Null String Theory

M.M. Sheikh-Jabbari

IPM, Tehran

Based on our recent work in collaboration with
Hossein Yavartanoo,

arXiv:2605.12414 , arXiv: 2605.25817 , arXiv: 2605.26822,
arXiv:2607.02970, arXiv: 2609.04844, arXiv:2609.10432,
arXiv:2610.03495

6th BRICS-AGAC Conference Hangzhou
November 8, 2026

Introduction and Motivation

- Standard textbook tensile string theory is described by a worldsheet theory, a $2d$ field theory.
- String worldsheet theory is a geometric action governing how strings move in a target space.
- The worldsheet theory is
  - a $2d$ σ-model in which worldsheet is a $2d$ Lorentzian or Euclidean space;
  - all worldsheets related by $2d$ diffeomorphisms and Weyl scalings should be identified: $2d$ diffeos and Weyl scaling are gauged.
- Quantum consistency conditions (cancellation of anomalies of gauge symmetries) specify which target spaces strings can move on.

Introduction and Motivation

- While we are more used to discussing Lorentzian or Euclidean geometries, one can also imagine zero signature, Carrollian geometries.
- Null surfaces (like cosmological or black hole horizons, and null infinity of asymptotically flat spaces) are prime examples of Carrollian geometries.
- The question we explore in this talk is

  What is null string theory (i.e. strings with Carrollian worldsheets) ?

- We provide a first principle definition of null string theory (NST) and,
- take first steps in analyzing the theory.

Outline of the talk

- Introduction to Carrollian geometries.
- Weyl and Carroll-Weyl scalings.
- Minimal Null String Theory.
- Basic classical analyses of the minimal NST
- Null String Holography
- Discussions and Outlook.

Carrollian Geometry

- A D dimensional Carrollian manifold $\mathcal{M}$ is described by a degenerate positive semi-definite metric $h_{\mu\nu}$ and a nowhere vanishing kernel vector V^μ, denoted in (h, V) pair,

$$h_{\mu\nu} V^\nu = 0, \qquad \text{sign}(h) = (0, +, \cdots, +).$$

- If V^μ is the kernel, λV^μ (for any non-vanishing function λ) is also a kernel.
- We may deal with ruled Carrollian geometries which are equipped with the Ehresmann connection U_μ, (h, V, U), s.t.,

$$U \cdot V = U_\mu V^\mu = 1$$

- U_μ is also called clock one-form.

Geometric setup

- Using U one can define an inverse metric, $h^{\mu\nu}$, s.t.

$$h^{\mu\nu}U_\nu = 0, \qquad h^{\mu\rho}h_{\rho\nu} = \delta^\mu_\nu - V^\mu U_\nu.$$

- One can also define ruled Carrollian spaces through embedding them in $D+1$ dimensional Lorentzian manifolds.
- Connection and covariant derivatives on Carrollian manifolds can also be defined.
- Connection can be (h, V)-compatible, torsional or non-torsional and/or invariant under kernel flow or not.
- One can define curvature and other geometric invariants using these data.

  See [MMShJ, H. Yavartanoo, arXiv:2610.03495] and references therein.

Carrollian scalings

- Given the Carrollian structure (h, V), one can envisage two scaling transformations:

  Weyl Scaling: $\quad h_{\mu\nu} \to e^{2\psi} h_{\mu\nu}, \qquad V^\mu \to e^{-\psi} V^\mu$

  under which the volume form of the manifold scales by $e^{D\psi}$.

  Carroll-Weyl (CW) Scaling: $\quad h_{\mu\nu} \to e^{\frac{\chi}{2(D-1)}} h_{\mu\nu}, \quad V^\mu \to e^{\chi} V^\mu$

  under which the volume form of the manifold does not scale.

- For the ruled cases,

  Weyl scaling: $\quad U \to e^{\psi} U, \qquad$ CW scaling: $\quad U \to e^{-\chi} U$

Carrollian worldsheets

- We parametrize the $2d$ worldsheet with $\sigma^a = (\tau, \sigma)$.
- The $2d$ Carrollian worldsheet is hence described by (ℓ_a, v^a), s.t.

$$h_{ab} = \ell_a \ell_b, \qquad \ell_a v^a = 0$$

- We may add clock form n_a and vector ℓ^a s.t.

$$n_a v^a = 1, \quad n_a \ell^a = 0, \quad \ell_a \ell^a = 1, \quad \ell_a v^a = 0, \quad \ell^a \ell_b + v^a n_b = \delta^a_b$$

- The volume-form of the worldsheet will be denoted by

$$\epsilon_{ab} = \ell_a n_b - \ell_b n_a, \qquad \epsilon^{ab} = \ell^a v^b - \ell^b v^a$$

s.t. $\epsilon_{ab}\epsilon^{bc} = -\delta^c_a$

Carrollian worldsheets, the two scalings

- Weyl Scaling:

$$\ell_a \to e^{\psi}\ell_a, \qquad v^a \to e^{-\psi}v^a$$

under which $\ell^a \to e^{-\psi}\ell^a, \quad n_a \to e^{\psi}n_a, \quad \epsilon_{ab} \to e^{2\psi}\epsilon_{ab}.$

- Carroll-Weyl (CW) Scaling:

$$\ell_a \to e^{\chi}\ell_a, \quad v^a \to e^{\chi}v^a$$

$$\ell^a \to e^{-\chi}\ell^a, \quad n_a \to e^{-\chi}n_a, \quad \epsilon_{ab} \to \epsilon_{ab}.$$

- Weyl invariant kernel $\mathcal{V}^a = \sqrt[4]{h}\, v^a$, where $h_{ab} = \text{diag}(0\,, h)$, s.t.

Weyl Scaling: $\mathcal{V}^a \to \mathcal{V}^a$, CW Scaling: $\mathcal{V}^a \to e^{\chi}\mathcal{V}^a$

Note: $\mathcal{V}^a$ is similar to $\sqrt{-\gamma}\gamma^{ab}$ for standard worldsheet metric γ_{ab}, which is Weyl scaling invariant.

Carrollian worldsheets

- Under diffeomorphisms, $\sigma^a \to \sigma^a + \xi^a(\sigma^b)$,

$$\delta_\xi \mathcal{V}^a = \xi^b \partial_b \mathcal{V}^a - \mathcal{V}^b \partial_b \xi^a + \frac{1}{2}\mathcal{V}^a(\partial_b \xi^b)$$

- The embedding string coordinates $X^\mu(\sigma)$ are invariant under Weyl Scaling, just like $\mathcal{V}^a$, while under $2d$ diffeomorphisms,

$$\delta_\xi X^\mu = \xi^a \partial_a X^\mu$$

- So, if we use $\mathcal{V}^a, X^\mu, \partial_a X^\mu$ in constructing the action, it will be automatically Weyl invariant.

- We have $\mathcal{V}^a$ which should be naturally summed ∂_a. While $\mathcal{V}^a \partial_a$ is Weyl+diff. invariant, it is not CW invariant.

Minimal Null String Theory

- To make a 2nd derivative action CW invariant, one must consider:

$$X^\mu \to e^{-\chi} X^\mu, \qquad \mathcal{V}^a \to e^{\chi} \mathcal{V}^a$$

  while replacing the derivative with the CW covariant derivative:

$$D_a X^\mu = \partial_a X^\mu + \mathcal{W}_a X^\mu, \qquad \delta_\chi \mathcal{W}_a = \partial_a \chi$$

- With the above

$$\delta_\chi (D_a X^\mu) = -\chi (D_a X^\mu), \qquad \delta_\chi (\mathcal{V}^a D_a X^\mu) = 0$$

- Minimal null string (MNS) action on flat Minkowski background is

$$S_{\rm MNS}[\eta; \mathcal{V}] = \frac{\kappa}{2} \int_\Sigma \mathrm{d}^2\sigma \, \mathcal{V}^a \mathcal{V}^b \eta_{\mu\nu} D_a X^\mu D_b X^\nu$$

Minimal NST, basic analysis

- The CW gauge field $\mathcal{W}_a$ is non-dynamical and it appears only in $\mathcal{V}^a\mathcal{W}_a$ combination.
- One can gauge-fix the CW gauge field by adapting,

$$\mathcal{V}^a\mathcal{W}_a = 0$$

- This gauge-fixing should be supplemented with the associated Gauss-law, EoM of $\mathcal{W}_a$:

$$\frac{\delta S}{\delta \mathcal{W}_a} = \mathcal{V}^a C_3 = 0, \qquad C_3 = X^\mu \mathcal{V}^a D_a X_\mu.$$

- The gauge-fixed action is the diff+Weyl invariant ILST theory:

$$S_{\rm ILST}[\eta; \mathcal{V}] = \frac{\kappa}{2}\int_\Sigma \mathrm{d}^2\sigma\, \mathcal{V}^a\mathcal{V}^b\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$$

which should be supplemented with CW Gauss-law $C_3 = 0$.

Minimal NST, basic analysis

- So, ILST theory is gauge-fixed minimal NST.
- The ILST theory is invariant under the residual part of CW gauge symmetry, which is generated by χ subject to,

$$\mathcal{V}^a \partial_a \chi = 0$$

- C_3 is the Noether charge associated with the residual CW gauge symmetries.
- $\mathcal{V}^a$ of the gauge-fixed NST (ILST theory) is nondynamical.
- Its EoM is hence a constraint:

$$C_a := \frac{\delta S}{\delta \mathcal{V}^a} = \partial_a X_\mu \mathcal{V}^b \partial_b X^\mu = 0,$$

Minimal NST, basic analysis

- One may fix the diff+Weyl in the temporal gauge:

$$\mathcal{V}^a = (1, 0)$$

- This gauge-fixing leaves us with residual gauge transformations whose Noether charges are C_a.
- Thus, the fully gauge-fixed NST is

$$S_{\text{gauge-fixed}} = \frac{\kappa}{2} \int_\Sigma \mathrm{d}^2\sigma\, \eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu, \qquad \dot{X}^\mu := \partial_\tau X^\mu := P^\mu.$$

- The above action must be supplemented with three constraints:

$$C_1 = P \cdot P = 0, \qquad C_2 = P \cdot X' = 0, \qquad C_3 = P \cdot X = 0$$

Minimal NST, Constraint algebra

- One can compute canonical Poisson bracket of the constraints.
- If we denote Fourier modes of $C_1(\sigma), C_2(\sigma), C_3(\sigma)$ respectively by M_n, L_n, S_n, the nonzero brackets are

$$[L_m, L_n] = (m-n)L_{m+n}$$

$$[L_m, M_n] = (m-n)M_{m+n}$$

$$[L_m, S_n] = -nS_{m+n}$$

$$[S_m, M_n] = 2M_{m+n}$$

- The above is BMS_3 algebra extended by a vector field (S_n).
- BMS_3 algebra is coming from $C_1(\sigma), C_2(\sigma)$ and the extension from $C_3(\sigma)$ associated with CW gauge symmetry.
- This gauge-fixing leaves us with residual gauge transformations whose Noether charges are C_a.

Summary of minimal NST, so far

- EoM of NST on D dim. flat space

$$\ddot{X}^\mu = 0 \qquad \Longrightarrow \qquad X^\mu = P^\mu(\sigma)\tau + Q^\mu(\sigma)$$

- P^μ, Q^μ are subject to three constraints

$$C_1 = P \cdot P = 0, \qquad C_2 = P \cdot Q' = 0, \qquad C_3 = P \cdot Q = 0$$

- Physical P^μ, Q^μ are defined up to residual gauge transf.

$$\delta_\eta X^\mu = \big(h + (f' - 2\chi)\tau\big)\dot{X}^\mu + fX'^\mu + \chi X^\mu$$

  where h, f, χ are arbitrary functions of σ.
- $2D$ dim. phase space is subject to $3+3$ constraints.
- The unconstrained physical phase space is $2(D-3)$ dim.

Fixing the lightcone gauge (LCG)

- To explicitly construct the physical phase space and solve and impose the constraints, one can fix the LCG:

$$X^+ = p^+\tau, \qquad Q^+ = 0, \quad P^+ = p^+$$

  The above fixes h, χ residual gauge transf.

- Constraints C_1, C_3 may be solved as

$$P^- = \frac{P_i^2}{2p^+} \qquad Q^- = \frac{P_i Q_i}{p^+}$$

- Separate Q_i, P_i into $(Q_1, Q^I), (P_1, P^I)$, $I = 2, \cdots, D-2$.
- C_2 and f can be used to determine P_1, Q_1:

$$Q_1 = q_1, \qquad P_1 = -\frac{1}{q_1}\int^\sigma P'_I Q_I + p_1$$

Minimal NST in the LCG

- So, the physical phase space is described by P_I, Q^I, $I = 2, \cdots, D-2$ and three parameters, p^+, q_1, p_1, subject to the level matching condition

$$\int_0^{2\pi} P_I' Q_I = 0$$

- Lightcone Hamiltonian in terms of the physical parameters is,

$$H_{\rm LC} := \frac{1}{2\pi}\int_0^{2\pi} d\sigma\, P^-(\sigma) = \frac{1}{2p^+}\left[(p_1)^2 + p_I p_I + \mathcal{M}^2\right].$$

$$\mathcal{M}^2 = \mathcal{G}_{IJ}\,\mathbb{P}^{IJ} + \frac{1}{(q_1)^2}\left(2q^I \mathbb{\Phi}_I + \mathbb{\Phi}\right)$$

Minimal NST in the LCG

- p_I, q^I are zero modes of physical modes and Q^I_n, P^I_n their non-zero Fourier modes, and

$$\mathcal{G}^{IJ} := \delta^{IJ} + \frac{q^I q^J}{(q_1)^2}, \quad \mathbb{P}^{IJ} := \sum_{n\neq 0} P^I_{-n} P^J_n, \quad \mathbb{P}\Phi_I := \sum_{n\neq 0} P^I_{-n}\Phi_n,$$

$$\Phi\Phi := \sum_{n\neq 0} \Phi_{-n}\Phi_n, \qquad \Phi_n := \sum_{r\neq 0,n} \frac{r}{n} P^I_r Q^I_{n-r}, \qquad n \neq 0\,.$$

- $H_{\rm LC}$ contains non-local interaction terms, induced from the constraints, in particular the C_3, despite having free tensionless strings.
- $\mathbb{P}\Phi_I$ and $\Phi\Phi$ are cubic and quartic terms in powers of Q^I, P_I. They yield a nontrivial spectrum for the null strings.
- LCG fixing is a consistent one (in the Hamiltonian analysis sense). [arXiv:2605.26822 & our paper today]

Physics Discussion

- A null string is congruence of null geodesics in target space, subject to C_2, C_3 constraints and subject to residual gauge identifications, [arXiv:2609.04844]
- Despite the fact that it is a congruence of null geodesics, the null string center of mass is NOT moving on a null geodesic.
- The CW scaling of the worldsheet is accompanied by target space scaling $X^\mu \sim e^\chi X^\mu$.
- This induces effective "interactions" among different points in the congruence.
- One may rescale P, Q by $q_1, 1/q_1$ respectively to bring q_1 dependence in $H_{\rm LC}$ as an overall factor.
- While null string does not have a "fundamental scale/tension", q_1 appears as an effective scale for the null string.

Outlook

- We have reloaded ILST theory, into a true string theory, Null String Theory (NST).
- We reviewed basics of classical minimal NST on flat background.
- Established that null strings probe $D-3$ dimensions in D dimensional target space.
- There are much more to be done and explored. Essentially, all textbook material for tensile strings should be explored.
- Here we discuss some of these.....

Outlook

Null strings on non-flat backgrounds [arXiv:2609.10432.]

- Consider a manifold $\mathcal{M}$ with metric $G_{\mu\nu}$. The action,

$$S_{\rm MNS}[G;\mathcal{V}] = \frac{\kappa}{2}\int_{\Sigma} \mathrm{d}^2\sigma\, \mathcal{V}^a\mathcal{V}^b G_{\mu\nu} D_a X^\mu D_b X^\nu$$

with

$$D_a X^\mu = \partial_a X^\mu + \mathcal{W}_a \rho^\mu$$

describes a null string theory, iff the background is homothetic:

$$\nabla_\mu \rho_\nu + \nabla_\nu \rho_\mu = 2G_{\mu\nu}$$

- Here, the CW scaling acts as

$$\delta_\chi \mathcal{V}^a = -\chi \mathcal{V}^a, \qquad \delta_\chi X^\mu = \chi \rho^\mu, \qquad \delta_\chi \mathcal{W}_a = -\partial_a \chi$$

Outlook

Null strings on non-flat backgrounds [arXiv:2609.10432.]

- For this case we again have three constraints:

$$C_1 = P \cdot P = 0, \qquad C_2 = P \cdot X' = 0, \qquad P \cdot \rho = 0,$$

where $X \cdot Y = G_{\mu\nu} X^\mu Y^\nu$ and X^μ satisfy null geodesic eq. for background metric.

- Recalling that ρ^μ is a homothetic Killing vector,

$$\rho^\mu \partial_\mu := \partial_s, \qquad \rho \cdot \rho = e^{2s}$$

$$h_{\mu\nu} = e^{-2s} G_{\mu\nu}, \qquad \partial_s h_{\mu\nu} = 0$$

- One can rewrite the action in terms of CW invariant quantities:

$$\mathcal{U}^a = e^s \, \mathcal{V}^a, \qquad X^\mu = (s; Y^i)$$

Outlook

Null Strings Holography [arXiv:2609.10432.]

- Note that

$$\delta_\chi s = \chi, \qquad \delta_\chi Y^i = 0, \qquad \delta_\chi \mathcal{U}^a = 0$$

- Minimal NST on $\mathcal{M}$ takes the form

$$S_{\text{ILST}}[h;\mathcal{U}] = \frac{\kappa}{2}\int_\Sigma \mathrm{d}^2\sigma\, \mathcal{U}^a\mathcal{U}^b\mathcal{H}_{ij}\,\partial_a Y^i \partial_b Y^j$$

where

$$\mathcal{H}_{ij} = h_{ij} - \frac{h_{is}h_{sj}}{h_{ss}}, \qquad h_{ss} \neq 0$$

($h_{ss} = 0$ means ρ^μ is null and the target space is Carrollian)

- $\mathcal{H}_{ij}$ is the metric on the projective boundary of $\mathcal{M}$.

Outlook

Examples of Null Strings Holography [arXiv:2609.10432.]

- Null string holography, like AdS/CFT, works patchwise.
- However, now instead of causal boundary we deal with projective boundary.
- Projective boundary of $\text{Mink}_D = \mathcal{J}^{\pm} \cup dS_{D-1} \cup H^{D-1}$.
- Projective boundary of AdS_D in Poincare patch is its causal boundary Mink_{D-1}.
- Projective boundary of dS_D in cosmological patch is Euclidean constant time slices.

Outlook

Null Strings on Carrollian target spaces [arXiv:2610.03495]

- The above Null String Holography is true when $\mathcal{M}$ is a non-degenerate manifold, with non-zero signature.
- One should reanalyze the case when $\mathcal{M}$ is a Carrollian manifold $(h_{\mu\nu}, V^{\mu})$.
- Carrollian manifolds enjoy a CW scaling.
- If $h_{\mu\nu}$ has a homothetic Killing vector, one can define (minimal) NST on $(h_{\mu\nu}, V^{\mu})$.
- One can show that ILST theory and NST theories are classically equivalent on a Carrollian target space.

Outlook

Quantized Null strings [arXiv:2607.02970 & our paper today].

- There are various ways to quantize a gauged system, e.g.
  - Ghosts and BRST [papers by Bin Chen et al & Duary & Maji papers]
  - Canonical quantization in the LCG.
  - Schrödinger quantization in the LCG.
- Carrying out the latter for free bosonic null strings on flat space, we have found that in regular Hilbert space sectors
  - there is no critical dimensions
  - spectrum has no tachyons and is discrete

Outlook

- **Future directions.**
  - Going away from minimality
  - Integrability of the classical and quantum null strings
  - Interacting null strings
  - Strings probing BH or cosmological horizons

and MANY MORE .....

Thank you for your attention.