Characteristic initial value problem and formation of black hole

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This talk is based on following joint work with P. Zhao

1. P. Zhao, X. Wu, CQG, 2025

2. P. Zhao, X. Wu, Science China : Mathematics, 2026

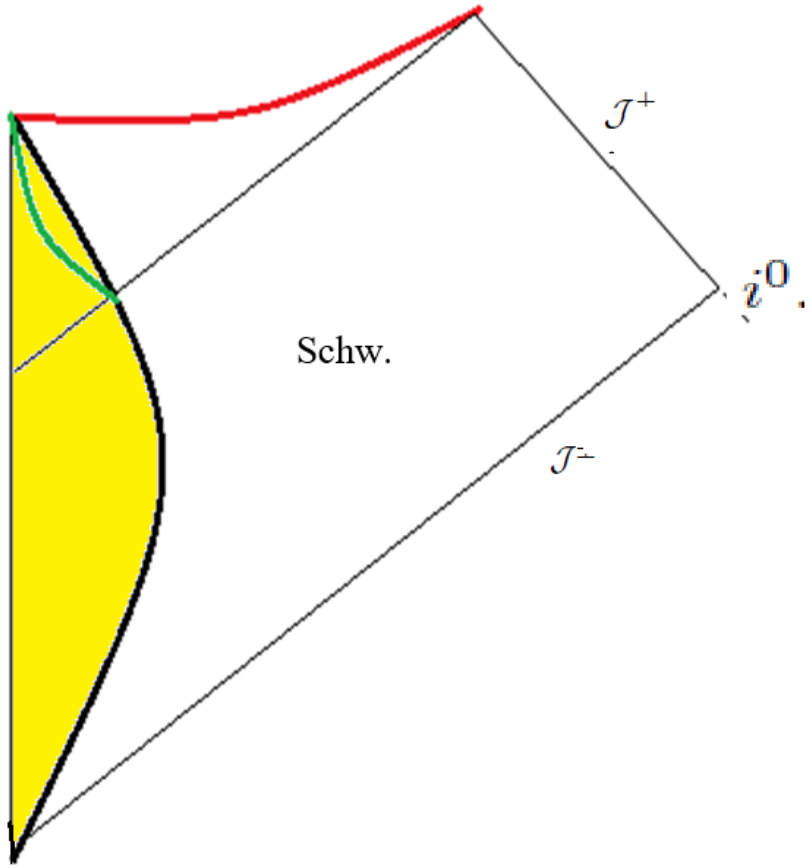
3. P. Zhao, X. Wu, arXiv : 2509.14525

➢ Motivation

- Formation of black hole is one of the most interesting progress in gravitational physics
- Trapped surface formation: a key phenomenon in black hole formation
- Core Conditions of the Singularity Theorems—Existence of compact trapped surface

➢Exact solutions

- Oppenheimer-Snyder solution



- Critical gravitational collapse：

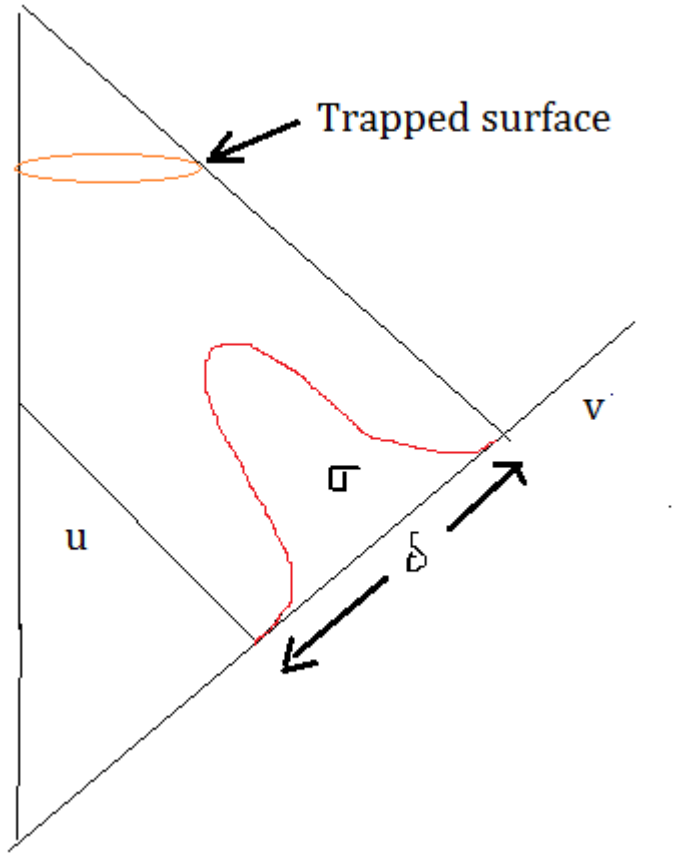
1. M. Choptuki, Universality and scaling in gravitational collapse of massless scalar field, PRL, 70, 1993, 9

2. D. Christodoulou, Examples of naked singularity formation in the gravitational collapse of a scalar field, Ann. Math. 140, 1994, 607

3. . A. Wang, H. P. de Oliveira, Critical phenomena of collapsing massless scalar wave packets, PRD 56, 1997, 753

Question： How to describe the formation of black hole without symmetric assumption？

- Christodoulou’s breakthrough



➢ Characteristic initial value problem (CIVP)

- Newman-Penrose Formulism

  Null tetrad $(l,\ n,\ \ m,\ \bar{m})$

$D := \nabla_l,\ D' := \nabla_n,\ \delta := \nabla_m$ and $\bar{\delta} := \nabla_{\bar{m}}$,

	$< m, \nabla l >$	$< n, \nabla l >$	$< \bar{m}, \nabla m >$	$< \bar{m}, \nabla n >$
D	$-\kappa$	$-(\varepsilon + \bar{\varepsilon})$	$\varepsilon - \bar{\varepsilon}$	π
D'	$-\tau$	$-(\gamma + \bar{\gamma})$	$\gamma - \bar{\gamma}$	ν
δ	$-\sigma$	$-(\bar{\alpha} + \beta)$	$-\bar{\alpha} + \beta$	μ
$\bar{\delta}$	$-\rho$	$-(\alpha + \bar{\beta})$	$\alpha - \bar{\beta}$	λ

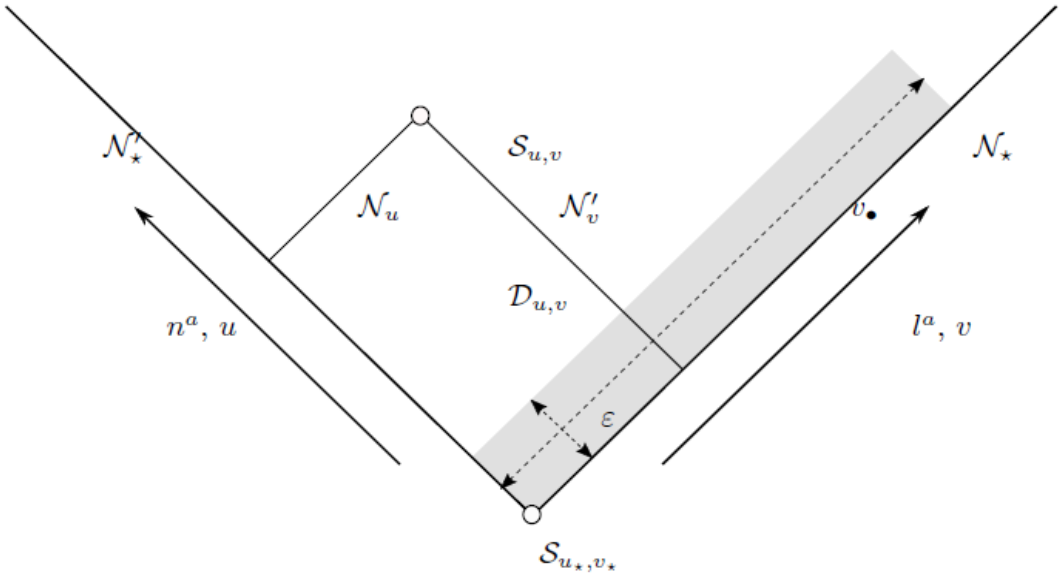
$$
\begin{aligned}
\Psi_0 &:= C_{abcd} l^a m^b l^c m^d, \\
\Psi_1 &:= C_{abcd} l^a n^b l^c m^d, \\
\Psi_2 &:= \frac{1}{2} C_{abcd} l^a n^b (l^c n^d - m^c \bar{m}^d), \\
\Psi_3 &:= C_{abcd} n^a l^b n^c \bar{m}^d, \\
\Psi_4 &:= C_{abcd} n^a \bar{m}^b n^c \bar{m}^d,
\end{aligned}
$$

- The choice of gauge and coordinates

$$l = \partial_v + C^{\mathcal{A}}\partial_{\mathcal{A}}, \qquad n = Q\partial_u, \qquad m = P^{\mathcal{A}}\partial_{\mathcal{A}},$$

$$\kappa = \nu = \gamma = 0, \quad \rho = \bar{\rho}, \quad \mu = \bar{\mu}, \quad \pi = \alpha + \bar{\beta}$$

$$\epsilon - \bar{\epsilon} = 0 \quad on \quad \mathcal{D}_{u,v} \cap \mathcal{N}_\star.$$



- Cartan structure equations

$$\Delta C^{\mathcal{A}} = -(\bar{\tau} + \pi) P^{\mathcal{A}} - (\tau + \bar{\pi}) \bar{P}^{\mathcal{A}},$$
$$\Delta P^{\mathcal{A}} = -\mu P^{\mathcal{A}} - \bar{\lambda} \bar{P}^{\mathcal{A}},$$
$$D P^{\mathcal{A}} - \delta C^{\mathcal{A}} = (\rho + \epsilon - \bar{\epsilon}) P^{\mathcal{A}} + \sigma \bar{P}^{\mathcal{A}},$$
$$D Q = -(\epsilon + \bar{\epsilon}) Q,$$
$$\bar{\delta} P^{\mathcal{A}} - \delta \bar{P}^{\mathcal{A}} = (\alpha - \bar{\beta}) P^{\mathcal{A}} - (\bar{\alpha} - \beta) \bar{P}^{\mathcal{A}},$$
$$\delta Q = (\tau - \bar{\pi}) Q.$$

$$\{\not{p}, \not{p}', \not{\partial}, \not{\partial}'\}\Gamma = \Gamma\Gamma + \Psi_i + \Phi_{jk},$$

- Scalar equations

$$\varphi_0 \equiv l^a \nabla_a \varphi, \quad \varphi_2 \equiv n^a \nabla_a \varphi, \quad \varphi_1 \equiv m^a \nabla_a \varphi, \quad \bar{\varphi}_1 \equiv \bar{m}^a \nabla_a \varphi,$$

$$\begin{aligned}
\text{þ}'\varphi_0 &= \mathrm{i}\mathfrak{e}\left(\varphi_2 A_0 + \varphi_0 A_2 - \bar{\varphi}_1 A_1 - \varphi_1 \bar{A}_1\right) \\
&\quad + \mathfrak{e}^2 A_0 A_2 \varphi - \mathfrak{e}^2 A_1 \bar{A}_1 \varphi + \varphi_0 \underline{\omega} - \varphi_0 \bar{\mu} + \varphi_2 \rho - \bar{\varphi}_1 \tau - \varphi_1 \bar{\tau} + \eth' \varphi_1, \\
\text{þ}\varphi_1 &= -\varphi_0 \vartheta - \varphi_2 \kappa + \varphi_0 \bar{\pi} + \varphi_1 \bar{\rho} + \bar{\varphi}_1 \sigma + \eth \varphi_0, \\
\text{þ}'\varphi_1 &= \varphi_2 \vartheta - \bar{\varphi}_1 \bar{\lambda} - \varphi_1 \mu + \varphi_0 \bar{\nu} - \varphi_2 \tau + \eth \varphi_2, \\
\text{þ}\varphi_2 &= \mathrm{i}\mathfrak{e}\left(\varphi_2 A_0 + \varphi_0 A_2 - \bar{\varphi}_1 A_1 - \varphi_1 \bar{A}_1\right) \\
&\quad + \mathfrak{e}^2 A_0 A_2 \varphi - \mathfrak{e}^2 A_1 \bar{A}_1 \varphi - \varphi_0 \mu + \varphi_1 \pi + \bar{\varphi}_1 \bar{\pi} + \varphi_2 \bar{\rho} - \varphi_2 \omega + \eth \bar{\varphi}_1.
\end{aligned}$$

- Lorentz Gauge $\quad A_0 \equiv l^a A_a, \quad A_2 \equiv n^a A_a, \quad A_1 \equiv m^a A_a.$

$$\begin{aligned}
\text{þ}' A_0 &= -\Phi_1 + A_0 \underline{\omega} - A_0 \bar{\mu} + A_2 \rho - \bar{A}_1 \tau - A_1 \bar{\tau} + \eth' A_1, \\
\text{þ} A_1 &= \Phi_0 - A_0 \vartheta - A_2 \kappa + A_0 \bar{\pi} + A_1 \bar{\rho} + \bar{A}_1 \sigma + \eth A_0, \\
\text{þ}' A_1 &= -\bar{\Phi}_2 + A_2 \vartheta - \bar{A}_1 \bar{\lambda} - A_1 \mu + A_0 \bar{\nu} - A_2 \tau + \eth A_2, \\
\text{þ} A_2 &= \bar{\Phi}_1 - A_0 \bar{\mu} + A_1 \pi + \bar{A}_1 \pi + A_2 \rho - A_2 \omega + \eth' A_1.
\end{aligned}$$

- Maxwell equations

$$\nabla_a F^{ab} = \mathrm{i}\mathfrak{e}\left(\varphi^* D^b\varphi - \varphi(D^b\varphi)^*\right)$$

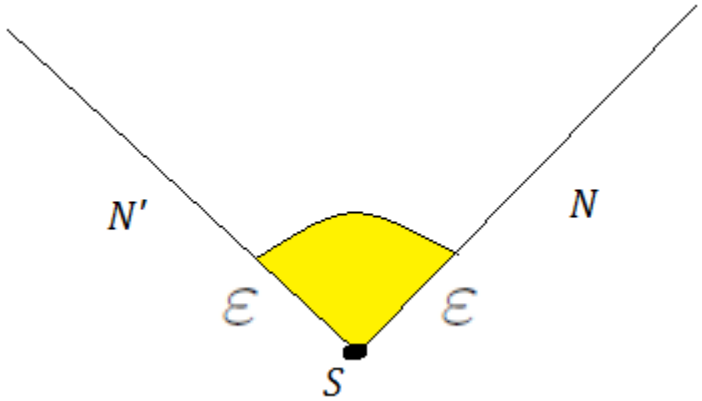
$$\begin{aligned}
\text{Þ}'\Phi_0 &= \frac{\mathrm{i}\mathfrak{e}}{2}\left(-\bar{\varphi}_1\varphi + \varphi_1\varphi^*\right) - \left(\mathfrak{e}^2 A_1\varphi\varphi^* - \Phi_0\underline{\omega} + \Phi_0\mu - \Phi_2\sigma + 2\Phi_1\tau - \eth\Phi_1\right), \\
\text{Þ}\Phi_1 &= \frac{\mathrm{i}\mathfrak{e}}{2}\left(\varphi_0^*\varphi - \varphi_0\varphi^*\right) + \left(\mathfrak{e}^2 A_0\varphi\varphi^* - \Phi_0\vartheta - \Phi_2\kappa + \Phi_0\pi + 2\Phi_1\rho + \eth'\Phi_0\right), \\
\text{Þ}'\Phi_1 &= \frac{\mathrm{i}\mathfrak{e}}{2}\left(-\varphi_2^*\varphi + \varphi_2\varphi^*\right) - \left(\mathfrak{e}^2 A_2\varphi\varphi^* - \Phi_2\vartheta + 2\Phi_1\mu - \Phi_0\nu + \Phi_2\tau - \eth\Phi_2\right), \\
\text{Þ}\Phi_2 &= \frac{\mathrm{i}\mathfrak{e}}{2}\left(\varphi_1\varphi - \bar{\varphi}_1\varphi^*\right) + \left(\mathfrak{e}^2 \bar{A}_1\varphi\varphi^* - \Phi_0\lambda + 2\Phi_1\pi + \Phi_2\rho - \Phi_2\omega + \eth'\Phi_1\right).
\end{aligned}$$

- Bianchi Identities

$$\{\text{Þ}, \text{Þ}', \eth, \eth'\}\Psi_i + \{\text{Þ}, \text{Þ}', \eth, \eth'\}\Phi_{jk} = \Gamma\Psi_i + \Gamma\Phi_{jk}$$

- Local existence of CIVP

1. Analytic case： H. Friedrich, 1980, Proc. Roy. Soc. Lond. A
2. General case： A. Rendell, 1990, Proc. Roy. Soc. Lond. A

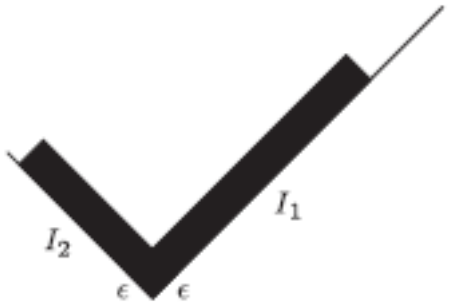


Free data：

$$\Psi_0,\quad \Phi_0,\quad A_0,\quad \varphi_0,\quad \epsilon+\bar{\epsilon}\quad on\quad \mathcal{N}_\star,$$
$$\Psi_4,\quad \Phi_2,\quad A_2,\quad \varphi_2\ on\ \mathcal{N}'_\star,$$
$$\lambda,\quad \sigma,\quad \mu,\quad \rho,\quad \pi,\quad \varphi,\quad A_1,\quad \Phi_1,\quad P^{\mathcal{A}}\quad on\quad \mathcal{S}_\star.$$

- Semi-Local existence of CIVP：

1. D. Christodoulou, Formation of Black hole, 2009
2. J. Luk，IMRN，2011



3. J. b. Li, X. p. Zhu, JDG, 2014
4. P. Zhao, D. Hilditch, J. A. Valiente-Kroon, GRG, 2021

- Semi-Local existence of CIVP of Einstein-Maxwell-Scalar (EMS) system: (P. Zhao, X. Wu, Science China : Mathematics, 2026)

Theorem 3.1 (***Improved local existence for the CIVP for the EMS system***). *Given regular initial data for the Einstein-Maxwell-complex scalar system* (1.1) *as constructed in Lemma 2.5 on the null hypersurfaces* $\mathcal{N}_\star \cup \mathcal{N}'_\star$ *for* $\{0 \leqslant v \leqslant I\}$, *there exists* $\varepsilon > 0$ *such that a unique smooth solution to the Einstein-Maxwell-complex scalar system exists in the region such that* $0 \leqslant v \leqslant I$ *and* $0 \leqslant u \leqslant \varepsilon$. *The number* ε *can be chosen to depend only on the initial data*

$$\Delta_{e_\star}, \quad \Delta_{\Gamma_\star}, \quad \Delta_{\varphi_\star}, \quad \Delta_{A_\star}, \quad \Delta_{\Phi_\star}, \quad \Delta_{\Psi_\star}.$$

Moreover, in this spacetime, the following holds:

$$\Delta_{\Gamma(\mathcal{S})} + \Delta_\varphi + \Delta_A + \Delta_\Phi + \Delta_\Psi \leqslant C(\Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\varphi_\star}, \Delta_{A_\star}, \Delta_{\Phi_\star}, \Delta_{\Psi_\star}).$$

$$\begin{aligned}\Delta_{\Psi_\star} \equiv & \sup_{\mathcal{S}\subset\mathcal{N}_\star\cup\mathcal{N}'_\star} \sup_{\Psi\in\{\Psi_0,\dots\Psi_4\}} \max\{1, \sum_{i=0}^{1} \|\mathcal{D}^i\Psi\|_{L^4(\mathcal{S})}, \sum_{i=0}^{2} \|\mathcal{D}^i\Psi\|_{L^2(\mathcal{S})}\} \\ & + \sum_{i=0}^{3} \left(\sup_{\Psi\in\{\Psi_0,\dots,\Psi_3\}} \|\mathcal{D}^i\Psi\|_{L^2(\mathcal{N}_\star)} + \sup_{\Psi\in\{\Psi_1,\dots,\Psi_4\}} \|\mathcal{D}^i\Psi\|_{L^2(\mathcal{N}'_\star)} \right).\end{aligned}$$

- Semi-Local existence of CIVP of Einstein-Dirac system: (P. Zhao, X. Wu, CQG, 2025）

Lemma 1 (***freely specifiable data for the CIVP***). *Working under the coordinate choice 3.1, initial data for the Einstein-Dirac system on* $\mathcal{N}_\star \cup \mathcal{N}'_\star$ *can be computed (near* $\mathcal{S}_\star$*) from a reduced data set* $\mathbf{r}_\star$ *consisting of:*

$$\begin{aligned} &\Psi_0, \quad \phi_0, \quad \chi_0, \quad \epsilon + \bar{\epsilon} \quad on \quad \mathcal{N}_\star, \\ &\Psi_4, \quad \phi_1, \quad \chi_1, \ on \ \mathcal{N}'_\star, \\ &\lambda, \quad \sigma, \quad \mu, \quad \rho, \quad \pi, \quad P^{\mathcal{A}} \quad on \quad \mathcal{S}_\star. \end{aligned}$$

$$\phi_0 \equiv \phi_A o^A, \quad \phi_1 \equiv \phi_A \iota^A, \quad \chi_0 \equiv \chi_A o^A, \quad \chi_1 \equiv \chi_A \iota^A.$$

Theorem 2 (***Improved local existence for the CIVP for the Einstein-Dirac system***). *Given regular initial data for the Einstein-Dirac system as constructed in Lemma 1 on the null hypersurfaces $\mathcal{N}_\star \cup \mathcal{N}'_\star$ for $\{0 \leq v \leq I\}$, there exists $\varepsilon > 0$ such that a unique smooth solution to the Einstein-Dirac system exists in the region where $\{0 \leq v \leq I\}$ and $0 \leq u \leq \varepsilon$ defined by the null coordinates (u, v). The number ε can be chosen to depend only on the initial data*

$$\Delta_{e_\star}, \quad \Delta_{\Gamma_\star}, \quad \Delta_{\psi_\star}, \quad \Delta_{\Upsilon_\star}, \quad \Delta_{\Psi_\star}.$$

Moreover, in this spacetime, the following holds

$$\Delta_{\Gamma(\mathcal{S})} + \Delta_\psi + \Delta_\Upsilon + \Delta_\Psi \leq C(\Delta_{e_\star}, \Delta_{\Gamma_\star}, \Delta_{\psi_\star}, \Delta_{\Upsilon_\star}, \Delta_{\Psi_\star}).$$

- Improvement of semi-local existence of CIVP of Einstein-Dirac system

Theorem 1 (***Existence result***). *Given a positive number $\mathcal{I}$, there exists a sufficiently large $a_0 = a_0(\mathcal{I})$ such that for $a \geq a_0 \geq 0$ and initial data*

$$\mathcal{I}_0 \equiv \sum_{j=0}^{1} \sum_{i=0}^{15} \frac{1}{a^{\frac{1}{2}}} \| b^j (|u_\infty| \mathcal{D})^i (\sigma, \phi_0) \|_{L^2(\mathcal{S}_{u_\infty, v})} \leq \mathcal{I}$$

along the outgoing initial null hypersurface $u = u_\infty$, and Minkowskian initial data along ingoing initial null hypersurface $v = 0$. Then the Einstein-Weyl spinor system admits a unique solution in

$$\mathbb{D} = \{(u, v) | u_\infty \leq u \leq -a/4, \quad 0 \leq v \leq 1\}.$$

- Significance of the semi-local existence：

1. Insure the double null coordinate region is large enough to allow the formation of trapped surface

2. Obtain sufficiently sharp estimates of the NP quantities to solve the Raychaudhuri equation and precisely characterize the formation of trapped surfaces

- Formation of black hole in Einstein-Dirac system

Theorem 2 (***Trapped surface formation***). *Given $\mathcal{I}$, there exists a sufficiently large $a = a(\mathcal{I})$ such that for $a \geq a_0 \geq 0$ an initial data set with*

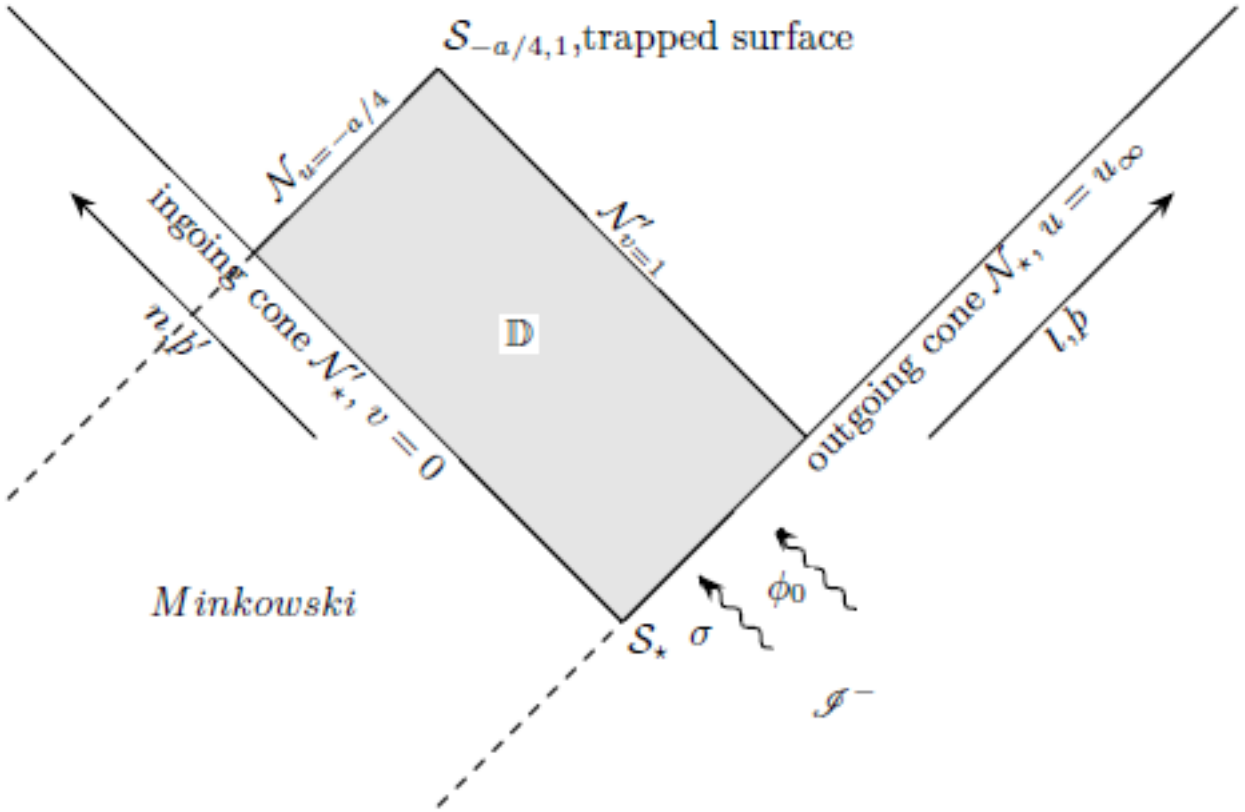
$$\mathcal{I}_0 \equiv \sum_{j=0}^{1} \sum_{i=0}^{15} \frac{1}{a^{\frac{1}{2}}} \| \not{p}^j (|u_\infty| \mathcal{D})^i (\sigma, \phi_0) \|_{L^2(\mathcal{S}_{u_\infty, v})} \leq \mathcal{I}$$

along the outgoing initial null hypersurface $u = u_\infty$ and Minkowskian initial data along the ingoing initial null hypersurface $v = 0$ such that

$$\int_0^1 \left(|u_\infty|^2 (\sigma\bar{\sigma} + 2\mathrm{i}(\phi_0 \not{p} \bar{\phi}_0 - \bar{\phi}_0 \not{p} \phi_0)) | \right) (u_\infty, v') \mathrm{d}v' \geq a$$

holds uniformly for any point on the initial outgoing null hypersurface $u = u_\infty$, it follows that $\mathcal{S}_{-a/4,1}$ is a trapped surface.

- Sketch of proof :



Aim： Solve Raychaudhuri equation, find a two dim space-like surface $S_{(u,v)}$, such that $\rho(S) > 0$ (Note： by definition, $\rho = -\theta_l$)

$$\not{b}\,\rho = 2\,\mathrm{i}\,\bar{\zeta}_0\,\phi_0 - 2\,\mathrm{i}\,\zeta_0\,\bar{\phi}_0 + \rho^2 + \sigma\,\bar{\sigma},$$

1. Estimates on the tetrad

$$|Q| - 1 \lesssim \frac{\mathcal{O}}{|u|}, \quad |P^{\mathcal{A}}(u,v)| \lesssim |P^{\mathcal{A}}(u,0)| + \frac{a^{\frac{1}{2}}}{|u|}\mathcal{O}, \quad |C^{\mathcal{A}}| \lesssim \frac{a^{\frac{1}{2}}}{|u|^2}\mathcal{O}.$$

2. Initial data :

$$\int_0^1 \left(|u_\infty|^2(\sigma\bar{\sigma} + 2\mathrm{i}\bar{\zeta}_0\phi_0 - 2\mathrm{i}\zeta_0\bar{\phi}_0)|\right)(u_\infty, v')\mathrm{d}v' \geq a.$$

3. Solving Raychaudhuri equation

$$\rho(u\ ,v,x^2,x^3)=\rho(u\ ,0,x^2,x^3)+\int_0^1\frac{\partial\rho}{\partial v}\mathrm{d}v=-\frac{1}{|u\ |}+\int_0^v b\rho\mathrm{d}v'$$

$$=-\frac{1}{|u\ |}+\int_0^v\left(\sigma\bar{\sigma}+2\,\mathrm{i}\,\bar{\zeta}_0\,\phi_0-2\,\mathrm{i}\,\zeta_0\,\bar{\phi}_0\right)|_{(\ u\ ,v',x^2,x^3)}\mathrm{d}v'$$

define $F\equiv|u|^2Q^{-1}(\sigma\bar{\sigma}+2\mathrm{i}\bar{\zeta}_0\phi_0-2\mathrm{i}\zeta_0\bar{\phi}_0).$

Combine equations on $b'\sigma,\ b'\phi_0,\ b'\zeta_0\ \ b'Q,$ we have

$$\partial_u F=b'F-C^{\mathcal{A}}\partial_{\mathcal{A}}F\equiv I_1+I_2.$$

Using the estimates from the semi-local existence of Einstein-Dirac system,

$$|I_1| \lesssim \frac{a}{|u|^2}. \qquad |I_2| \leq \frac{a^{\frac{3}{2}}}{|u|^2},$$

The control equation for F is

$$|\partial_u F| \leq \frac{a^{\frac{3}{2}}}{|u|^2} \ll \frac{a^{\frac{7}{4}}}{|u|^2}.$$

Careful choose the value of a,

$$\int_0^1 \left(|u|^2 Q^{-1}(\sigma\bar{\sigma} + 2\mathrm{i}\bar{\zeta}_0\phi_0 - 2\mathrm{i}\zeta_0\bar{\phi}_0)|\right)(u, v', x^2, x^3)\mathrm{d}v'$$

$$\geq \int_0^1 \left(|u_\infty|^2(\sigma\bar{\sigma} + 2\mathrm{i}\bar{\zeta}_0\phi_0 - 2\mathrm{i}\zeta_0\bar{\phi}_0)|\right)(u_\infty, v', x^2, x^3)\mathrm{d}v' - \frac{a^{\frac{7}{4}}}{|u|} \geq a - \frac{a^{\frac{7}{4}}}{|u|} \geq a - \frac{4a^{\frac{7}{4}}}{a} \geq \frac{a}{2}.$$

Especially, at $u = -a/4$,

$$\int_0^1 \left(Q^{-1}(\sigma\bar{\sigma} + 2\mathrm{i}\bar{\zeta}_0\phi_0 - 2\mathrm{i}\zeta_0\bar{\phi}_0)\right)|_{(-\frac{a}{4},v',x^2,x^3)}\mathrm{d}v' \geq \frac{8}{a}.$$

4. Check there is no trapped surface on initial light cone N_*

$$\begin{aligned}\rho(u_\infty, v, x^2, x^3) &= \rho(u_\infty, 0, x^2, x^3) + \int_0^1 \frac{\partial\rho}{\partial v}\mathrm{d}v = -\frac{1}{|u_\infty|} + \int_0^v b\rho\mathrm{d}v' \\ &= -\frac{1}{|u_\infty|} + \int_0^v \left(\sigma\bar{\sigma} + 2\,\mathrm{i}\,\bar{\zeta}_0\,\phi_0 - 2\,\mathrm{i}\,\zeta_0\,\bar{\phi}_0\right)|_{(-\frac{a}{4},v',x^2,x^3)}\mathrm{d}v' \\ &\lesssim -\frac{1}{|u_\infty|} + \frac{a}{|u_\infty|^2} < 0,\end{aligned}$$

5. Formation of trapped surface：

Consider the final light cone $v = 1$, on $S_{(-\frac{a}{4},1)}$

$$\rho(-\frac{a}{4},1,x^2,x^3) = \rho(-\frac{a}{4},0,x^2,x^3) + \int_0^1 \frac{\partial \rho}{\partial v}\mathrm{d}v = -\frac{4}{a} + \int_0^1 Q^{-1} b\rho \mathrm{d}v'$$

$$\geq -\frac{4}{a} + \int_0^1 Q^{-1}(\sigma\bar{\sigma} + 2\mathrm{i}\,\bar{\zeta}_0\,\phi_0 - 2\mathrm{i}\,\zeta_0\,\bar{\phi}_0)|_{(-\frac{a}{4},v',x^2,x^3)}\mathrm{d}v'$$

$$\geq -\frac{4}{a} + \frac{8}{a} = \frac{4}{a} > 0.$$

6. On the shape of $S_{(u,v)}$

Gaussian curvature $K = \mathcal{K} + \bar{\mathcal{K}}$ $\mathcal{K} = \mathrm{i}\bar{\zeta}_1\phi_1 - \mathrm{i}\zeta_1\bar{\phi}_1 + \mathrm{i}\bar{\tilde{\zeta}}_4\phi_0 - \mathrm{i}\tilde{\zeta}_4\bar{\phi}_0 - \Psi_2 + \mu\rho - \lambda\sigma.$

$$||(a^{\frac{1}{2}}\mathcal{D})^k\mathcal{K}||_{L^2_{sc}(\mathcal{S}_{u,v'})} \lesssim ||(a^{\frac{1}{2}}\mathcal{D})^k\Psi_2||_{L^2_{sc}(\mathcal{S}_{u,v'})} + \frac{a^{\frac{1}{2}}}{|u|}||(a^{\frac{1}{2}}\mathcal{D})^k\tilde{\zeta}_4||_{L^2_{sc}(\mathcal{S}_{u,v'})} + 1.$$

- Some recent developments on black hole formation

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4. X. An, Q. Han, Anisotropic dynamical Horizon arising in gravitational collapse, 2020

5. C. Kehle, R. Unger, Formation of extremal R-N black hole, 2022

6. Y. Chen, J. Zhu, S. T. Yau, New solution of Einstein-Yang-Mills equation, 2023

7. X. An, Naked singularity censoring with anisotropic apparent horizon, 2024

8. J. b. Li, Interior instability of naked singularity of scalar field, 2025

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Thank you