Bianchi IX Universe dynamics with a phantom field — new type of BKL oscillations

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The Belinskii-Khalatnikov-Lifshitz (BKL) oscillations have been attracting a lot of attention ever since their discovery, mainly because they are assumed to be the general regime of approaching the cosmological singularity. Some interesting mathematical results have been established, especially considering the chaotic properties of this regime. More precisely, the regime in question can be represented by a sequence of Bianchi I Kasner epochs, and this discrete sequence (referred to as the Mixmaster map) is shown to be chaotic, though a rigorous mathematical proof that the Mixmaster map is indeed an attractor for a general vacuum Bianchi IX dynamics is still absent.

Studies of a Bianchi IX dynamics with an exotic scalar field is a reasonable generalization of the standard BKL scenario. This program has been started rather recently with investigation of Bianchi IX Universe in Horndeski theory. In the paper (A. Starobinsky, S. Sushkov and M. Volkov, 2020) it was claimed that numerical integration had shown non-monotonic behavior of the mean scale factor, i.e. a bounce. This means that the volume initially decreasing, while going to the past, turns to an increasing stage. This is in a sharp contrast to the usual BKL picture where the volume decreases linearly towards the singularity.

Arguably, the simplest case of an exotic matter is a massless scalar field with a wrong sign of its kinetic term, a so-called phantom field. A special numerical study of Bianchi IX dynamics with a phantom field has been done in (Volkov, 2025), and volume oscillations have been observed. One of the goals of this presentation is to explain this phenomenon.

The Einstein field equations together with the scalar field equation yield the following system (dots denote d/dt):

$$(\dot{a}bc)^{\cdot} = \frac{1}{2abc}\left[(b^2 - c^2)^2 - a^4\right], \tag{1a}$$

$$(a\dot{b}c)^{\cdot} = \frac{1}{2abc}\left[(a^2 - c^2)^2 - b^4\right], \tag{1b}$$

$$(ab\dot{c})^{\cdot} = \frac{1}{2abc}\left[(a^2 - b^2)^2 - c^4\right], \tag{1c}$$

and the scalar field equation $\Box\phi = 0$,

$$\dot{\phi}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right) + \ddot{\phi} = 0, \tag{2}$$

together with its constraint equation

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\epsilon}{2}\dot{\phi}^2 = 0, \tag{3}$$

where $\epsilon = +1$ for the usual scalar field, and $\epsilon = -1$ for the phantom field.

We introduce logarithmic scale factors α, β, γ as

$$a = e^{\alpha(\tau)}, \qquad b = e^{\beta(\tau)}, \qquad c = e^{\gamma(\tau)}, \tag{4}$$

and pass to another time variable τ defined by

$$dt = abc\, d\tau \tag{5}$$

Under this change of variables (5), the equations (1) take the form

$$2\alpha_{,\tau\tau} = (b^2 - c^2)^2 - a^4 = e^{4\beta} + e^{4\gamma} - 2e^{2(\beta+\gamma)} - e^{4\alpha}, \tag{6a}$$

$$2\beta_{,\tau\tau} = (c^2 - a^2)^2 - b^4 = e^{4\gamma} + e^{4\alpha} - 2e^{2(\gamma+\alpha)} - e^{4\beta}, \tag{6b}$$

$$2\gamma_{,\tau\tau} = (a^2 - b^2)^2 - c^4 = e^{4\alpha} + e^{4\beta} - 2e^{2(\alpha+\beta)} - e^{4\gamma}. \tag{6c}$$

The commas denote the derivatives: $\alpha_{,\tau} \equiv d\alpha/d\tau$, etc. The scalar field equation (2) becomes

$$\phi_{,\tau\tau} = 0, \tag{7}$$

and its constraint (3) reads

$$\alpha_{,\tau\tau} + \beta_{,\tau\tau} + \gamma_{,\tau\tau} + \frac{\epsilon}{2}\phi_{,\tau}^2 = 2[\alpha_{,\tau}\beta_{,\tau} + \alpha_{,\tau}\gamma_{,\tau} + \gamma_{,\tau}\beta_{,\tau}]. \tag{8}$$

Equation (7) integrates immediately to

$$\phi_{,\tau} = \tilde{q} = \text{const}, \tag{9}$$

where $\tilde{q}$ is the conserved scalar momentum.

Bianchi I Kasner solution reads $a \sim t^{p_1}, \quad b \sim t^{p_2}, \quad c \sim t^{p_3}$, where

$$p_1 + p_2 + p_3 = 1, \tag{10}$$

and

$$p_1^2 + p_2^2 + p_3^2 + \epsilon q^2 = 1, \tag{11}$$

with $q \equiv \tilde{q}/\sqrt{2}$.

$$p_1' = \frac{-p_1}{1+2p_1}, \qquad p_2' = \frac{p_2+2p_1}{1+2p_1}, \qquad p_3' = \frac{p_3+2p_1}{1+2p_1}, \qquad q' = \frac{q}{1+2p_1}. \tag{12}$$

Note however that the value of $\tilde{q}$ depends on the chosen time variable τ, while after a bounce we will rescale the time for better correspondence to the Bianchi I approximation parameters.
For the numerical simulations, one might define

$$p_1 = \frac{\alpha_{,\tau}}{\alpha_{,\tau}+\beta_{,\tau}+\gamma_\tau}, \qquad p_2 = \frac{\beta_{,\tau}}{\alpha_{,\tau}+\beta_{,\tau}+\gamma_\tau}, \qquad p_3 = \frac{\gamma_{,\tau}}{\alpha_{,\tau}+\beta_{,\tau}+\gamma_\tau} \tag{13}$$

and

$$q = \frac{\phi_{,\tau}/\sqrt{2}}{\alpha_{,\tau}+\beta_{,\tau}+\gamma_\tau}. \tag{14}$$

In relation to the discussion above, let us note again that the Kasner exponents are unambiguously defined (in the physical time) for the Bianchi I solutions only. In order to keep track of them outside the Bianchi I epochs, we introduce the definitions (13). Note also that, in a general Bianchi IX dynamics with such a definition, q is no longer constant.
An important difference from the standard BKL is that, with a phantom, there might be TWO negative exponents, $p_1 \leqslant p_2 < 0$

Having neglected the third dimension, we get the Bianchi IX equations (6)

$$2\alpha_{,\tau\tau} = e^{4\beta} - e^{4\alpha}, \qquad 2\beta_{,\tau\tau} = e^{4\alpha} - e^{4\beta}, \qquad 2\gamma_{,\tau\tau} = (e^{2\alpha} - e^{2\beta})^2 \tag{15}$$

as well as the scalar field motion (7) and the constraint (8) as before,

$$\phi_{,\tau\tau} = 0, \qquad 2\Big[\alpha_{,\tau}\beta_{,\tau} + \alpha_{,\tau}\gamma_{,\tau} + \beta_{,\tau}\gamma_{,\tau}\Big] = \alpha_{,\tau\tau} + \beta_{,\tau\tau} + \gamma_{,\tau\tau} - \frac{1}{2}(\phi_{,\tau})^2. \tag{16}$$

From the equations (15, 16) we have $\alpha_{,\tau\tau} + \beta_{,\tau\tau} = 0$, and can choose the origin of time such that

$$\alpha + \beta = p_+\tau, \tag{17}$$

where $p_1 + p_2 = p_+ < 0$ is a constant.
Now, we give a name to the difference of α and β, and introduce yet another time variable η:

$$\chi \equiv \alpha - \beta, \qquad \ln(-2p_+\eta) = p_+\tau$$

so that

$$2\alpha = \ln(-2p_+\eta) + \chi, \qquad 2\beta = \ln(-2p_+\eta) - \chi. \tag{18}$$

After a simple algebra we get

$$\chi_{,\eta\eta} + \frac{\chi_{,\eta}}{\eta} + \frac{1}{2}\sinh(2\chi) = 0. \tag{19}$$

$$4\,\frac{\gamma_{,\eta}}{\eta} = -\frac{Q}{\eta^2} + \chi_{,\eta}^2 + \frac{1}{2}\left(\cosh 2\chi - 1\right). \tag{20}$$

where $Q \equiv 1 + \frac{\tilde{q}^2}{p_+^2}$ and $\phi_{,\tau} = \tilde{q}$ as before.

In Case of $\chi \gg 1$, we get $a \gg b$ and, *given also that* $p_1 < p_2$, we are fully back to the case of one negative exponent (12).
In Case of $\chi \ll -1$, we get $b \gg a$ and, *if the inequality is strong enough to initiate a bounce before the factor a takes over due to* $p_2 > p_1$, then we get a bounce driven by the dimension b only, so the new result is

$$p_2' = \frac{-p_2}{1+2p_2}, \qquad p_1' = \frac{p_1+2p_2}{1+2p_2}, \qquad p_3' = \frac{p_3+2p_2}{1+2p_2}, \qquad q' = \frac{q}{1+2p_2}. \tag{21}$$

In other words, α and β do switch places.
In order to avoid switching between two sets of formulas, we can agree to use (23) only, but to *define* p_1 as the index corresponding to the scale factor actually inducing the bounce in question, regardless of its numerical value.

So that, the information in Mixmaster map is NOT sufficient to reproduce the dynamics – we need information about actual values of scale factors. We need full equations!

For $|\chi| \ll 1$, the system of equations (19, 20) reduces to

$$\chi_{,\eta\eta} + \frac{\chi_{,\eta}}{\eta} = -\chi, \qquad \frac{\gamma_{,\eta}}{\eta} = -\frac{Q}{4\eta^2} + \frac{\chi_{,\eta}^2}{4} + \frac{\chi^2}{4}. \tag{22}$$

The first equation (22) can be solved exactly as an arbitrary combination of J_0 and Y_0 Bessel functions.
A possible solution would be

$$\chi(\eta) = \frac{\pi(p_1 - p_2)}{2(p_1 + p_2)} \cdot Y_0(\eta) + C \cdot J_0(\eta). \tag{23}$$

So that, if the two scale factors are very close to each other, it is more natural to consider not a sequence of Kasner eras, but a continuous evolution (23). This regime is also known for the standard BKL case (Belinsky, Lifshitz and Khalatnikov, 1970) as small oscillations near the Taub solution (corresponding to $p_1 = p_2 = 0$, $p_3 = 1$). In contrast to that, now p_1 and p_2 need not be small. Due to this, the volume derivative changes its sign near the point of minimum of the 3-rd scale factor.

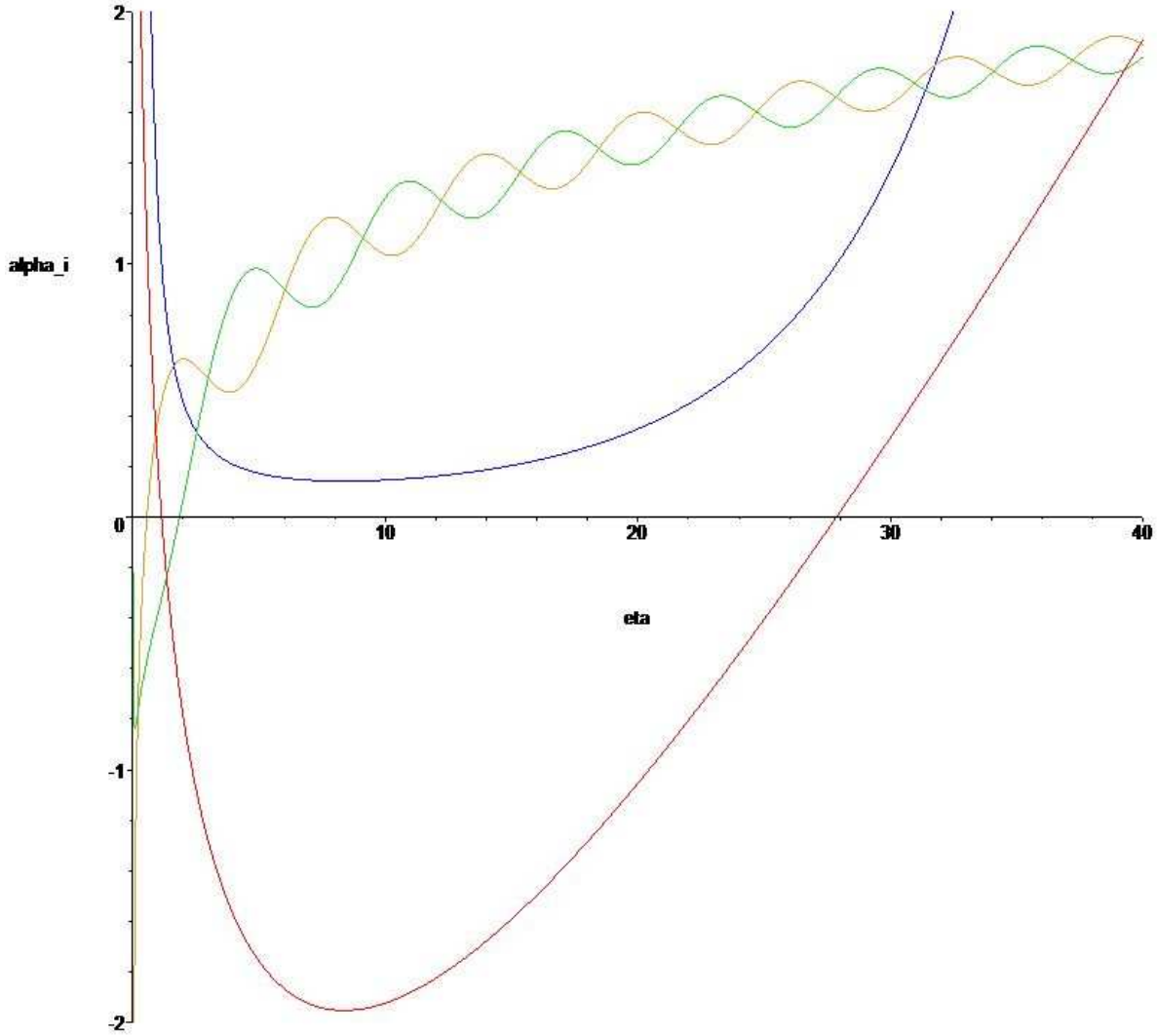


Figure: Exponents; α, β, γ (yellow, green and red) drawn with the volume (blue) in the case of $\chi \ll 1$. The volume evolution show that the volume rate of change flip sign after certain time, around the time when γ flips sign also.

A crucial feature of the BKL dynamics with a phantom is the fact that the least possible value of Kasner index can become large and negative. It leads to important consequences. To understand them, let us recall the structure of the Mixmaster map (12). As after the bounce

$$\alpha_{,\tau} = -p_1, \qquad \beta_{,\tau} = p_2 + 2p_1, \qquad \gamma_{,\tau} = p_3 + 2p_1,$$

we see that

$$\frac{dV}{d\tau} = \frac{d(abc)}{d\tau} = V \cdot (1 + 2p_1). \tag{24}$$

Getting the linear dependence $dV/dt = 1 + 2p_1$ with the standard equation $p_1' + p_2' + p_3' = 1$ after the bounce is achieved by returning to the cosmic time $dt = abcd\tau$.
Note, however, that if $p_1 < -1/2$, the volume changes its behavior (24), and a contraction (when integrating back in time) turns to expansion. In a vacuum case this situation is impossible since $p_1 > -1/3$ while for the phantom case this possibility appears when $q^2 > 3/8$.

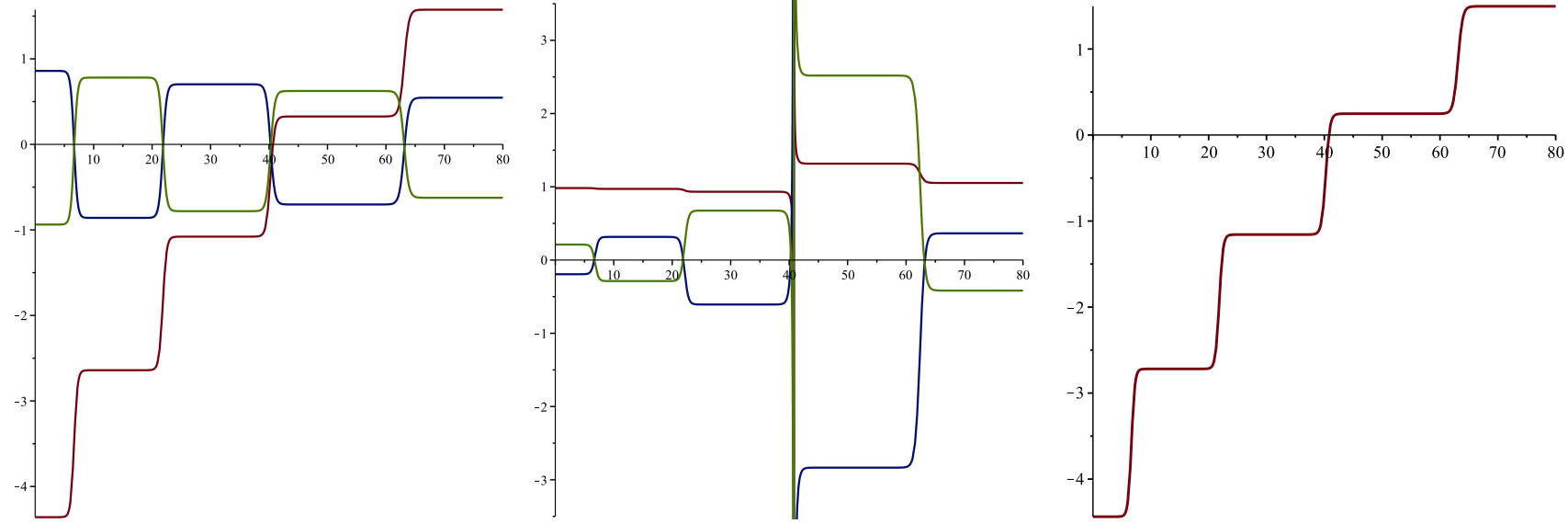


Figure: Time evolution of $\alpha_{,\tau}$ (brown), $\beta_{,\tau}$ (blue) and $\gamma_{,\tau}$ (green) – left panel, corresponding evolution of p_1 (brown), p_2 (blue), p_3 (green) – central panel, volume derivative dV/dt – right panel

It is easy to see that the transition from expansion to contraction can not be explained the same way. Indeed, for our definition of p_i the denominator of Mixmaster map at expansion can not be negative since the bounce is induced by a positive index. However, it is known that even a vacuum Bianchi IX Universe must have a point of maximal volume, and after that a collapse starts (X. Lin and R. Wald, 1989). The same situation appears in the phantom case also, and near this point the Mixmaster approximation breaks down.
So, we see that volume extrema have different origin. While volume maximum is typical for Bianchi IX evolution for the vacuum case as well, the volume minima appear only due to the presence of phantom field.

Conclusions

The presence of a phantom field induces several important modifications of the standard BKL scenario.
First, two of the three Kasner indices can become negative. The consequence is that now the information encoded in the Mixmaster map is not enough to trace evolution of the full system since a bounce can be induced by the curvature terms from both scale factors corresponding to negative indices.
The second effect of two negative indices is that when corresponding scale factors are close to each other, the potential walls are no longer exponential, and instead of the sequence of Kasner epochs we have a smooth evolution. For the phantom case, it is accompanied with the volume expansion rate changing its sign.
We show also that two negative indices are not necessary to get the volume extrema. We argue that they appear when the least index drops below $-1/2$, which is impossible without a phantom. After that, the denominator of the standard Mixmaster map changes its sign. Our numerical studies confirms that after that the volume expansion rate changes its sign as well. Since volume maxima is known to exist even in the vacuum case, these two facts lead to volume oscillations detected in (Volkov, 2025).
This bring us to a conclusion that for a phantom case the Mixmaster map is a transient regime realized with some finite accuracy and *it cannot be considered as an attractor for a general Bianchi IX evolution*