Paths to Solving the Strong CP Problem and the Footprints Left Behind

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Strong CP Problem

QCD Lagrangian for strong interactions allows

$$\mathcal{L}_\theta = \theta \frac{g_s^2}{32\pi^2} G^{a\mu\nu} \widetilde{G}^a_{\mu\nu}$$

explicitly violating **CP** symmetry.

The physical strong CP phase : $\bar{\theta} \equiv \theta - \arg\det\left(M_u M_d\right)$

The current upper bound on the neutron electric dipole moment

$$\Rightarrow |\bar{\theta}| < 10^{-11}$$

Why is $\bar{\theta}$ so small ??

Some shifts of $\bar{\theta}$ would not provide a visible change in our world.

Potential Solutions

- **Massless up quark**

  Existing global U(1) axial symmetry can rotate the θ term to zero.

  Ruled out by the lattice result.

- **Axion**

  Axion dynamically cancels the θ term at the minimum of its potential.

- **Spontaneous CP (or P) violation**

  CP is an exact symmetry of the Lagrangian but broken spontaneously at the vacuum.

  CKM phase is generated without reintroducing the θ term.

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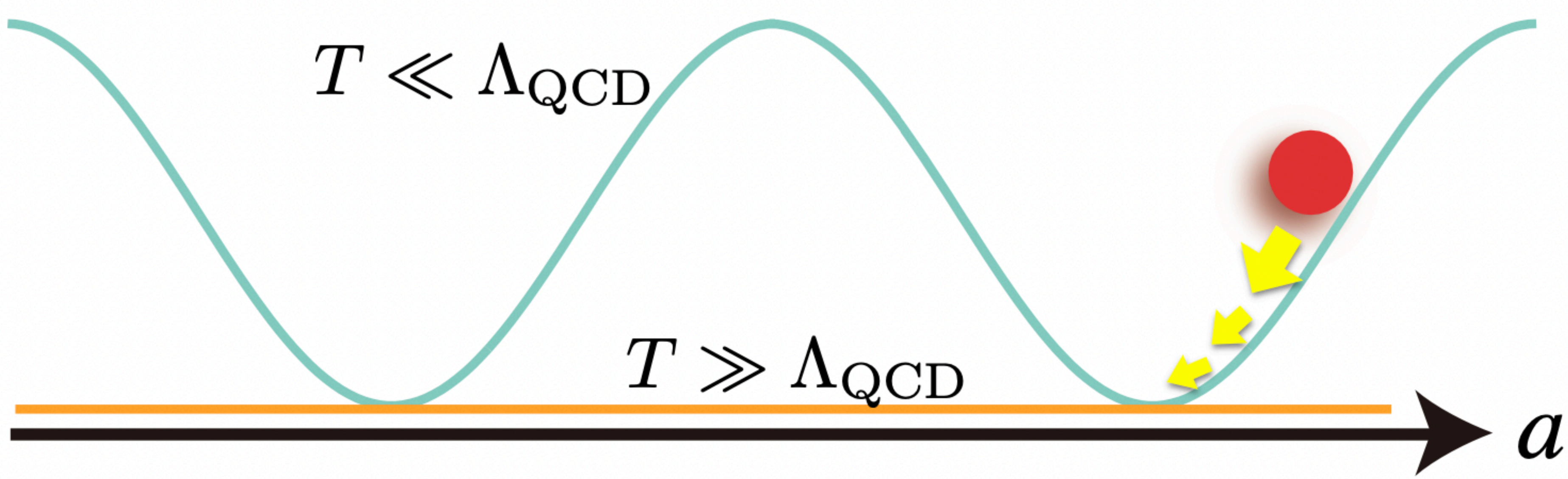
Axion Solution

The most common explanation is **the Peccei-Quinn mechanism** that the strong CP phase is promoted to a dynamical variable.

$$\mathcal{L}_\theta = \left(\theta + \frac{a}{f_a}\right) \frac{g_s^2}{32\pi^2} G^{a\mu\nu} \widetilde{G}^a_{\mu\nu}$$



Fuminobu Takahashi slide

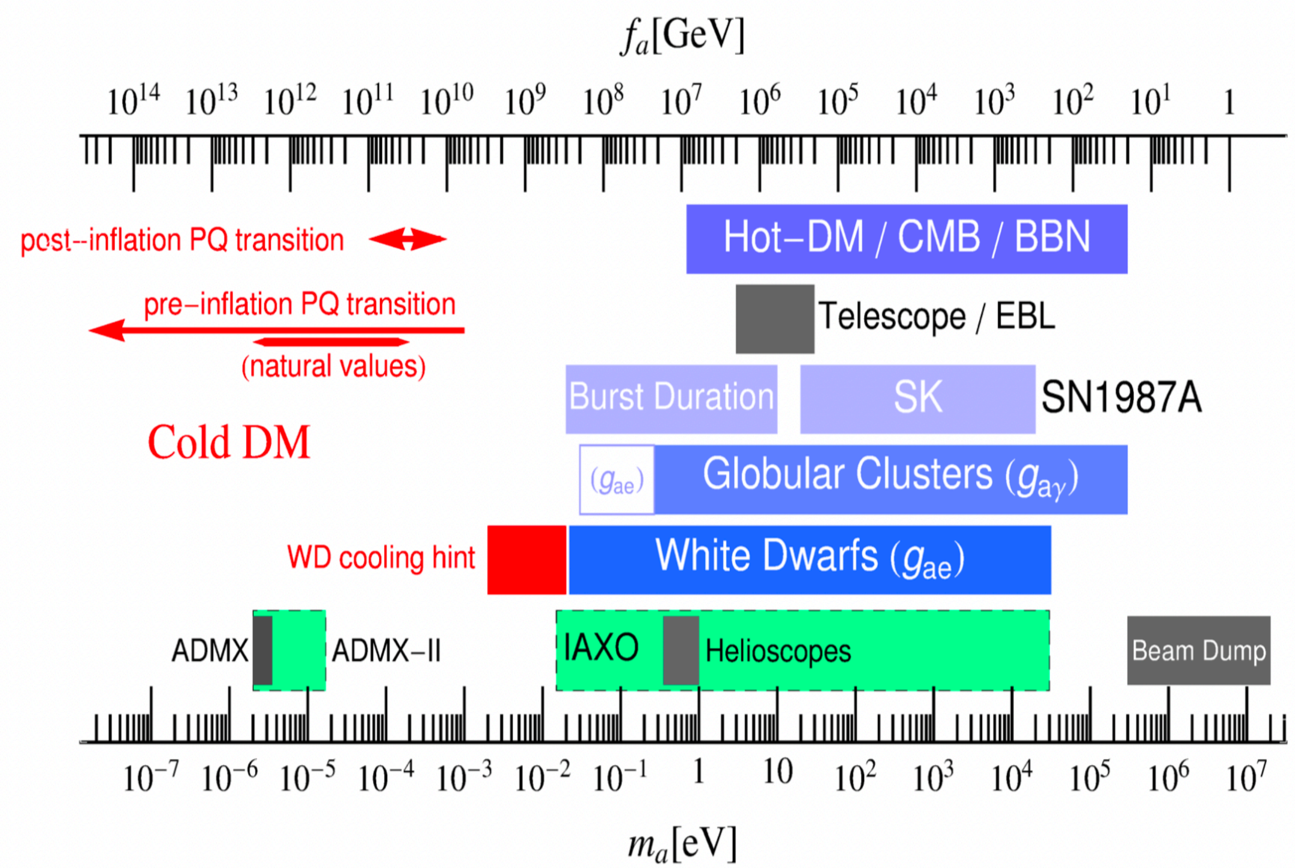
The **axion *a*** dynamically cancels the strong CP phase !

Axion Solution

Axion is a pseudo-Nambu-Goldstone boson associated with spontaneous breaking of **a global U(1)$_{PQ}$ symmetry.**

Non-perturbative QCD effects break the U(1)$_{PQ}$ explicitly and generate the axion potential :

$$V(a) \sim m_\pi^2 f_\pi^2 \cos\left(\theta + \frac{a}{f_a}\right)$$



PDG

Axion can be a dominant component of **dark matter** :

$$10^8\ \mathrm{GeV} \lesssim f_a \lesssim 10^{12}\ \mathrm{GeV}$$

Naturalness problem of the intermediate scale

Composite Axion

Axion emerges as **a pion-like bound state** of a new gauge dynamics.

Chiral symmetry breaking :

$$\mathrm{SU}(4)_L \times \mathrm{SU}(4)_R \to \underline{\mathrm{SU}(4)_V}$$

$$\supset \mathrm{SU}(3)_c$$

One of pNGBs is identified as axion.

Fields	$[SU(3)_c]$	$[SU(N)]$	$U(1)_{\mathrm{PQ}}$
Q	$\mathbf{3}$	$\mathbf{N}$	1
η	$\mathbf{1}$	$\mathbf{N}$	-3
$\bar{Q}$	$\bar{\mathbf{3}}$	$\bar{\mathbf{N}}$	0
$\bar{\eta}$	$\mathbf{1}$	$\bar{\mathbf{N}}$	0

Anomaly coefficient :

$$A(\mathrm{U}(1)_{\mathrm{PQ}} - \mathrm{SU}(3)_c{}^2) = \underline{N}$$ ➡ Axion coupling to QCD

Domain wall number

Dimensional transmutation naturally explains <u>the hierarchically small PQ breaking scale</u>.

Axion Cosmology

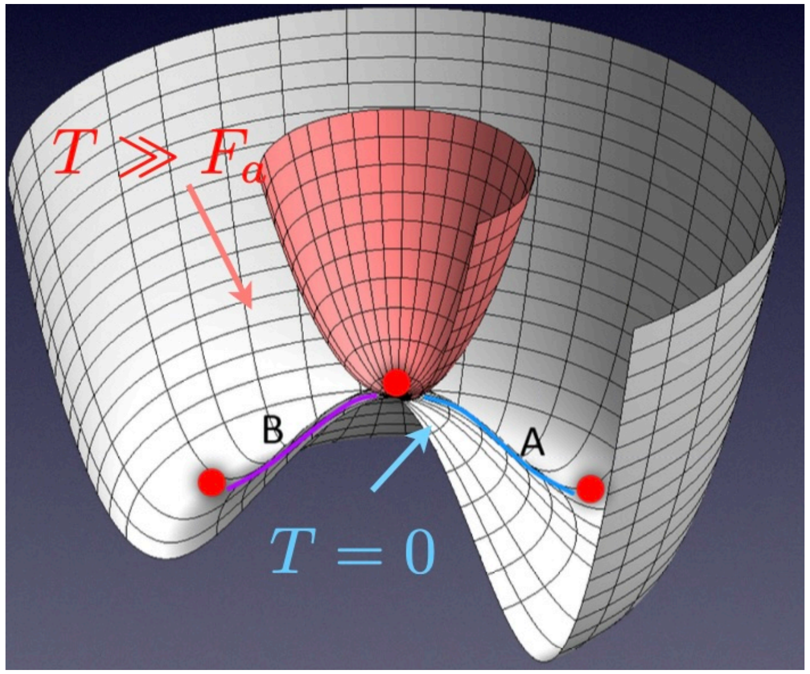
- Cosmological consequences :

★ U(1)PQ breaking **before** inflation → **Isocurvature perturbations**

Constraint on the inflation scale

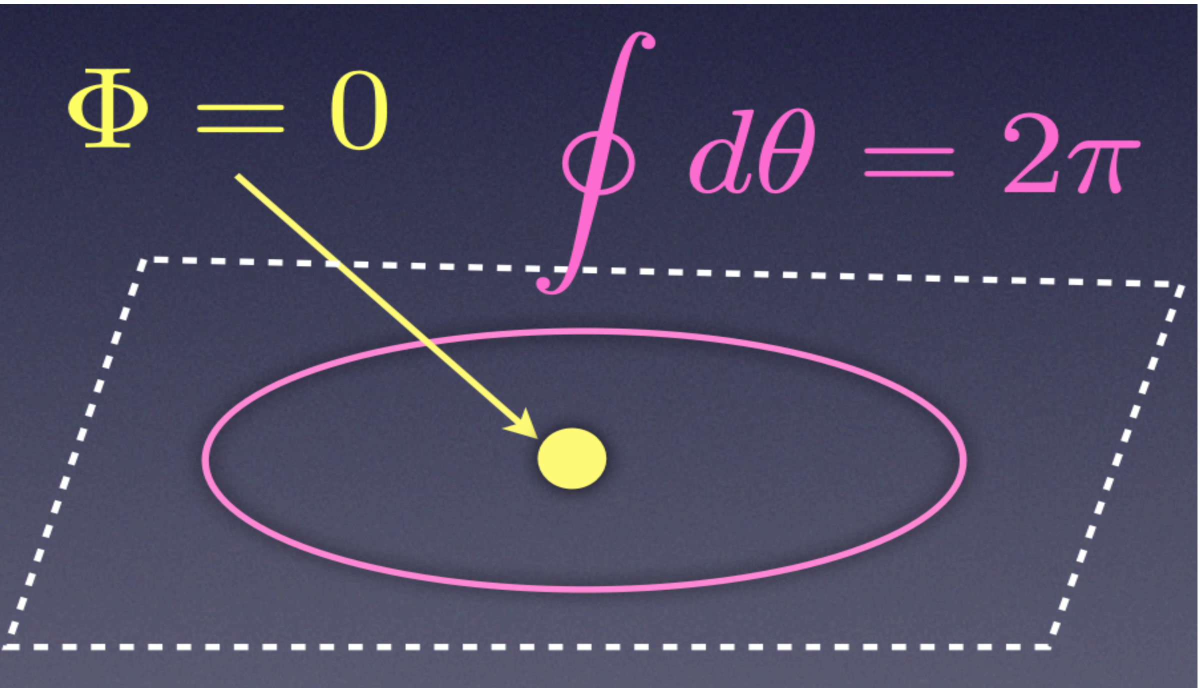
★ U(1)PQ breaking **after** inflation

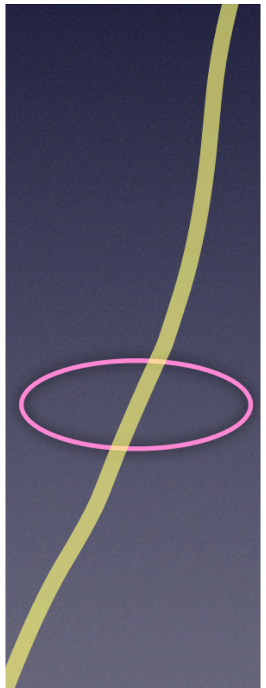
Axion field acquires **spatial variations** across the Universe.



Masahiro Kawasaki slide

$$\Phi \sim v_{\mathrm{PQ}} e^{i\theta_a}$$





Formation of **cosmic strings**

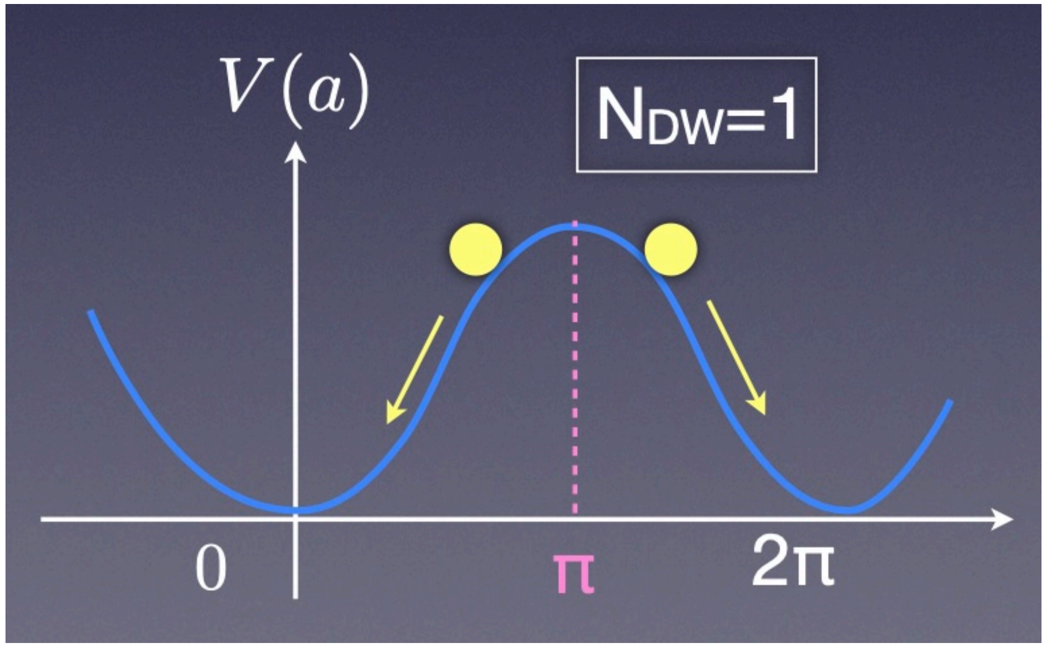
Domain Walls

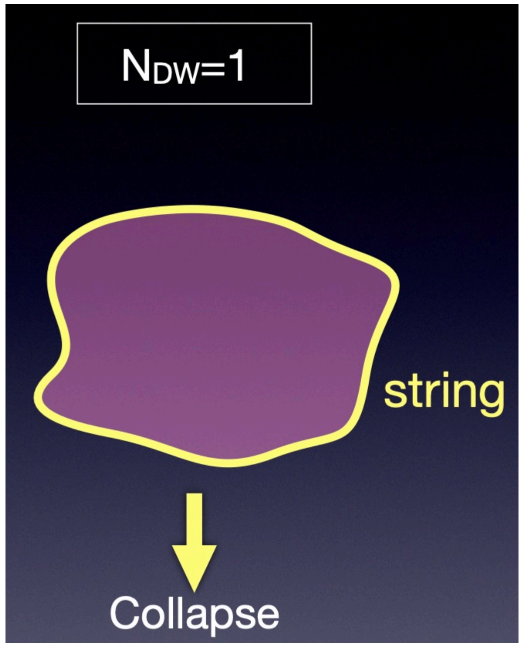
As the Universe expands and cools,
the axion gets a potential with $\underline{N_{\rm DW}}$ minima.

Domain wall number (associated with color anomaly)

$\boxed{\mathbf{N_{DW} = 1}}$

Domain walls form as disk-like structures attached to strings and eventually <u>collapse due to their tension</u>.





Masahiro Kawasaki slide

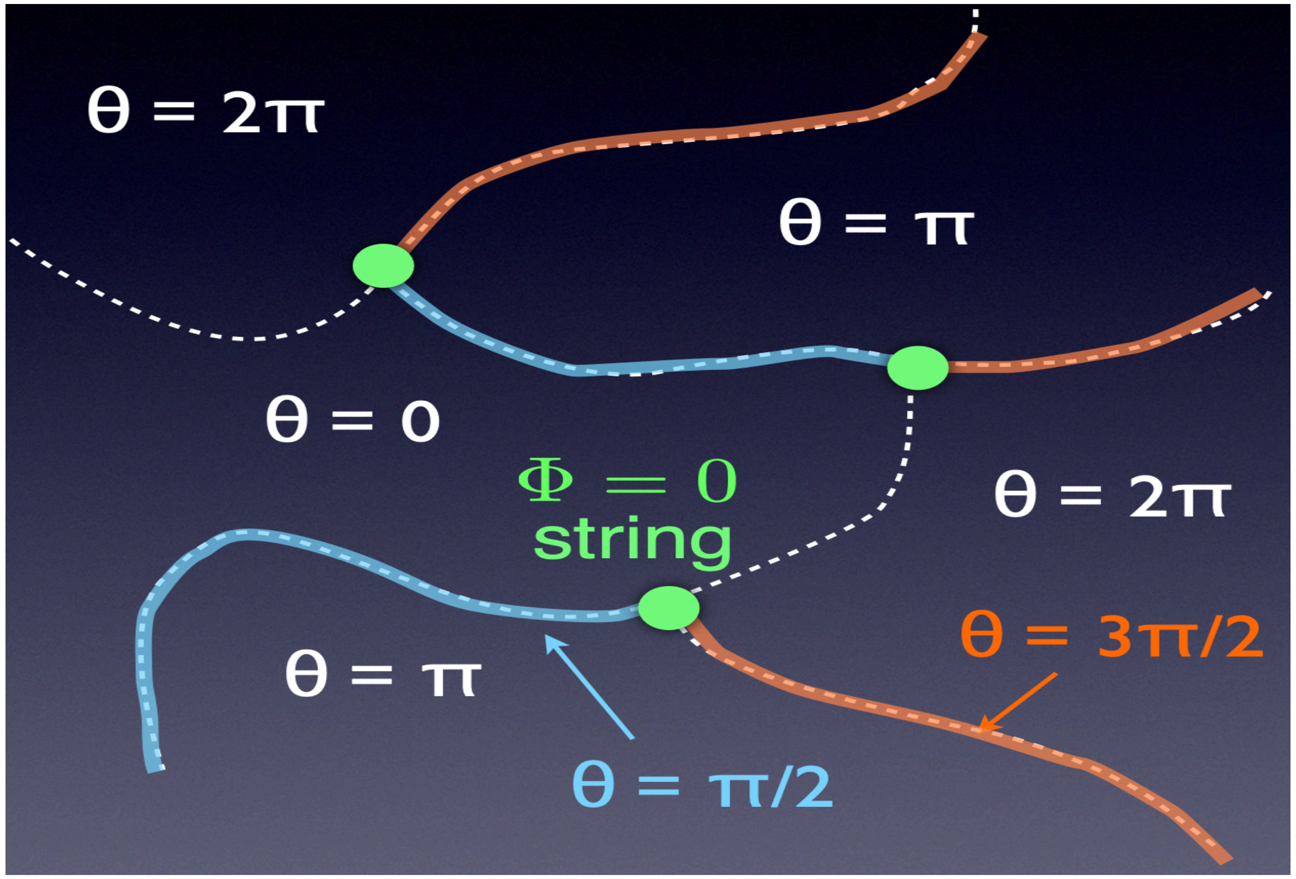
Decay of the string-domain wall network produces a lot of axions which <u>dominate the axion DM abundance</u>.

Domain Walls

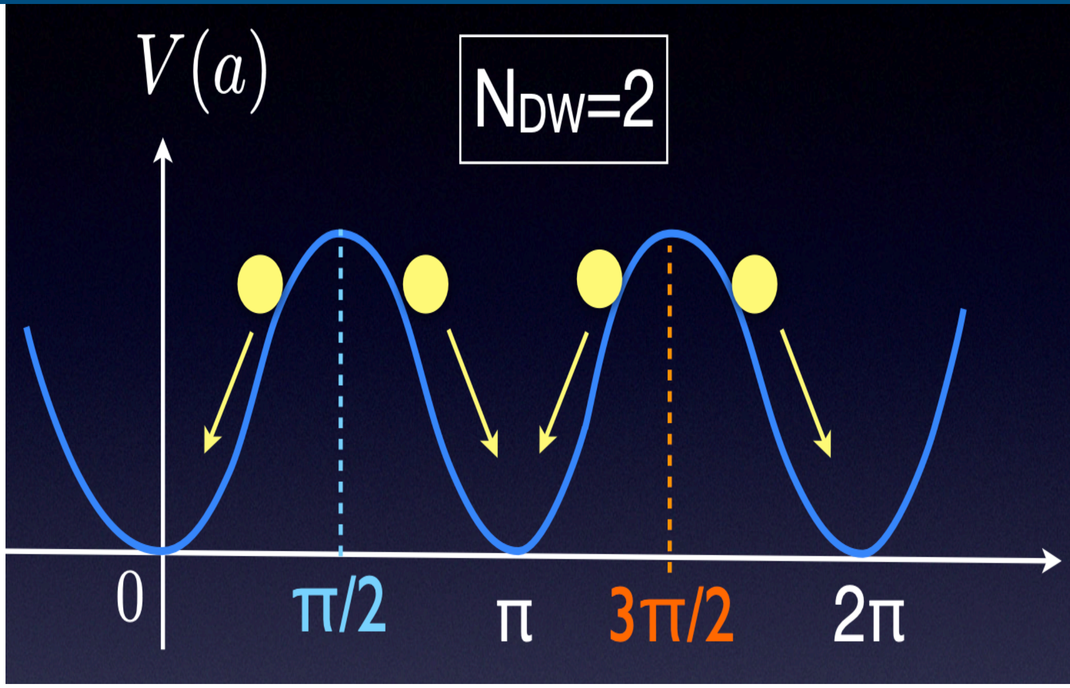
$N_{DW} > 1$

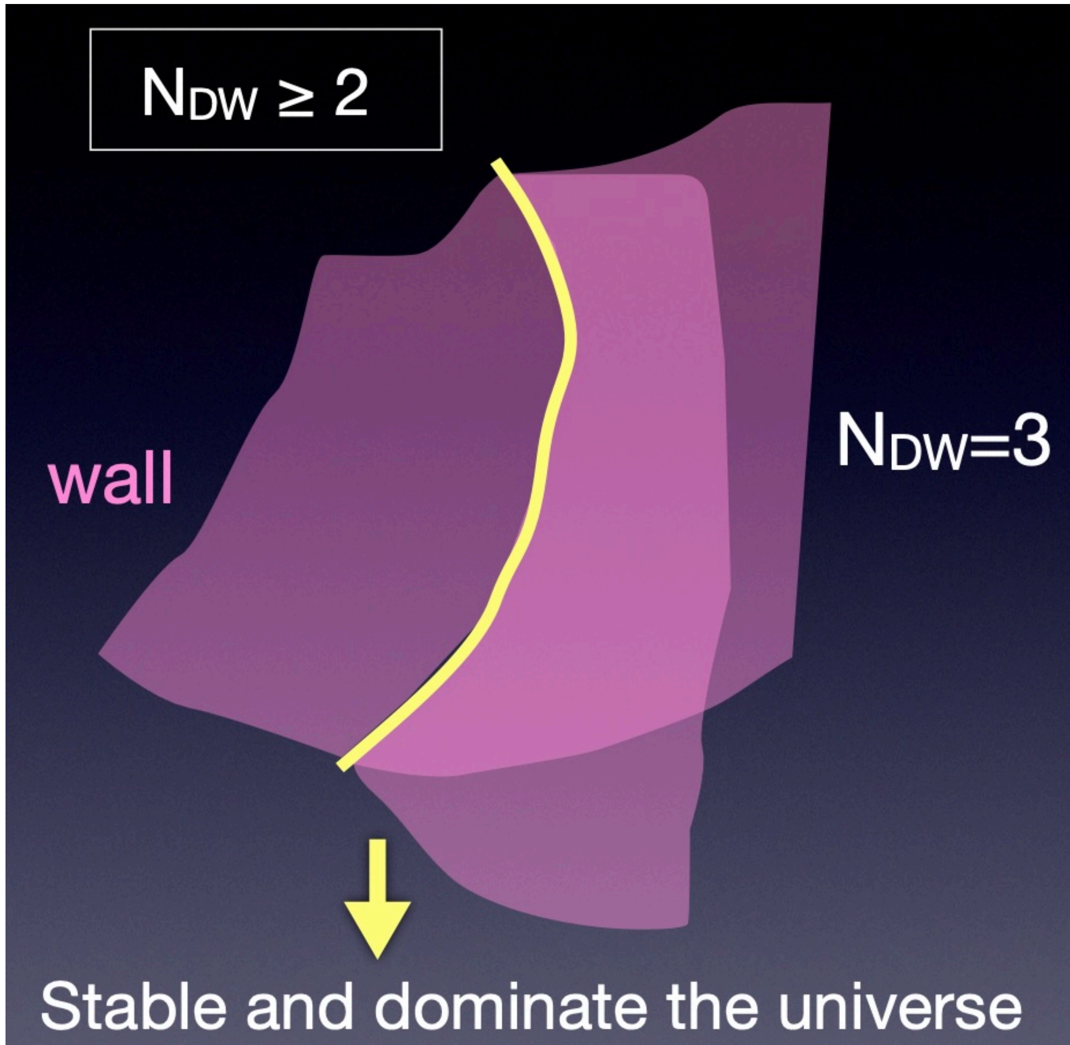
Domain walls form a **stable** network that eventually dominates the Universe.

➡ **Domain wall problem**



Masahiro Kawasaki slide

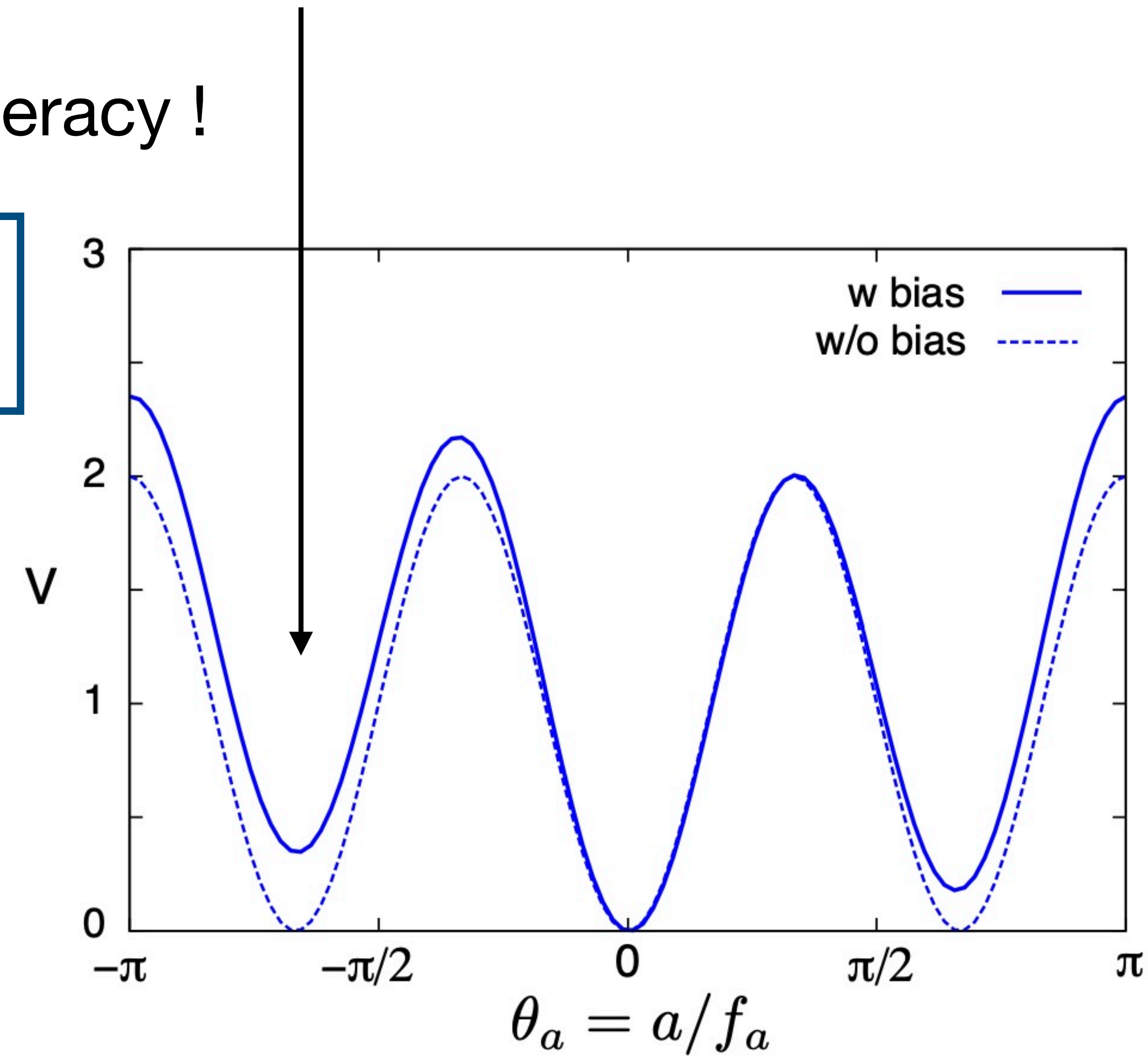




Bias Term

A potential solution to the domain wall problem is to introduce a small explicit breaking of the $U(1)_{PQ}$ symmetry (**Bias term**).

Bias term lifts the vacuum degeneracy !



Domain walls collapse before they dominate the Universe.

★ Axion DM abundance is changed.

However …

★ **Ad hoc introduction** of the bias term is unsatisfactory.

Potential Solutions

- **Special embedding** Hor, YN, Suzuki, Xu (2025)

  The SM gauge group is identified as a **special subgroup** of a UV group to obtain **small instanton effects** resolving the vacuum degeneracy of the axion potential.

- **Bias with a timer** Hao, Nakagawa, YN, Suzuki (2025)

  New scalar field interaction with the axion induces **a time-dependent bias term** only effective in the early Universe.

- **PQ symmetry non-restoration** Nakagawa, YN, Qiu, Wang, Wang (2025)

  PQ symmetry remains broken in the entire history of the Universe, avoiding the formation of axion strings and domain walls.

  …

Potential Solutions

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Our focus

- **Bias with a timer** Hao, Nakagawa, YN, Suzuki (2025)

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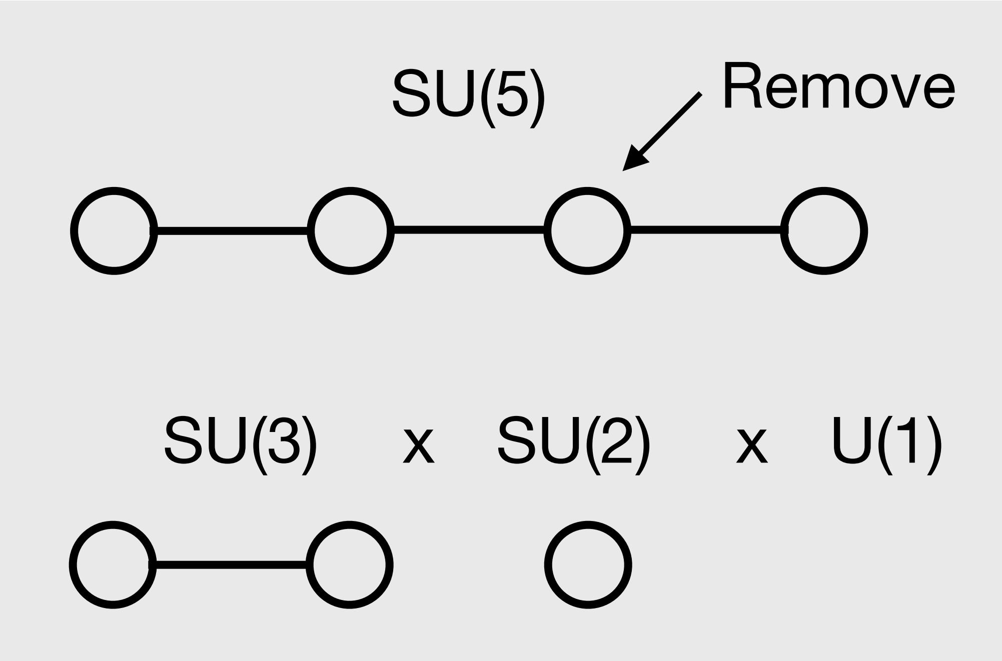
Special Subalgebra

Simple Lie algebras possess not only **regular** subalgebras but also **special** subalgebras.

Regular subalgebras : systematically obtained by removing nodes from Dynkin diagrams.



Special subalgebras :

do not follow this scheme !

Representation contents and group-theoretic indices are **reorganized** in a non-trivial manner.

Special Embedding

We focus on a gauge symmetry breaking : $SU(MN) \to SU(N)$

$T^m_{\rm UV}\ (m = 1, \ldots, (MN)^2 - 1)$: SU(MN) generators

$T^a_{\rm IR}\ (a = 1, \ldots, N^2 - 1)$: SU(N) generators

$$\boxed{T^a_{\rm IR} = \underbrace{\mathcal{O}^{am}}_{\text{Coefficients}} T^m_{\rm UV}}$$

r rep. of SU(N) is embedded into the fundamental rep. of SU(MN)

$$\mathrm{tr}(T^m_{\rm UV} T^n_{\rm UV}) = \frac{1}{2}\delta^{mn} \qquad \mathrm{tr}(T^a_{\rm IR} T^b_{\rm IR}) = \underbrace{T_{\rm IR}(\mathbf{r})}_{\text{Dynkin index}}\delta^{ab}$$

$$\Rightarrow \mathcal{O}^{am}\mathcal{O}^{bn}\delta_{mn} = c\delta^{ab} \qquad \boxed{c \equiv \frac{T_{\rm IR}(\mathbf{r})}{1/2}}$$

Special embedding corresponds to c > 1.

Special Embedding

Consider **a Weyl fermion** that transforms as the fundamental rep. of SU(MN) but behaves as the **r** rep. of the SU(N) subgroup :

$$\mathcal{D}_\mu \psi = \partial_\mu \psi - i g_{\mathrm{UV}} A^m_{\mathrm{UV},\mu} (T^m_{\mathrm{UV}}) \psi$$

$$\supset \partial_\mu \psi - i g_{\mathrm{IR}} A^a_{\mathrm{IR},\mu} (T^a_{\mathrm{IR}}) \psi$$

A part of SU(MN) gauge field is expressed in terms of SU(N) gauge field :

$$A^l_{\mathrm{UV},\mu} = \frac{g_{\mathrm{IR}}}{g_{\mathrm{UV}}} \underline{A^a_{\mathrm{IR},\mu}} (\mathcal{O})^{al}$$

Canonically normalized kinetic term $\Rightarrow$ $\boxed{g_{\mathrm{IR}} = g_{\mathrm{UV}}/\sqrt{c}}$

Theta term : $\displaystyle\int \frac{g^2_{\mathrm{UV}}}{8\pi^2} \mathrm{tr}(F_{\mathrm{UV}} \wedge F_{\mathrm{UV}}) = \int \frac{c g^2_{\mathrm{IR}}}{8\pi^2} \mathrm{tr}(F_{\mathrm{IR}} \wedge F_{\mathrm{IR}})$

UV Completion

Hor, YN, Suzuki, Xu (2026)

Gauge symmetry breaking :

$$\underline{SU(4N)} \times SU(3)_2 \times SU(N)_2 \times U(1)_X \to \underline{SU(N) \times SU(3)_c} \times U(1)_Y$$

$\supset SU(4)_1 \times SU(N)_1$: **Special embedding** **Diagonal subgroup**

- All SM matter fields are charged under $SU(3)_2 \times U(1)_X$

- **A vector-like pair of PQ-charged fermions** transform as (anti-)fundamental reps. under SU(4N), so that **N_{DW} = 1**.

  After gauge symmetry breaking …

$$\Psi : (\mathbf{N}, \mathbf{3}) + (\mathbf{N}, \mathbf{1}) \ , \quad \bar{\Psi} : (\overline{\mathbf{N}}, \overline{\mathbf{3}}) + (\overline{\mathbf{N}}, \mathbf{1})$$

- The apparent vacuum degeneracy is lifted by **small instanton effects** on the axion potential that operates as <u>a PQ-violating bias term</u>.

Small Instanton

QCD θ-vacuum : superposition of *n*-vacua (energy degenerate but topologically distinct)

$$|\theta\rangle = \sum_{n=-\infty}^{\infty} e^{-in\theta}|n\rangle = \cdots + |0\rangle + e^{-i\theta}|1\rangle + \cdots$$

Instanton describes the tunneling effect between degenerate *n*-vacua

Instanton : localized object in Euclidean spacetime, satisfying Euclidean EOM with non-trivial topology and minimize Euclidean action

SU(2) BPST instanton solution with Q = 1 :

$$\frac{g^2}{32\pi^2}\int d^4x\, F^{a\mu\nu}\widetilde{F}^a_{\mu\nu}\Big|_{\text{inst.}} = Q \quad (Q \in \mathbb{Z})$$

$$A^a_\mu(x)\Big|_{1-\text{inst.}} = 2\eta_{a\mu\nu}\frac{(x-x_0)_\nu}{(x-\underline{x_0})^2+\underline{\rho^2}}$$

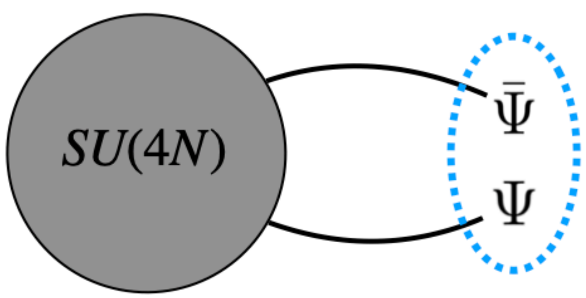
Position Instanton size

Small Instanton

Instantons contribute to the axion potential $\propto \exp\left(-\frac{8\pi^2}{g^2(1/\rho)}\right)$

In QCD, large-size instantons dominate the axion potential due to asymptotic freedom.

UV gauge interaction beyond QCD ➡ **Small instanton effects**



$$\sim \exp\begin{pmatrix} i\frac{a(x)}{f_{\mathrm{PQ}}}\mathbf{1}_{3\times 3} & 0 \\ 0 & -i\frac{3a(x)}{f_{\mathrm{PQ}}} \end{pmatrix}$$

't Hooft vertex

$$\Rightarrow \quad V_{4N} \sim \Lambda_{\mathrm{cut}}\Psi\bar{\Psi}e^{-8\pi^2/(g_{SU(4N)}(\Lambda_{\mathrm{cut}}))^2}$$

Λ_{cut} : cut-off scale

Post-Inflation Axion

PQ symmetry is spontaneously broken after reheating.

We focus on the scenario where the axion field starts to oscillate due to the axion mass originated from **the bias term** :

$$m_{\rm bias}^2 = \frac{1}{f_{\rm PQ}^2}\frac{\partial^2 V_{\rm bias}}{\partial\theta_a^2}$$

Temperature when the oscillation starts :

$$m_a(T_1) \approx m_{\rm bias} = 3H(T_1) \qquad H(T_1)^2 \approx \frac{\pi^2}{90 M_{\rm Pl}^2} g_* T_1^4$$

$$T_1 \; > \; \underline{T_{1,{\rm QCD}}} \equiv 0.98 \ {\rm GeV} \left(\frac{f_a}{10^{12} \ {\rm GeV}}\right)^{-0.19}$$

Temperature when the axion would start to oscillate with non-perturbative QCD effects if there was no bias term

Domain walls decay in a similar way as the standard $N_{DW} = 1$ case.

Post-Inflation Axion

Axion abundance :

$$\Omega_a h^2 \approx (2^{+2}_{-1}) \times 10^{-12} \frac{f_{\rm PQ}}{T_1}$$

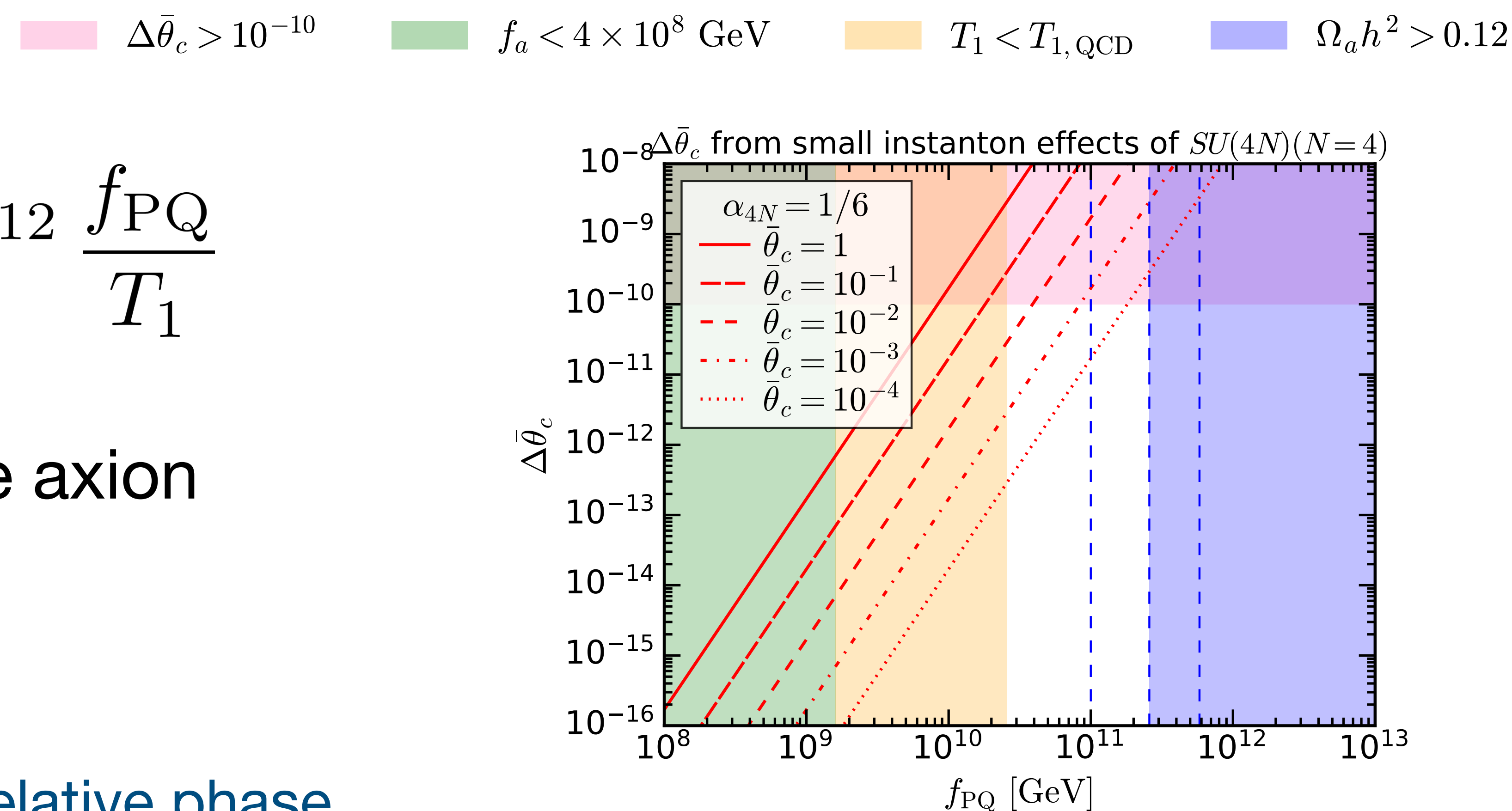
A large bias term shifts the axion potential minimum :

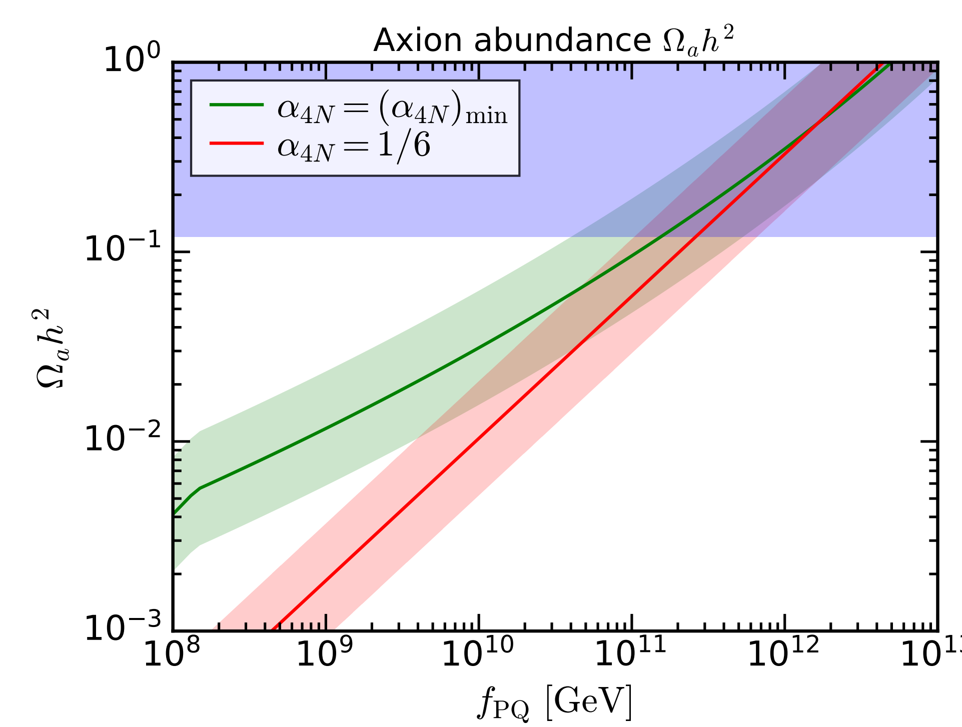
$$\Delta\bar{\theta}_c \approx \frac{V_{\rm bias}}{V_{\rm QCD}} \underline{\bar{\theta}_c}$$ Relative phase

Neutron EDM

$$\Rightarrow \Delta\bar{\theta}_c \lesssim 10^{-10}$$

Axion does not appear to explain the correct DM abundance…





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  Existing global U(1) axial symmetry can rotate the θ term to zero.

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- **Spontaneous CP (or P) violation**

  CP is an exact symmetry of the Lagrangian but broken spontaneously at the vacuum.

  CKM phase is generated without reintroducing the θ term.

…

Spontaneous CPV

- Spontaneous CP violation (SCPV) provides **an axionless solution** to the strong CP problem.

- CP is an exact symmetry of the Lagrangian but <u>broken spontaneously at the vacuum</u>

→ Generation of the CKM phase <u>without reintroducing a nonzero strong CP phase</u>

Nelson-Barr mechanism

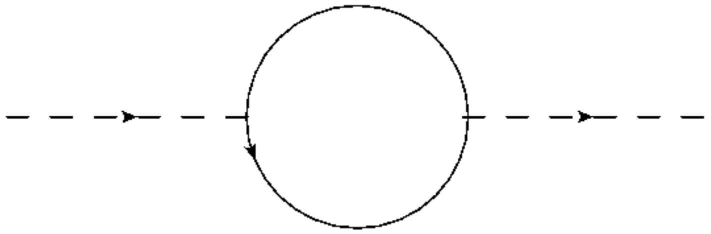
<u>A vector-like pair of heavy quarks</u> is introduced, so that the extended quark mass matrix transmits SCPV into the CKM matrix.

$$\mathcal{L} = \mu \underline{\bar{q}q} + a_{a\bar{f}} \underline{\eta_a} \bar{d}_{\bar{f}} q + y_{f\bar{f}} H Q_f \bar{d}_{\bar{f}} \quad \Rightarrow \quad \mathcal{M} = \begin{pmatrix} \mu & a_{a\bar{f}}\eta_a \\ \boxed{0} & m_d \end{pmatrix}$$

Heavy quarks **CP breaking field**

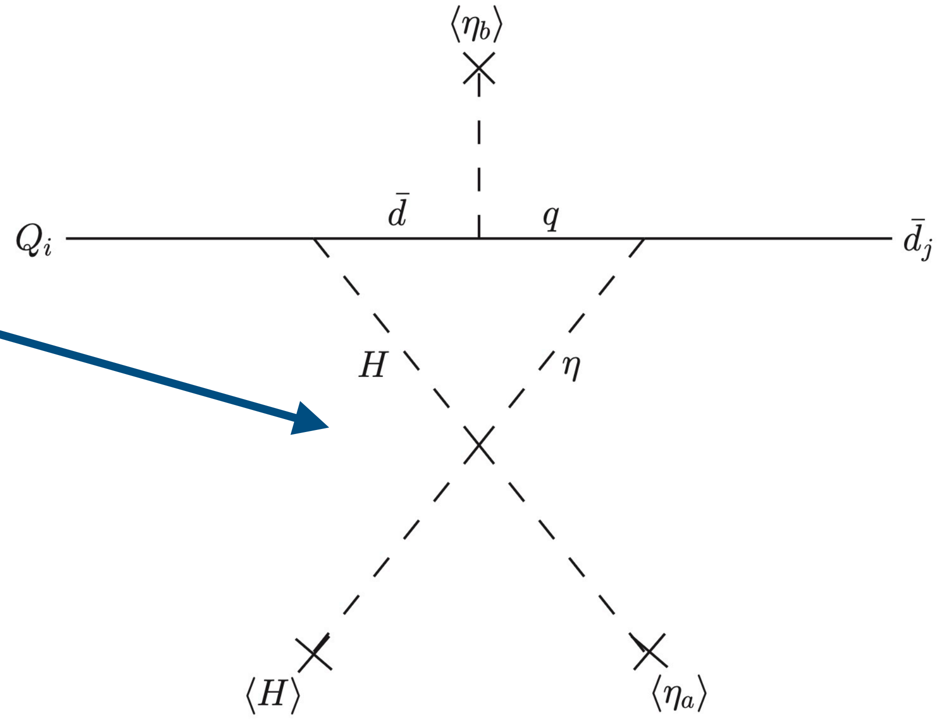
Challenges for SCPV

- **Naturalness problem**: the mechanism requires new scalar fields whose VEVs break CP much below the Planck scale.



- Sensitivity to **higher-dim. operators** and **radiative corrections**, regenerating a strong CP phase.

$$\mathcal{L} \supset \frac{\eta_b^*}{\Lambda}\eta_a \bar{q} q + \frac{\eta_a^*}{\Lambda} H Q \bar{q} + \underline{\gamma_{ab}\eta_a^* \eta_b H^\dagger H}$$



Dine, Draper (2015)

Challenges for SCPV

- **Naturalness problem**: the mechanism requires new scal ds whose VEVs break CP much below the Planck

Supersymmetry (SUSY) offers a natural framework to address these difficulties !

- Sensi im. operators and radiative corrections, regene g a strong CP phase.

✓ SUSY stabilizes the SCPV scale in much the same way it stabilizes the EW scale.

✓ SUSY can forbid or strongly suppress dangerous higher-dim. operators.

It's reasonable to consider SCPV in SUSY.

For non-SUSY approach, see *e.g.* Girmohanta, Lee, YN, Suzuki (2022).

SCPV in SUSY

The physical strong CP phase :

$$\bar{\theta} \equiv \theta - \arg\det\left(M_u M_d\right) - 3\arg\left(\underline{m_{\tilde{g}}}\right)$$

Gluino mass

To set the cosmological constant to zero

$$\Rightarrow \underline{\langle W \rangle} \sim \underline{m_{3/2}} M_{\rm Pl}^2$$

Gravitino mass

Complex phase generates a gluino mass phase via 1-loop anomaly med.

Constraint on $\bar{\boldsymbol{\theta}}$ $\Rightarrow$ $\dfrac{\alpha_s}{4\pi}\dfrac{m_{3/2}}{m_{\tilde{g}}} < 10^{-10}$

Gravitino mass must be sufficiently small.

SCPV in SUSY

- **Flat directions** are ubiquitous in SUSY.

  Valleys in field space along which the potential is exactly flat.

- Flat directions naturally contain a point to spontaneously break CP.

How to stabilize such a point ?

Giving positive masses for all scalar fields around the minimum.

- Two qualitatively different ways for the stabilization:

  ✓ Purely in a **supersymmetric** way ← No low-energy mode …

  ✓ Through **SUSY breaking effects**

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- Two qualitatively different ways for the stabilization:

  ✓ Purely in a **supersymmetric** way

  ✓ Through **SUSY breaking effects** ← **Let's focus !**

SCPV via ~~SUSY~~

Liu, Nakagawa, YN, Wang (2025)

- **Supersymmetric potential**

$$V_F = \lambda^2 |\phi_1\phi_2 - v^2|^2 + \underline{\lambda^2}|X|^2(|\phi_1|^2 + |\phi_2|^2)$$

Real parameter

$$\Rightarrow \langle X \rangle = 0, \quad \langle \phi_1 \rangle = v_1 e^{i\theta}, \quad \langle \phi_2 \rangle = v_2 e^{-i\theta} \quad v_1 v_2 \equiv v^2$$

Not uniquely determined

SCPV scale

$$\phi_1(x) = \left(v_1 + \frac{\sigma_1(x)}{\sqrt{2}}\right)\exp\left[i\left(\theta + \frac{\pi_1(x)}{\sqrt{2}v_1}\right)\right], \quad \phi_2(x) = \left(v_2 + \frac{\sigma_2(x)}{\sqrt{2}}\right)\exp\left[i\left(-\theta + \frac{\pi_2(x)}{\sqrt{2}v_2}\right)\right]$$

Massless modes :

$$s(x) = \frac{1}{f_a}\left(v_1\sigma_1(x) - v_2\sigma_2(x)\right), \quad a(x) = \frac{1}{f_a}\left(v_1\pi_1(x) - v_2\pi_2(x)\right)$$

$$f_a \equiv \sqrt{v_1^2 + v_2^2}$$

SCPV via ~~SUSY~~

Two terms with <u>different periodicity</u> are needed, *e.g.* $V = \mathcal{C}\cos\theta + \mathcal{D}\cos(2\theta)$

- **SUSY breaking**

$$V_{\text{soft}} = \left(\frac{1}{2}b_1\phi_1^2 + \frac{1}{2}b_2\phi_2^2 + \text{h.c.}\right) + m_1^2\phi_1^*\phi_1 + m_2^2\phi_2^*\phi_2$$

The potential is bounded from below. ➡ $|b_1| \leq m_1^2 \ , \quad |b_2| \leq m_2^2$

- **Non-perturbative effect**

SU(N) dark QCD with quarks coupled to SCPV fields.

$$\Rightarrow \quad V_{\text{dyn}} = \frac{\left(\kappa_1^2 + \kappa_2^2\right)\Lambda^{6-\frac{2}{N}}}{\left|\kappa_1\phi_1 + \kappa_2\phi_2\right|^{2-\frac{2}{N}}}$$

Real coupling constants

Λ : dark QCD dynamical scale

SCPV via ~~SUSY~~

$$V_{\rm tot} = V_F + V_{\rm soft} + V_{\rm dyn}$$

$$= b_1\left(v_1^2 + \frac{b_2}{b_1}\frac{v^4}{v_1^2}\right)\underline{\cos(2\theta)} + m_1^2\left(v_1^2 + \frac{m_2^2}{m_1^2}\frac{v^4}{v_1^2}\right) + \frac{(\kappa_1^2+\kappa_2^2)\Lambda^{6-\frac{2}{N}}}{[\kappa_1^2 v_1^2 + \kappa_2^2 v^4/v_1^2 + 2\kappa_1\kappa_2 v^2\,\underline{\cos(2\theta)}]^{1-\frac{1}{N}}}$$

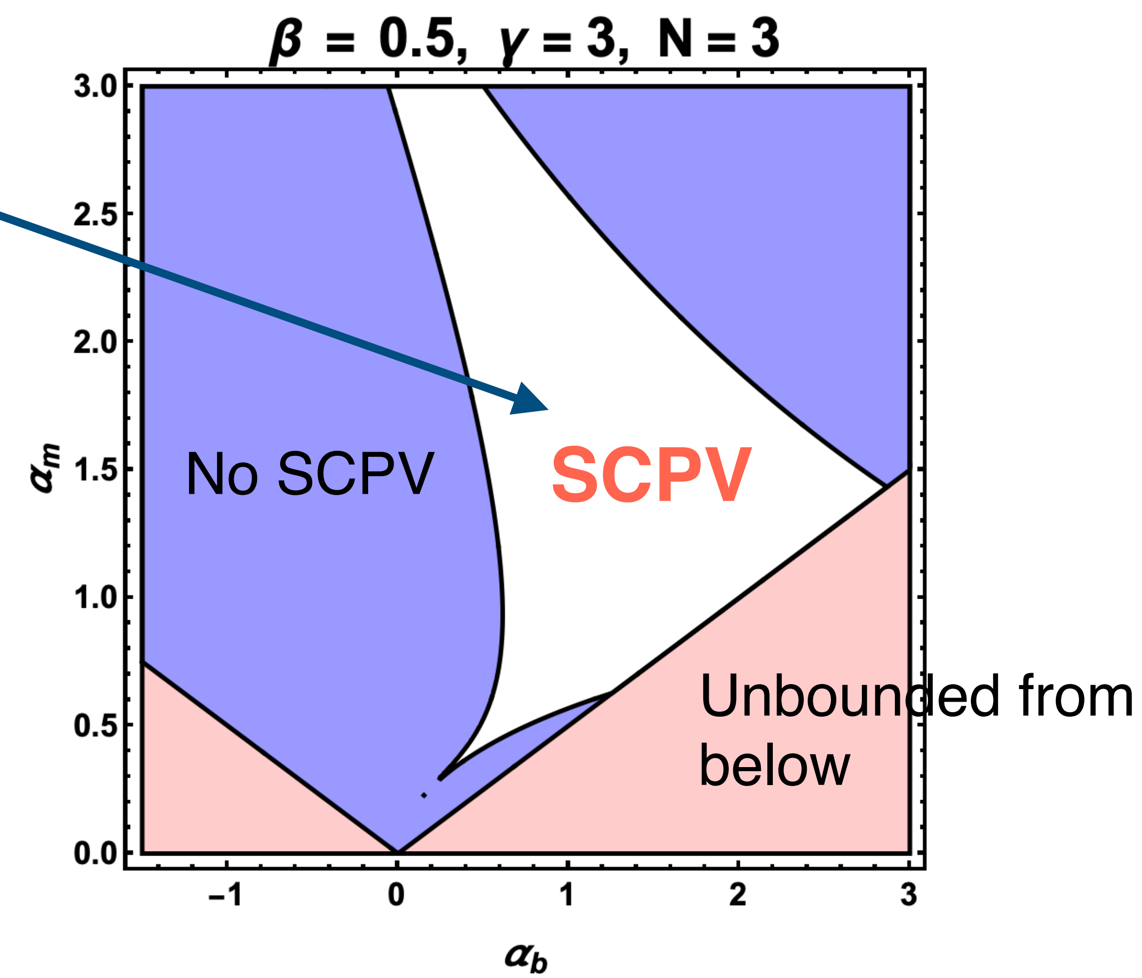
Stabilized at a non-trivial value $(\neq 0, \pi)$

All directions properly stabilized.

Existence of light modes :

$$m_a^2 \sim m_s^2 \lesssim m_{\rm soft}^2$$

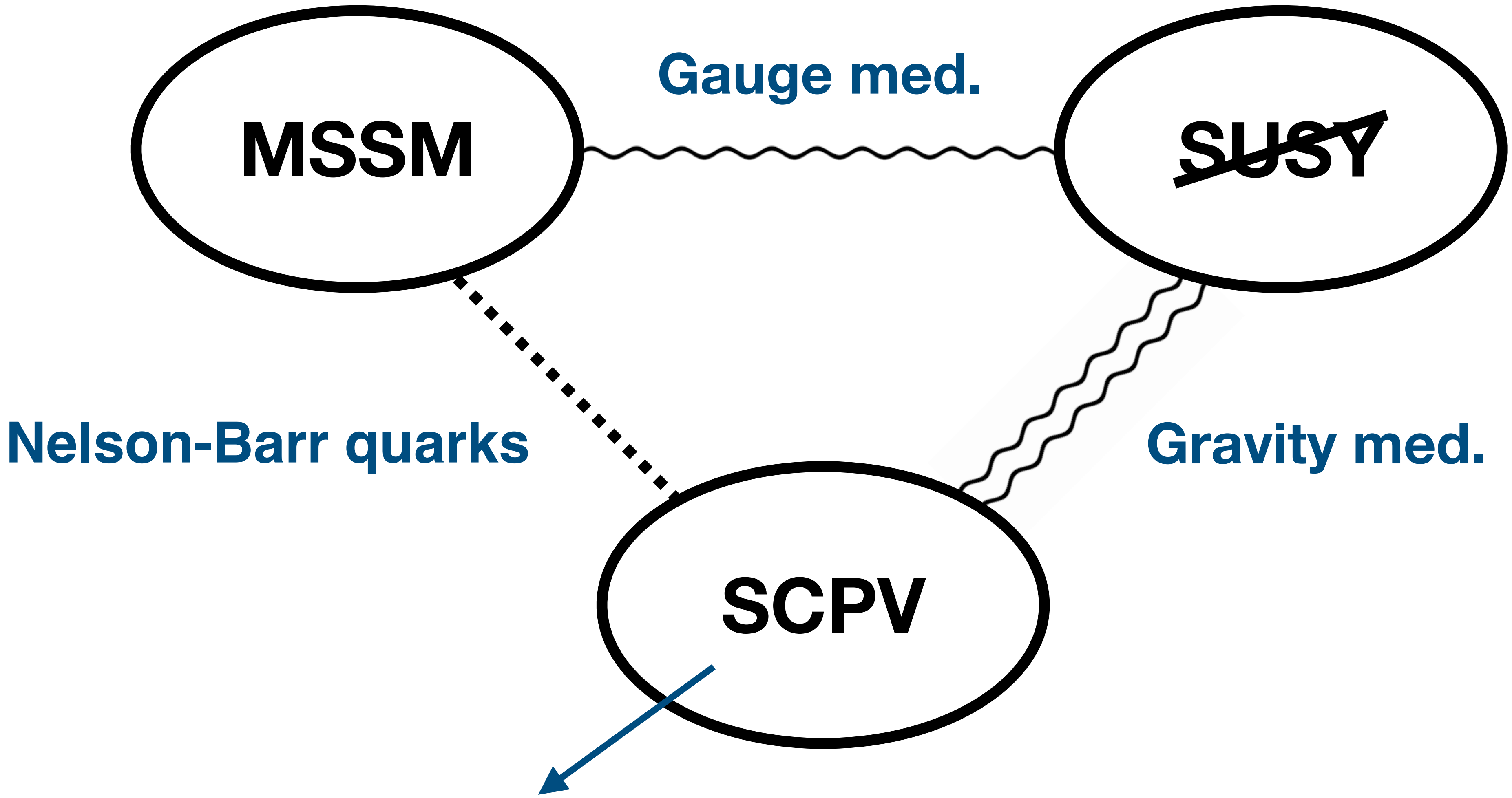
Much smaller than SCPV scale !



$\alpha_b \equiv b_2/b_1, \alpha_m \equiv m_2^2/m_1^2, \beta \equiv b_1/m_1^2, \gamma \equiv \Lambda^{6-\frac{2}{N}}/(m_1^2 v^{4-\frac{2}{N}})$

SCPV via ~~SUSY~~

Solving the strong CP problem via SCPV



$$m_a^2 \sim m_s^2 \sim m_{3/2}^2 < (10 - 100\,\mathrm{keV})^2$$

Light particles feebly interacting with SM are predicted !

Detailed phenomenology & cosmology will be explored in a future study.

Baryogenesis

Fujikura, YN, Sato, Yamada (2022)

Considering SCPV …

- What is **the source of CPV** for baryogenesis?
- High reheating temperature for thermal leptogenesis causes **the overproduction of gravitinos**.

Affleck-Dine (AD) mechanism

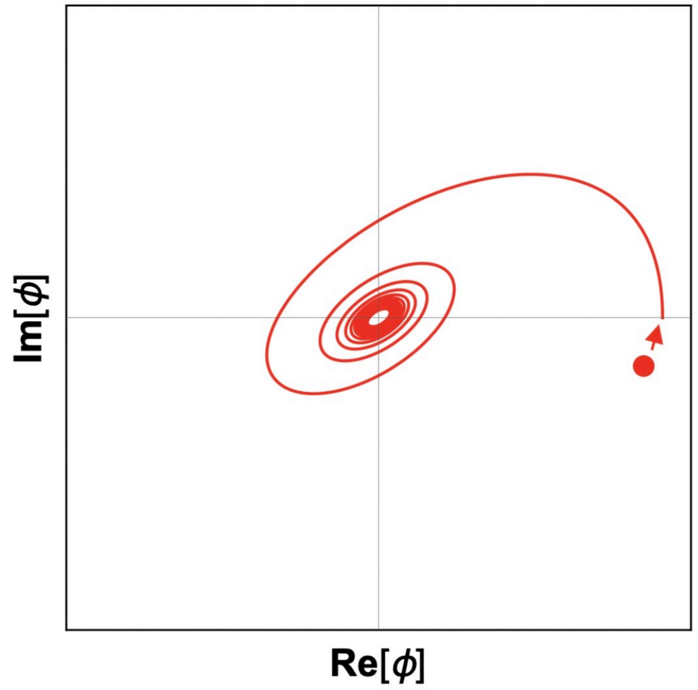
realized in SUSY and compatible with a low reheating temperature.

✓ **AD field** (ϕ) coherent rotational motion leads to baryon asymmetry.

$$\phi^3 \approx Q\bar{q}L$$

$$\phi = \frac{\varphi}{\sqrt{2}}e^{i\theta}, \quad n_{B-L} = q_{B-L}\varphi^2\dot{\theta}$$

✓ Explicit CPV is not needed.



Baryogenesis

Fujikura, YN, Sato, Yamada (2022)

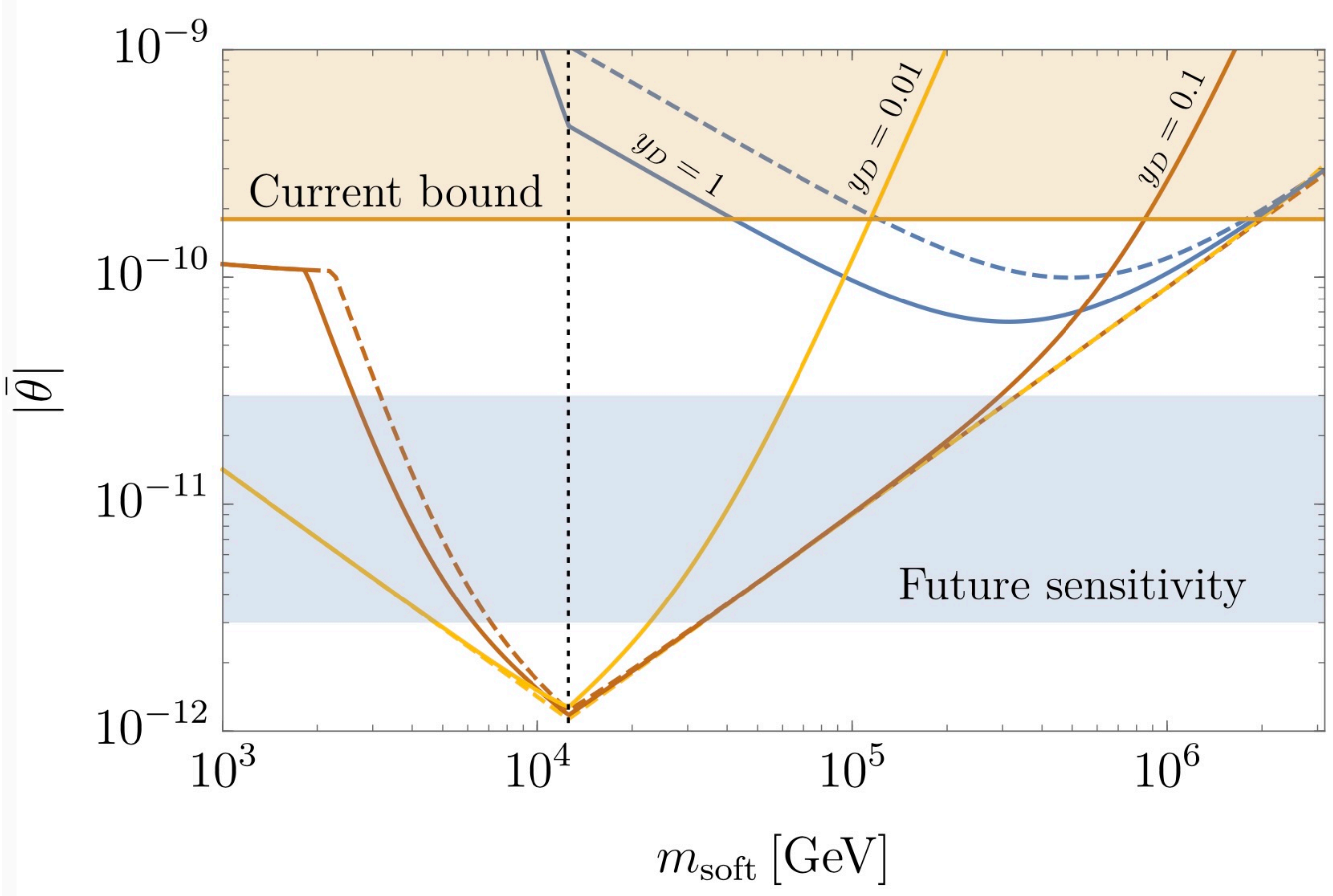
If **gravitino** gives DM …

Lyman-α constraint → $m_{3/2} \gtrsim 5.3\,\mathrm{keV}$

Smallest value of $\bar{\theta}$

SCPV scale and reheating temperature are chosen to obtain the observed asymmetry and DM.

Consistency with SCPV via ~~SUSY~~ will be explored in a future study.



Neutron EDM is within the reach of near-future experiments !

Summary

Composite axion

- UV completion via **special embedding** leads to domain wall # = 1.
- **Small instanton effects** provide a bias term for the domain wall decay.

Spontaneous CP violation (SCPV) in SUSY

- SCPV vacuum stabilization via **SUSY breaking** predicts **light particles feebly interacting with SM**.
- **Neutron EDM** is within the reach of near-future experiments.

Thank you.

Backup

Special Embedding

Consider a gauge sym. breaking : $SU(MN) \to SU(M) \times SU(N)$

Embed **bi-fundamental** representation of SU(M) × SU(N) into **fundamental** representation of SU(MN).

$$\mathbf{v} \ \to \ e^{i\beta_M^a \hat{T}_M^a} \, e^{i\beta_N^\alpha \hat{\tau}_N^\alpha} \, \mathbf{v}$$

$$\hat{T}_M^a \equiv T_M^a \otimes 1_{N\times N} = \begin{pmatrix} T_M^a & \mathbf{0} & \cdots \\ \mathbf{0} & T_M^a & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Generators of SU(M) in fundamental rep.

$$\mathrm{tr}(T_M^a T_M^b) = \frac{1}{2}\delta^{ab}$$

$$\hat{\tau}_N^\alpha \equiv 1_{M\times M} \otimes \tau_N^\alpha = \begin{pmatrix} 1_{M\times M}(\tau_N^\alpha)_{11} & 1_{M\times M}(\tau_N^\alpha)_{12} & \cdots \\ 1_{M\times M}(\tau_N^\alpha)_{21} & 1_{M\times M}(\tau_N^\alpha)_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Generators of SU(N) in fundamental rep.

$$\mathrm{tr}(\hat{T}_M^a \hat{\tau}_N^\alpha) = 0$$

Special Embedding

Embed SU(M) × SU(N) generators into those of SU(MN) :

$$\hat{T}^a_M = \underbrace{\mathcal{O}^{am}_M}_{\text{Coefficients}} T^m_{\mathrm{UV}} \,, \qquad \hat{\tau}^\alpha_N = \underbrace{\mathcal{O}^{\alpha m}_N}_{\text{Coefficients}} T^m_{\mathrm{UV}}$$

$$\mathrm{tr}(\hat{T}^a_M \hat{T}^b_M) = N \cdot \frac{1}{2}\delta^{ab} \qquad \mathrm{tr}(\hat{\tau}^\alpha_N \hat{\tau}^\beta_N) = M \cdot \frac{1}{2}\delta^{\alpha\beta}$$

$$\mathrm{tr}(T^m_{\mathrm{UV}} T^n_{\mathrm{UV}}) = \frac{1}{2}\delta^{mn}$$

$$\mathcal{O}^{am}_M \mathcal{O}^{bn}_M \delta_{mn} = c_M\, \delta^{ab} \,, \qquad c_M = N$$

$$\mathcal{O}^{\alpha m}_N \mathcal{O}^{\beta n}_N \delta_{mn} = c_N\, \delta^{\alpha\beta} \,, \qquad c_N = M$$

Special Embedding

Consider **a fermion** ψ which transforms as fundamental rep. of SU(MN) and bi-fundamental rep. of SU(M) × SU(N) :

$$\mathcal{D}_\mu \psi = \partial_\mu \psi - i g_{\mathrm{UV}} A^m_{\mathrm{UV},\mu} T^m_{\mathrm{UV}} \psi$$

$$\supset \partial_\mu \psi - i g_{\mathrm{IR},M} A^a_{\mathrm{IR},M,\mu} \hat{T}^a_M \psi - i g_{\mathrm{IR},N} A^\alpha_{\mathrm{IR},N,\mu} \hat{\tau}^\alpha_N \psi$$

Matching the gauge fields :

$$A^m_{\mathrm{UV},\mu} = \frac{g_{\mathrm{IR},M}}{g_{\mathrm{UV}}} \underline{A^a_{\mathrm{IR},M,\mu}} \mathcal{O}^{am}_M , \qquad A^m_{\mathrm{UV},\mu} = \frac{g_{\mathrm{IR},N}}{g_{\mathrm{UV}}} \underline{A^\alpha_{\mathrm{IR},N,\mu}} \mathcal{O}^{\alpha m}_N$$

Canonically normalized kinetic term

$$\Rightarrow \boxed{ g_{\mathrm{IR},M} = \frac{g_{\mathrm{UV}}}{\sqrt{c_M}} , \qquad g_{\mathrm{IR},N} = \frac{g_{\mathrm{UV}}}{\sqrt{c_N}} }$$

Special Embedding

Consider **a fermion** ψ which transforms as fundamental rep. of SU(MN) and bi-fundamental rep. of SU(M) × SU(N) :

$$\mathcal{D}_\mu \psi = \partial_\mu \psi - i g_{\mathrm{UV}} A^m_{\mathrm{UV},\mu} T^m_{\mathrm{UV}} \psi$$

$$\supset \partial_\mu \psi - i g_{\mathrm{IR},M} A^a_{\mathrm{IR},M,\mu} \hat{T}^a_M \psi - i g_{\mathrm{IR},N} A^\alpha_{\mathrm{IR},N,\mu} \hat{\tau}^\alpha_N \psi$$

Theta term

$$\int \frac{g^2_{\mathrm{UV}}}{8\pi^2} \mathrm{tr}(F_{\mathrm{UV}} \wedge F_{\mathrm{UV}}) \supset \int c_N \frac{g^2_{\mathrm{IR},N}}{8\pi^2} \mathrm{tr}(F_{\mathrm{IR},N} \wedge F_{\mathrm{IR},N}) + \int c_M \frac{g^2_{\mathrm{IR},M}}{8\pi^2} \mathrm{tr}(F_{\mathrm{IR},M} \wedge F_{\mathrm{IR},M})$$

Domain wall # = 1 in UV gauge theory

➡ **Nontrivial domain wall # in IR gauge theory !**