A LQG inspired effective action in spherically symmetric background and its applications

Hongwei Tan

Hunan Institute of Technology

2611.xxxx

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- Motivation
- A LQG-inspired effective action
- Wald entropy and the first law of black hole thermodynamics
- A two-parameter Birkhoff-type theorem
- Conclusions and outlooks

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Motivation

Loop quantum gravity (LQG) is a competitive candidate for quantum gravity.

The q-OS geometry(Oppenheimer-Snyder) provides a LQG-corrected exterior metric. [J. Lewandowski, et al. (2023)]

$$\mathrm{d}s^2 = -f_{\mathrm{q-OS}}\mathrm{d}t^2 + f_{\mathrm{q-OS}}^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

$$f_{\mathrm{q-OS}} = 1 - \frac{2M_0}{r} + \frac{4\zeta^2 M_0^2}{r^4}$$

A covariant Hamiltonian construction motivates an explicit action formulation.

[C. Zhang and Z. Cao (2025)]

We seek an action that admits q-OS geometry and supports a consistent thermodynamic analysis.

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Spherical reduction and dilaton gravity

$$\mathrm{d}s^2 = h_{ab}\mathrm{d}x^a\mathrm{d}x^b + r^2\mathrm{d}\Omega^2, \quad X := r^2, \quad Z := h^{ab}\nabla_a X \nabla_b X$$

$$S_{\mathrm{GR}} = \int \mathrm{d}^2x\sqrt{-h}\left[\frac{X}{4}R[h] + \frac{(\nabla X)^2}{8X} + \frac{1}{2}\right] + S_{\mathrm{bdy}}$$

- Angular integration leaves a two-dimensional covariant theory.

$$S_{\mathrm{eff}}[h, X] = \int_{\mathscr{M}} d^2x\sqrt{-h}\ \left[F(X)R[h] + W(X, Z)\right] + + S_{\mathrm{bdy}}$$

[D. Grumiller, et al. (2002)]

Equations of motion

$$\partial_X F(X) R + \partial_X W(X,Z) - 2\,\nabla_a \left(\partial_Z W(X,Z)\, \nabla^a X \right) = 0$$

$$-\nabla_a \nabla_b F(X) + h_{ab} \Box F(X) + \partial_Z W(X,Z)\, \partial_a X\, \partial_b X - \frac{1}{2} h_{ab} W(X,Z) = 0$$

- For the static metric ansatz:

$$\mathrm{d}s_2^2 = -f(r)\mathrm{d}t^2 + g(r)\mathrm{d}r^2$$

$$\partial_Z W = \partial_X^2 F + \frac{\partial_X F}{2X} \implies f(r)g(r) = C \xrightarrow{\text{Asymptotic flatness}} f(r)g(r) = 1$$

The classical GR limit

$$F_{\mathrm{GR}}(X) = \frac{X}{4}, \quad W_{\mathrm{GR}}(X, Z) = \frac{Z}{8X} + \frac{1}{2}$$

$$f = g^{-1}, \quad r\,\partial_r f + f = 1$$

$$f(r) = 1 - \frac{2m}{r}, \quad S_{\mathrm{SCH}} = 4\pi F_{\mathrm{GR}}(r_h^2) = \frac{A_h}{4}$$

[R. M. Wald (1993)]

A LQG-inspired effective action

$$S_{\mathrm{LQG}}[h, X, M, A] = \int \mathrm{d}^2x\sqrt{-h}\left[F_{\mathrm{LQG}}(X)R[h] + W_{\mathrm{LQG}}(X, Z, M) + A^a\nabla_a M\right] + S_{\mathrm{bdy}}$$

The theory admits the q-OS metric as a vacuum solution.

It gives an area term and a logarithmic correction to the black hole entropy.

The curvature coupling and potential

$$F_{\mathrm{LQG}}(X) = \frac{X}{4} + \alpha \log \frac{X}{X_0}$$

A polynomial of ζ^2

$$W_{\mathrm{LQG}}(X, Z, M) = B(X)Z + U(X, M), \quad B(X) = \frac{1}{8X} - \frac{\alpha}{2X^2}$$

$$U(X, M) = \frac{1}{2} - \frac{2\alpha}{X} + \frac{8\alpha M}{X^{3/2}} - \frac{6\zeta^2 M^2}{X^2} - \frac{40\alpha\zeta^2 M^2}{X^3}$$

The equations of motion

$$(\partial_X F_{\rm LQG})R[h] + \partial_X W_{\rm LQG} - 2\nabla_a\Big[(\partial_Z W_{\rm LQG})\nabla^a X\Big] = 0$$

$$-\nabla_a\nabla_b F_{\rm LQG} + h_{ab}\Box F_{\rm LQG} + (\partial_Z W_{\rm LQG})\nabla_a X\nabla_b X$$

$$-\frac{1}{2}h_{ab}\Big[W_{\rm LQG} + A^c\nabla_c M\Big] = 0$$

$$M = M_0$$

$$\nabla_a M = 0, \quad \nabla_a A^a = \partial_M W_{\rm LQG}$$

Static field equations in Asymptomatic flatness

$$X = r^2, \;\; M = M_0, \;\; f_{\mathrm{LQG}} = g_{\mathrm{LQG}}^{-1}, \;\; R[h] = -\,\partial_r^2 f_{\mathrm{LQG}}$$

tt component

$$W_{\mathrm{LQG}} = 2 f_{\mathrm{LQG}}\, \partial_r^2 F_{\mathrm{LQG}} + (\partial_r f_{\mathrm{LQG}})\, \partial_r F_{\mathrm{LQG}}$$

rr component

$$W_{\mathrm{LQG}} = (\partial_r f_{\mathrm{LQG}})\, \partial_r F_{\mathrm{LQG}} + 8 f_{\mathrm{LQG}} X\, \partial_Z W_{\mathrm{LQG}}$$

Dilaton equation

$$-(\partial_X F_{\mathrm{LQG}})\, \partial_r^2 f_{\mathrm{LQG}} + \partial_X W_{\mathrm{LQG}} - 4\, \partial_r \left[r f_{\mathrm{LQG}}\, \partial_Z W_{\mathrm{LQG}} \right] = 0$$

q-OS solves the metric equations

$$f_{\mathrm{LQG}} = f_{\mathrm{q-OS}} = 1 - \frac{2M_0}{r} + \frac{4\zeta^2 M_0^2}{r^4}$$

$$\partial_Z W_{\mathrm{LQG}} = \partial_X^2 F_{\mathrm{LQG}} + \frac{\partial_X F_{\mathrm{LQG}}}{2X}$$

$$\frac{r^2 + 4\alpha}{2r}\partial_r f_{\mathrm{q-OS}} + \left(\frac{1}{2} - \frac{2\alpha}{r^2}\right) f_{\mathrm{q-OS}}$$

$$= \frac{1}{2} - \frac{2\alpha}{r^2} + \frac{8\alpha M_0}{r^3} - \frac{6\zeta^2 M_0^2}{r^4} - \frac{40\alpha\zeta^2 M_0^2}{r^6} = U(r^2, M_0)$$

Both independent metric equations are satisfied.

q-OS solves the remaining equations

$$-(\partial_X F_{\mathrm{LQG}})\,\partial_r^2 f_{\mathrm{q-OS}} + \partial_X W_{\mathrm{LQG}} - 4\,\partial_r\left[r f_{\mathrm{q-OS}}\,\partial_Z W_{\mathrm{LQG}}\right] = 0$$

Stationary condition → $\nabla_a A^a = \partial_r P(r) = \partial_M W_{\mathrm{LQG}}$

$$P(r) = -\frac{4\alpha}{r^2} + \frac{4\zeta^2 M_0}{r^3} + \frac{16\alpha\zeta^2 M_0}{r^5}$$

$$M = M_0, \quad \nabla_a M = 0, \quad A^t = \lambda_0(r), \quad A^r = P(r) + C_A$$

The horizon boundary condition

$$A^r = P(r) + C_A, \quad P(r) = -\frac{4\alpha}{r^2} + \frac{4\zeta^2 M_0}{r^3} + \frac{16\alpha\zeta^2 M_0}{r^5}$$

- A stationary field smooth at the bifurcation horizon requires:

$$A^r(r_+) = 0, \quad A^r(r) = P(r) - P(r_+)$$

$$\Downarrow \qquad r \to \infty$$

$$A^r(\infty) = -P(r_+)$$

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Covariant phase space

$$\delta \mathscr{L}_{\text{tot}} = \boldsymbol{E}\delta\Phi + \mathbf{d}\boldsymbol{\Theta}(\Phi, \delta\Phi), \quad \Phi = (h_{ab}, X, M, A^a)$$

$$\boldsymbol{J}_\xi = \boldsymbol{\Theta}(\Phi, \mathscr{L}_\xi \Phi) - \xi \cdot \boldsymbol{L}(\Phi) \approx \mathrm{d}\boldsymbol{Q}_\xi$$

$$\delta H_\xi = \int_{\partial\Sigma} \Big[\delta \boldsymbol{Q}_\xi - \xi \cdot \boldsymbol{\Theta}(\Phi, \delta\Phi)\Big]$$

[J. Lee and R. M. Wald (1990); V. Iyer and R. M. Wald (1994)]

The gravitational Noether charge

$$\boldsymbol{Q}_{\xi,\mathrm{grav}} = -E^{abcd}\epsilon_{ab}\nabla_{[c}\xi_{d]} - 2\epsilon_{ab}\xi_{[c}\nabla_{d]}E^{abcd}$$

$$E^{abcd} := \frac{\partial(FR[h])}{\partial R[h]_{abcd}} = Fh^{a[c}h^{d]b}$$

$$\delta H^{\infty}_{\chi,\,\mathrm{grav}} = \int_{\infty}\left[\delta\boldsymbol{Q}^{\infty}_{\chi,\,\mathrm{grav}} - \chi\cdot\boldsymbol{\Theta}^{\infty}_{\mathrm{grav}}(\Phi,\delta\Phi)\right] = \delta M$$

Auxiliary charge and boundary variation

$$\boldsymbol{\beta} := A \cdot \boldsymbol{\epsilon}$$

The Killing vector defines the bifurcation Killing horizon

$$\boldsymbol{\Theta}_A = \delta M \boldsymbol{\beta}, \quad \boldsymbol{J}_{\chi,A} = (\chi \cdot \boldsymbol{\beta}) \mathrm{d}M \approx 0$$

- Smoothness and $\chi = 0$ at the bifurcation horizon fix the auxiliary charge.

$$Q_{\chi,A} = 0 \text{ (has no contribution to the entropy!)}$$

$$\delta H^{\infty}_{\chi,A} = A^r(\infty)\delta M = -P(r_+)\delta M$$

[R. M. Wald (1990); V. Iyer and R. M. Wald (1994)]

First law of Black hole thermodynamics

$$\delta H^{\infty}_{\chi,\mathrm{grav}} = \delta M, \quad \delta H^{\infty}_{\chi,A} = -P(r_+)\delta M$$

$$\delta H_{\chi} = \delta H_{\chi,\mathrm{grav}} + \delta H_{\chi,\mathrm{A}} = 0$$

$$\left[1 - P(r_+)\right]\delta M = T_{\mathrm{H}}\delta S_{\mathrm{q-OS}}$$

$$S_{\mathrm{q-OS}} = \frac{A}{4} + 4\pi\alpha \log\frac{A}{A_0}$$

Fixed couplings and a stationary, nonextremal branch

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Local staticity from the field equations

$$\text{Kodama vector: } K^a := \frac{1}{2}\epsilon^{ab}\nabla_b r, \quad K^a \nabla_a r = 0$$

$$g_{ab}K^a K^b < 0$$

$$\text{Vacuum EOMs} \longrightarrow \nabla_{(a}K_{b)} = 0$$

Spcetime is static!

[H. Kodama (1980)]

A two-parameter vacuum family

The EOMs indicate (aasymptomatic flatness):

$$\frac{r^2+4\alpha}{2r}\,\partial_r f_{\mathrm{LQG}} + \left(\frac{1}{2} - \frac{2\alpha}{r^2}\right) f_{\mathrm{LQG}} = U(r, M_0)$$

$$f_{\mathrm{LQG}}(r) = g^{-1}_{\mathrm{LQG}}(r) = 1 - \frac{2M_0}{r} + \frac{4\zeta^2 M_0^2}{r^4} + \frac{r}{r^2+4\alpha} D$$

- D is independent integration constant.
-

Integrability in the two-parameter space

$$\boldsymbol{k}: = \left[1 - P(r_+)\right]\boldsymbol{\delta M} - \frac{1}{2}\boldsymbol{\delta D} = T_{\mathrm{H}}\delta S_{\mathrm{q-OS}}$$

δH_χ^∞

⬇ phase-space exterior derivative

$$\delta\boldsymbol{k} = \frac{\partial P(r_+)}{\partial D}\boldsymbol{\delta M} \wedge \boldsymbol{\delta D} \neq 0 \ \text{ generically}$$

⬇

k is not exact in general

⬇

The integrability condition of H_χ^∞ forces $D = 0$.

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Conclusions

- A covariant dilaton action admits the q-OS geometry as a vacuum branch.
- The Wald entropy contains an area term and a logarithmic correction.
- Auxiliary boundary data supply the correction required by the first law.
- **The nondegenerate vacuum sector admits a two-parameter Birkhoff-type family.**

Outlooks

- Is it possible to add additional boundary terms to preserve to integrality rather than require $D = 0$?
- How do the extra parameter and effective dynamics affect stability and observations?
- More LQG-inspired geometry by coupling matter fields.

THANK YOU!

The current for a general generator

$$\boldsymbol{\Theta}_W = 2\,\partial_Z W \delta X\ \nabla X \cdot \boldsymbol{\epsilon}, \quad \mathscr{L}_\xi X = \xi^b \nabla_b X$$

$$j^a_{\xi,\mathrm{grav}} = F(-2\nabla^a \nabla_c \xi^c + \nabla_b \nabla^a \xi^b + \nabla_b \nabla^b \xi^a)$$

$$-(\nabla_b F)(\nabla^a \xi^b + \nabla^b \xi^a) + 2(\nabla^a F)\nabla_c \xi^c$$

$$+ 2\,\partial_Z W \nabla^a X\, \xi^b \nabla_b X - (FR + W)\xi^a$$

[V. Iyer and R. M. Wald (1994)]

Boundary improvement under reduction

$$\boldsymbol{L}_{\mathrm{GR}} = \overline{\boldsymbol{L}}_{\mathrm{EH}} + \mathrm{d}\boldsymbol{\mu}, \quad \boldsymbol{\mu} = r\ \nabla r \cdot \boldsymbol{\epsilon}$$

$$\boldsymbol{\Theta}_{\mathrm{GR}} = \overline{\boldsymbol{\Theta}}_{\mathrm{EH}} + \delta\boldsymbol{\mu}, \quad \boldsymbol{Q}_{\chi,\mathrm{GR}} = \overline{\boldsymbol{Q}}_{\chi,\mathrm{EH}} + \chi \cdot \boldsymbol{\mu}$$

$$\boldsymbol{Q}_{\chi,\mathrm{GR}} = F\,\partial_r f - 2f\,\partial_r F = \frac{m}{2} - rf, \quad \boldsymbol{J}_{\chi,\mathrm{GR}} = -\,\mathbf{d}\boldsymbol{r}$$

The paired boundary shifts cancel in the Hamiltonian variation.

$$\delta H_\chi^\infty = \delta m$$

[V. Iyer and R. M. Wald (1994)]

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