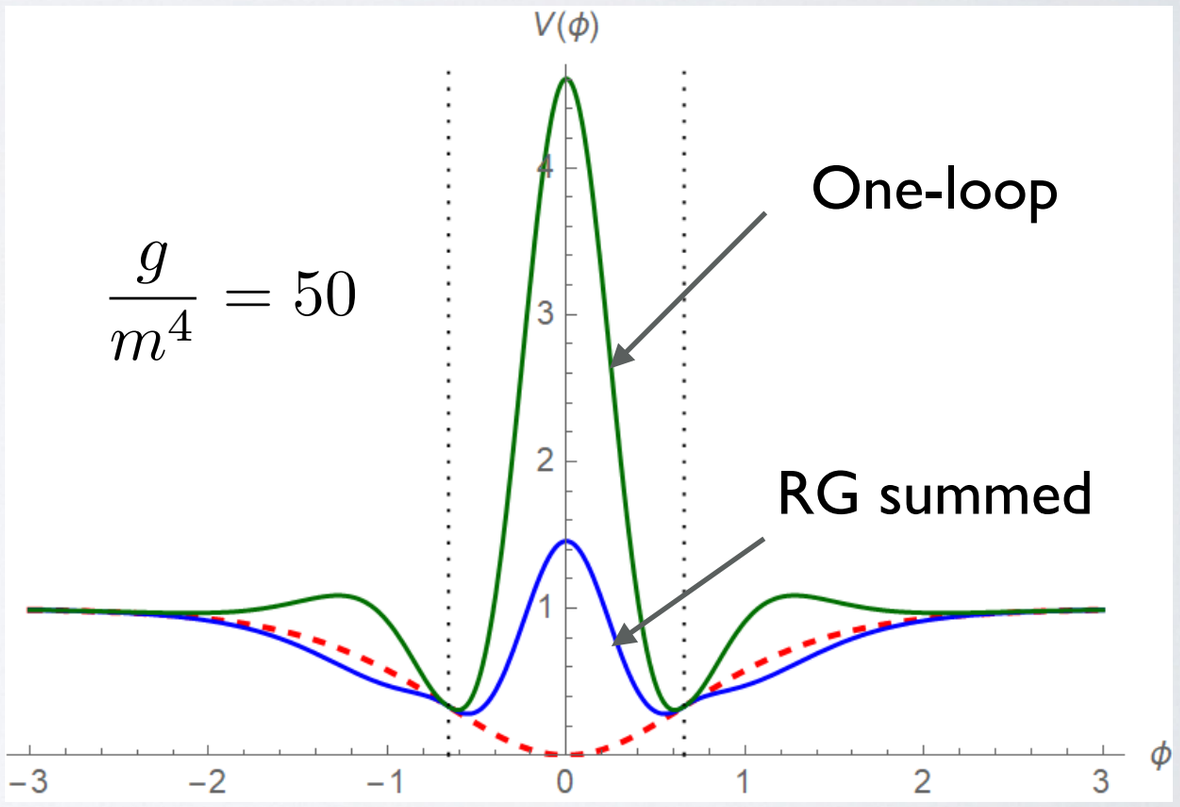
ArXiv: 2308.03872

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$$gV_0 = g\tanh^2(\phi/m)$$

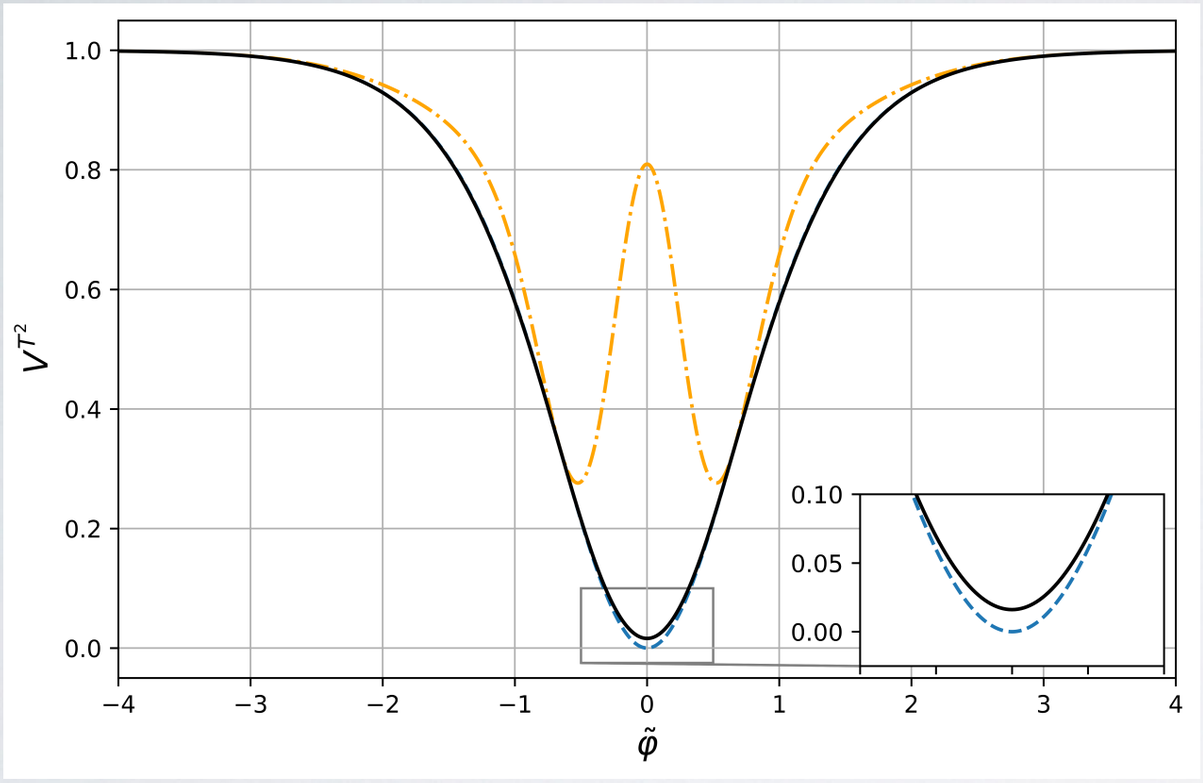
$$V_{eff}(g,\phi) = g\Sigma(z,\phi)\big|_{z\to -\frac{g}{16\pi^2}\log gv_2/\mu^2}\cdot \qquad v_2(\phi)\equiv\frac{d^2V_0(\phi)}{d\phi^2}$$





- Peak at the origin
- Additional minima

Lift of the Potential at the Minima - Origin of the Cosmological Constant



Kazakov, Iakhibbaev, Tolkachev, Filippov 24,25

ArXiv: 2405.18818

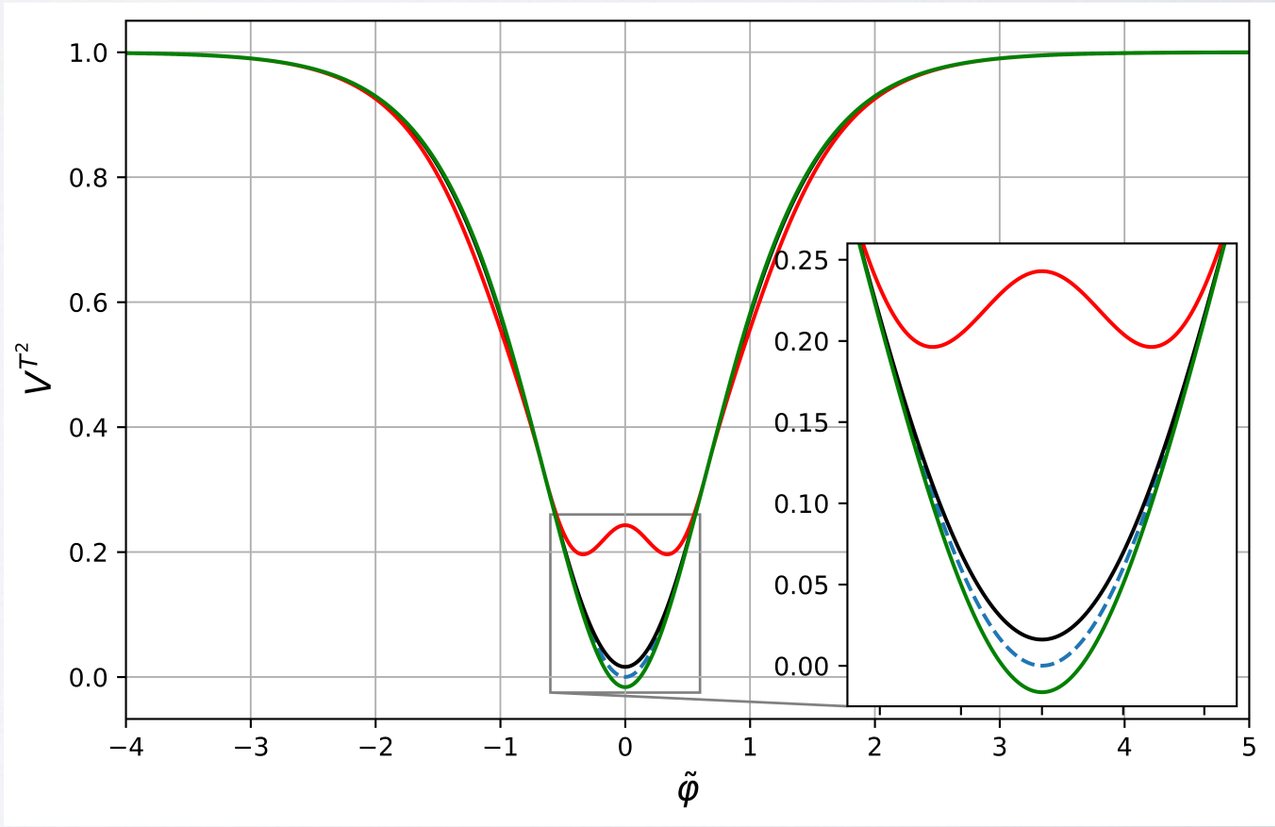
Comparison of the classical T2-model potential (blue dashed line), the one-loop correction (orange dashed line), and the RG summed potential (black solid line) for $g \sim 1, \mu < M_{PL}$

T2-model potential: variation of μ . The classical potential (blue dashed line), the RG summed potential (solid lines) for

$$\mu < M_{Pl} \quad \mu \ll M_{Pl} \quad \mu > M_{Pl}$$

black line, red line, green line

$$g = 2, \ \alpha = 1$$



Cosmological implications

Action for scalar theory of cosmological inflation:

$$S[g_{\mu\nu},\varphi]=\int d^4x\sqrt{-g}\left(\frac{1}{2}F(\varphi)R+\frac{1}{2}g_{\mu\nu}\partial^\mu\varphi\partial^\nu\varphi-V(\varphi)\right),$$

where

$$F(\varphi)=\frac{1}{2}(1+\xi\varphi^2).$$

Karčiauskas, Díaz' 22

Slow roll parameters:

$$\epsilon=\frac{1}{2\mathcal{K}}\left(\frac{V_\varphi}{V}-2\frac{F_\varphi}{F}\right)^2,\qquad \eta=\frac{1}{\mathcal{K}}\left[2\frac{F_{\varphi\varphi}}{F}-\frac{V_{\varphi\varphi}}{V}-2\frac{F_\varphi^2}{F^2}+\frac{V_\varphi^2}{V^2}+\frac{\mathcal{K}_\varphi}{2\mathcal{K}}\left(\frac{V_\varphi}{V}-2\frac{F_{,\varphi}}{F}\right)\right],$$

where

$$\mathcal{K}(\varphi)=\frac{1}{F}+\frac{3}{2}\left(\frac{F_\varphi}{F}\right)^2.$$

The CMB tilt of scalar perturbations

$$n_{\rm s}-1\simeq-2(\epsilon+\eta)$$

the tensor-to-scalar ratio

$$r\simeq 16\epsilon$$

The number of e-folds

$$N\simeq\int_{\varphi}^{\varphi_{\rm end}}\frac{\mathcal{K}}{2\frac{F_\varphi}{F}-\frac{V_\varphi}{V}}{\rm d}\varphi$$

Comparison with observed data

Iakhibbaev, Tolkachev, Filippov 25

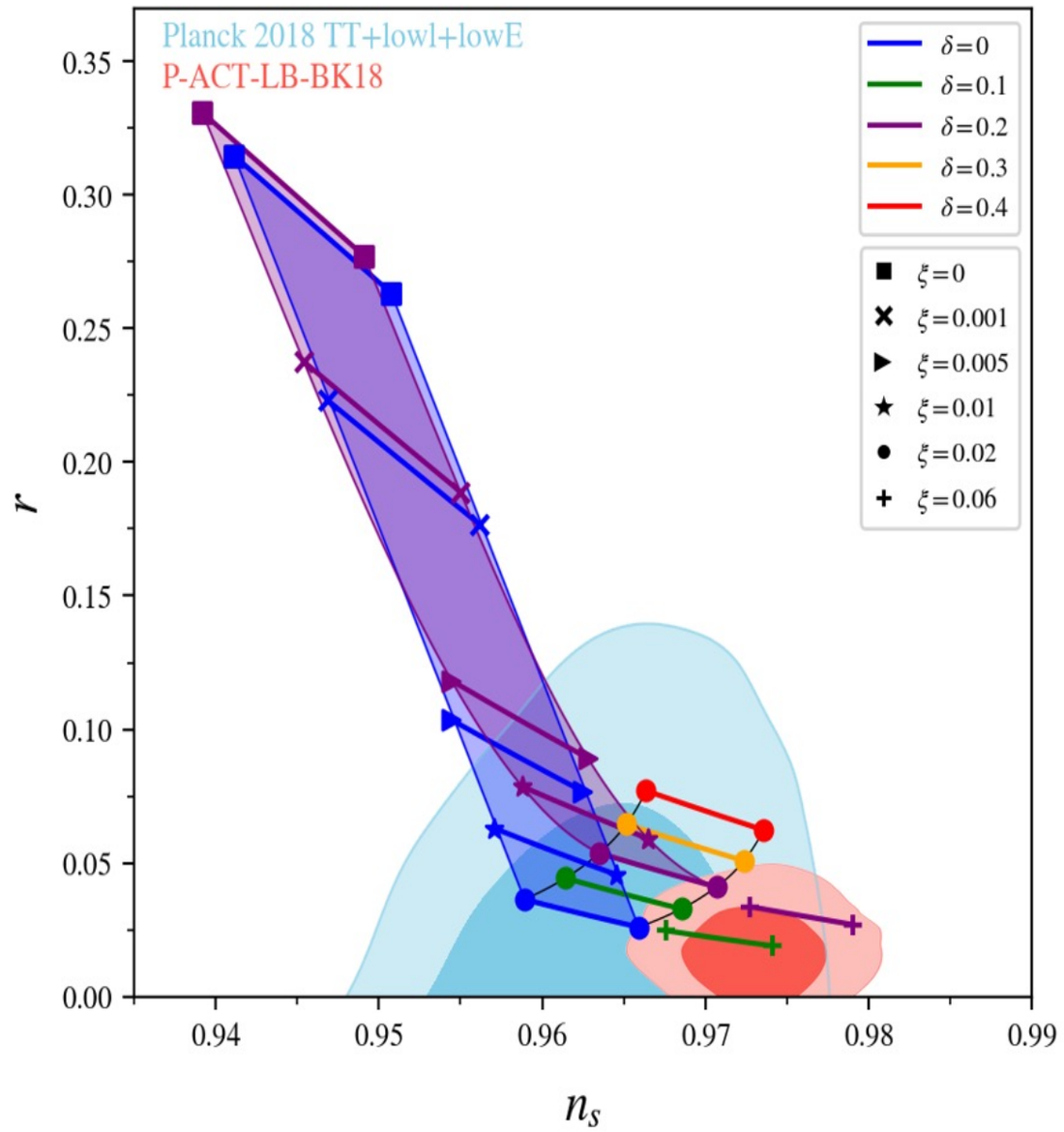


Fig.11 Example of the cosmological observables (n_s, r) for different resummed scalar potentials for various δ and ξ-coupling values in the $N_e = 50$ (left point) and 60 (right point) e-foldings.

- Within the models under consideration, parameters were selected to successfully match the PLANCK 2018 and ACT 2025 data

Resume

- **The property of locality of Quantum Field Theory equally applicable to renormalizable and non-renormalizable interactions states that the UV divergences are local and can be removed by local counter terms**

- **Based on locality of the counter terms due to the Bogoliubov-Parasiuk theorem one can construct the recurrence relations for the leading, subleasing, etc divergences in all loops divergences starting from one-, two-, three-, etc loop diagrams**

- **The recurrence relations can be converted into the generalized RG equations just like in renormalizable theories**

- **The RG equations allow one to sum up the leading (subleading, etc) divergences in all loops and define the high-energy/field behaviour**