Timelike entanglement first law and linearized field equations in higher curvature gravity

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Contents

1. Background and Lovelock setup
2. Linearized Lovelock field equations
3. Cubic Lovelock gravity
4. The general first-law argument
5. Complex surfaces and conclusions

Entanglement entropy

For a pure state on $A \cup B$, the reduced density matrix and entropy are

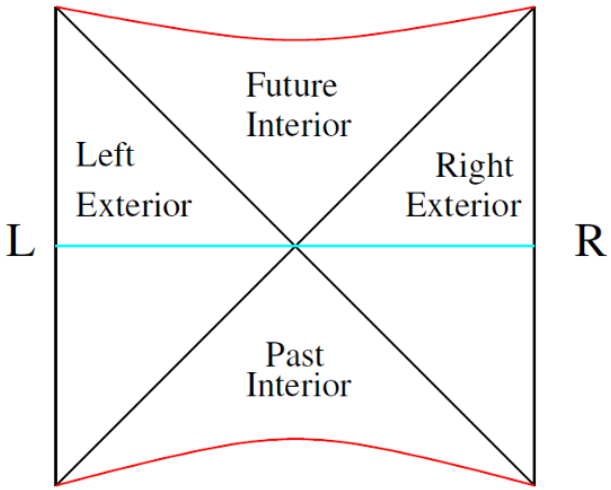
$$\rho_A = \mathrm{tr}_B\, \rho, \qquad \boxed{S_A = -\,\mathrm{tr}_A(\rho_A \ln \rho_A).}$$

d=2: at critical point (zero T), the EE is

$$S_A = \frac{c}{3} \ln\left[\frac{L}{\pi\epsilon} \sin\left(\frac{\pi\ell}{L}\right)\right] + c_1' \xrightarrow{\ell \ll L} \frac{c}{3} \ln \frac{\ell}{\epsilon}$$

d>2: EE∼are law+sub-leading corrections.

$$S_{\mathrm{EE}} = \frac{A}{\epsilon^2} \propto S_{\mathrm{BH}} = \frac{A}{4G_N}$$



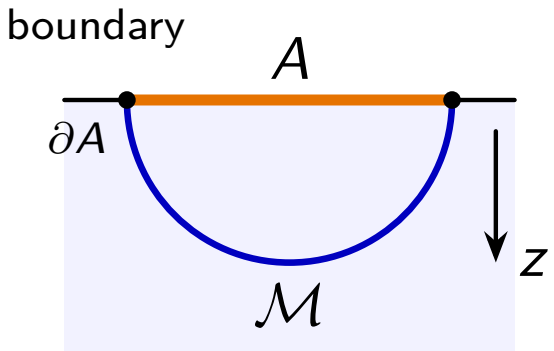
ER=EPR

Calabrese–Cardy (2004); Bombelli et al. (1986); Jacobson (1994); Maldacena–Susskind (2013).

Holographic entanglement entropy

In static Einstein gravity, the Ryu–Takayanagi prescription gives

$$\boxed{S_A = \frac{\text{Area}(\mathcal{M})}{4G_{d+1}}, \qquad \partial\mathcal{M} = \partial A.}$$



$\mathcal{M}$ is a codimension-two bulk minimal surface, anchored on ∂A and homologous to A.

$$S_A = -\,\partial_n\big(\ln Z[n] - n \ln Z[1]\big)\big|_{n=1} = \frac{\text{Area}(\mathcal{M})}{4G_N}.$$

Ryu–Takayanagi (2006); Lewkowycz–Maldacena (2013).

Entanglement first law and gravitational dynamics

The relative entropy measures the distance between two states in the same Hilbert space

$$S(\rho_1|\rho_0) = \mathrm{tr}(\rho_1 \ln \rho_1) - \mathrm{tr}(\rho_1 \ln \rho_0).$$

At the first order variation, the relative entropy is stationary at $\rho_1 = \rho_0$, which gives

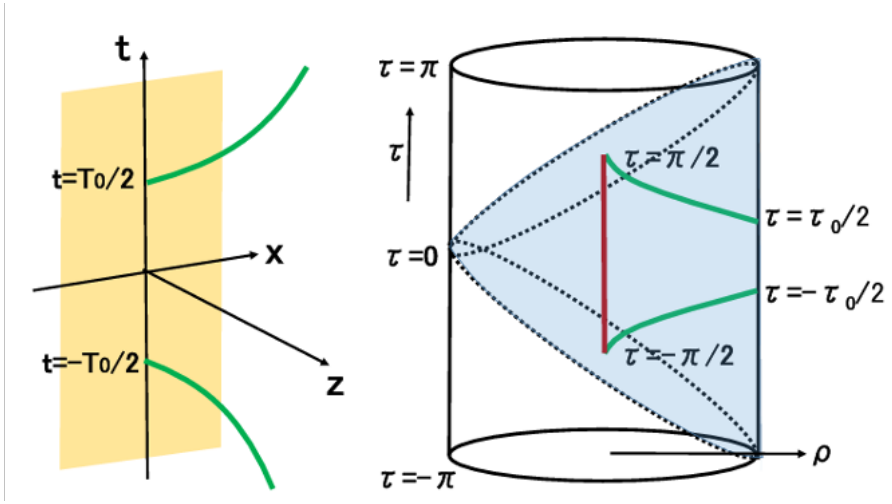
$$\boxed{\Delta S = \Delta \langle H \rangle \quad \text{to first order.}}$$

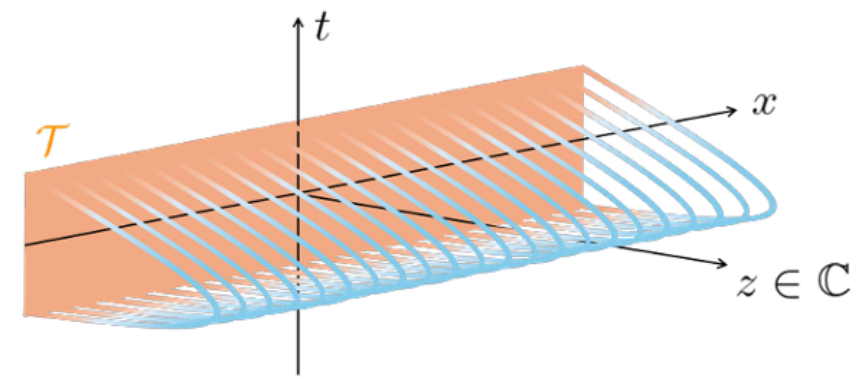
where H is modular Hamiltonian which is defined by $\rho = \frac{e^{-H}}{\mathrm{Tr}(e^{-H})}$.
Spherical subregion: The 1st law like relation was shown to be equivalent to the 1st order perturbative Einstein equations in asymptotically AdS spacetime.
Faulkner et al. (2013);Lashkari et al. (2014); Nozaki et al. (2013);Bhattacharya–Takayanagi (2013);Czech et al. (2017)

Holographic timelike entanglement entropy

- **Wick rotation**: $t_E = it \to$ Poincaré EAdS_{d+1}: ordinary HEE. Geometric configuration: two spacelike surfaces + a timelike surface.
- **Complexified bulk geometry:**

$$S_{\mathcal{T}} = \frac{A(\gamma_{\mathcal{T}})}{4G_{d+1}}.$$





Wang et al. (2018); Li et al. (2022); Doi et al. (2023); Heller et al. (2024)

Hyperbolic region and the modular Hamiltonian

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 - dt^2 + dx_{d-1}^2 + d\mathbf{x}^2\right).$$

Hyperbolic region: $-t^2 + \mathbf{x}^2 \leq -T_0^2$ at $x_{d-1} = 0$.
Double Wick rotation:

$$t \longrightarrow -i\tau, \qquad x_{d-1} \longrightarrow i x_{d-1}.$$

The local modular-energy variation is

$$\boxed{\Delta\langle H\rangle = 2\pi \int_{\mathcal{D}'} d\tau\, d^{d-2}\mathbf{x}\, \frac{T_0^2 - \xi^2}{2T_0}\, \Delta T_{(d-1)(d-1)}.}$$

linearized Einstein equations $\longleftrightarrow$timelike entanglement first law

- The flow kernel is fixed by the region and conformal symmetry.
- Higher curvature dependence enters through the holographic stress tensor.

Casini et al. (2011); Li et al. (2025).

Lovelock gravity

In $d+1$ bulk dimensions,

$$I = \frac{1}{16\pi G_{d+1}} \int d^{d+1}x\sqrt{-g} \sum_{m=0}^{\lfloor (d+1)/2 \rfloor} c_m L^{2m-2} \mathcal{L}_m(R),$$

where $c_0 = d(d-1)$, $c_1 = 1$, and

$$\mathcal{L}_m = \frac{1}{2^m} \delta^{A_1 \cdots A_{2m}}_{B_1 \cdots B_{2m}} \prod_{q=1}^{m} R^{B_{2q-1}B_{2q}}{}_{A_{2q-1}A_{2q}}.$$

Vacuum solution:

$$ds^2 = \frac{\tilde{L}^2}{z^2}\left(dz^2 - dt^2 + dx_{d-1}^2 + d\mathbf{x}^2\right).$$

The bare length L and effective AdS radius $\tilde{L}$ are related by $\tilde{L}^2 = \frac{L^2}{f_\infty}$.

The AdS vacuum condition is $1 = f_\infty - \sum_{m=2}^{\lfloor d/2 \rfloor} \lambda_m f_\infty^m$, where $\lambda_m = (-1)^m \frac{(d-2)!}{(d-2m)!} c_m$. Select the small-coupling branch approaching $f_\infty = 1$ as $\lambda_m \to 0$.

Lovelock (1970); Lovelock (1971).

Jacobson–Myers functional

The Lovelock holographic entropy functional is

$$S = \frac{1}{4G_{d+1}} \int d^{d-1}x \sqrt{\gamma}\, \mathcal{S} + S_{\text{surf}},$$

$$\mathcal{S} = 1 + \sum_{m=2}^{\lfloor (d+1)/2 \rfloor} m c_m L^{2m-2} \mathcal{L}_{m-1}(\mathcal{R}).$$

Wald's expression uses projected bulk curvature. Gauss–Codazzi gives

$$R^{\parallel}_{ijkl} = \mathcal{R}_{ijkl} - \eta_{\hat{m}\hat{n}} \left(\mathfrak{K}^{\hat{m}}_{ik} \mathfrak{K}^{\hat{n}}_{jl} - \mathfrak{K}^{\hat{m}}_{il} \mathfrak{K}^{\hat{n}}_{jk} \right).$$

- Generic entangling surfaces need not have $\mathfrak{K}^{\hat{m}}_{ij} = 0$.
- The JM functional reproduces the relevant anomaly checks.

Jacobson–Myers (1993); Hung–Myers–Smolkin (2011).

The Lovelock field equations

The Lovelock field equations are

$$\sum_{m=0}^{\lfloor d/2 \rfloor} c_m L^{2m-2} \mathcal{E}_{AB}^{(m)} = 0,$$

with

$$\mathcal{E}_{AB}^{(m)} = -\frac{g_{AC}}{2^{m+1}} \delta^{CA_1 \cdots A_{2m}}_{BB_1 \cdots B_{2m}} \prod_{q=1}^{m} R^{B_{2q-1}B_{2q}}{}_{A_{2q-1}A_{2q}}.$$

Around the pure AdS background, the linearized Lovelock equations collapse to the Einstein form

$$\boxed{\delta G_{AB} = \left(1 - \sum\nolimits_{m=2}^{\lfloor d/2 \rfloor} m\lambda_m f_\infty^{m-1}\right) \delta G^E_{AB}},$$

where $\delta G^E_{AB} = \delta R_{AB} - \frac{1}{2}\bar{g}_{AB}\delta R + \frac{d}{\tilde{L}^2}\delta g_{AB}$.

The effective gravitational coupling

The overall factor in linearized field equations can be regarded as effective gravitational coupling

$$\boxed{G_{\rm eff}^{-1} = \mathcal{C}_{\rm h}\, G_{d+1}^{-1}.}$$

Here $\mathcal{C}_{\rm h} = \left(1 - \sum_{m=2}^{\lfloor d/2 \rfloor} m\lambda_m f_\infty^{m-1}\right)$.

- In cubic Lovelock gravity, the factor is

$$\mathcal{C}_{\rm h} = 1 - 2\lambda_2 f_\infty - 3\lambda_3 f_\infty^2.$$

- If the factor is nonzero, the linearized Lovelock equations reduce to $\delta G_{AB}^{E} = 0$.

Cubic Lovelock gravity: thermal setup

Take $d \geq 6$, $\lambda = \lambda_2$ and $\mu = \lambda_3$, with

$$1 = f_\infty - \lambda f_\infty^2 - \mu f_\infty^3.$$

A low-energy thermal excitation has the near-boundary metric

$$ds^2 = \frac{\tilde{L}^2}{z^2}\left[\frac{dz^2}{f(z)} - f(z)dt^2 + dx_{d-1}^2 + d\mathbf{x}^2\right],$$

where $f(z) = 1 - m_z z^d, m_z T_0^d \ll 1$.
After double Wick rotation, the vacuum surface obeys

$$\xi^2 + z^2 \leq T_0^2, \qquad \xi^2 = \tau^2 + \mathbf{x}^2.$$

First-order entropy variation

Evaluate the metric variation of the JM functional on this vacuum surface. Stationarity keeps the shape of extremal surface.

Cubic Lovelock gravity: the first-law check

The entropy variation is

$$\Delta S = -\mathcal{C}_{\rm h}\frac{\Omega_{d-2}\tilde{L}^{d-1}m_z T_0^d}{8G_{d+1}(d^2-1)} = \mathcal{C}_{\rm h}\,\Delta S^{\rm E}.$$

Using $\Delta T_{(d-1)(d-1)} = -\mathcal{C}_{\rm h}\frac{\tilde{L}^{d-1}m_z}{16\pi G_{d+1}}$ and integrating the modular kernel gives

$$\Delta\langle H\rangle = \frac{2\pi\Omega_{d-2}T_0^d}{d^2-1}\Delta T_{(d-1)(d-1)} = \mathcal{C}_{\rm h}\,\Delta\langle H\rangle^{\rm E}.$$

Reduction to the Einstein equivalence

If $\mathcal{C}_{\rm h}\neq 0$, then

$$\begin{gathered}[\Delta S = \Delta\langle H\rangle \iff \delta G_{AB} = 0] \\ \Updownarrow \\ \left[\Delta S^{\rm E} = \Delta\langle H\rangle^{\rm E} \iff \delta G^{\rm E}_{AB} = 0\right].\end{gathered}$$

Fefferman–Graham perturbations

Write the bulk metric as

$$ds^2 = \frac{\tilde{L}^2}{z^2}\left[dz^2 + (\eta_{\mu\nu} + \delta\tilde{h}_{\mu\nu})dx^\mu dx^\nu\right].$$

Consider the compatible FG and TT sector:

$$\delta g_{zA} = 0, \qquad \bar{g}^{AB}\delta g_{AB} = 0, \qquad \bar{\nabla}^A \delta g_{AB} = 0.$$

The near-boundary asymptotic expansion of the boundary metric is given by

$$\delta\tilde{h}_{\mu\nu} = \delta\tilde{h}^{(0)}_{\mu\nu} + \cdots + z^d\delta\tilde{h}^{(d)}_{\mu\nu} + z^d \log z^2 \delta\hat{h}^{(d)}_{\mu\nu} + \ldots.$$

Fixing the flat boundary source and imposing the physical requirement $\delta\tilde{h}^{(0)}_{\mu\nu} = 0$ give

$$\boxed{\delta\tilde{h}_{\mu\nu} = z^d\delta\tilde{h}^{(d)}_{\mu\nu} + \cdots.}$$

de Haro et al. (2000); Skenderis et al. (2000).

The stress-tensor I

An auxiliary-field expression suggests

$$\tilde{I}_{\rm surf} = \frac{1}{4\pi G_{d+1}} \int \sqrt{-h}\, \Psi^{AB} K_{AB}, \qquad \Psi^{AB} = \frac{\partial \mathcal{L}}{\partial R_{ACBD}} n_C n_D.$$

In the flat-slice sector, Ψ itself depends on K. Varying ΨK therefore includes $K\,\delta\Psi$.

Correct boundary completion

The Myers term obeys the homogeneity relation

$$K_{AB} \frac{\partial I^{(m)}_{\rm surf}}{\partial K_{AB}} = \tilde{I}^{(m)}_{\rm surf}, \qquad \boxed{I^{(m)}_{\rm surf} = \frac{\tilde{I}^{(m)}_{\rm surf}}{2m-1}.}$$

Liu–Lü–Pope (2017).

The stress-tensor II

The stress-tensor is $\tilde{T}_{\mu\nu} = 2\pi_{\mu\nu} + T^{\rm ct}_{\mu\nu}$. After the vacuum divergences cancel against counterterms, $\Delta\tilde{T}_{\mu\nu} = 2h_{\nu\rho}\,\delta\pi_{\mu}{}^{\rho}$, where

$$\delta\pi^{\mu}{}_{\nu} = \frac{1}{16\pi G_{d+1}}\left(1 - \sum\nolimits_{m=2}^{\lfloor d/2\rfloor} m\lambda_m f_\infty^{m-1}\right)(\delta K\delta^{\mu}{}_{\nu} - \delta K^{\mu}{}_{\nu}).$$

In the normalizable FG sector,

$$\delta K^{\mu}{}_{\nu} = -\frac{d}{2\tilde{L}}z^d\delta\tilde{h}^{(d)\mu}{}_{\nu} + \cdots, \qquad \delta K = 0 \quad \text{at this order.}$$

The conformal boundary limit is

$$\Delta T_{\mu\nu} = \lim_{z\to 0}\left(\frac{\tilde{L}}{z}\right)^{d-2}\Delta\tilde{T}_{\mu\nu}.$$

$$\boxed{\Delta T_{\mu\nu} = \frac{d\tilde{L}^{d-1}}{16\pi G_{d+1}}\left(1 - \sum\nolimits_{m=2}^{\lfloor d/2\rfloor} m\lambda_m f_\infty^{m-1}\right)\delta\tilde{h}^{(d)}_{\mu\nu}.}$$

The modular Hamiltonian

For the hyperbolic region $\mathcal{D}'$, the flow kernel is fixed:

$$\Delta\langle H\rangle = \frac{\pi}{T_0}\int_{\mathcal{D}'} d\tau\, d^{d-2}\mathbf{x}\,(T_0^2-\tau^2-\mathbf{x}^2)\Delta T_{(d-1)(d-1)}.$$

Substituting the stress-tensor response gives

$$\begin{aligned}\Delta\langle H\rangle = &\left(1-\sum_{m=2}^{\lfloor d/2\rfloor} m\lambda_m f_\infty^{m-1}\right)\frac{d\tilde{L}^{d-1}}{16G_{d+1}T_0}\\ &\times\int_{\mathcal{D}'} d\tau\, d^{d-2}\mathbf{x}\,(T_0^2-\tau^2-\mathbf{x}^2)\delta\tilde{h}^{(d)}_{(d-1)(d-1)}.\end{aligned}$$

The general modular-energy variation

$$\boxed{\Delta\langle H\rangle = \left(1-\sum_{m=2}^{\lfloor d/2\rfloor} m\lambda_m f_\infty^{m-1}\right)\Delta\langle H\rangle^E.}$$

The extremal surface at arbitrary Lovelock order I

For a rotationally symmetric surface, parameterize

$$\xi = f(u/T_0)\cos(u/T_0), \qquad z = f(u/T_0)\sin(u/T_0).$$

The induced metric has the warped-product form

$$d\hat{s}^2 = \gamma_{uu}\, du^2 + e^{2F} d\Omega_{d-2}^2,$$

where $\gamma_{uu} = \frac{\tilde{L}^2}{\sin^2(u/T_0)}\left[(\partial_u \log f)^2 + \frac{1}{T_0^2}\right]$ and $e^{2F} = \tilde{L}^2 \cot^2(u/T_0)$.
The remaining task is to remove the f'' terms arising from $\partial_u \gamma_{uu}$.
Using

$$(\gamma^{uu})^l \partial_u \frac{1}{\sqrt{\gamma_{uu}}} = \frac{1}{2l+1}\partial_u\left((\gamma_{uu})^{-(2l+1)/2}\right),$$

the endpoint contribution cancels against $S_{\rm surf}$, leaving no f'' dependence.

The extremal surface at arbitrary Lovelock order II

The reduced density depends on f only through $(\partial_u \log f)^2$. Therefore,

$$\frac{d}{du}\left[\frac{\partial \mathcal{S}}{\partial(\partial_u \log f)}\right] = 0,$$

admits $f = \text{const}$; boundary anchoring fixes $f = T_0$.

The spherical extremum

$$\xi^2 + z^2 = T_0^2, \qquad \mathfrak{K}_{ij}^{\hat{1}} = \mathfrak{K}_{ij}^{\hat{2}} = 0.$$

The induced geometry is maximally symmetric with radius $\tilde{L}$, and Wald and JM entropy agree on this background.

The variation of entropy I

Using

$$\bar{\mathcal{L}}_{m-1}=\frac{(d-1)!}{(d+1-2m)!}\left(-\frac{1}{\tilde{L}^2}\right)^{m-1},$$

and

$$\delta(\sqrt{\gamma}\,\mathcal{L}_{m-1})=\sqrt{\bar{\gamma}}\left[\frac{1}{2}\bar{\mathcal{L}}_{m-1}\delta\gamma+\left(-\frac{1}{\tilde{L}^2}\right)^{m-2}\frac{(m-1)(d-3)!}{(d+1-2m)!}\delta\mathcal{R}\right],$$

we can obtain the entropy variation is

$$\Delta S=\left(1-\sum\nolimits_{m=2}^{\lfloor d/2\rfloor}m\lambda_m f_\infty^{m-1}\right)\Delta S^E+\Delta S^{\mathrm{D}}_{m-1}+\Delta S_{\mathrm{surf}},$$

here $\Delta S^E=\frac{1}{8G_{d+1}}\int_{\mathcal{M}}d^{d-1}x\,\sqrt{\bar{\gamma}}\,\delta\gamma$. The remaining terms are the curvature-variation flux and the explicit surface term. Both vanish in the conformal limit.

The variation of entropy II

Curvature-variation flux term:

$$\Delta S^{\mathrm{D}}_{m-1} = \frac{1}{4G_{d+1}} \sum_{m=2}^{\lfloor (d+1)/2 \rfloor} m c_m L^{2m-2} \frac{(-1)^{m-2}}{\tilde{L}^{2m-4}} \frac{(m-1)(d-3)!}{(d+1-2m)!}$$
$$\times \int_{\partial \mathcal{M}} d^{d-2}x \sqrt{\sigma} \left(\sigma^{ab} \delta \Gamma^{\xi}_{ab} - \gamma^{\xi\xi} \delta \Gamma^{a}_{a\xi}\right) n_{\xi}.$$

At $z = \epsilon$, we have

$$\sqrt{\sigma} = O(\epsilon^{-(d-2)}), \qquad \sigma^{ab} \delta \sigma_{ab} = O(\epsilon^{d}),$$
$$\delta \gamma_{\xi\xi} = O(\epsilon^{d-2}).$$

The unit normal and derivatives give a flux density of order ϵ^d:

$$\boxed{\Delta S^{\mathrm{D}}_{m-1} = O(\epsilon^2) \longrightarrow 0.}$$

The variation of entropy III

Surface term: The surface term is

$$\Delta S_{\text{surf}} = \frac{1}{G_{d+1}} \int_{\partial} d^{d-2}x \sum_{m=2}^{[\frac{d+1}{2}]} m c_m L^{2m-2} \frac{1}{2^{m-1}} (m-1) \sum_{l=0}^{m-2} \frac{(m-2)!(-2)^l}{(m-2-l)!l!(2l+1)}$$
$$\times \delta \left[\delta^{a\cdots a_{2m-4}}_{b\cdots b_{2m-4}} \sqrt{\sigma} \prod_{k=1}^{m-2-l} \mathcal{R}^{\partial b_{2k-1} b_{2k}}{}_{a_{2k-1} a_{2k}}[\sigma] \prod_{i=1}^{l} \mathcal{K}^{b_{2(m-2-l)+2i-1}}_{a_{2(m-2-l)+2i-1}} \mathcal{K}^{b_{2(m-2-l)+2i}}_{a_{2(m-2-l)+2i}} \mathcal{K}^{b}_{a} \right].$$

The variation gives the linear combination of the following terms

$$\sqrt{\sigma}\sigma^{ab}\delta\sigma_{ab},\ \sqrt{\sigma}\delta\tilde{\sigma},\ \sqrt{\sigma}\delta\mathcal{K}.$$

Here $\delta\mathcal{K} = O(\epsilon^d)$ and $\delta\tilde{\sigma} = O(\epsilon^d)$.

Cutoff scaling of each varied monomial

$$\epsilon^{-(d-2)}\epsilon^d = \epsilon^2 \quad \Longrightarrow \quad \boxed{\Delta S_{\text{surf}} = O(\epsilon^2).}$$

Timelike first law and the linearized equations

The two variations contain exactly the same factor:

$$\Delta S - \Delta\langle H\rangle = \left(1 - \sum\nolimits_{m=2}^{\lfloor d/2\rfloor} m\lambda_m f_\infty^{m-1}\right) \times \left(\Delta S^E - \Delta\langle H\rangle^E\right).$$

If this factor is nonzero, for the required family of hyperbolic regions,

$$\begin{aligned} \Delta S = \Delta\langle H\rangle \quad &\Longleftrightarrow \quad \Delta S^E = \Delta\langle H\rangle^E \\ &\Longleftrightarrow \quad \delta G_{AB} = 0 \end{aligned}$$

Result

The equivalence extends to Lovelock gravity in the normalizable, hyperbolic perturbative sector with fixed boundary sources.

Comparison with complex extremal surfaces

The vacuum hyperbolic surface can also be written in complex coordinates:

$$\rho^2 - z^2 = T_0^2, \qquad z = T_0 \sinh y, \quad \rho = T_0 \cosh y$$

For excited states, a real hyperbolic-angle integral has additional noncompact divergences which can be removed by regularization.

$$\frac{\Delta S}{\mathrm{Vol}(\mathbb{H}^{d-2})\, T_0^{d-2}} = \frac{\tilde{L}^{d-1} T_0^2 m_z}{8 G_{d+1}} \frac{1}{d^2-1}(1 - 2\lambda f_\infty)$$

Matched continuation and subtraction

Continue $\psi = i\theta$, $T_0 \to -iT_0$ and obtains, in Gauss–Bonnet gravity,

$$\frac{\Delta S}{\Omega_{d-2} T_0^{d-2}} = -\frac{m_z T_0^2 \tilde{L}^{d-1}}{8 G_{d+1}(d^2-1)}(1 - 2\lambda f_\infty).$$

Conclusions

Timelike entanglement first law $\Longleftrightarrow$ linearized field equation of Lovelock gravity

Hyperbolic timelike subregions, linear order around AdS:

$$\Delta S = \mathcal{C}_{\mathrm{h}} \Delta S^{E},$$
$$\Delta \langle H \rangle = \mathcal{C}_{\mathrm{h}} \Delta \langle H \rangle^{E},$$
$$\sum_{m} c_m L^{2m-2} \delta \mathcal{E}^{(m)}_{AB} = \mathcal{C}_{\mathrm{h}} \delta G^{E}_{AB}.$$

with one common prefactor $\mathcal{C}_{\mathrm{h}} \equiv 1 - \sum_{m=2}^{\lfloor d/2 \rfloor} m \lambda_m f_{\infty}^{m-1}$, which reduces to 1 (Einstein gravity) when the Lovelock couplings vanish.

- **Assumes** the double Wick rotation and the continued JM functional; not a Lorentzian replica derivation.
- **Surface term vanishes** for normalizable perturbations in FG gauge, hyperbolic slicing and a fixed conformally flat boundary.

Thanks!