Thermodynamics and chaotic orbits for semi-classical black hole with quantum anomaly correction

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Based on arXiv： 2410.23783， 2512.19163， 2607.26493 @BRICS-AGAC， HangZhou

Quantum anomaly triggers the violation of scaling laws in gravitational system

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Scaling laws for critical phenomena take pivotal status in almost all branches of physics. However, as scaling laws are commonly guaranteed by the renormalization group theory, systems that violate them have rarely been found. In this letter, we demonstrate that gravitational system can break scaling laws. We derive this result through investigating phase transition and critical phenomenon in a gravitational system with quantum anomaly. For the first time, we outline the key conditions to violate the scaling laws in generic gravitational system viewed from the equation of state $P = P(T, V)$. Our results indicate that quantum effects can magnify the distinctiveness of gravity, which may be significant to understand the microscopic structure of spacetime.

2410.23783 on thermodynamics

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Letter

Probing quantum anomaly corrections on black hole physics through chaos

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2512.19163 on chaotic dynamics

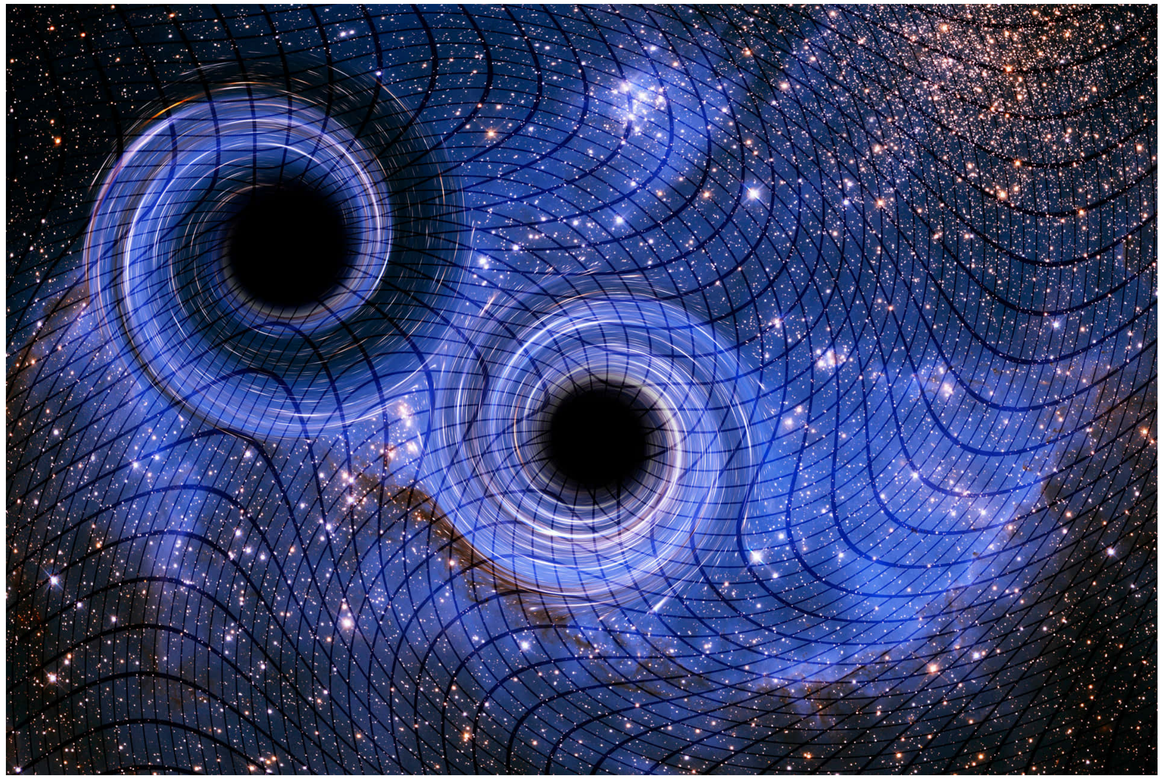
Chaotic Imprints in Gravitational Waves from Conformal-Anomaly-Corrected Extreme-Mass-Ratio Inspirals

Wei-Hao Zhang[a], Yu-Sen An[a,b,1]

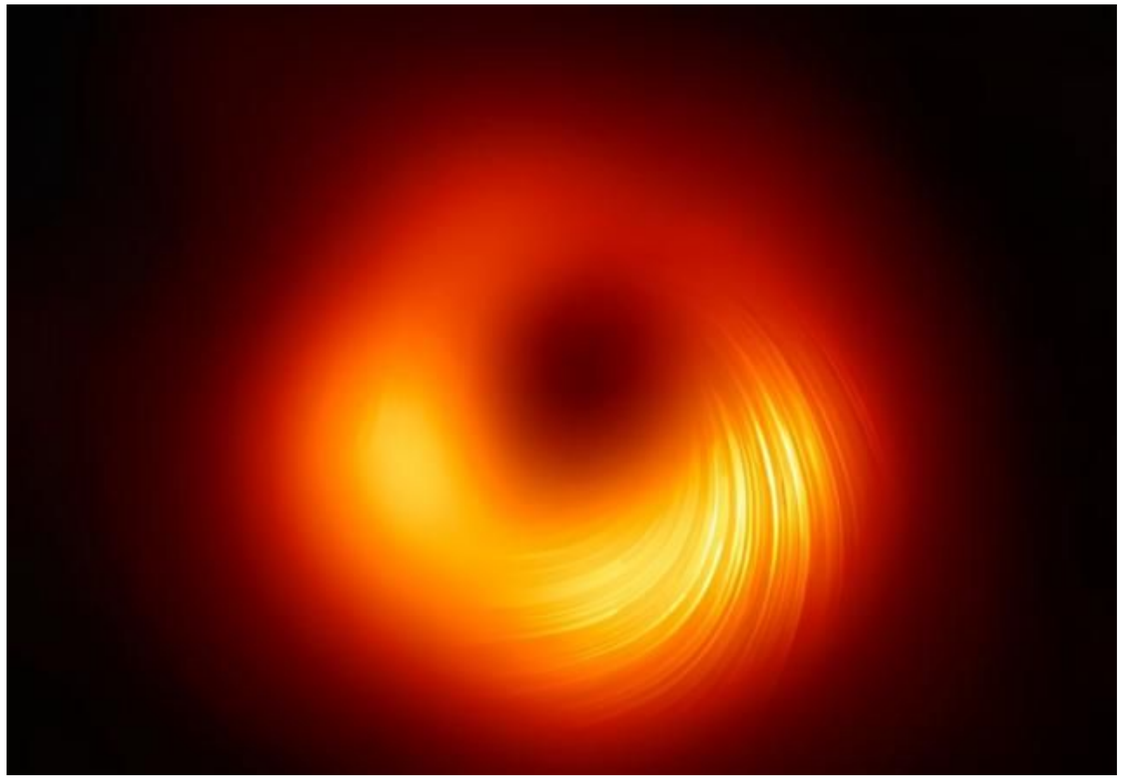
[a] *Center for the Cross-disciplinary Research of Space Science and Quantum-technologies (CROSS-Q), College of Physics, Nanjing University of Aeronautics and Astronautics, Nanjing, 210016, China,*
[b] *MIIT Key Laboratory of Aerospace Information Materials and Physics, Nanjing University of Aeronautics and Astronautics, Nanjing, 210016, China,*

2607.26493 on gravitational waves

Black hole： A crucial prediction by GR



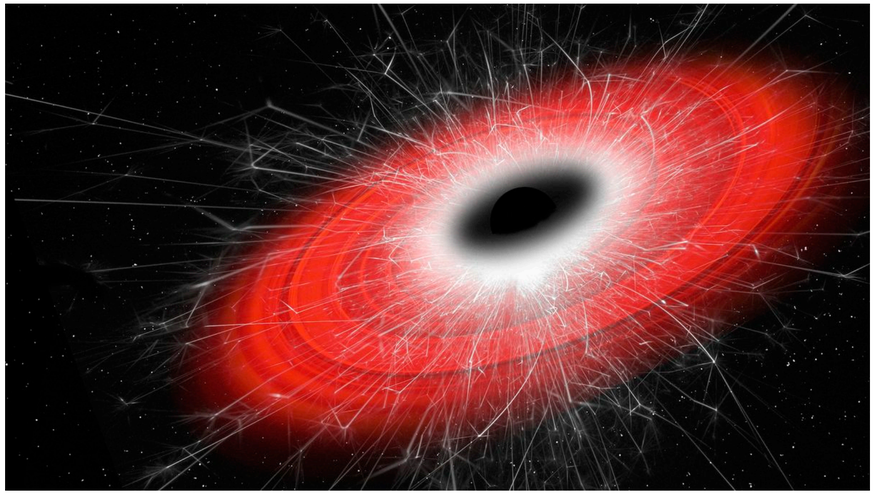
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Black hole image of M87*

Black hole is a real astrophysical object !

Black hole's physical properties:



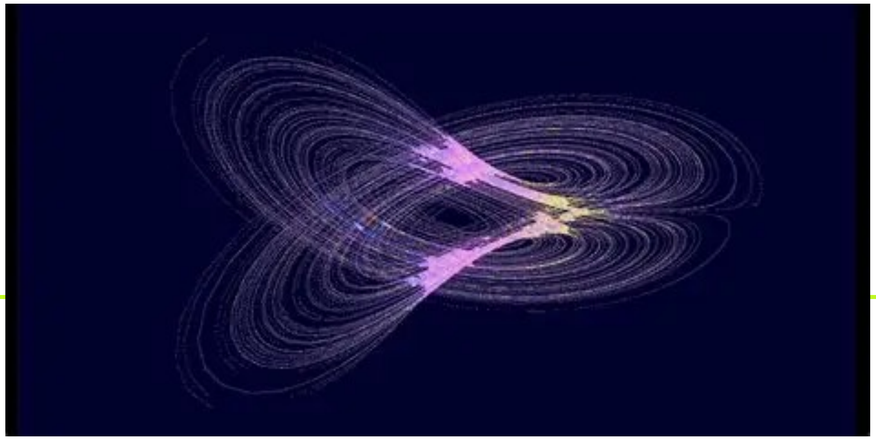
Black hole is a thermodynamical system.

Hawking radiation, Bekenstein entropy and thermodynamic laws.



Black hole is the nest of chaos.

Single-body chaos [Hashmoto,2017], many-body chaos [Shenker, 2013].



Thermodynamics:

Black hole thermodynamics:

Einstein gravity: $T = \frac{\kappa}{2\pi}, \quad S = \frac{A}{4G}$

Semi-classical approximation: matter field is quantized, spacetime is classical.

No back-reaction by QFT in curved spacetime.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle_{HH}$$

How to explore quantum backreaction on the black hole ?

Conformal anomaly:

Four dimensional conformal anomaly: : $\langle T^{\mu}_{\mu} \rangle_a = \beta I_4 - \alpha_c E_4$

α_c, β : Type A& B central charge, positive constant

$I_4 = C_{\mu\nu\lambda\delta} C^{\mu\nu\lambda\delta}$ $E_4 = R_{\mu\nu\lambda\delta} R^{\mu\nu\lambda\delta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$

Considering back-reaction of $\langle T_{\mu\nu} \rangle_a$

Spherical symmetric ansatz:

$$ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) .$$

Both β, α_c non-zero: only series solutions exist.

Switch off type-B anomaly $\beta = 0$: analytical solution. [Cai, 2010,2014]

Analytical BH with anomaly correction:

Anomaly corrected black hole: $$f(r) = 1 - \frac{r^2}{4\alpha_c}\left(1 - \sqrt{1 - 8\alpha_c\left(\frac{2M}{r^3} - \frac{Q^2}{r^4} - \frac{1}{l^2}\right)}\right)$$

Q: U(1) charge of underlying QFT

Finite volume singularity: $$1 - 8\alpha_c\left(\frac{2M}{r_s^3} - \frac{Q^2}{r_s^4} - \frac{1}{l^2}\right) = 0$$

$\alpha_c \to 0$: AdS-Schwarzschild black hole

$\alpha_c \to -\alpha/2$: 4d EGB black hole solution [Glavan,2020]

[Same effective action !] $$S_{eff} = \int d^4x \left(R - 2\Lambda + a\left(\sigma E_{(4)} - 4G^{\mu\nu}\nabla_\mu\sigma\nabla_\nu\sigma - 4(\nabla^2\sigma)(\nabla\sigma)^2 - 2(\nabla\sigma)^4\right)\right)$$

Used to prove a-theorem [Komargodski, 2011]

BH Thermodynamics with anomaly correction:

Mass and temperature: $M = \frac{3r_h^4/l^2 + 3r_h^2 + 3Q^2 - 6\alpha_c}{6r_h}$ $T = \frac{3r_h^4/l^2 + r_h^2 - Q^2 + 2\alpha_c}{4\pi r_h^3 - 16\pi\alpha_c r_h}$

Logarithmic correction of entropy:

$$S = \int \frac{dM}{T} = \frac{A}{4} - 4\pi\alpha_c \ln(\frac{A}{A_0})$$ Matches micro-state counting in next leading order

Black hole enjoys rich thermodynamic phase structure [Kubiznak,2012]

RN-AdS black hole  Van de Waals Liquid-gas system

Criticality： Mean field theory universality class

$$\alpha = 0, \quad \beta = 1/2, \quad \gamma = 1, \quad \delta = 3.$$

$$C_V = T(\frac{\partial S}{\partial T})_V \propto |t|^{-\alpha}$$

$$\eta = \frac{V_l - V_s}{V_c} \propto |t|^{\beta}$$

$$\kappa_T = -\frac{1}{V}(\frac{\partial V}{\partial P})_T \propto |t|^{-\gamma}$$

$$\tilde{P} - 1 \propto \omega^{\delta}$$

Whether conformal anomaly influence it ?

Black hole critical phenomenon with conformal anomaly:

Equation of State:

$$P = \frac{T}{2r_h} - \frac{1}{8\pi r_h^2} - \frac{2\alpha_c T}{r_h^3} + \frac{-2\alpha_c + Q^2}{8\pi r_h^4}$$

Critical Point:

$$\frac{\partial P}{\partial V} = 0, \quad \frac{\partial^2 P}{\partial V^2} = 0$$

$$V_c = \frac{4\pi}{3}(-12\alpha_c + 3Q^2 \pm K)^{3/2}, \quad T_c = \frac{3Q^2 \mp K}{48\pi\alpha_c\sqrt{-12\alpha_c + 3Q^2 \pm K}}, \quad P_c = \frac{-18\alpha_c + 6Q^2 \pm K}{24\pi(-12\alpha_c + 3Q^2 \pm K)^2},$$

$$K^2 \equiv 192\alpha_c^2 + 9Q^4 - 96\alpha_c Q^2 = 3(Q^2 - 8\alpha_c)(3Q^2 - 8\alpha_c)$$

$\alpha_c < \frac{Q^2}{8}$ Two critical points instead of one.

Black hole critical phenomenon with conformal anomaly:

Expansion of E.O.S:

$$\tilde{P}(t,w) = 1 + \sum_{i\geq 1} a_{i0} t^i + \sum_{j\geq 3} a_{0j} w^j + \sum_{i,j\geq 1} a_{ij} t^i w^j$$

$$t = T/T_c - 1$$

$$w = V/V_c - 1$$

Coefficient:

$$a_{10} = \frac{4(-4\alpha_c + K + 3Q^2)}{-22\alpha_c + 9Q^2}, \quad a_{11} = \frac{-48\alpha_c - 10K + 6Q^2}{-66\alpha_c + 27Q^2}, \quad a_{03} = \frac{40\alpha_c + K - 15Q^2}{27(-22\alpha_c + 9Q^2)}$$

$a_{11} \neq 0$ for $0 \leqslant \alpha_c < \frac{Q^2}{8}$, $a_{11} = 0$ for $\alpha_c = \frac{Q^2}{8}$.

$a_{11} \neq 0$ case: traditional mean field theory result.

$a_{11} = 0$ case: A new kind of critical point

Black hole critical phenomenon with conformal anomaly:

$$\tilde{P} = 1 + a_{10}t + a_{12}tw^2 + a_{21}t^2w + a_{03}w^3 + \dots \qquad \tilde{P}(t,w) = 1 + \frac{8}{5}t - \frac{4}{15}tw^2 - \frac{8}{135}w^3 + \mathcal{O}(tw^3, w^4)$$

α:

$$S = A/4 + c\log(A)$$

$$V = \frac{4\pi r_h^3}{3}$$

$$\longrightarrow \quad C_V \equiv 0 \quad \longrightarrow \quad \alpha = 0$$

β:

Maxwell Construction:

$$\tilde{P}_l = \tilde{P}_g \qquad G_l = G_g$$

$$\longrightarrow \quad w_l = \mathrm{A}_1 t \qquad w_s = \mathrm{B}_1 t \quad \longrightarrow \quad \beta = 1$$

$$C_V = T(\frac{\partial S}{\partial T})_V \propto |t|^{-\alpha}$$

$$\eta = \frac{V_l - V_s}{V_c} \propto |t|^{\beta}$$

$$\kappa_T = -\frac{1}{V}(\frac{\partial V}{\partial P})_T \propto |t|^{-\gamma}$$

$$\tilde{P} - 1 \propto \omega^{\delta}$$

γ:

$$\kappa \propto -(\frac{\partial \tilde{P}}{\partial \omega})_t^{-1} \propto t^{-2}$$

$$\gamma = 2$$

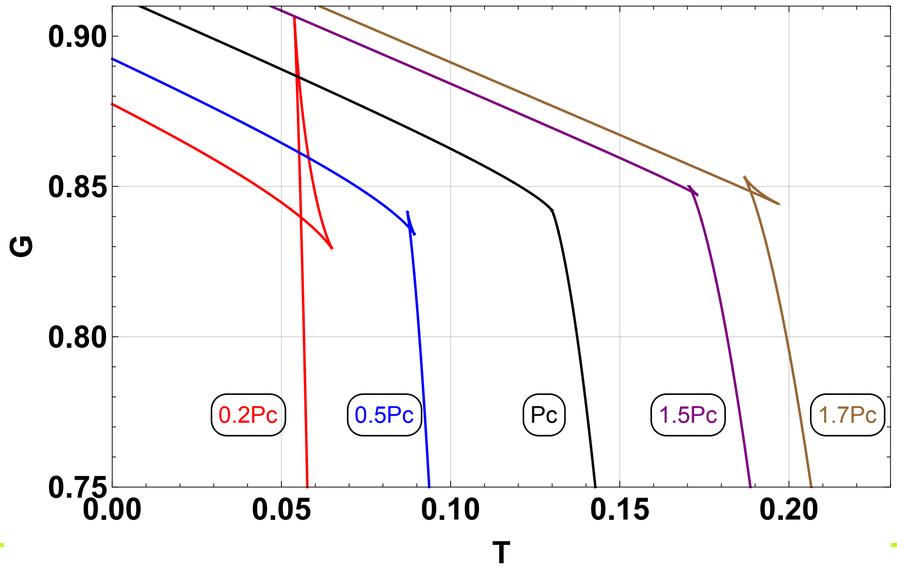
δ:

$$\tilde{P} - 1 \propto \omega^3$$

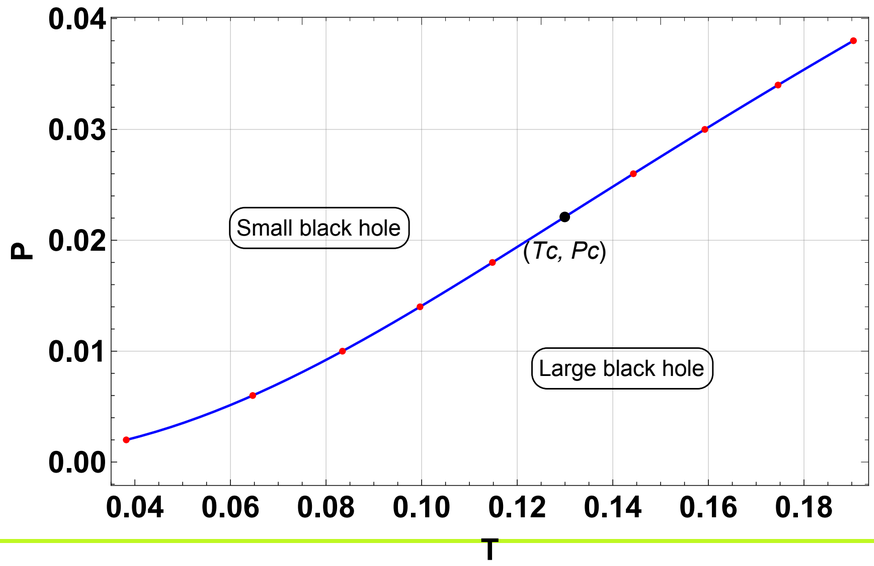
$$\delta = 3$$

Black hole critical phenomenon with conformal anomaly:

- $\alpha = 0, \quad \beta = 1, \quad \gamma = 2, \quad \delta = 3.$ Beyond mean field theory
- $\alpha + 2\beta + \gamma \neq 2, \quad \alpha + \beta(1 + \delta) \neq 2$ Violate scaling law
- First Violation of scaling law in 4d physical system. (Key condition:$a_{11} = 0$)
- Isolated critical point



Swallowtail behavior



Coexistence curve

Chaos:

Black hole single-body chaos

Black hole lead to integrable-chaotic transition of particle motions [Hashmoto 2016, Dalui 2019]

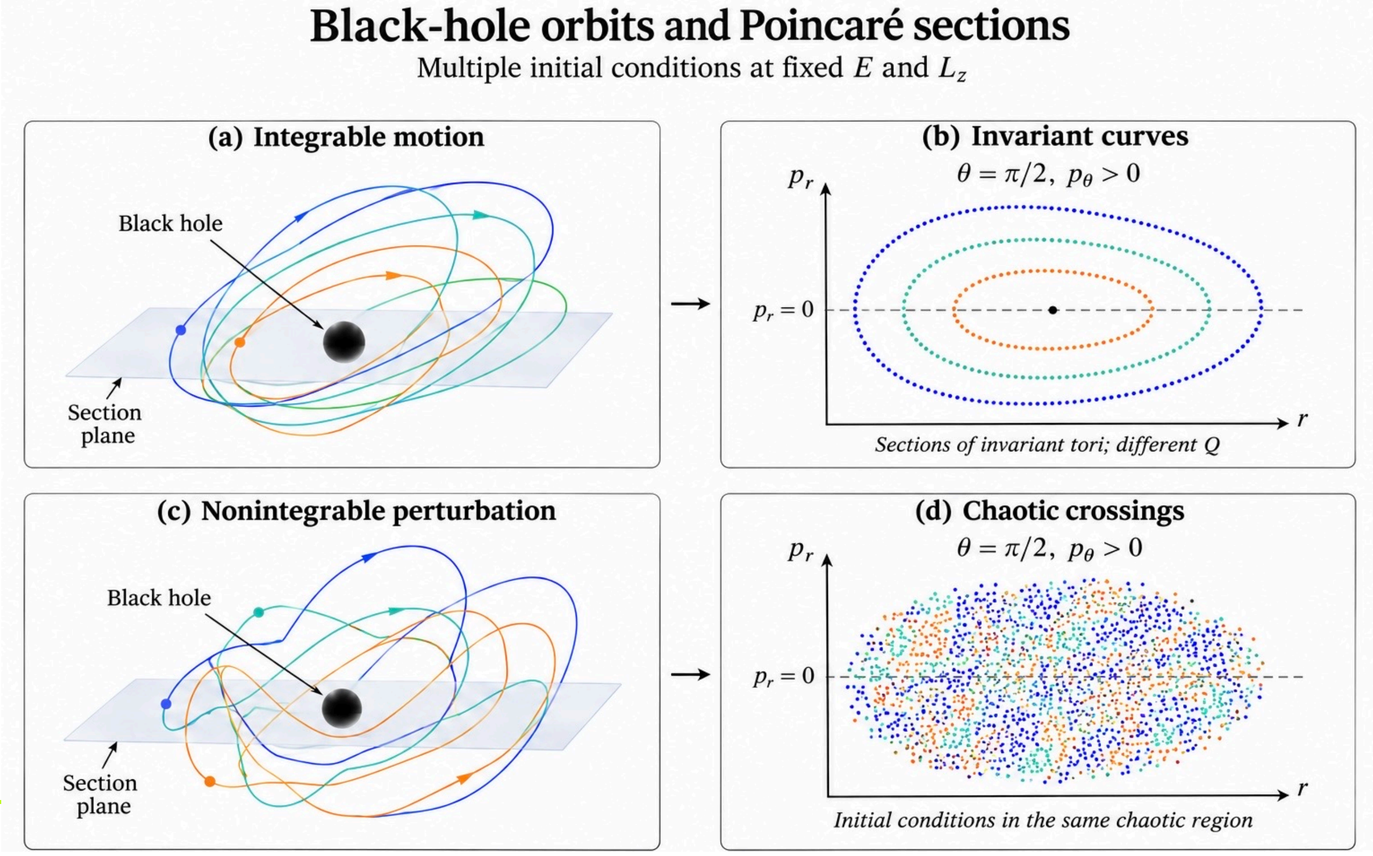
Classical chaos indicator:

Poincare section, Lyapunov exponent etal.

Distinct from the quantum many body

chaos in [Shenker, 2013] which is

characterized by OTOC, random matrix etal.



Black hole single-body chaos

Metric: $ds^2 = -f(r)dt^2 + 2\sqrt{1-f(r)}dtdr + dr^2 + r^2 d\Omega^2$

$$f(r) = 1 - \frac{r^2}{4\alpha_c}\left(1 - \sqrt{1 - 8\alpha_c\left(\frac{2M}{r^3} - \frac{Q^2}{r^4} - \frac{1}{l^2}\right)}\right)$$

Hamiltonian: $$E = -\sqrt{1-f(r)}p_r + \sqrt{p_r^2 + \frac{p_\theta^2}{r^2} + m^2} + \frac{1}{2}K_r(r-r_c)^2 + \frac{1}{2}K_\theta r_h^2(\theta-\theta_c)^2$$

→ External potential

EOM:

$$\dot{r} = -\sqrt{1-f(r)} + \frac{p_r}{\sqrt{p_r^2 + \frac{p_\theta^2}{r^2} + m^2}}$$

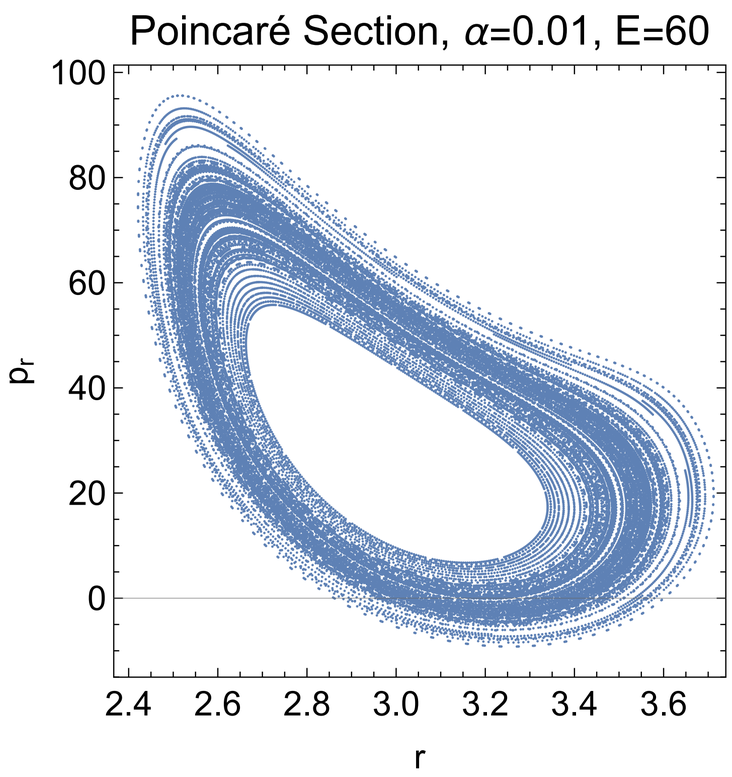
$$\dot{p}_r = -\frac{f'(r)}{2\sqrt{1-f(r)}}p_r + \frac{p_\theta^2/r^3}{\sqrt{p_r^2 + \frac{p_\theta^2}{r^2} + m^2}} - K_r(r-r_c)$$

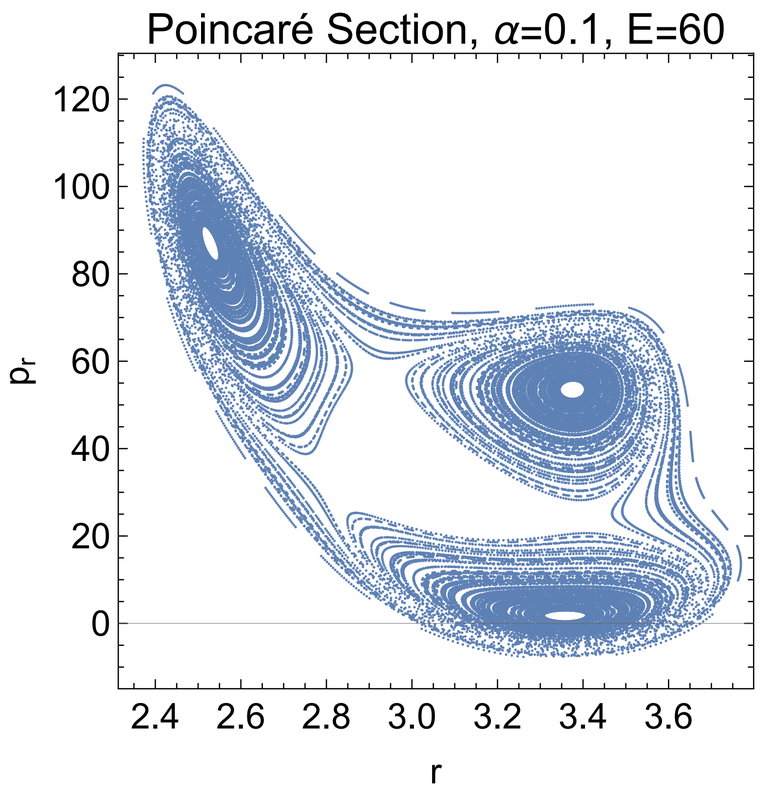
$$\dot{\theta} = \frac{p_\theta/r^2}{\sqrt{p_r^2 + p_\theta^2/r^2 + m^2}}$$

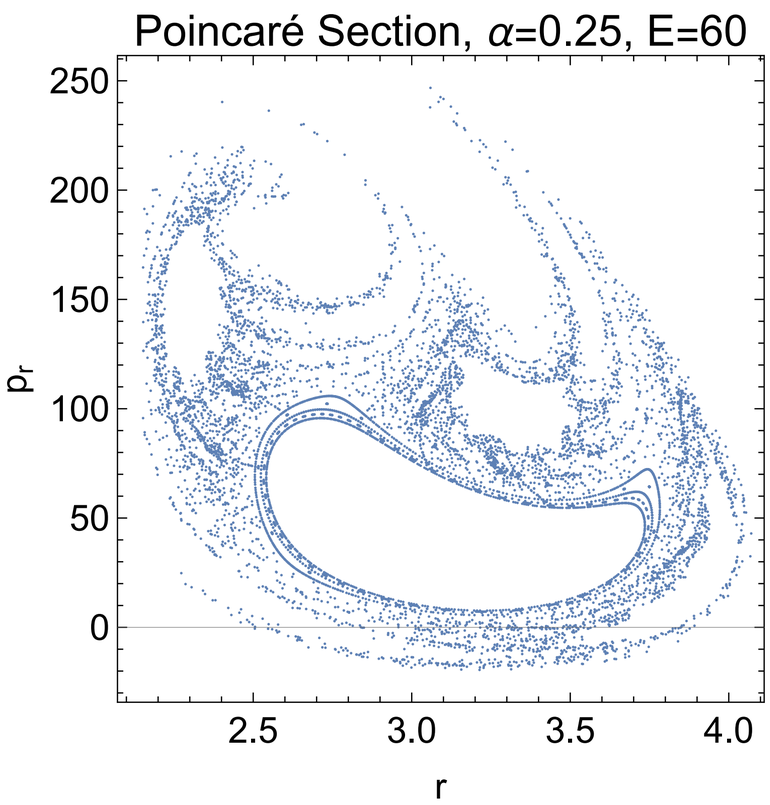
$$\dot{p}_\theta = -K_\theta r_h^2(\theta-\theta_c)$$

Chaotic dynamics:

Poincare section for $r-\theta$ plane: (For Massive neutral particles)







As conformal anomaly increases, the Poincare section transit from regular KAM tori to scattered points.

Chaotic dynamics:

Lyapunov exponent : $\lambda_L = \lim_{t\to\infty} \frac{1}{t} \ln(\frac{\delta r(t)}{\delta r(0)})$

(For massive charged particles)

Charged particles: $L = \frac{1}{2f(r)^{3/2}}\dot{r}^2 - V_{eff}(r), \qquad V_{eff}(r) = \sqrt{f(r)} + \frac{e}{m}\psi_e(r)$

Perturbations around horizon

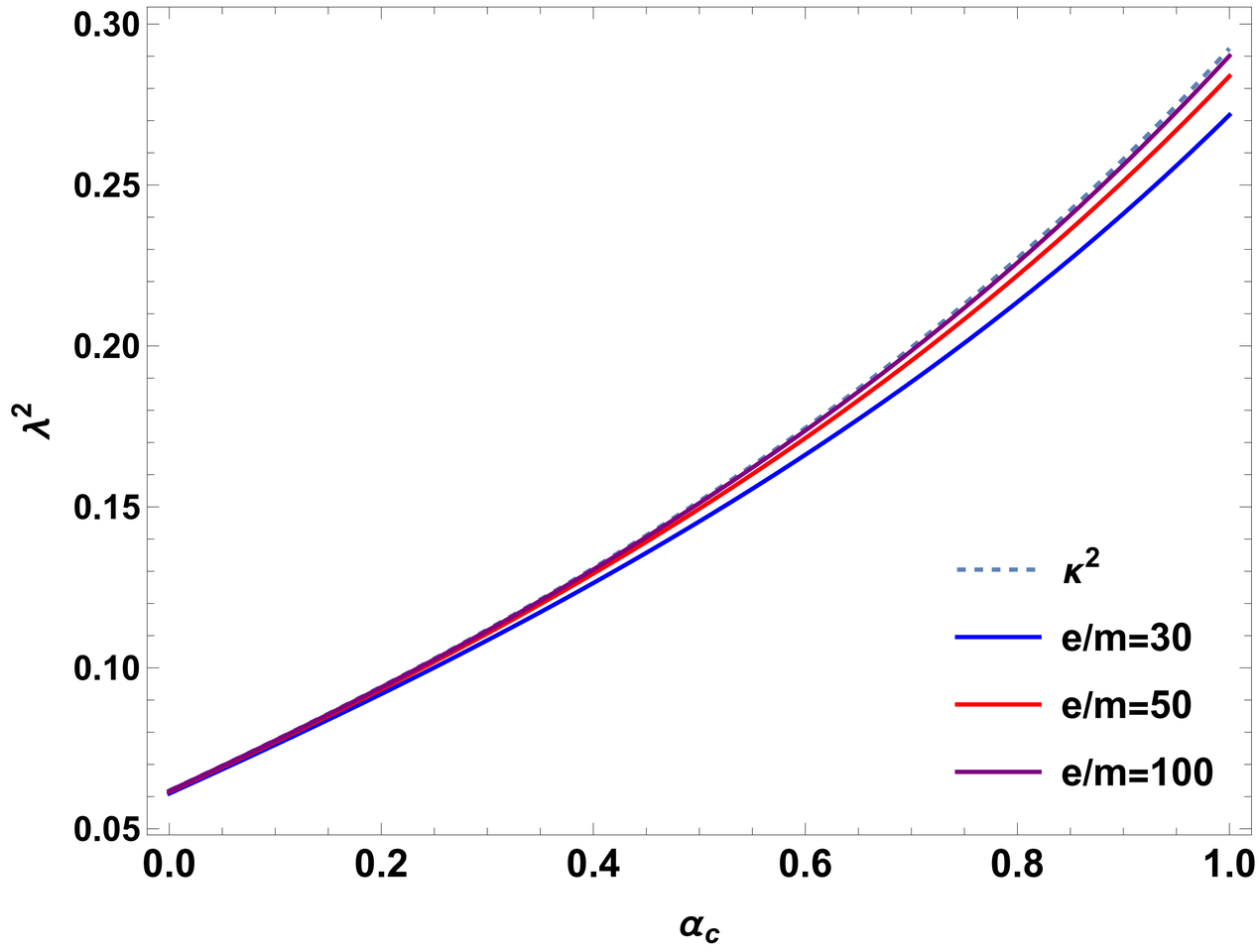
$$f(r) = f_1(r - r_h) + f_2(r - r_h)^2 + \dots \qquad \psi_e(r) = \psi_{e0} + \psi_{e1}(r - r_h) + \psi_{e2}(r - r_h)^2 \dots$$

Perturbed solutions:

$$\frac{e}{m} = -\frac{\sqrt{f_1}}{2\psi_{e1}\sqrt{r_0 - r_h}} + \frac{-3f_2\psi_{e1} + 4f_1\psi_{e2}}{4\sqrt{f_1}\psi_{e1}^2}\sqrt{r_0 - r_h}$$

$$\lambda^2 = \kappa^2 + \gamma(r_0 - r_h) + \mathcal{O}((r_0 - r_h)^2), \quad \gamma = 4\kappa^2\frac{\psi_{e2}}{\psi_{e1}}$$

Chaotic dynamics:



Lyapunov exponent

From Poincare section and Lyapunov exponent, conformal anomaly universally enhances chaos of particles.[An,2025]

Gravitational wave signals:

Extreme-Mass-Ratio inspirals： central object as background, small object as particle around it

Gravitational wave depends on particle trajectory: chaotic orbit leads to distinct GW signal.

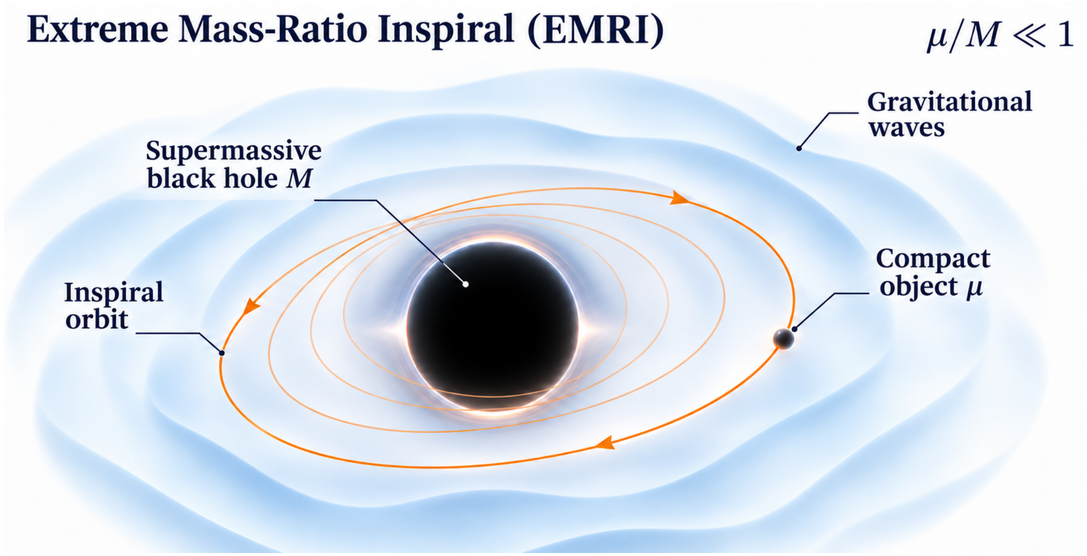
Numerical kludge method:

Quadrupole formula

$$\bar{h}^{ij} = \frac{2G}{r}\left[\ddot{I}^{ij}(t')\right]_{t'=t-|x-x'|}$$

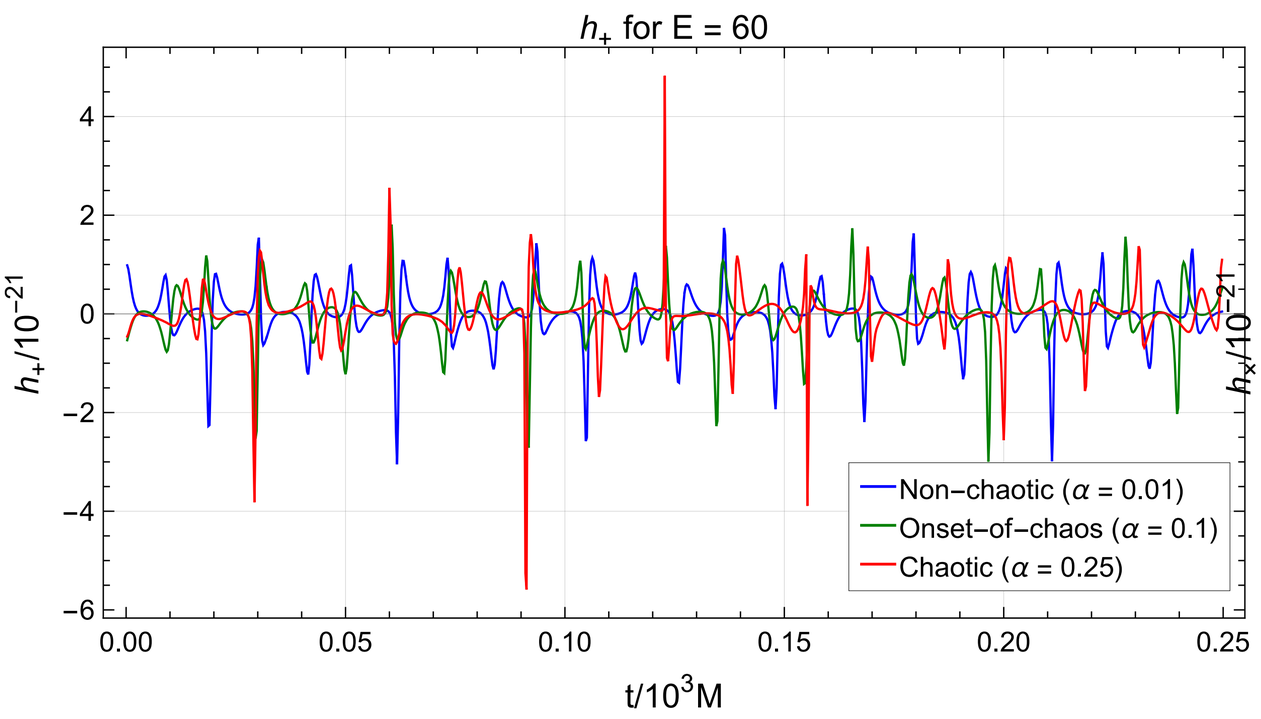
$$T^{tt}(t, x^i) = m\delta^3(x^i - Z^i(t))$$

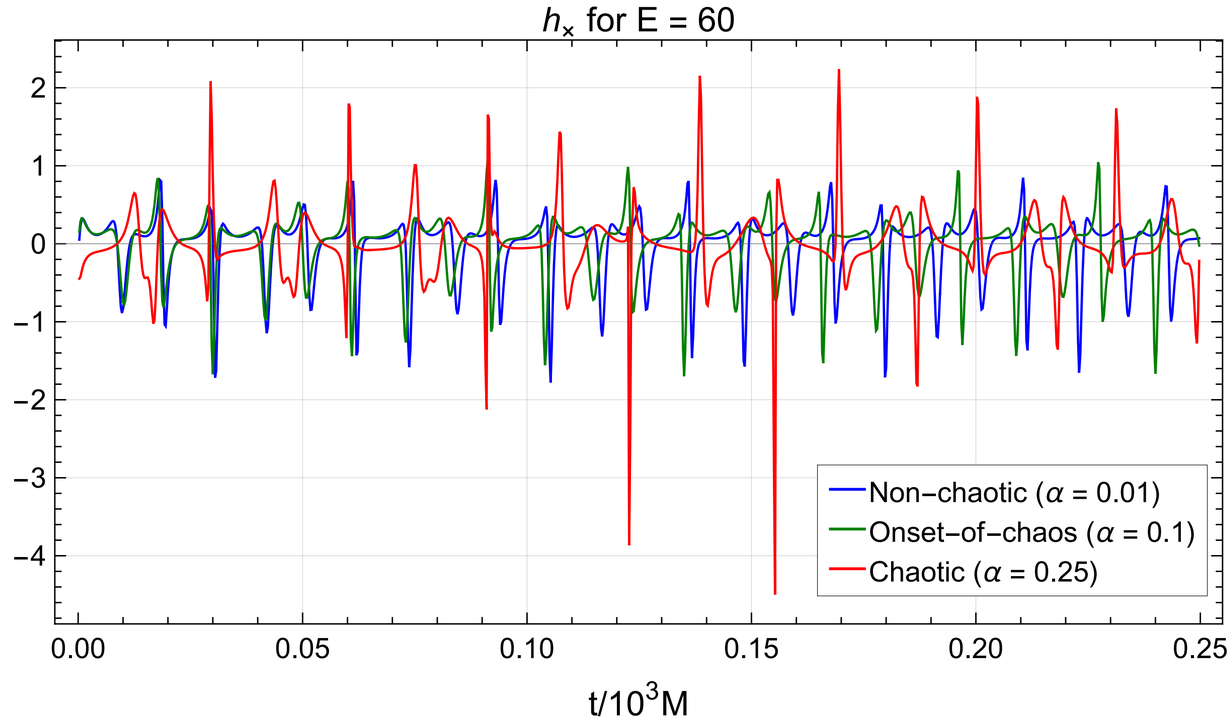
$$h_{ij} = \frac{2G}{D_L}\frac{d^2 I_{ij}}{dt^2} = \frac{2Gm}{D_L}(a_i x_j + a_j x_i + 2v_i v_j) \xrightarrow{\text{detector frame}} h_+, \quad h_\times$$



[Das etal, 2025]

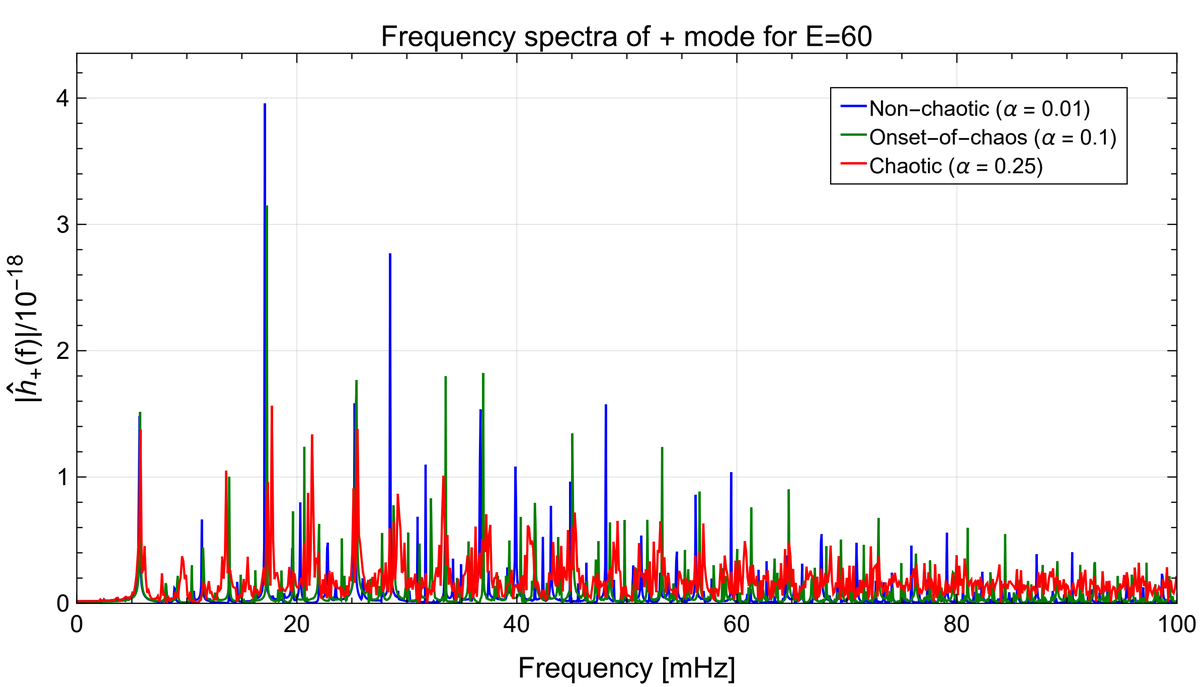
Gravitational wave signals:

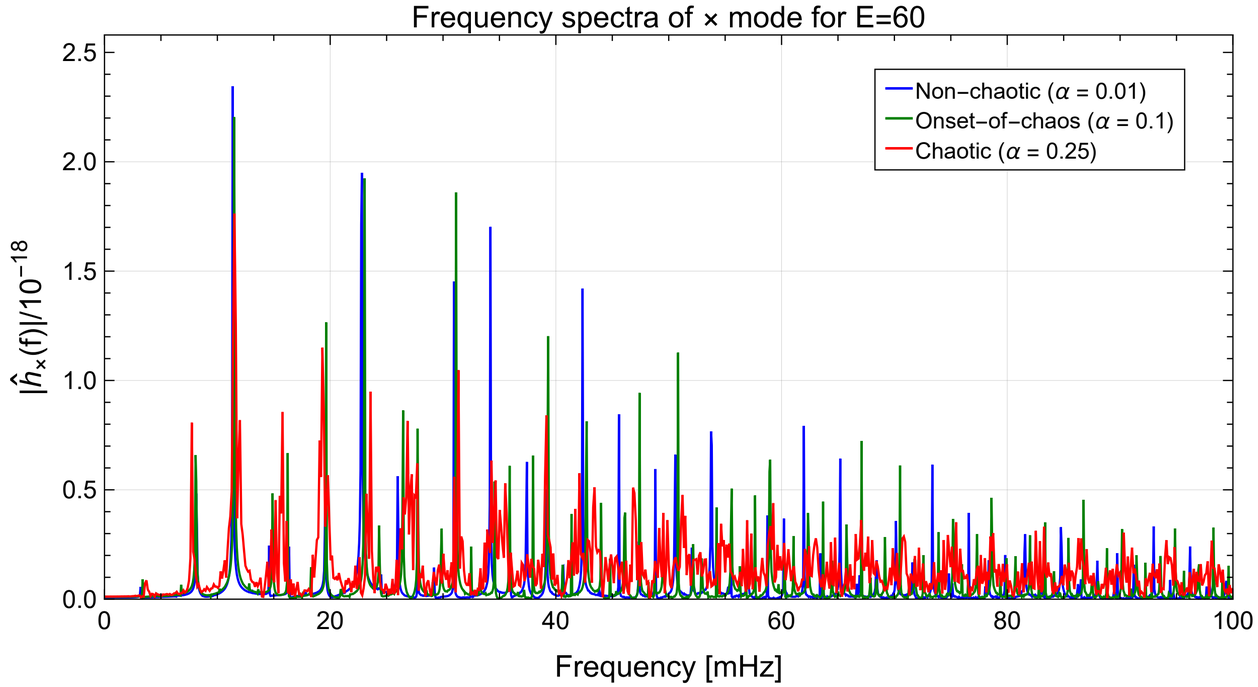




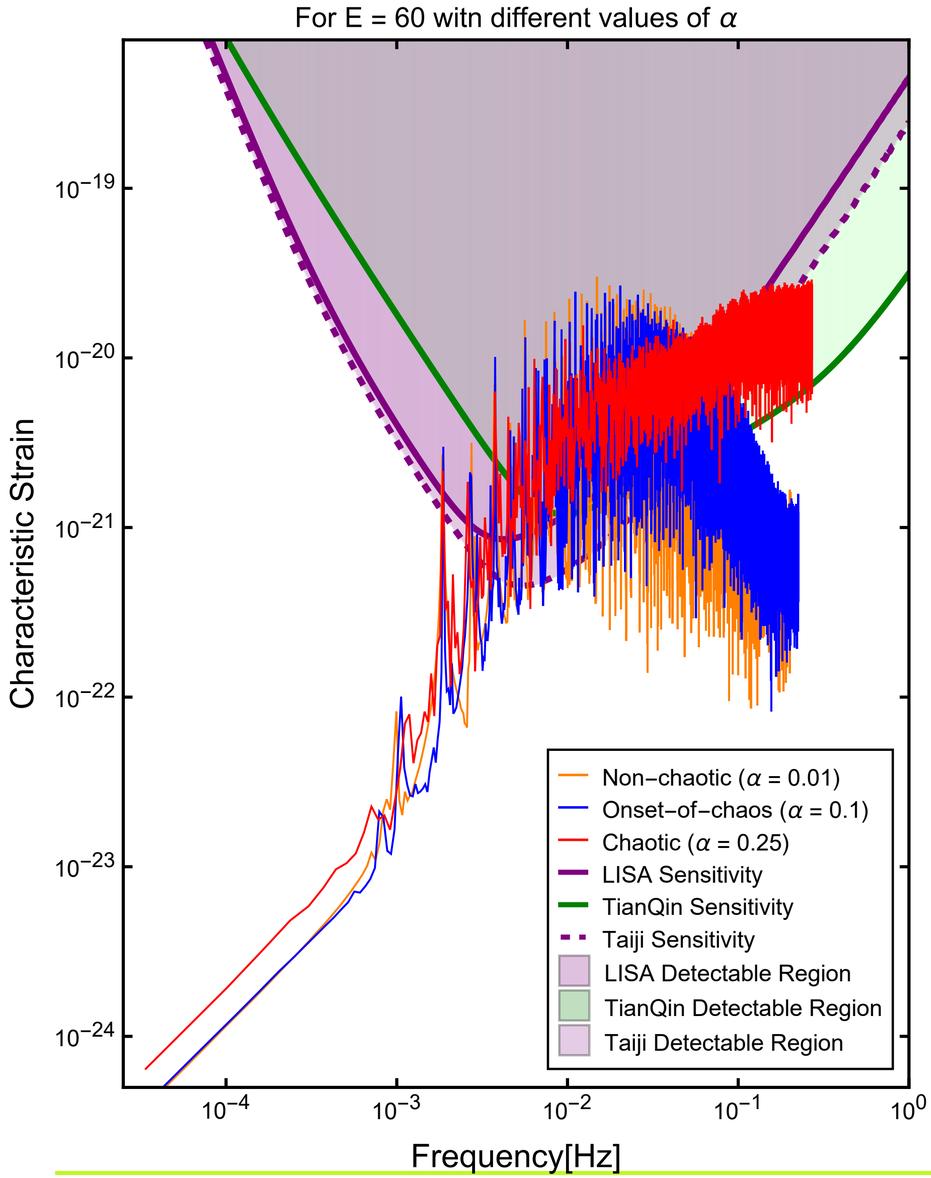
Anomaly coefficient → Chaotic orbit → Gravitational waveform!

Gravitational wave signals:





Extended frequency domain



Characteristic strain: $h_c(f) = 2|f|\sqrt{|\hat{h}_+(f)|^2 + |\hat{h}_\times(f)|^2}$

Sensitivity Curve of space GW detector $h_n(f) = \sqrt{fS_n(f)}$

LISA: $$S_n^L(f) = \frac{10}{3L^2}\left(P_{OMS} + \frac{2P_{acc}}{(2\pi f)^2}(1+\cos^2(\frac{f}{f^*}))\right) \times \left(1 + 0.6(\frac{f}{f^*})^2\right),$$

Tianqin: $$S_N(f) = \frac{10}{3L^2}\left(S_x + \frac{4S_a}{(2\pi f)^4}(1+\frac{10^{-4}\text{Hz}}{f}\right) \times \left(1 + 0.6(\frac{f}{f_*})^2\right)$$

Taiji: $$S_n^T(f) = \frac{10}{3L^2}\left(P_{dp} + \frac{2P_{acc}}{(2\pi f)^4}(1+\cos^2(\frac{f}{f^*}))\right) \times \left(1 + 0.6(\frac{f}{f^*})^2\right),$$

Can be detected in future observation.

Future direction

- Implications of scaling law violation (from holographic side ?)
- Rotating black hole [P.G.Fernandes,2023].
- General solutions with anomaly correction.

A lot to be explored.....

Thanks