Conserved mass and thermodynamics of rotating black holes in Bertotti–Robinson universe

胡立

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Hu, Cai, Wang 2026 ***PRD Letter***

Hu, Cai, Jiang, Wang *2611.03123*

Observational evidence

Faraday rotation

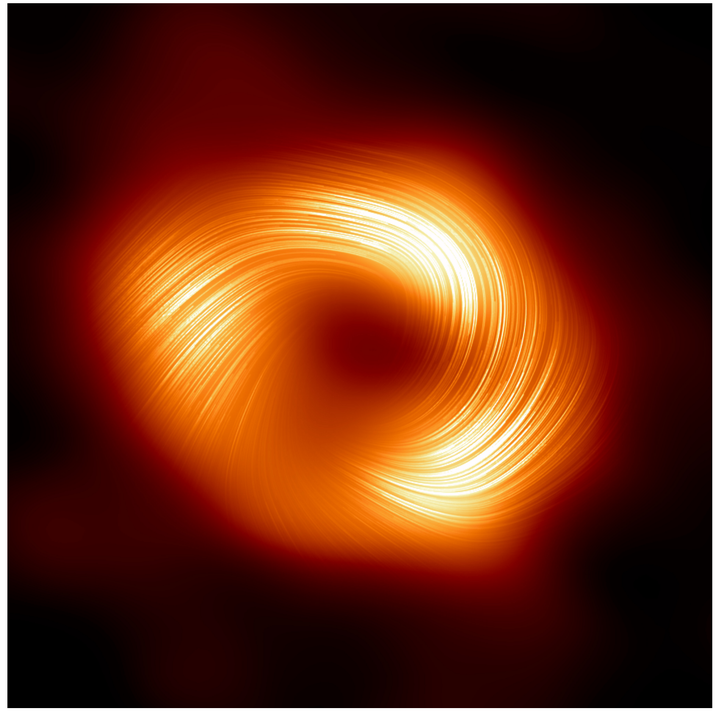
A newly discovered pulsar close to Sgr A*

With unusually large Faraday rotation

Indicates a magnetic field near the SMBH

Eatough et al. 2013 ***Nature***

Horizon-scale polarimetry

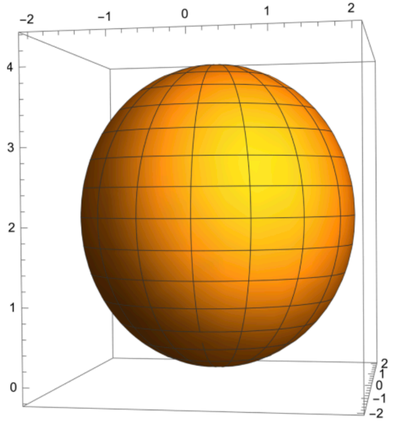


EHT Collaboration 2024 ***APJL***

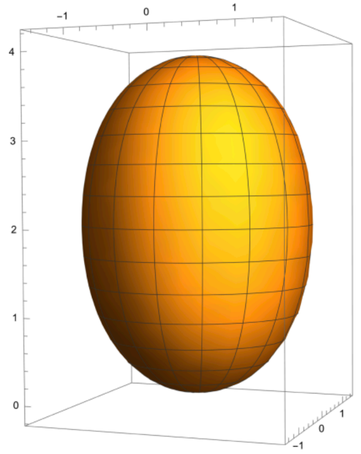
Exact solutions

Schwarzschild-BR-BM

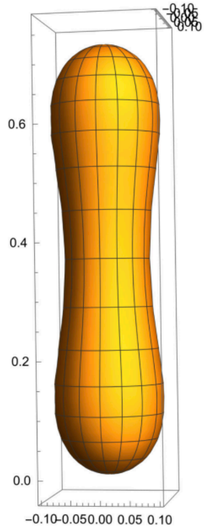
Astorino 2025 ***PRD***



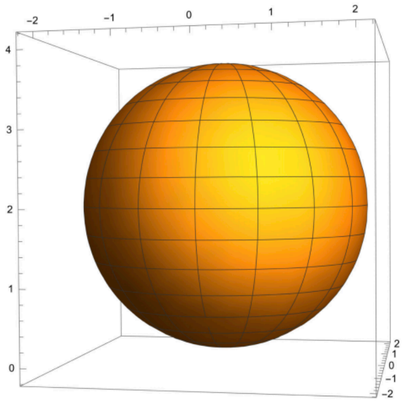
$m = 1,\ B = 0.1,\ b = 0.1$



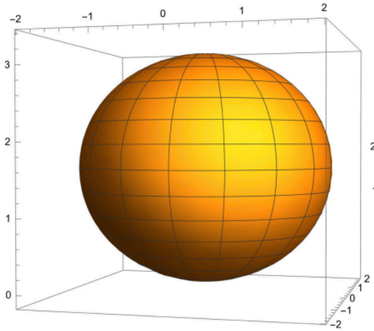
$m = 1,\ B = 0.4,\ b = 0.1$



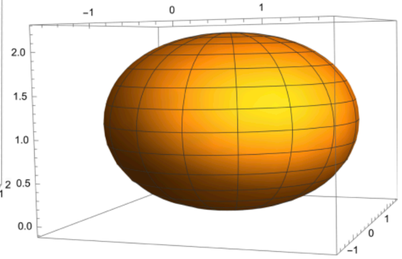
$m = 1,\ B = 2,\ b = 0.1$



$m = 1,\ B = 0,\ b = 0$



$m = 1,\ B = 0.3,\ b = -0.4$



$m = 1,\ B = 0.5,\ b = -0.6$

Magnetic Universe

Bonnor-Melvin

1954, 1964

BM black hole

Ernst 1976 ***JMP***

Bertotti-Robinson

1959

Kerr-BR black hole

2025 ***PRL***

KN-BR black hole

Ovcharenko, Podolsky 2026 ***PRL***

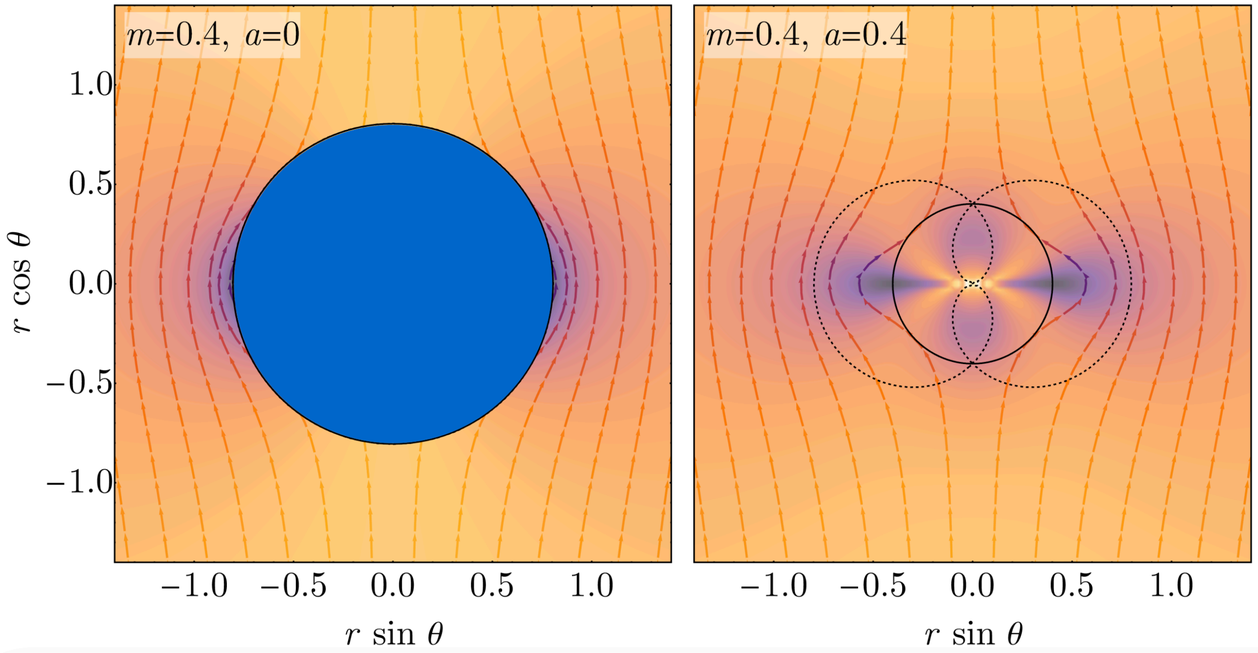
Kerr-BR black hole

Line element & gauge potential

Podolsky & Ovcharenko 2025 ***PRL***

$$\mathrm{d}s^2 = \frac{1}{\Sigma^2}\left[-\frac{\beta}{\rho^2}\left(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi\right)^2 + \frac{\rho^2}{\beta}\mathrm{d}r^2 + \frac{\rho^2}{P}\mathrm{d}\theta^2 + \frac{P}{\rho^2}\sin^2\theta\left(a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi\right)^2\right]$$

$$A_\mu \mathrm{d}x^\mu = \frac{\mathrm{e}^{\mathrm{i}\gamma}}{2B}\left[\Sigma_{,r}\frac{a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi}{r + ia\cos\theta} + (\Sigma - 1)\mathrm{d}\phi + \frac{i\Sigma_{,\theta}}{\sin\theta}\frac{\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi}{r + ia\cos\theta}\right]$$



$$\rho^2 = r^2 + a^2\cos^2\theta\,,$$

$$P = 1 + B^2\left(m^2\frac{I_2}{I_1^2} - a^2\right)\cos^2\theta\,,$$

$$\beta = \left(1 + B^2 r^2\right)\Delta\,,$$

$$\Sigma^2 = \left(1 + B^2 r^2\right) - B^2\Delta\cos^2\theta\,,$$

$$\Delta = \left(1 - B^2 m^2\frac{I_2}{I_1^2}\right)r^2 - 2m\frac{I_2}{I_1}r + a^2\,,$$

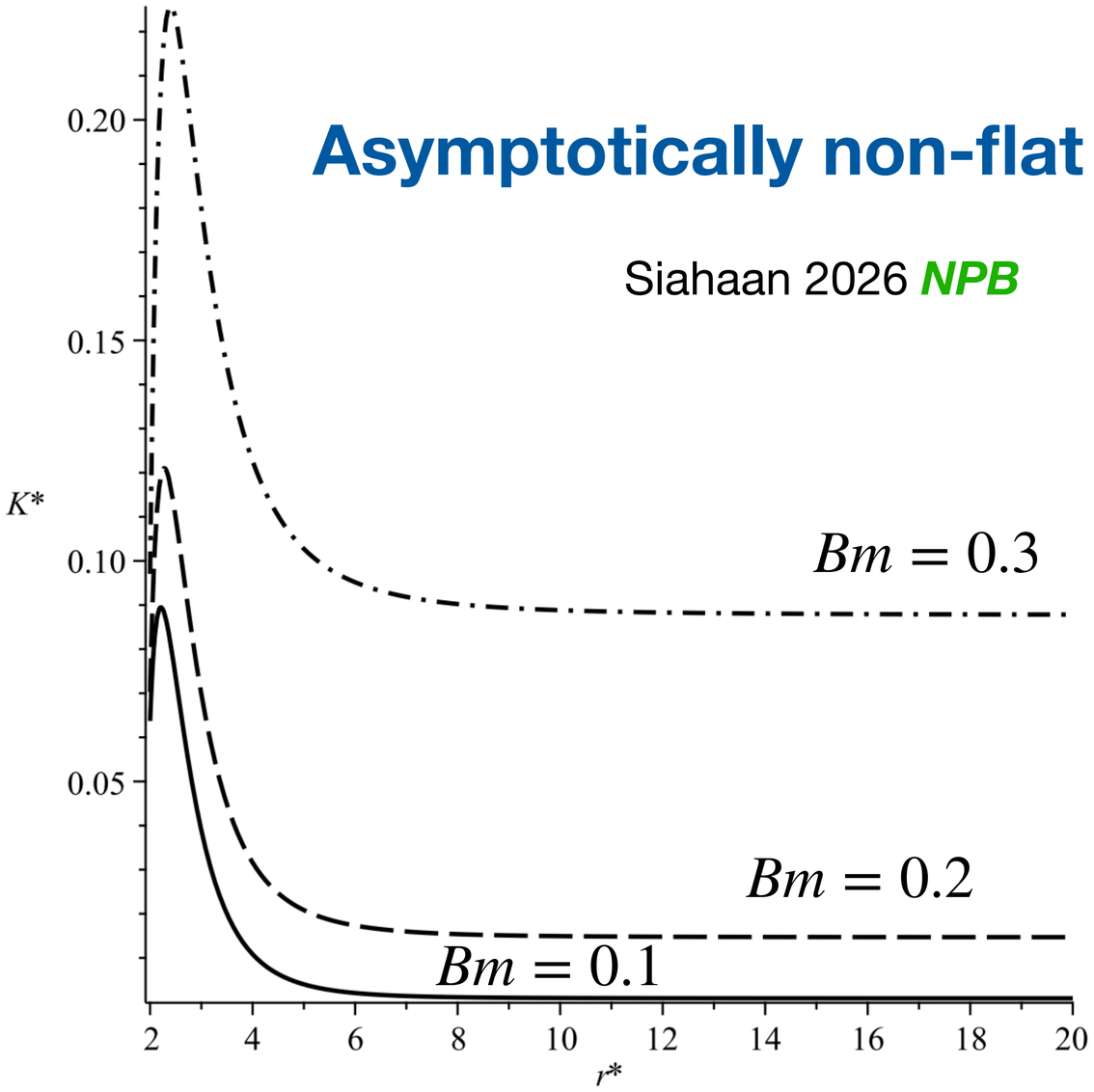
$$I_1 = 1 - \frac{1}{2}B^2a^2\,,$$

$$I_2 = 1 - B^2a^2\,.$$

Kerr-BR black hole

Asymptotically non-flat

Siahaan 2026 ***NPB***



Petrov type-D

	Ψ_0	Ψ_1	Ψ_2	Ψ_3	Ψ_4
Type I	0	$\neq 0$	$\neq 0$	1	0
Type II	0	0	$\neq 0$	1	0
Type III	0	0	0	1	0
Type D	0	0	$\neq 0$	0	0
Type N	0	0	0	0	1

Kerr-BR black hole

Komar mass

$$M_K = -\frac{1}{8\pi}\int_S \varepsilon_{abcd}\nabla^c \xi^d$$

stationary

ADM mass

$$E = \frac{1}{16\pi}\lim_{r\to\infty}\int_S (\partial_j h_{ij} - \partial_i h_{jj})\, N^i \mathrm{d}S$$

asymptotically flat

Kerr-BR black hole

Einstein-Maxwell action

$$S = \frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\,(R - F_{\mu\nu}F^{\mu\nu})$$

Killing equations

$$\mathscr{L}_\xi g_{\mu\nu} = 0$$
$$\mathscr{L}_\xi A_\mu + \partial_\mu \lambda = 0$$

Conserved charge

Barnich & Brandt 2002 ***NPB***

$$\mathscr{Q}_{(\xi,\lambda)}[g, A; \bar{g}, \bar{A}] = \int_{\partial C}\int_{\bar{g}}^{g}\int_{\bar{A}}^{A} k_{(\xi,\lambda)}[\delta g', \delta A'; g', A']$$

Kerr-BR black hole

Surface charge

$$k_{(\xi,\lambda)}[\delta g, \delta A; g, A] = k^{grav}_{\xi} + k^{F}_{(\xi,\lambda)}$$

$$k^{grav}_{\xi} = \frac{1}{8\pi}(\mathrm{d}^2x)_{\mu\nu}\left\{ \xi^{\nu}\nabla^{\mu}h - \xi^{\nu}\nabla_{\rho}h^{\mu\rho} + \xi_{\rho}\nabla^{\nu}h^{\mu\rho} + \frac{1}{2}h\nabla^{\nu}\xi^{\mu} - h^{\rho\nu}\nabla_{\rho}\xi^{\mu} \right\}$$

$$k^{F}_{(\xi,\lambda)} = \frac{1}{4\pi}(\mathrm{d}^2x)_{\mu\nu}\left\{ -F^{\mu\nu}\delta A_{\rho}\xi^{\rho} - 2\xi^{\mu}F^{\nu\rho}\delta A_{\rho} + \left[2h^{\mu\rho}F_{\rho}{}^{\nu} - \delta F^{\mu\nu} - \frac{1}{2}hF^{\mu\nu}\right]\left(A_{\sigma}\xi^{\sigma} + \lambda\right) \right\}$$

Integrability condition

$$\int_{\partial C}\left(\delta_1 k_{(\xi,\lambda)}[\delta_2 g, \delta_2 A; g, A] - \delta_2 k_{(\xi,\lambda)}[\delta_1 g, \delta_1 A; g, A]\right) = 0$$

Kerr-BR black hole

Angular momentum & electric charge

Hu, Cai, Wang 2026 ***PRD Letter***

$$J = \mathcal{Q}_{(-\partial_\varphi,0)} = \frac{2ma(2-B^2a^2)^3}{(1-B^2a^2)\left[4+B^4a^4+4B^2(m^2-a^2)\right]^2},$$

$$Q = \mathcal{Q}_{(0,-1)} = \frac{2Bma(-2+B^2a^2)}{\sqrt{1-B^2a^2}\left[4+B^4a^4+4B^2(m^2-a^2)\right]}$$

infinitesimal conserved charge

$$\omega = \delta\mathcal{Q}_{(\partial_t,0)} = c_m\delta m + c_a\delta a + c_B\delta B$$

Non-integrable! $\partial_a c_m \neq \partial_m c_a$

$$c_m = \frac{4I_1\left(4-4B^2a^2-8B^2m^2+8B^4m^2a^2+B^4a^4\right)}{I_2\left[4+B^4a^4+4B^2(m^2-a^2)\right]^2}$$

$$c_a = \frac{2B^2ma}{I_2\left[4+B^4a^4+4B^2(m^2-a^2)\right]^2}\times\left(16-8B^2m^2-20B^2a^2+12B^4m^2a^2+8B^4a^4-B^6a^6\right)$$

$$c_B = \frac{4Bm}{I_2\left[4+B^4a^4+4B^2(m^2-a^2)\right]^2}\times\left[12(a^2-m^2)+B^2a^2(14m^2-16a^2+7B^2a^4-B^4a^6)\right]$$

Kerr-BR black hole

Generators and linear combination

$$(\xi, \lambda) = \left\{ (\partial_t, 0), (-\partial_\varphi, 0), (0, -1) \right\}$$

$$\alpha(\partial_t + \Omega_{int}\partial_\varphi, \Phi_{int})$$

Integrable mass

$$\delta M = \alpha \left(\omega - \Omega_{int}\delta J - \Phi_{int}\delta Q \right)$$

$$\partial_m M = \alpha \left(c_m - \Omega_{int}\partial_m J - \Phi_{int}\partial_m Q \right)$$

$$\partial_a M = \alpha \left(c_a - \Omega_{int}\partial_a J - \Phi_{int}\partial_a Q \right)$$

$$\partial_B M = \alpha \left(c_B - \Omega_{int}\partial_B J - \Phi_{int}\partial_B Q \right)$$

Kerr-BR black hole

Christodoulou-Ruffini mass relation

Christodoulou & Ruffini 1971 ***PRD***

$$M^2(S,J,Q) = \frac{S}{4\pi} + \frac{Q^2}{2} + \frac{\pi(Q^4+4J^2)}{4S}$$

Conserved mass

$$M(m,a,B) = 2m(2-B^2a^2) \times \sqrt{\frac{\left(4+B^6a^6-3B^4a^4+4B^4m^2a^2\right)}{(1-B^2a^2)\left[4+B^4a^4+4B^2(m^2-a^2)\right]^3}}$$

Hu, Cai, Wang 2026 ***PRD Letter***

Two dark clouds

Uniqueness?

Thermodynamic dependence!

Kerr-Newman-BR black hole

Line element & gauge potential

Ovcharenko & Podolsky 2026 ***PRL***

$$\mathrm{d}s^2 = \frac{1}{\Sigma^2}\left[-\frac{\beta}{\rho^2}\big(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi\big)^2 + \frac{\rho^2}{\beta}\mathrm{d}r^2 + \frac{\rho^2}{P}\mathrm{d}\theta^2 + \frac{P}{\rho^2}\sin^2\theta\,\big(a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi\big)^2\right]$$

$$A_\mu \mathrm{d}x^\mu = \frac{a\,\mathrm{d}t}{\Sigma}\left[-u\,B + \frac{r}{\rho^2}(m\,B\cos^2\theta - k)\right] + \frac{\mathrm{d}\varphi}{P_0 B}\,\frac{s + ameB^3}{1+e^2B^2}$$

$$+\frac{\mathrm{d}\varphi}{P_0\Sigma}\left[u\,a^2B + \frac{r}{\rho^2}\big(-m\,B(r^2+a^2)\cos^2\theta + k\,a^2\sin^2\theta\big) - kr - \frac{1}{B}\right]$$

$$\rho^2 = r^2 + a^2\cos^2\theta\,, \qquad P = 1 + B^2\zeta^2\cos^2\theta\,,$$
$$\beta = I\Delta\,, \qquad \Sigma^2 = I - B^2\Delta\cos^2\theta\,,$$
$$I = (1+k\,B\,r)^2 + B^2r^2\,, \qquad P_0 = 1 + B^2\zeta^2\,,$$
$$\Delta = u\,a^2 - 2m\,r + (u - e^2/a^2)\,r^2\,,$$

Auxiliary parameters

$$u = 1 + k^2\,, \quad k = \frac{e\,s - a\,m\,B}{a\,(1+e^2B^2)}\,,$$
$$\zeta^2 = m^2 - u^2a^2 + u\,e^2\,,$$
$$s^2 = 1 + (e^2 - a^2)B^2 - (m^2+e^2)a^2B^4\,.$$

Kerr-Newman-BR black hole

Angular momentum & electric charge

$$J = \mathcal{Q}_{(-\partial_\varphi,0)} = \frac{ma}{P_0^2}, \qquad Q = \mathcal{Q}_{(0,-1)} = \frac{e}{P_0}$$

infinitesimal conserved charge

$$\omega = \delta\mathcal{Q}_{(\partial_t,0)} = c_m\delta m + c_a\delta a + c_e\delta e + c_B\delta B$$

Integrable mass

$$\delta(M^2) = \lambda\,\omega + \eta\,\delta J + \tau\,\delta(Q^2)$$

Local coordinate change

$$e \to k$$

Before

c_i contains $\sqrt{s^2} = \sqrt{1 + (e^2 - a^2)B^2 - (m^2 + e^2)a^2B^4}$

After

$$\widetilde{c}_m = c_m + \frac{ec_e}{2p}\,\partial_m p = \frac{\Xi_m}{D^2},$$

$$\widetilde{c}_a = c_a + \frac{ec_e}{2p}\,\partial_a p = \frac{\Xi_a}{D^2},$$

$$\widetilde{c}_k = \frac{ec_e}{2p}\,\partial_k p = \frac{\Xi_k}{D^2},$$

$$\widetilde{c}_B = c_B + \frac{ec_e}{2p}\,\partial_B p = \frac{\Xi_B}{D^2},$$

Ξ_i just polynomial !!!

$$p \equiv e^2 = \frac{a^2(k + mB)^2}{Z}$$

Hu, Cai, Jiang, Wang *2611.03123*

Local coordinate change

Polynomials

$$\Xi_m = Z\left[1-(2m^2+ua^2)B^2-2kuma^2B^3\right],$$

$$\Xi_a = uaB\left\{k+3mB-5km^2B^2-m\left[3m^2+2u(3u-1)a^2\right]B^3-kua^2(2m^2+u^2a^2)B^4+u^3ma^4B^5\right\},$$

$$\Xi_k = a^2B\left\{2u-1+3kmB-\left[(7u-5)m^2+2u^2a^2\right]B^2-km\left[3m^2+2u(4u-1)a^2\right]B^3\right.$$
$$\left.+\left[2um^2a^2+u^3(2-u)a^4\right]B^4+3ku^3ma^4B^5\right\},$$

$$\Xi_B = kua^2+m(4ua^2-3m^2)B-9kum^2a^2B^2-u^2(7u-1)ma^4B^3+ku^2a^4(2m^2-u^2a^2)B^4+2u^4ma^6B^5.$$

$$Z = 1-ua^2B^2 \qquad D = ZP_0 = 1+(m^2-2ua^2)B^2+2kuma^2B^3+u^3a^4B^4$$

Master equations

$$\partial_m M^2 = \lambda\, \tilde{c}_m + \eta\, \partial_m J + \tau\, \partial_m Q^2\,,$$
$$\partial_a M^2 = \lambda\, \tilde{c}_a + \eta\, \partial_a J + \tau\, \partial_a Q^2\,,$$
$$\partial_k M^2 = \lambda\, \tilde{c}_k + \eta\, \partial_k J + \tau\, \partial_k Q^2\,,$$
$$\partial_B M^2 = \lambda\, \tilde{c}_B + \eta\, \partial_B J + \tau\, \partial_B Q^2\,.$$

four unknowns

$$(M^2, \lambda, \eta, \tau)$$



$$V(M^2) = 0\,, \quad V = A_m \partial_m + A_a \partial_a + A_k \partial_k + A_B \partial_B\,.$$

Hu, Cai, Jiang, Wang *2611.03123*

Master equations

$$
\begin{aligned}
A_m &= -kuma^2 + m^2\left[3m^2 - (1+2u)a^2\right]B + kma^2\left[(1+9u)m^2 - u^2a^2\right]B^2 \\
&\quad + m^2a^4u(1-7u+9u^2)B^3 - kma^4u\left[m^2 + u^2(1-3u)a^2\right]B^4\,, \\
A_a &= kua^3 + ma\left[m^2 + (1-2u)a^2\right]B + ka^3\left[(3u-1)m^2 - 3u^2a^2\right]B^2 \\
&\quad + ma^5u(-1-u+3u^2)B^3 + ka^5u\left[m^2 + u^2(1+u)a^2\right]B^4\,, \\
A_k &= -m^2 + u(1-u)a^2 + km(m^2 - 2ua^2)B + m^2a^2u(3u-1)B^2 \\
&\quad + kma^4u^2(3u-1)B^3 - a^4u^2\left[m^2 + u^2(1-u)a^2\right]B^4\,, \\
A_B &= m + k(u-1)a^2B - m\left[2m^2 + (2-u)a^2\right]B^2 - ka^2\left[(6u-2)m^2 - u^2a^2\right]B^3 \\
&\quad + ma^2\left[m^2 + (-u+7u^2-6u^3)a^2\right]B^4 + ka^4u\left[m^2 + u^2(1-2u)a^2\right]B^5\,.
\end{aligned}
$$

Cauchy–Kovalevskaja theorem

$$V(M^2) = 0\,, \quad V = A_m\partial_m + A_a\partial_a + A_k\partial_k + A_B\partial_B\,.$$

$A_B|_{B\to 0} = m$

$A_m\,, A_a\,, A_k\,, A_B$ are polynomials

$M^2(m, a, k, 0) = m^2$

There is a **unique** real analytic solution defined in a neighborhood of the origin.

$$M^2(m, a, k, B) = m^2 + \sum_{j\geq 1} f_j(m, a, k)B^j$$

Mass construction

Rescaled function

$$W_n = D^n M^2 = m^2 + \sum_{\ell \geq 1} w_\ell^{(n)} B^\ell$$

Master equation

$$V(W_n) - n\Gamma W_n = 0\,, \qquad V(D) = \Gamma D$$

Collecting the terms order by order

$$w_1^{(n)} = 2kuma^2$$

$$w_2^{(n)} = (n-3)m^4 + (4 + 5k^2 - 2nu)m^2a^2 + uk^4a^4$$

$$\widetilde{c}_m = c_m + \frac{ec_e}{2p}\,\partial_m p = \frac{\Xi_m}{D^2}\,,$$

$$\widetilde{c}_a = c_a + \frac{ec_e}{2p}\,\partial_a p = \frac{\Xi_a}{D^2}\,,$$

$$\widetilde{c}_k = \frac{ec_e}{2p}\,\partial_k p = \frac{\Xi_k}{D^2}\,,$$

$$\widetilde{c}_B = c_B + \frac{ec_e}{2p}\,\partial_B p = \frac{\Xi_B}{D^2}\,,$$

Mass construction

Recurrence relation

Hu, Cai, Jiang, Wang *2611.03123*

$$w_\ell^{(n)} = -\frac{1}{\ell m}\sum_{j=0}^{\min(4,\ell-1)}\Big[A_{m,j}\partial_m w_{\ell-j-1}^{(n)} + A_{a,j}\partial_a w_{\ell-j-1}^{(n)}$$

$$+A_{k,j}\partial_k w_{\ell-j-1}^{(n)} + \big((\ell-j-1)b_{j+1} - n\gamma_j\big)w_{\ell-j-1}^{(n)}\Big]\,.$$

depends only on **five** preceding coefficient functions!!!

$w_{\ell-1}^{(n)}, \ldots, w_{\ell-5}^{(n)}$

Where $\quad A_\psi = \sum_{j=0}^{4} A_{\psi,j}B^j \quad (\psi = m, a, k)\,,$

$$A_B = m + \sum_{j=1}^{5} b_j B^j\,, \qquad \Gamma = \sum_{j=1}^{4}\gamma_j B^j\,.$$

Mass construction

Recurrence relation

Hu, Cai, Jiang, Wang *2611.03123*

$$w_9^{(n)} = (n-3)\,\widehat{w}_9^{(n)}(m,a,k)$$

suggests the rescaling $W_3 = D^3M^2$, but not enough

$$w_9^{(3)} = w_{10}^{(3)} = w_{11}^{(3)} = w_{12}^{(3)} = w_{13}^{(3)} = 0$$

Induction therefore proves that the entire tail vanishes

Mass construction

W_3 has finite term

Hu, Cai, Jiang, Wang *2611.03123*

$$M^2 = W_3/D^3 = \frac{m^2 + 2kuma^2B + B^2(ue^4 + m^2e^2 + um^2a^2)}{P_0^3}$$

A derivation independent of thermodynamics!!!

Subcase KBR

$$e \to \frac{maB}{\sqrt{1 - a^2B^2}} \quad \text{and re-definition} \quad m \to m\,\frac{1 - a^2B^2}{1 - \frac{1}{2}a^2B^2}$$

$$M(m, a, B) = 2m(2 - B^2a^2) \times \sqrt{\frac{\left(4 + B^6a^6 - 3B^4a^4 + 4B^4m^2a^2\right)}{(1 - B^2a^2)\left[4 + B^4a^4 + 4B^2(m^2 - a^2)\right]^3}}$$

Thermodynamics

Parameter change

$$\delta(M^2) = \lambda\,\omega + \eta\,\delta J + \tau\,\delta(Q^2) \quad \Longleftrightarrow \quad \delta M = \alpha(\omega - \Omega_{\rm int}\delta J - \Phi_{\rm int}\delta Q)$$

$$\alpha = \lambda/(2M)\,, \qquad \Omega_{\rm int} = -\,\eta/\lambda\,, \qquad \Phi_{\rm int} = -\,2Q\tau/\lambda$$

$$\alpha = \frac{ZL}{MD^2}\,,$$

$$\Omega_{\rm int} = \frac{aB(k+mB)P_0[-1+kmB+u^2a^2B^2]}{L}\,,$$

$$\Phi_{\rm int} = -\,\frac{eB(k+mB)(m+kua^2B)}{L}\,,$$

where $\quad L = m + k^3a^2B - 2ma^2B^2 + km^2a^2B^3 + mu^2a^4B^4$

Thermodynamics

Thermodynamical potentials

$$r_{\pm} = \frac{m \pm \zeta}{u - e^2/a^2}$$

$$\Omega_H = -\left.\frac{g_{t\varphi}}{g_{\varphi\varphi}}\right|_{r=r_+} = \frac{aP_0}{r_+^2 + a^2}\,,$$

$$\Phi_H = -\left. A_\mu \chi^\mu \right|_{r=r_+} = \frac{e\, r_+}{r_+^2 + a^2}\,,$$

$$T_H = \frac{\kappa}{2\pi} = \frac{\zeta}{2\pi}\frac{(1 + k\,B\,r_+)^2 + B^2 r_+^2}{r_+^2 + a^2}\,,$$

$$S = \frac{\mathscr{A}}{4} = \frac{\pi}{P_0}\frac{r_+^2 + a^2}{(1 + k\,B\,r_+)^2 + B^2 r_+^2}\,,$$

Thermodynamics

Potential redefinition

$$T = \alpha T_H\,, \qquad \Omega = \alpha(\Omega_H - \Omega_{\rm int})\,, \qquad \Phi = \alpha(\Phi_H - \Phi_{\rm int})$$

First law & Smarr formula

$$\delta M = T\delta S + \Omega\delta J + \Phi\delta Q$$

established!

$$M = 2TS + 2\Omega J + \Phi Q$$

Conclusions and discussions

Kerr-BR

Thermodynamic input

Integrability conditions are insufficient for determining mass

The Christodoulou-Ruffini mass relation fixes the mass

Naturally satisfy the laws of thermodynamics

Kerr-Newman-BR

Integrability condition

The extra variation gave a master equation for the conserved mass

Cauchy–Kovalevskaja theorem ensures there is a unique real analytic solution

A terminating recurrence gives the full KN–BR expression.

The KN-BR mass derivation independently confirms the thermodynamic laws.