The Complex Conformal Kähler Structure of Type-D Spacetimes

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Frame Title

This is based on my work with Prof. Lars Andersson, KTH.

Catalog

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2. Fundamentals of Conformal c-Kähler structure
3. Conformal c-Kähler structure
4. Duality symmetry
5. The moment polytope of Lorentzian type-D
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Introduction

In 1970's, Penrose found that with Lorentzian signature, a type-D Einstein metric is conformally related with a complexified Kähler metric. Then Atiyah-Hitchin-Singer gave the version in Riemannian geometry, which shows there's a complex structure on a self-dual 4-manifold. The mean method they used to proof this result is the spinor and twistor theory.

Introduction

After that, many people studied the conformal Kähler structure for some special kinds of 4-manifolds with both signature (actually these two case are more or less very similar). For example, Derdzinski and LeBrun studied the self-dual Kähler and Einstein 4-manifolds and showed that the conformal factor here is a power function of the rescaling of the scalar curvature of the metric. These years, more and more people are interested in a kind of metric with "toric" symmetry, which means there's two commutative Killing vector fields. The type-D spacetime, is one of such classes of metrics.

Introduction

Some basic results of Kähler geometry are also used. Flaherty firstly found there is a complexified complex structure in Lorentzian 4-manifold in 1976, which means the metrics, connections, curvatures and coordinates are complex valued. In such a condition, one can build a theory of complex geometry, as well as the Kähler geometry and study the related structures of GR. In 2022, Steffen and Bernardo proved that metrics in Lorentzian Plebański-Demiański family have a complexified conformal Kähler structure and they gave a specific form of the Kähler potential, claimed the duality symmetry of this family and how to construct new solutions by transforming the Kähler potential.

Introduction

In our work, we would give a continuous (which is not given in Steffen-Bernardo) duality transformation in whole Kerr-Taub-NUT class and obtained a Monge–Ampère system to reconstruct the type-D family, which is another method distinguish to the Ernst method.

Introduction

Another important result in Kähler geometry we will use is the Delzant theory, which told us: for a compact symplectic manifold (so our Kähler manifold) with a T^k effective Hamilyonian action, the image of the moment map is a Delzant's polytope, which is a convex polytope with the vertexes the classes of fixed points of the T^k action and the edges the stabilizers.

Introduction

- Biquard and Gauduchon found this result also suitbal for Einstein metrics though the related spaces are non-compact. One just need to change one of the edge into the direction goes to infinity, then there's also a "polytope" with one edge refers to the asymptotically infinity. Use the reduction above, they proved that all of the toric ALF (asymptotically locally flat) Riemannian 4-manifold should be one of these up to a differmorphism: Kerr, Taub-NUT, Taub-bolt or Chen-Teo (which is a 5 parameter class and is also conformal Kähler, but it can only be well defined in Riemannian case).
- In our work, we give the version of Lorentzian case, and show that the duality transformation is actually related with the deformation of the image of the moment map, which gives a new method to understand the type-D class, and even try to find solutions beyond P-D class.

Complex spacetime

Definition

A complex spacetime is a spacetime $(M, g_{\mathbb{C}})$ with the complexified Riemannian metric $g_{\mathbb{C}} = g \otimes \mathbb{C} : (TM \otimes \mathbb{C}) \otimes (TM \otimes \mathbb{C}) \to \mathbb{C}$. It is complex dimension 4, which is real dimension 8.

Lorentzian Hodge star operator

Definition

The Lorentzian Hodge star operator on 4-manifold M is an endmorphism:

$$* : \Omega^2(M) \to \Omega^2(M)$$

with eigenvalues i and $-i$. For a 2-form $\alpha \in \Omega^2(M)$, it is called self-dual (SD) [resp. anti-self-dual (ASD)] if $*\alpha = i$[resp. $-i$]α.

Spinors

According to Penrose and Windler, one can use two spinors (the sections of Weyl bundle) to represent a world tensor in the complex spacetime

$$v^a = o^A o^{A'}$$

We have decomposed $\Lambda^2(M, \mathbb{C}) = \Lambda^2_+ \oplus \Lambda^2_-$, the SD and ASD parts. And $\Lambda^2(M, \mathbb{C}) = \mathrm{so}(3,1) \otimes \mathbb{C} = sl(2, \mathbb{C}) + sl(2, \mathbb{C})$, the double cover shows why we can write a world vector into two spinors. Specially, for metric we have

$$g_{ab} = \varepsilon_{AB} \varepsilon_{A'B'}$$

where ε_{AB} the 2-dim Livi-Civita symbol, which is skew-symmetry and with norm 2.

type-D and Einstein condition

Recall that the curvature operator of the 4-manifold is an endmorphism: $\mathcal{R} : \Lambda^2 \to \Lambda^2$, and related to the decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, it is a block matrix as follow:

$$\mathcal{R} = \begin{bmatrix} W^+ & B \\ B^* & W^- \end{bmatrix}$$

where $W^\pm$ the SD and ASD parts of the Weyl curvature and B the Bach tensor. The metric is Einstein iff $B \equiv 0$. The metric is type-D if the Weyl curvatures are traceless and has a order 1 eigenvalue and a order 2 eigenvalue, i.e., for example,

$$W^+ = \begin{bmatrix} \lambda & & \\ & -\frac{\lambda}{2} & \\ & & -\frac{\lambda}{2} \end{bmatrix}$$

Conformal Kähler structure

According to Penrose and Newman, one have the Newman-Penrose formalism and then choose a null frame $l^a, n^a, m^a, \bar{m}^a$ and the related spinors $\{o^A, \bar{o}^{A'}, \iota^A, \bar{\iota}^{A'}\}$ satisfies

$$l^a = o^A \bar{o}^{A'}, \quad n^a = \iota^A \bar{\iota}^{A'}, \quad m^a = o^A \bar{\iota}^{A'}, \quad \bar{m}^a = \iota^A \bar{o}^{A'}$$

and

$$l^a n_a = 1, \qquad m^a \bar{m}_a = -1$$

and then one can write the vacuum Einstein equations into a system of equations in N-P formalism. What's more, the type-D condition here is the Weyl spinor

$$\Psi^+_{ABCD} = \psi_2 o_{(A} o_B \iota_C \iota_{D)}$$

similar for $\Psi^-_{A'B'C'D'}$.

Existence of conformal c-Kähler structure

According to the discussion above, use vacuum Einstein equations, the first Bianchi identities and the twistor equations in N-P formalism, one can prove that there's a conformal Killing-Yano (CKY) spinor $\omega_{AB} = i\psi_2^{-1/3} o_{(A}\iota_{B)}$ satisfies the CKY equation:

$$\nabla_{A'(A}\omega_{BC)} = 0 \tag{1}$$

and the related CKY tensor

$$y_{ab} = i\psi_2^{-1/3} o_{(A}\iota_{B)}\varepsilon_{A'B'}. \tag{2}$$

And the related complex structutre is

$$J_a^b = i(o_A\iota^B + \iota_A o^B)\varepsilon_{A'}^{B'} = i(l_a n^b - n_a l^b + \bar{m}_a m^b - m_a \bar{m}^b). \tag{3}$$

Existence of conformal c-Kähler structure

Actually using spinor twistor equations to compare with the CKY equation one can prove that the $\omega_{ab} = \omega_{AB}\varepsilon_{A'B'}$ is more or less a conformal c-Kähler form and is SD, and $\hat{\omega}_{ab} = io_{(A}\iota_{B)}\varepsilon_{A'B'}$ is a c-Kähler form. So there's the c-Kähler metric

$$\hat{g}_{ab} = J_a^c \hat{\omega}_{bc} \tag{4}$$

it is to say, the metric g_{ab} is conformally related with a c-Kähler metric $\hat{g}_{ab}$ with the conformal factor $\Omega = \psi_2^{-1/3}$, i.e. $\hat{g}_{ab} = \psi_2^{2/3} g_{ab}$ is a complex valued Kähler metric.
Then we obtain the first important result.

Conformal c-Kähler structure of Lorentzian type-D Einstein metrics

Theorem

Suppose (M, g) is a type-D Einstein 4-manifold and g is a Riemannian metric with Lorentzian signature. Denote J the complex structure such that the SD 2-form $\omega = g(J\cdot, \cdot)$ is an eigenform of W^+ for the eigenvalue ψ_2. Then the complex valued metric $\hat{g} = \Omega^{-2} g = \psi_2^{2/3} g$ is c-Kähler, i.e. the following staments are equivalent:

- $d\hat{\omega} = d(\psi_2^{1/3}\omega) = 0$;
- *the complex structure J is compatible with the metric $\hat{g}$:*

$$\nabla_{\hat{g}} J = 0,$$

where $\nabla_{\hat{g}}$ is the Livi-Civita connection w.r.t the c-Kähler metric $\hat{g}$.

Existence of the frame

By Darboux theory, there' a set of frames $\{z^i, \tilde{z}^i\}$. For P-D class

$$
\begin{aligned}
z^0 &= \tau - (r^* - iax^*) \\
\tilde{z^0} &= \tau + (r^* - iax^*) \\
z^1 &= \phi - (ar^{\#} - ix^{\#}) \\
\tilde{z^1} &= \phi + (ar^{\#} - ix^{\#})
\end{aligned} \tag{5}
$$

where $dr^* = \dfrac{a^2 + r^2}{\Delta_r} dr, \quad dx^* = \dfrac{1 - x^2}{\Delta_x} dx, \quad dr^{\#} = \dfrac{1}{\Delta_r} dr,$

$dx^{\#} = \dfrac{1}{\Delta_x} dx.$

Conformal c-Kähler potential

Denote $\partial_i = \dfrac{\partial}{\partial z^i}$ and $\tilde{\partial}_i = \dfrac{\partial}{\partial \tilde{z}^i}$ and then according to the $\partial\bar{\partial}$-lemma (here is $\tilde{\partial}_i$ because in Lorentzian case it's not the complex conjugate, but similar), there's a scalar potential K locally, satisfies

$$g_{i\tilde{j}} = \partial_i \tilde{\partial}_j K$$

and there's a gauge under Kähler transformation like this

$$K \to K' = K + F(z) + \tilde{F}(\tilde{z})$$

P-D class

Then let's study the Kähler potential of P-D class. The metric is given as

$$
\begin{aligned}
g_{\tau\tau} &= \frac{\Delta_r - a^2\Delta_x}{\Pi\Sigma} \\
g_{\tau\phi} &= \frac{a\left[(r^2+a^2)\Delta_x - (1-x^2)\Delta_r\right]}{\Pi\Sigma} \\
g_{\phi\phi} &= \frac{a^2(1-x^2)^2\Delta_r - (r^2+a^2)^2\Delta_x}{\Pi\Sigma} \\
g_{xx} &= -\frac{\Sigma}{\Pi\Delta_x} \\
g_{xy} &= 0 \\
g_{yy} &= -\frac{\Sigma}{\Pi\Delta_r}
\end{aligned}
\tag{6}
$$

where

$$
y \equiv r, \quad \Sigma = r^2 + a^2x^2, \quad a = \text{const},
$$

$$
\Pi = \Pi(r;x), \quad \Delta_x = \Delta_x(x), \quad \Delta_r = \Delta_r(r)
$$

P-D class

There are two KVFs ∂_t and ∂_ϕ and $[\partial_t, \partial_\phi] = 0$, which means there's a toric structure here, with a free action $\mathbb{R} \times S^1$. According to what we showed above, it's straightforward to obtain the Kähler form is

$$\hat{\omega} = \frac{i}{(r - iax)^2}\{-d\phi \wedge [a(1-x^2)dr - i(r^2+a^2)dx] + d\tau \wedge (dr - iadx)\} \quad (7)$$

and the Hamiltonians are

$$\begin{aligned} H_0 &= \int \frac{i}{(r - iax)^2}(dr - iadx) = -\frac{i}{r - iax} \\ H_1 &= i\frac{a + irx}{r - iax} \end{aligned} \quad (8)$$

c-Kähler potential of Plebański-Demiański class

Then the Kähler potential of Plebański-Demiański class is

$$
\begin{aligned}
K &= -4i \int (H_0 d\psi^0 + H_1 \psi^1) \\
&= \frac{(-4i)i}{r - iax} \int r^2 \Delta_r^{-1} dr - rx\Delta_x^{-1} dx - (irax\Delta_x^{-1} dr + iax^2 \Delta_x^{-1} dx) \\
&= 4\left(\int \frac{r}{\Delta_r} \frac{r - iax}{r - iax} dr - \int \frac{x}{\Delta_x} \frac{r - iax}{r - iax} dr\right) \\
&= 4\left(\int \frac{r}{\Delta_r} dr - \int \frac{x}{\Delta_x} dx\right) \qquad (9)
\end{aligned}
$$

so now we have the second important result.

c-Kähler potential of Plebański-Demiański class

- Kähler potential of Plebański-Demiański class

$$K = 4\left(\int \frac{r}{\Delta r} dr - \int \frac{x}{\Delta x} dx\right) \tag{10}$$

With frames

$$\begin{aligned} z^0 &= \tau - (r^* - iax^*) \\ \tilde{z^0} &= \tau + (r^* - iax^*) \\ z^1 &= \phi - (ar^{\#} - ix^{\#}) \\ \tilde{z^1} &= \phi + (ar^{\#} - ix^{\#}) \end{aligned} \tag{11}$$

where $dr^* = \frac{a^2 + r^2}{\Delta_r} dr, \quad dx^* = \frac{1 - x^2}{\Delta_x} dx, \quad dr^{\#} = \frac{1}{\Delta_r} dr,$

$dx^{\#} = \frac{1}{\Delta_x} dx.$

c-Kähler potential of Plebański-Demiański class

- Kähler potential of Schwarzschild solution

$$K_{Sch} = 4(-\frac{r}{2m} + \ln \sin \theta) \tag{12}$$

- Kähler potential of Kerr solution

$$K_{Kerr} = 4(-\frac{r - ia\cos\theta}{2m} + \ln \sin \theta) \tag{13}$$

- The relationship between the Kähler potentials of two solutions is just Newman-Janis shift $r \to r - ia\cos\theta$.

This enables us to construct new solutions begin with a "seed solution" by doing a parameter analytic continuation, on can obtain a new c-Kähler potential, and then a new c-Kähler metric, then after a conformal rescaling, a new Einstein metric.

c-Kähler potential of Kerr-Taub-NUT family

The c-Kähler potential of Kerr-Taub-NUT family is

$$K = 4[\frac{-r + r^*}{2m} - \frac{a(\chi - \chi^*)}{2n}] \tag{14}$$

After a complex but straightforward computing, the specific form of c-Kähler potential is

$$K = \log\left(\Delta_r(1 - \chi^2)\right) + 2(m + in)I(R) \tag{15}$$

where $M^2 = m^2 + n^2, \quad R = r - m$ and

$$I(R) = \begin{cases} \frac{1}{2d}\log\frac{R-d}{R+d}, & d^2 = Q_* - a^2 > 0, \\ -R^{-1}, & Q_* = a^2, \\ \frac{1}{e}\arctan(R/e), & e^2 = a^2 - Q_* > 0. \end{cases}$$

Duality symmetry of KTN family

Denote the first term $\log(\Delta_r(1-\chi^2))$ as K_{ref}. There's a $SO(2)$ duality symmetry of the c-Kähler potential

$$\hat{K}_\beta = K_{\text{ref}} + 2Me^{i\beta}I(R), \tag{16}$$

and the reaction to the conformal rescaling is

$$K_\beta = (-CM)^{2/3}[e^{-2i\beta/3}K_{\text{ref}} + 2Me^{i\beta/3}I(R)]. \tag{17}$$

where C is a non-zero complex number. And then the Einstein metric $g = \partial\tilde{\partial}K_\beta$. In this way, we can obtain the whole KTN family once have a "seed metric", i.e. $\hat{K}_{\beta_0}$. It's a continuous $SO(2)$ duality symmetry.

The MA equation

We can also write the vacuum Einstein equation the KTN family satisfies into a hyperbolic MA equation on the non-compact c-Kähler manifold

$$\det K_{i\tilde{j}} = -\frac{1}{4} e^{K-2(m+in)I(R)} (K_{\tilde{z}^0})^4 \tag{18}$$

the existence of the solution of such an equation corresponding to the existence of metrics of KTN family.

The MA equation

This result can also be generalized into the whole type-D family. Denote $w = K_{z^0}$ and $v = K_{z^1}$.
Then let

$$s_w = w^{-1}, \quad t_w = \sqrt{w^{-2} + 4v/w},$$

$$q(w, v) = \frac{t_w + s_w}{2}, \quad p(w, v) = \frac{t_w - s_w}{2i}.$$

The MA equation for arbitrary type-D metric is

$$\det K_{i\tilde{j}} = -\frac{1}{4}(K_{\tilde{z}^0})^4 P(p(w, v))Q(q(w, v)) \tag{19}$$

where P and Q are real analytic functions in type-D metric.

Biquard-Gauduchon classfication theory

Biquard-Gauduchon studied the moment polytope of ALF toric Einstein metrics and classify all of them using the moment polytope. Li-Sun also gives a discussion on it. Both of their results are in Riemannian case, which are also known as "gravitational instantons".

Theorem (Biquard and Gauduchon)

A smooth Hermitian, non hyperKähler, ALF gravitational instanton, is either:

- *the Taub-NUT metric (with the orientation opposed to the hyperKähler orientation);*
- *the 1-parameter family of Kerr instantons;*
- *the Taub-bolt metric (with respect to both orientations);*
- *the 1-parameter family of Chen-Teo instantons.*

But next we will give a Lorentzian version of such results, which is related with the physical spacetimes.

Moment maps

Definition

Moment map The action ψ is a Hamiltonian action if there exists a map

$$\mu : M \longrightarrow \mathfrak{g}^*$$

satisfying:

For each $X \in \mathfrak{g}$, let

- $\mu^X : M \to \mathbb{R}$, $\mu^X(p) := \langle \mu(p), X \rangle$, be the component of μ along X,
- $X^\#$ be the vector field on M generated by the one-parameter subgroup $\{\exp tX \mid t \in \mathbb{R}\} \subseteq G$.

Then

$$d\mu^X = \iota_{X^\#}\omega$$

i.e., μ^X is a Hamiltonian function for the vector field $X^\#$. And μ^X is the comoment map, μ is the moment map.

Moment maps of Lorentzian KTN famliy

Denote $s = \psi_2^{-1/3}$ and then the two Hamiltonians are

$$u = \frac{i}{s}, \quad v = \frac{\chi(r + in) - ia}{s} = \chi + Bu$$

and the Kähler form is

$$\hat{\omega} = du \wedge dt + dv \wedge d\phi$$

after that, the reduced space is the product with orbit base

$$\mathcal{B} = (b, \infty) \times (-1, 1), \quad (R, \chi) \in \mathcal{B}$$

Moment maps of Lorentzian KTN famliy

Theorem

The map $(R, X) \mapsto (u, v)$ *is a smooth real embedding of* $\mathcal{B}$ *into* $\mathbb{C}^2$*, including when* $a = 0$*. Its image with the normalized horizon label determines* (m, n, a) *in this coefficient family.*

Each interior of the polytope gives an equivalence class of diffeomorphism metric, so we can use the moment polytope to classify different metrics in KTN family. Now what we do next is to choose a real-phase domain and then study the moment polytopes on the real slice of complex spacetime after a projection map.

Moment maps of Lorentzian KTN famliy

Take a constant β and set

$$\alpha_\beta = \mathrm{Re}(e^{-i\beta}\hat{\omega}), \qquad x = \frac{r\sin\beta - p\cos\beta}{\Sigma}, \qquad y = \chi\cos\beta + Bx.$$

where $p = n + a\chi$, $\Sigma = r^2 + p^2$ and $B = 2n\chi - a(1-\chi^2)$.
Then $\alpha_\beta = \mathrm{d}x \wedge \mathrm{d}t + \mathrm{d}y \wedge \mathrm{d}\phi$. Let

$$\delta_- = \min\left(0, \arctan\frac{n-a}{r_H}\right), \qquad \delta_+ = \max\left(0, \arctan\frac{n+a}{r_H}\right),$$

where $r_H = m + b$ on the horizon.

Moment maps of Lorentzian KTN famliy

Theorem

A faithful real-phase domain Suppose

$$2\delta_+ < \beta < \pi/2 + 2\delta_-.$$

With compactification $z = 1/r$, the map $(z, \chi) \mapsto (x, y)$ is an analytic diffeomorphism of the closed rectangle $[0, 1/r_H] \times [-1, 1]$ onto the quadrilateral

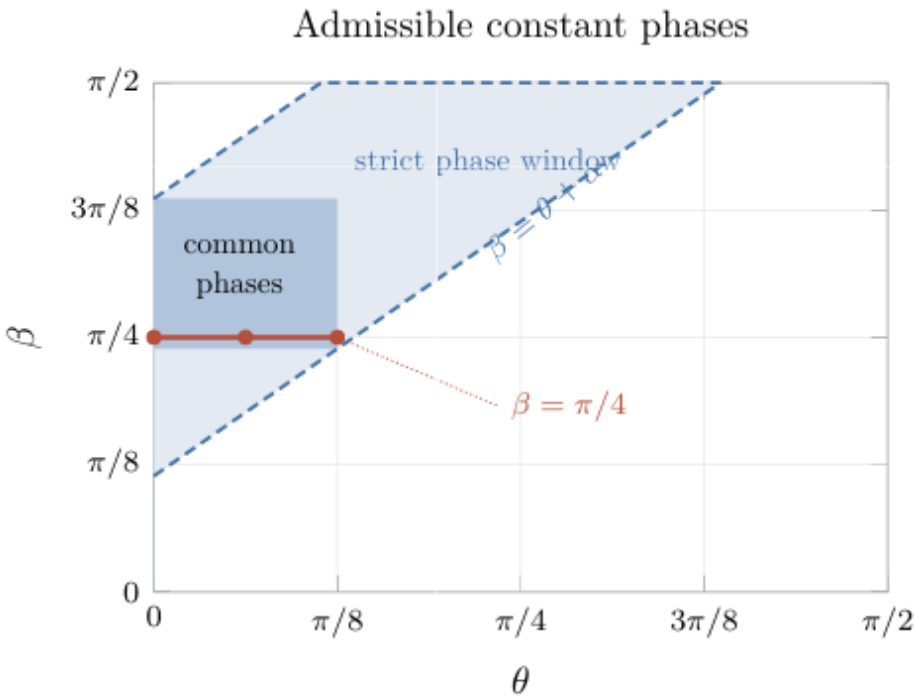
$$x \geq 0, \qquad -\cos\beta - 2nx \leq y \leq \cos\beta + 2nx, \qquad x + \Omega_H y \leq \mathrm{Re}(e^{-i\beta} c_H).$$

Its open interior is the moment image of $\mathcal{B}$ and α_β is symplectic on $\mathcal{B} \times A$.

Where

$$r_H = m + b, \qquad L_H = r_H^2 + a^2 + n^2, \qquad \Omega_H = \frac{a}{L_H}, \qquad c_H = \frac{\mathrm{i} r_H - n}{L_H}.$$

Moment maps of Lorentzian KTN famliy



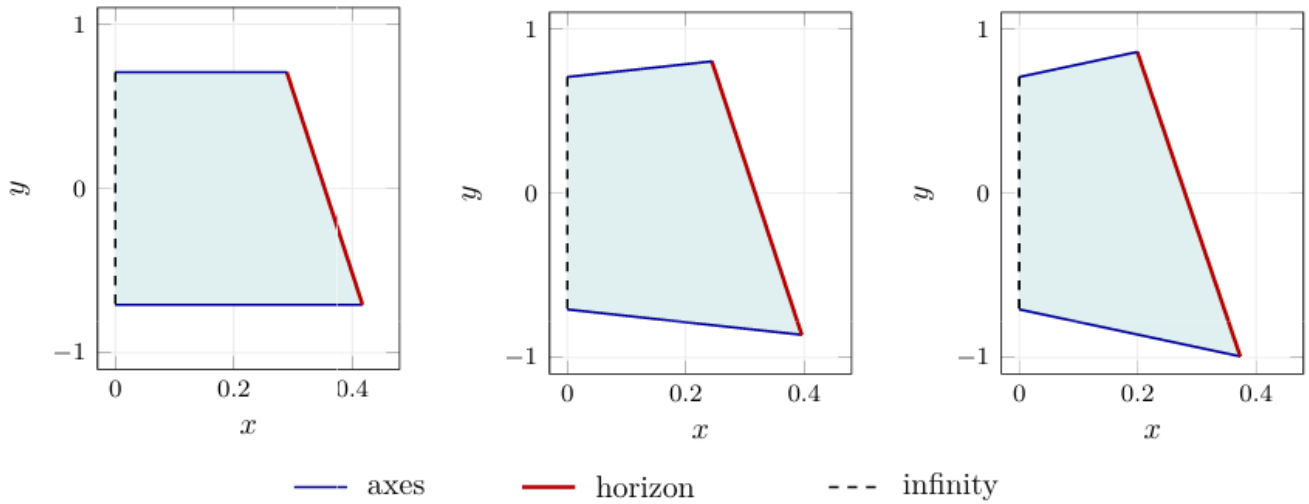
This domain is the main differnece between our Lorentzian case and the Riemannian case of Biquard-Gauduchon, we give such a domain restriction to make sure the moment image corresponding to a physical spacetime.

Moment maps of Lorentzian KTN famliy

Now we choose $\beta = \pi/4$, which is in the faithful real-phase domain. Then we ontain the vertexes

$$(0, -c_0), \quad (h_-, -c_0 - 2nh_-), \quad (h_+, c_0 + 2nh_+), \quad (0, c_0),$$

where $c_0 = \dfrac{1}{\sqrt{2}}$ and $h_\pm = \frac{c_0(r_H - n \mp a)}{r_H^2 + (n \pm a)^2}$. Then we have the moment "polytopes" of KTN class: the first one is Kerr and the other two are Kerr-Taub-NUT with different phase.



Moment maps of Lorentzian KTN famliy

The deformation of the moment "polytopes" of KTN class is actually coresposding to the duality transformation of c-Kähler potential we obtained before. Because every moment "polytope" gives a diffeomorphism class of metric, so it uniquely determines a Lorentzian KTN metric. When one changes the c-Kähler potential, one actually done a paramater transformation and so changes the Hamiltonians u and v, and then changes the vertexes of the image of the moment map, which is a continuous deformation of them.
So in this point of view:

A continuous deformation of the moment "polytope"

$$\Longleftrightarrow$$

A continuous duality transformation of c-Kähler potential

Open problems

- Can we use such a method to find new solutions with toric symmetry?
- Can we classify more Einstein 4-manifolds by classify some "seed metrics" (which are simple enough) and then use the deformation theory to prove all of the Einstein 4-manifolds are deformed from these "seed metrics" and then classify these Einstein 4-manifolds? (I think it may only works for toric case) What will happen if we study S^1 metrics? Without toric structure, it's even non-Hermitian;
- What's the moduli space $\mathcal{M}$ of such metrics and c-Kähler potentials? What's the relationship between $\mathcal{M}$ and the moduli space of Ernst potentials $\mathcal{N}$?
  We claim that: $\mathcal{M}/\text{symmetry of conformal structure} \simeq \mathcal{N}$
- May this help us to understand the stationary problems?
- The reconstruction and classification theorys are both locally, how to obtain a global theory?

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Thank You

Thank you for your attention!