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(De)constructing the Extra-dimensional Axion

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Axion 2026 @Qingdao

based on Hor, Nakai, Suzuki, JX, *arXiv:2606.02728*

Strong CP Problem & Axion Solution

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QCD Lagrangian allows the interaction which explicitly violates P and CP symmetry:

$$\mathcal{L}_\theta = \theta \frac{g_s^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a\mu\nu}$$

The physical strong CP Phase:

The upper bound on neutron EDM requires

$$\bar{\theta} = \theta + 2\Sigma(\alpha_i^u + \alpha_i^d) = \theta - \mathrm{Arg}\,\det(\mathcal{M}^u \mathcal{M}^d) \lesssim 10^{-10} \;\; \textbf{!!}$$

The axion solution introduces an additional chiral $U(1)$ symmetry, known as $U(1)_{\rm PQ}$, which is spontaneously broken and the axion is the corresponding pseudo-Nambu-Goldstone boson.

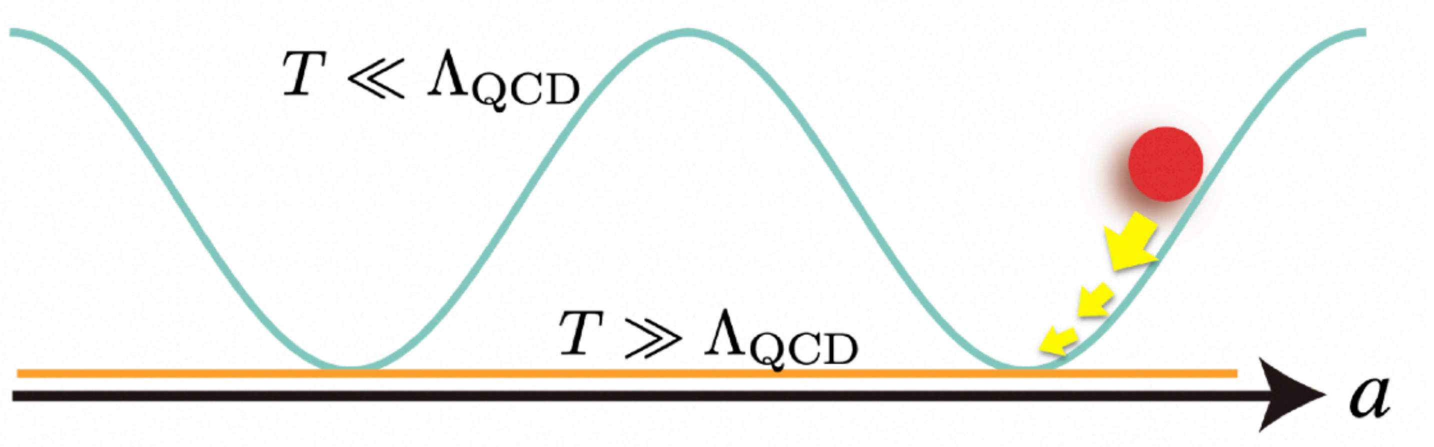
Peccei and Quinn (1977)

The strong CP phase is promoted to a dynamical field $\bar{\theta}_a(x)$

$$\mathcal{L} \supset \underbrace{\left(\bar{\theta} + N_{\rm DW}\frac{a(x)}{f_{\rm PQ}}\right)}_{\bar{\theta}_a(x)} \frac{g_s^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a\mu\nu}$$

After QCD confinement, the strong dynamics generates the axion potential and dynamically cancel the phase.

$$V(a) = m_a^2 f_a^2 \left(1 - \cos\left(\frac{a}{f_a}\right)\right)$$



Axion Quality Problem

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Additional axion potential induced by **non-QCD PQ-violation** can easily *spoil the axion solution!*

$$\delta\theta_a \simeq \frac{\delta V}{m_a^2 f_a^2} < 10^{-10}$$

Global symmetries are generically expected to be broken by the **Quantum Gravity** effects:

Dine, Seiberg (1986), Barr, Seckel (1992), Kamionkowski, March-Russell (1992) …

$$\mathcal{L}_{\rm PQV} = \frac{c}{M_{\rm Pl}^{n-4}}\Phi^n + \text{h.c.}$$

$$\Rightarrow \quad \delta V(a) = -2|c|M_{\rm Pl}^4\left(\frac{f_{\rm PQ}}{\sqrt{2}M_{\rm Pl}}\right)^n \cos\left(n\frac{a}{f_{\rm PQ}} + \delta\right)$$

For $|c|, \delta \sim \mathcal{O}(1)$, $f_{\rm PQ} = 10^{12}\text{GeV}$, we need $n \geq 14$!

Extra-dimensional Axion provides a possible solution to the quality problem!

Choi (2004), Reece (2025), Craig, Kongsore (2025), Choi, Lee, Shin (2026) …

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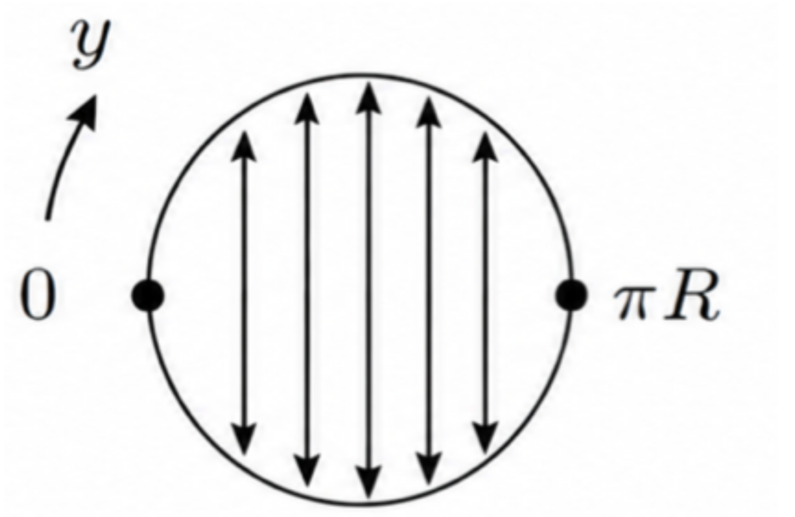
5D Orbifold Axion

Consider a $U(1) \times SU(3)$ gauge theory on S^1/Z_2



$y \in [-\pi R, \pi R)$ $\quad y \cong -y$

$$S_5^{\mathrm{gauge}} = \int \mathrm{d}^5 x \left[\underbrace{-\frac{1}{4g_5^2}\mathcal{F}_{MN}\mathcal{F}^{MN}}_{\text{Axion}} \underbrace{- \frac{1}{2g_{5,c}^2}\mathrm{tr}\left(\mathcal{G}_{MN}\mathcal{G}^{MN}\right)}_{\text{QCD}} \right] + S_5^{\mathrm{CS}} + S_5^{\mathrm{GF}}$$

Boundary conditions:

No light DoFs

$U(1)$ $\quad \cancel{A_\mu|_{0,\pi R} = 0}$ $\quad \partial_y A_5|_{0,\pi R} = 0$ $\quad SU(3)$ $\quad \partial_y G_\mu|_{0,\pi R} = 0$ $\quad \cancel{G_5|_{0,\pi R} = 0}$

Massless scalar: **axion** ⊕ Massless gauge field: **gluon** ⊕ Massive KK tower

Define the ***axion*** — the phase of the Wilson line in the 5th direction

$$\theta_5(x) \equiv \int_0^{\pi R} \mathrm{d}y\, A_5(x, y) = A_{5,0}(x)\pi R$$

zero mode of A_5 component

5D Orbifold Axion

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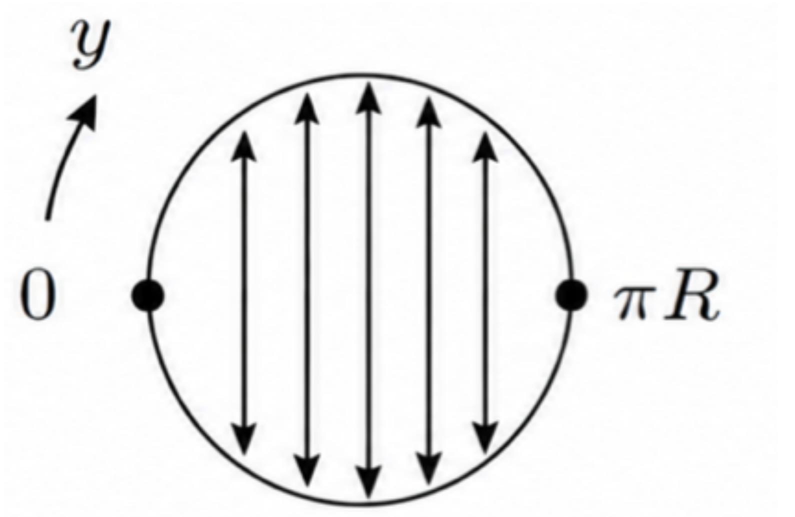
Consider a $U(1)\times SU(3)$ gauge theory on S^1/Z_2

aGG-coupling



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$y \in [-\pi R, \pi R)$ $\quad y \cong -y$

$$\boxed{-\int \mathrm{d}^5 x \frac{\kappa_{\text{CS}}}{32\pi^2}\epsilon^{MNPQR}A_M\,\mathrm{tr}\left(\mathcal{G}_{NP}\mathcal{G}_{QR}\right)}$$

Boundary conditions:

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$U(1)$ $\quad \cancel{A_\mu|_{0,\pi R} = 0}$ $\quad \partial_y A_5|_{0,\pi R} = 0$ $\quad SU(3)$ $\quad \partial_y G_\mu|_{0,\pi R} = 0$ $\quad \cancel{G_5|_{0,\pi R} = 0}$

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Axion Quality

The bulk $U(1)$ *gauge symmetry* highly constrains the **axion shift symmetry** violation.

$$A \to A + \mathrm{d}\alpha$$

$$A \to A + \lambda \qquad \theta_5 = \int_I A \to \theta_5 + \int_I \lambda$$

$\cong$ electric one-form symmetry

Craig, Kongsore (2025)

Two main $U(1)_{\mathrm{shift}}$-violating channels: Reece (2024)

- **Gauge theory instanton (Focus in this talk)**

  A couples to additional gauge sectors with CS term $\sim e^{-8\pi^2/g^2}$

"Take log of quality problem"

- **Charged bulk matter** Choi, Lee, Shin (2026)

  with boundary $U(1)$-violating operator, induced axion potential $\sim e^{-M\pi R}$

$$D_M Q \supset (\partial_5 + iA_5)Q$$

axion-dep. in KK masses

5D Gauge Theory is Non-Renormalizable

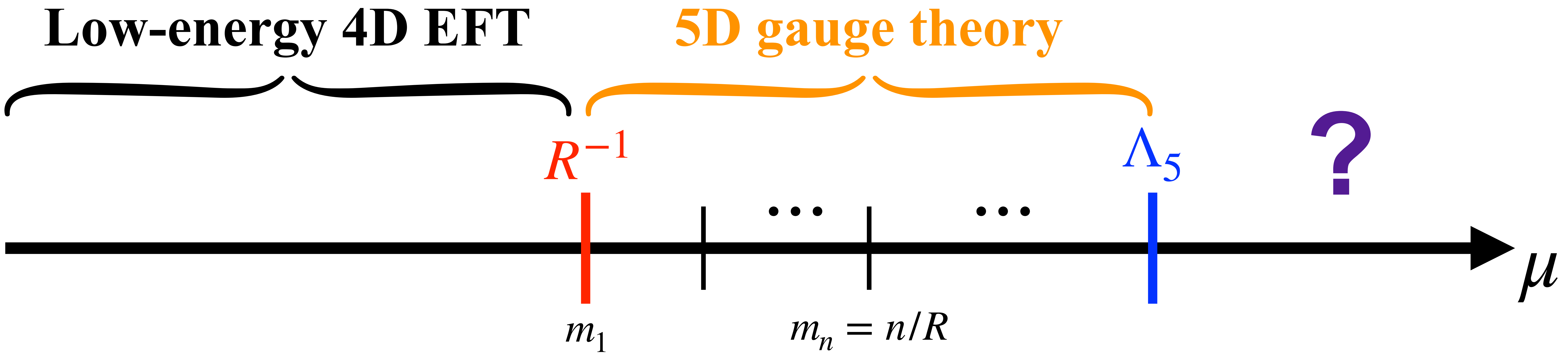
$$S_5^{\text{gauge}} = \int \mathrm{d}^5x \left[-\frac{1}{4g_5^2} \mathcal{F}_{MN}\mathcal{F}^{MN} - \frac{1}{2g_{5,c}^2} \operatorname{tr}\left(\mathcal{G}_{MN}\mathcal{G}^{MN}\right) \right] + S_5^{\text{CS}} + S_5^{\text{GF}}$$

Gauge coupling constants have *negative* mass dimension (generic for higher dim.)

$$[g_5] = [g_{5,c}] = -1/2 < 0$$

5D gauge theory breaks down when the coupling gets strong $\dfrac{g_5^2 E}{24\pi^3} \gtrsim 1$

→ UV cutoff $\Lambda_5 \lesssim \min\left\{ \dfrac{24\pi^3}{g_5^2}, \dfrac{24\pi^3}{g_{5,c}^2} \right\}$



Dimensional Deconstruction

Akani-Hamed, Cohen, Georgi (2001)

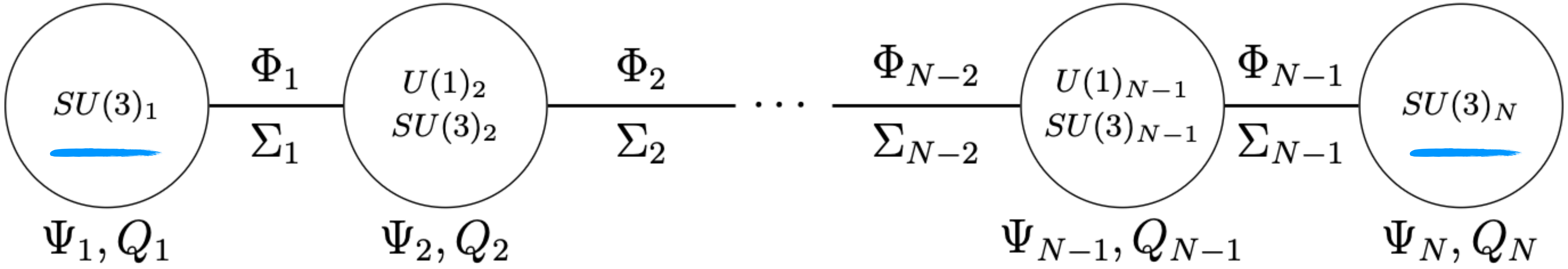
Discretize the 5th dim

5D field theory $\longrightarrow$ 4D field theory

$y \in [0, \pi R] \longrightarrow y_j = ja\,, j = 1, \cdots, N$ $\quad \pi R = Na$

Dirichlet BC

No $U(1)_1\,, U(1)_N$



Dimensional Deconstruction

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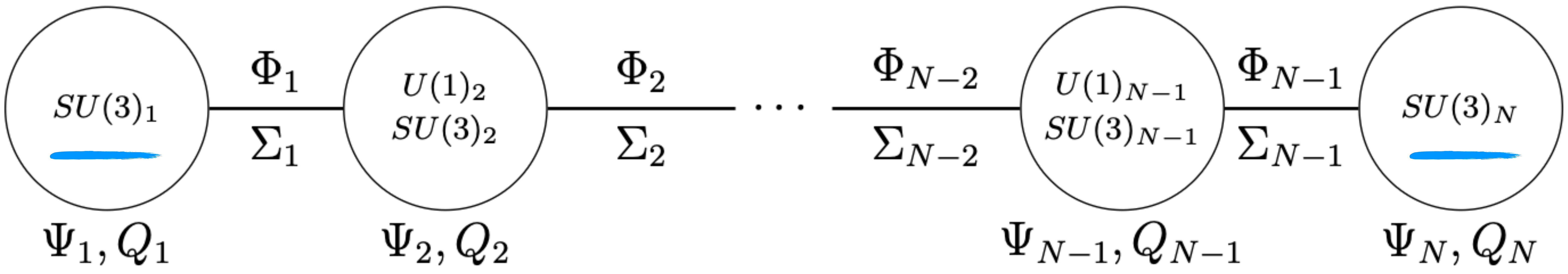
5D field theory —— **Discretize the 5th dim** ⟶ 4D field theory

$y \in [0, \pi R] \longrightarrow y_j = ja\,, j = 1, \cdots, N \qquad \pi R = Na$

Dirichlet BC ↓ No $U(1)_1\,, U(1)_N$



Link fields

	$U(1)_j$	$U(1)_{j+1}$	$SU(3)_j$	$SU(3)_{j+1}$
Φ_j	$+1$	-1	$\mathbb{1}$	$\mathbb{1}$
Σ_j	0	0	$\square$	$\overline{\square}$
Ψ_j	$+1$	0	$\square$	$\mathbb{1}$
Q_j	$+1$	0	$\mathbb{1}$	$\mathbb{1}$

Breaking pattern:

$$U(1)_2 \times \cdots \times U(1)_{N-1} \xrightarrow{\langle \Phi_j \rangle \equiv v/\sqrt{2}} 0$$

$$SU(3)_1 \times \cdots \times SU(3)_N \xrightarrow{\langle \Sigma_j \rangle \equiv v_3} SU(3)^{(0)}$$

Phase directions as **link variables** $\qquad \Phi_j \simeq (v/\sqrt{2})e^{i\chi_j/v} \qquad \chi_j/v \longleftrightarrow aA_{j,5}$

Deconstruction — Gauge Sector

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Deconstructed action

$$S_D^{U(1)} = \int \mathrm{d}^4x \left(-\frac{1}{4g^2} \sum_{j=2}^{N-1} \mathcal{F}_j^{\mu\nu} \mathcal{F}_{j,\mu\nu} + \sum_{j=1}^{N-1} D_\mu \Phi_j^* D^\mu \Phi_j - V(\Phi) \right)$$

Reproducing the 5D action:

$$\begin{aligned} S_D^{U(1)} &= \int \mathrm{d}^4x \left(-\frac{1}{4g^2} \sum_{j=2}^{N-1} \mathcal{F}_j^{\mu\nu} \mathcal{F}_{j,\mu\nu} + \frac{1}{2} v^2 \left[(\partial_\mu \chi_1/v - A_{2,\mu})^2 + (\partial_\mu \chi_{N-1}/v + A_{N-1,\mu})^2 \right] \right. \\ &\left. + \frac{1}{2} v^2 \sum_{j=2}^{N-2} (\partial_\mu \chi_j / v + A_{j,\mu} - A_{j+1,\mu})^2 \right) \\ &= \int \mathrm{d}^4x \left(-\frac{1}{4g^2} \sum_{j=2}^{N-1} \mathcal{F}_j^{\mu\nu} \mathcal{F}_{j,\mu\nu} + \frac{1}{2g^2} \sum_{j=1}^{N-1} [g \partial_\mu \chi_j + g v (A_{j,\mu} - A_{j+1,\mu})]^2 \right) \qquad (3.3) \\ &\leftrightarrow \underbrace{\int \mathrm{d}^4x \left(-\frac{a}{4g_5^2} \sum_{j=2}^{N-1} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu}(y_j) + \frac{a}{2g_5^2} \sum_{j=1}^{N-1} \left[\partial_\mu A_5(y_j) - \frac{A_\mu(y_j + a) - A_\mu(y_j)}{a} \right]^2 \right)}_{\text{Discretized 5D action}} , \end{aligned}$$

Dictionary

$$\begin{aligned} a \equiv \frac{\pi R}{N} &\longleftrightarrow (gv)^{-1} , \\ y_j / a &\longleftrightarrow j , \\ A_\mu(x, y_j) &\longleftrightarrow A_{j,\mu}(x) , \\ A_5(x, y_j) &\longleftrightarrow g \chi_j(x) , \\ \frac{a}{g_5^2} &\longleftrightarrow \frac{1}{g^2} , \end{aligned}$$

Continuum limit: $N \to +\infty, a \to 0 \longleftrightarrow v \to +\infty$ with πR fixed.

Deconstruction — Gauge Sector

$$\mathcal{L}_{gm} = \frac{1}{2} g^2 v^2 \left[\hat{A}_{2,\mu}^2 + \sum_{j=2}^{N-2} \left(\hat{A}_{j+1,\mu} - \hat{A}_{j,\mu} \right)^2 + \hat{A}_{N-1,\mu}^2 \right]$$

$$\equiv \frac{1}{2} \sum_{i,j} \hat{A}_{i,\mu} \left(\mathcal{M}_{U(1)}^2 \right)_{ij} \hat{A}_{j,\mu} \,. \qquad A_{j,\mu} = g\hat{A}_{j,\mu}$$

$$\mathcal{M}_{U(1)}^2 = g^2 v^2 \Delta_{N-2}^{\mathrm{D}}$$

$$\Delta_N^{\mathrm{D}} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

Discrete Laplacian

Spectrum

$$A_\mu^{(k)} = \frac{\sqrt{2}}{N-1} \sum_{j=2}^{N-1} \sin\left(\frac{(j-1)k\pi}{N-1} \right) A_{j,\mu}$$

$$\chi^{(k)} = \sqrt{\frac{2}{N-1}} \sum_{j=1}^{N-1} \cos\left(\frac{(j-\frac{1}{2})k\pi}{N-1} \right) \chi_j \,, \quad \text{with} \quad k = 1, \cdots, N-2 \,,$$

$N-2$ massive gauge bosons with eigenmasses

$$m_k^2 = 4g^2 v^2 \sin^2\left(\frac{k\pi}{2N-2} \right)$$

$$\xrightarrow{N \to \infty} g^2 v^2 \frac{k^2 \pi^2}{(N-1)^2} \leftrightarrow \frac{k^2}{R^2} \equiv m_{\mathrm{KK}}$$

One massless Goldstone left — the axion

$$a(x) \equiv \frac{1}{\sqrt{N-1}} \sum_{j=1}^{N-1} \chi_j(x) \longleftrightarrow a_5 = f_a \int \mathrm{d}y \, A_5(x, y)$$

KK spectrum reproduced in the continuum limit!

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$$\equiv \frac{1}{2}\sum_{i,j}\hat{A}_{i,\mu}\left(\mathcal{M}^2_{U(1)}\right)_{ij}\hat{A}_{j,\mu}\,. \qquad A_{j,\mu} = g\hat{A}_{j,\mu}$$

$$\mathcal{M}^2_{U(1)} = g^2v^2\Delta^{\mathrm{D}}_{N-2}$$

$$\Delta^{\mathrm{D}}_N = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

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KK spectrum reproduced in the continuum limit!

Similar stories for the $SU(3)$ sector:

$$G^{(0)}_\mu = \frac{1}{N}\sum_{j=1}^{N}G_{j,\mu}$$

Gluon with coupling const.

$$\frac{1}{g^2_{c,D}} = \frac{N}{g_c^2} \longleftrightarrow \frac{Na}{g^2_{5,c}} = \frac{\pi R}{g^2_{5,c}} = \frac{1}{g^2_{4,c}}$$

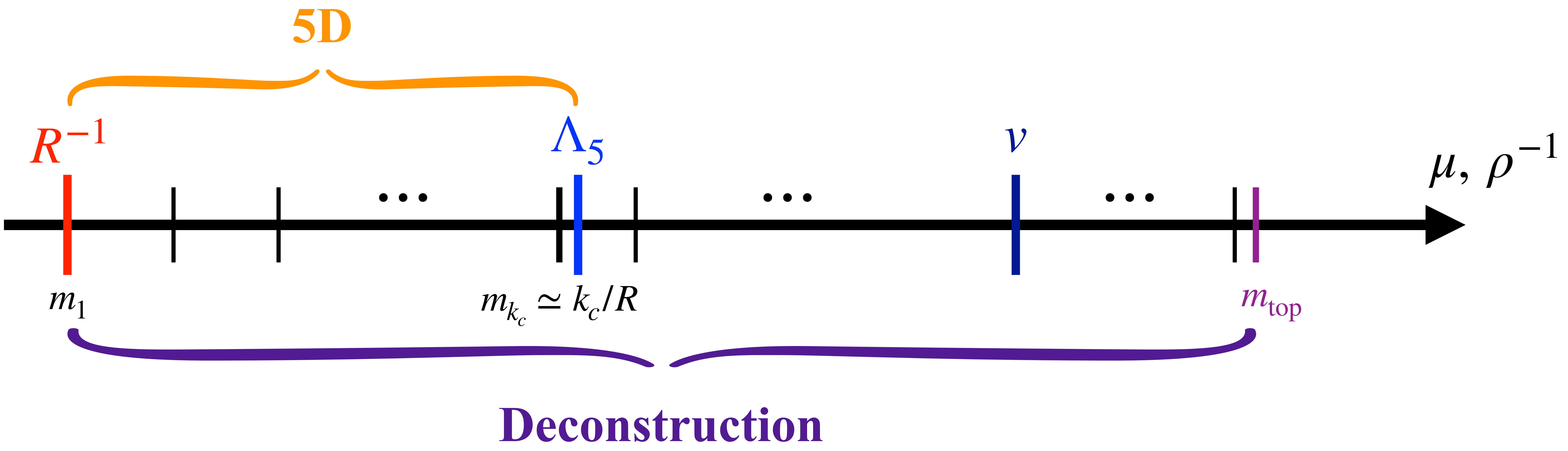
Deconstruction — Gauge Sector

$v \to \infty$ can only be a formal limit.

For deconstruction to be a successful UV completion, we require:

Below the 5D cutoff Λ_5, the mass spectrum should be correctly reproduced.

$$\underbrace{\Lambda_5 \gtrsim m_{k_c}}_{\text{critical level}} = 2gv\sin\left(\frac{k_c\pi}{2N}\right) \simeq \frac{k_c}{R} \quad \Longrightarrow \quad \Lambda_5 \simeq \frac{k_c}{R} \ll \frac{N}{\pi R} = \frac{1}{a} = gv$$



Gauged Wess-Zumino-Witten Term

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Axion-gluon coupling is provided by the **gauged WZW term**

$$S^{\rm mix}_{\rm WZW} = \sum_{i=1}^{N-1} \int_{R^4} \frac{1}{16\pi^2} \Bigg[\underline{\left(A_i + A_{i+1} + \tilde{\Phi}_i^* \mathrm{d}\tilde{\Phi}_i\right) \mathrm{tr}\left(G_{i+1}\mathrm{d}G_{i+1} + \frac{2}{3}G_{i+1}^3\right)}$$

$$-\frac{1}{3}A_i\,\mathrm{tr}\left(\tilde{\Sigma}_i \mathrm{d}\tilde{\Sigma}_i^\dagger\right)^3 + \mathrm{d}A_i\,\mathrm{tr}\left(\tilde{\Sigma}_i^\dagger \mathrm{d}\tilde{\Sigma}_i G_{i+1} + G_i \tilde{\Sigma}_i G_{i+1} \tilde{\Sigma}_i^\dagger + G_i \tilde{\Sigma}_i \mathrm{d}\tilde{\Sigma}_i^\dagger\right) - \mathrm{p.c.}\Bigg]$$

$$A_i \leftrightarrow A\,, A_{i+1} \leftrightarrow A + a\partial_5 A\,, \tilde{\Phi}_i \leftrightarrow 1 + aA_5\,,$$
$$G_i \leftrightarrow G\,, G_{i+1} \leftrightarrow G + a\partial_5 G\,, \tilde{\Sigma}_i \leftrightarrow 1 + aG_5$$

$$\Rightarrow \quad -\int_{R^4\times I} \frac{1}{8\pi^2} A\,\mathrm{tr}\left(\mathcal{G}^2\right) = -\int \mathrm{d}^4x \int_0^{\pi R} \mathrm{d}y \frac{1}{32\pi^2} \epsilon^{MNPQR} A_M\,\mathrm{tr}\left(\mathcal{G}_{NP}\mathcal{G}_{QR}\right)$$

5D CS term reproduced!

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$$-\frac{1}{3}A_i\,{\rm tr}\left(\tilde{\Sigma}_i {\rm d}\tilde{\Sigma}_i^\dagger\right)^3 + {\rm d}A_i\,{\rm tr}\left(\tilde{\Sigma}_i^\dagger {\rm d}\tilde{\Sigma}_i G_{i+1} + G_i\tilde{\Sigma}_i G_{i+1}\tilde{\Sigma}_i^\dagger + G_i\tilde{\Sigma}_i {\rm d}\tilde{\Sigma}_i^\dagger\right) - {\rm p.c.}\Big]$$

$$A_i \leftrightarrow A\,,\, A_{i+1} \leftrightarrow A + a\partial_5 A\,,\, \tilde{\Phi}_i \leftrightarrow 1 + aA_5\,,$$
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$$\rightarrow$$

$$-\int_{R^4\times I} \frac{1}{8\pi^2} A\,{\rm tr}\left(\mathcal{G}^2\right) = -\int {\rm d}^4x \int_0^{\pi R} {\rm d}y \frac{1}{32\pi^2}\epsilon^{MNPQR} A_M\,{\rm tr}\left(\mathcal{G}_{NP}\mathcal{G}_{QR}\right)$$

5D CS term reproduced!

$$\rightarrow \mathcal{L}_{aGG} = -\frac{1}{32\pi^2}\frac{a(x)}{\sqrt{N-1}v}{\rm tr}\left(\mathcal{G}_{1,\mu\nu}\widetilde{\mathcal{G}}_1^{\mu\nu} + 2\sum_{i=2}^{N-1}\mathcal{G}_{i,\mu\nu}\widetilde{\mathcal{G}}_i^{\mu\nu} + \mathcal{G}_{N,\mu\nu}\widetilde{\mathcal{G}}_N^{\mu\nu}\right)$$

$$= -\frac{1}{16\pi^2}\boxed{\frac{\sqrt{N-1}a(x)}{v}}{\rm tr}\left(\mathcal{G}^{(0)}_{\mu\nu}\widetilde{\mathcal{G}^{(0)}}^{\mu\nu}\right) + \cdots$$

Decay constant

$$f_a = \frac{v}{\sqrt{N-1}} = \frac{\sqrt{v/g}}{\sqrt{(N-1)(gv)^{-1}}} \leftrightarrow \frac{1}{g_5\sqrt{\pi R}} \equiv f_{a,5}$$

Dimensionless field

$$\theta(x) \equiv \frac{a(x)}{f_a} = \sum_{j=1}^{N-1}\frac{\chi_j(x)}{v} \leftrightarrow \sum_{j=1} aA_5(x,y_j) \equiv \theta_5(x)$$

Axion Quality Revisited

Since the 5D theory of axion is just a low-energy EFT, we want to see whether *the high-quality is preserved in the UV completion — deconstruction theory.*

Axion shift symmetry $a(x) \equiv \frac{1}{\sqrt{N-1}} \sum_{j=1}^{N-1} \chi_j(x) \longrightarrow a(x) + c$

	Φ_j	ψ_j	η_j	Q_j
$U(1)_{\text{shift}}$	$+1$	$-\frac{1}{2}$	$-\frac{1}{2}$	j

Chiral & Anomalous $\downarrow$ **aGG-coupling**

Charged bulk matter:
Deconst. calc. consistent with 5D result

Collective $\Pi_j U(1)_j$ gauge symmetry highly constrained the $U(1)_{\text{shift}}$-violating operators!

The one with lowest dimension:

Huge suppression for $N \gg 1$

$$V_{\text{shift}} = \frac{\kappa}{M_{\text{Pl}}^{N-5}} \prod_{j=1}^{N-1} \Phi_j + \text{h.c.} = 2|\kappa| M_{\text{Pl}}^4 \boxed{\left(\frac{v}{\sqrt{2} M_{\text{Pl}}}\right)^{N-1}} \cos\left(\frac{a}{f_a} + \delta_\kappa\right)$$

Instanton Effect

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Take the bulk $SU(3)$ as a prototype, the discussion can be easily generalized.

Ordinary QCD potential calculated from χPT — *strong coupling regime*

$$V_{\rm QCD}(a) = -m_\pi^2 f_\pi^2 \sqrt{1 - \frac{2m_u m_d}{(m_u+m_d)^2}\left[1-\cos\left(\frac{a}{f_a}-\bar{\theta}_c\right)\right]}$$

Instanton calculus *in the weak coupling regime*

single (BPST) instanton solution & dilute gas approximation

$$e^{-E\left[\frac{a}{f_a}-\theta\right]\mathcal{V}_4} = \lim_{\mathcal{V}_4\to\infty} \langle\theta|\, e^{-HT}\,|\theta\rangle = \lim_{\mathcal{V}_4\to\infty} \sum_{n,m} e^{-i(n-m)\theta} \langle n-m|\, e^{-HT}\,|0\rangle$$

$$= \sum_{n,\bar{n}=0}^{\infty} \frac{1}{n!\bar{n}!}(W_I[a])^n (W_{\bar{I}}[a])^{\bar{n}} = \exp\left(W_I[a] + W_{\bar{I}}[a]\right)$$

$$W_I[a] \equiv \lim_{\mathcal{V}_4\to\infty} e^{-i\theta}\langle 1|\, e^{-HT}\,|0\rangle = \frac{Z_{\Delta Q=1}}{Z_0}$$

$$G_\mu^{\rm BPST} = 2\rho^2 \frac{\bar{\eta}^a_{\mu\nu}(x-x_0)_\nu}{(x-x_0)^2((x-x_0)^2-\rho^2)^2} J^a_\Omega$$

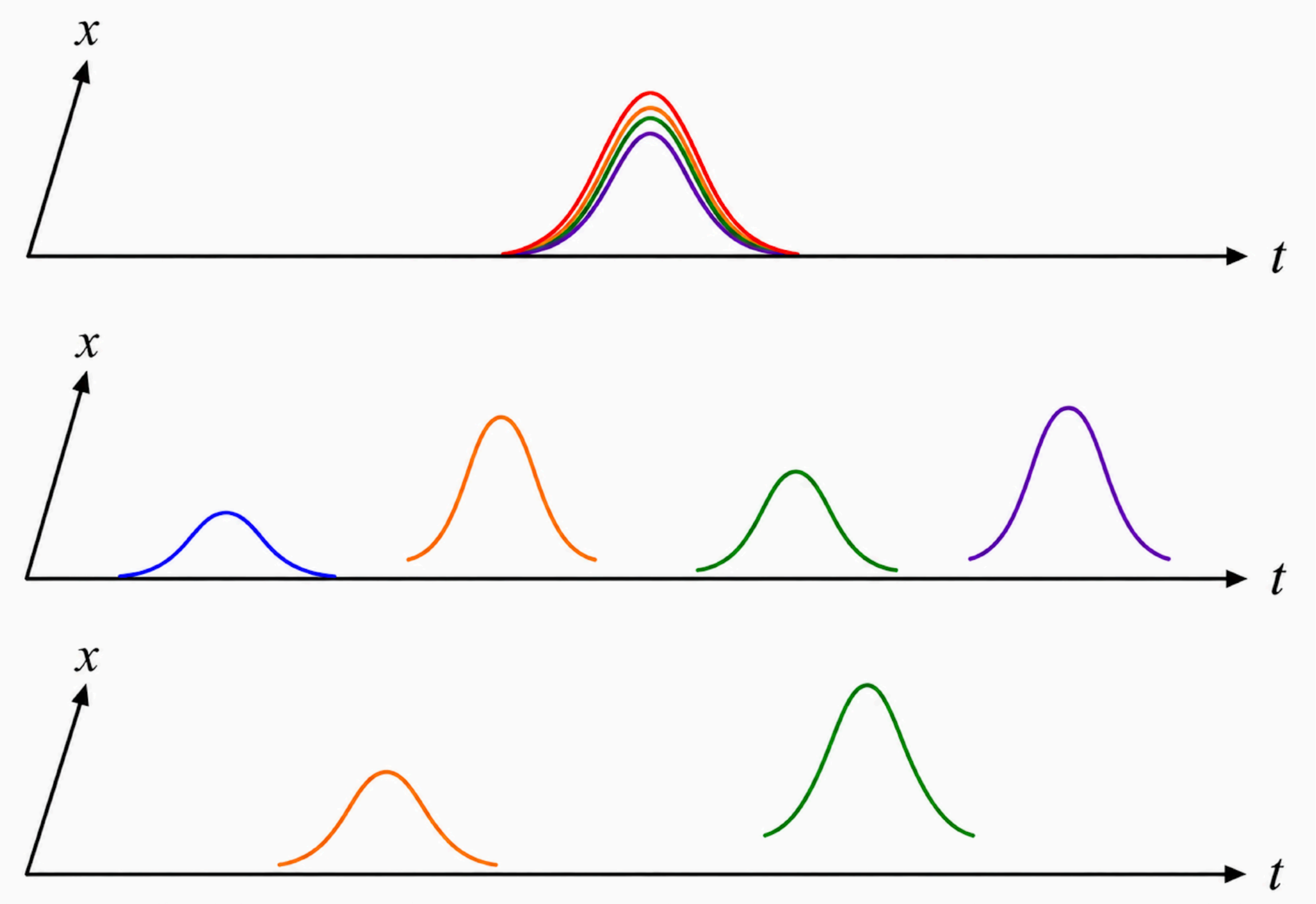
Asymptotically free → Weak coupling at high energy/ *small size of instanton* → **Small instanton effect** *(sub-dominant)*

Could be enhanced in extra dim & deconstruction!

Instanton Effect

$$\mathcal{L}_{aGG} = -\frac{1}{32\pi^2}\frac{a(x)}{\sqrt{N-1}v}\,\mathrm{tr}\left(\mathcal{G}_{1,\mu\nu}\widetilde{\mathcal{G}_1}^{\mu\nu} + 2\sum_{i=2}^{N-1}\mathcal{G}_{i,\mu\nu}\widetilde{\mathcal{G}_i}^{\mu\nu} + \mathcal{G}_{N,\mu\nu}\widetilde{\mathcal{G}_N}^{\mu\nu}\right)$$

Instanton configurations



j-th instanton number

$(1,1,\cdots,1)_{\mathrm{zero}}$ — all instantons aligned

Zero-mode instanton $G_\mu^{(0)}(x) = G_\mu^{\mathrm{BPST}}(x)$

$(1,1,\cdots,1)$ — all instantons exist but not aligned

$(0,1,1,0\cdots,0)$ — not all instantons exist

Constrained by mass mixing term

$\frac{1}{2}v_3^2\left(G_{j+1,\mu} - G_{j,\mu}\right)^2$

“Fractional” instanton

Instanton Effect

For extremely small instantons with $\rho \ll v_3^{-1}$, they don't feel the effects of Higgsing.

→ N independent gauge fields & instanton effects $\quad W_{I,\mathrm{tot}} = \prod_{j=1}^{N} (W_{I,j})^{n_j}$

Each is **asymptotically free** $\quad W_{I,j} \equiv \frac{Z_{\Delta Q=1,j}}{Z_{0,j}} \propto \exp\left(-8\pi^2/g^2(\rho_j^{-1})\right) \ll 1$

For instantons with $\rho \gtrsim v_3^{-1}$, the Higgsing becomes effective:

- Only $(1,1,\cdots,1)_{\mathrm{zero}}$ is the **exact** solution to the classical EoM:

$$G_{j,\mu}(x) = G_\mu^{\mathrm{BPST}}(x)\,, \quad j = 1, \cdots, N \quad \Longleftrightarrow \quad G_\mu^{(0)}(x) = G_\mu^{\mathrm{BPST}}(x)\,,$$

corresponding to *the 5D instanton — BPST instanton wrapping around 5th dim*

$$G_\mu(x,y) = G_\mu^{\mathrm{BPST}}(x)\,, \; G_5(x,y) = 0$$

Gherghetta, Khoze, Pomarol, Shirman (2020)

- Other configurations are *not* the saddle points of the classical action.

No 5D counterpart… → *Lattice artifacts??*

Poppitz, Shirman (2002)

Instanton Effect — Zero-mode

$$G_{j,\mu}(x) = G_\mu^{\rm BPST}(x)\,, \quad j = 1, \cdots, N \quad \Longleftrightarrow \quad G_\mu^{(0)}(x) = G_\mu^{\rm BPST}(x)\,, \quad \leftrightarrow SU(3)^{(0)}$$

Under $SU(3)^{(0)}$, all the massive modes $G_\mu^{(k)}$ transform as **adjoint representations**

$$\Delta_\alpha G_\mu^{(k\geq 1)} = \frac{\sqrt{2}}{N}\sum_{j=1}^{N}\cos\left(\frac{\left(j-\frac{1}{2}\right)k\pi}{N}\right)\Delta_\alpha G_{j,\mu} = \frac{\sqrt{2}}{N}\partial_\mu \alpha(x)\sum_{j=1}^{N}\cos\left(\frac{\left(j-\frac{1}{2}\right)k\pi}{N}\right) + i\left[\alpha(x), G_\mu^{(k)}\right] = i\left[\alpha(x), G_\mu^{(k)}\right]$$

and contribute to the effective action

$$e^{-S^I_{\rm eff}(R^{-1}<\rho^{-1}\lesssim v_3)} = \exp\left(-\frac{8\pi^2}{g^2_{c,D}(1/R)} + 11\ln\frac{\rho}{R} - \frac{3}{2}\sum_{n=1}^{K(\rho)}\ln(m_n\rho)\right)$$

$K(\rho)$: # of modes lighter than ρ^{-1}

$$V_I[a] = -\frac{1}{\mathcal{V}_4}\ln Z[a] = -2\mathcal{K}\cos\left(\frac{a}{f_a}\right)$$

$$\mathcal{K} \equiv \int\frac{d\rho}{\rho^5}C_3\left(\frac{2\pi}{\alpha_{c,D}}\right)^6 e^{-S^I_{\rm eff}(\rho)}$$

5D regime $R^{-1} \ll \rho^{-1} < \Lambda_5$, $m_n = n/R$, $K(\rho) \simeq R/\rho$

$$e^{-S^I_{\rm eff}(R^{-1}\ll\rho^{-1}\lesssim\Lambda_5)} = \exp\left(-\frac{8\pi^2}{g^2_{c,D}(1/R)} + \frac{3}{2}\frac{R}{\rho} + \frac{47}{4}\ln\frac{\rho}{R}\right)$$

Power-law enhancement

amplitude peaked at $\rho \simeq \Lambda_5^{-1}$

$$\mathcal{K}_5 \simeq C_3\left(\frac{2\pi}{\alpha_{c,D}(\Lambda_5)}\right)^6 \Lambda_5^4 e^{-S^{I,5}_{\rm eff}(\Lambda_5^{-1})}$$

Instanton Effect — Zero-mode

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$$G_{j,\mu}(x) = G_\mu^{\rm BPST}(x)\,, \quad j=1,\cdots,N \quad \Longleftrightarrow \quad G_\mu^{(0)}(x) = G_\mu^{\rm BPST}(x)\,, \quad \leftrightarrow SU(3)^{(0)}$$

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Power-law enhancement

amplitude peaked at $\rho \simeq \Lambda_5^{-1}$

$$\mathcal{K}_5 \simeq C_3\left(\frac{2\pi}{\alpha_{c,D}(\Lambda_5)}\right)^6 \Lambda_5^4 e^{-S^{I,5}_{\rm eff}(\Lambda_5^{-1})} \quad < \mathcal{K}_{\rm deconst}$$

Instanton Effect — “Fractional”

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For instantons damaged by the Higgsing/SSB, we can revive them by considering the **“constrained instantons”** — *non-trivial configuration of Higgs field* can realize an approximate saddle point. Affleck (1981)

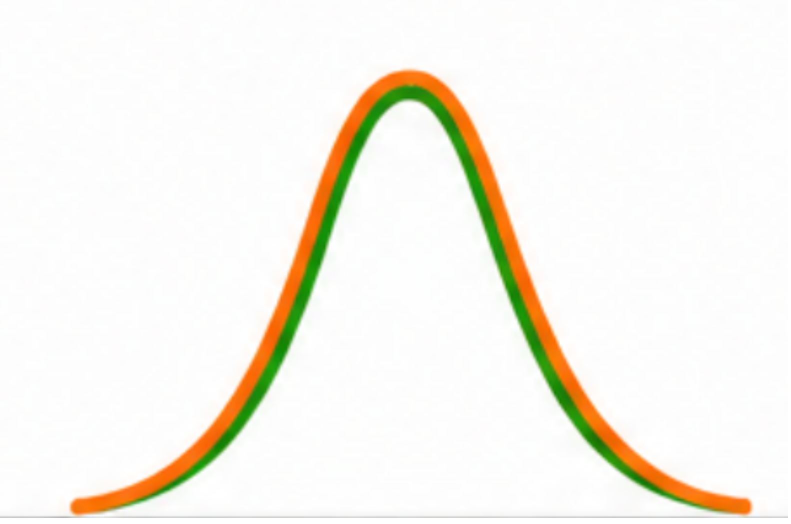
$$\phi_{in}(x) = \begin{cases} \left(\frac{x^2}{x^2+\rho^2}\right)^{1/2} \langle \phi_{in} \rangle & \text{for } i = 1, 2 \\ \langle \phi_{in} \rangle & \text{for } i = 3, \ldots, N_c \end{cases}$$

as solution to the classical EoM

$$D^2(A_{cl})\phi = 0$$

Csaki, Ruhdorfer, Shirman (1981)

→ Additional contribution to action $S_{\rm cl} = \frac{8\pi^2}{g^2} + 2\pi^2\rho^2 \sum_{i=1}^{2} \sum_{n=1}^{S} |\langle \phi_{in} \rangle|^2 + i\theta$



For simplicity, we consider the ***aligned*** *“fractional” instanton configuration*

$$(\underbrace{0, \cdots, 0}_{r}, \underbrace{1, 1, \cdots, 1}_{s}, \underbrace{0, \cdots, 0}_{N-r-s}) \,, \qquad G_{j,\mu}(x) = \begin{cases} G_\mu^{\rm BPST}(x) \,, & j = r+1, \cdots, r+s \,, \\ 0 \,, & j = 1, \cdots, r, r+s+1, \cdots, N \,. \end{cases}$$

Instanton Effect — "Fractional"

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We can define the sub-zero mode for such configuration

$$G_\mu^{(0)'} = \frac{1}{s}\sum_{j=r+1}^{r+s} G_{j,\mu}(x) \quad \leftrightarrow SU(3)^{(0)'}$$

$$(\underbrace{0,\cdots,0}_{r}\overset{\Sigma_r}{,}\boxed{\underbrace{1,1,\cdots,1}_{s}}\overset{\Sigma_{r+s}}{,}\underbrace{0,\cdots,0}_{N-r-s})\,,$$

The sub-diagonal group $SU(3)^{(0)'}$ is finally broken by the VEV of nearby link fields Σ_r, Σ_{r+s}.

$$\longrightarrow \quad \underline{W_I^{\mathrm{frac}} \propto e^{-8\pi^2\rho^2 v_3^2}}$$

Negligible for 5D region $\rho^{-1} < \Lambda_5 \ll v_3$ $\longleftrightarrow$ **No 5D counterpart**

It is **consistent** that *the "fractional" instantons are included in the deconstruction setup as a UV completion of the 5D theory*, which can be understood as *UV completing the Stueckelberg mechanism to the Higgs mechanism so that constrained instantons can be considered*.

Instanton Effect

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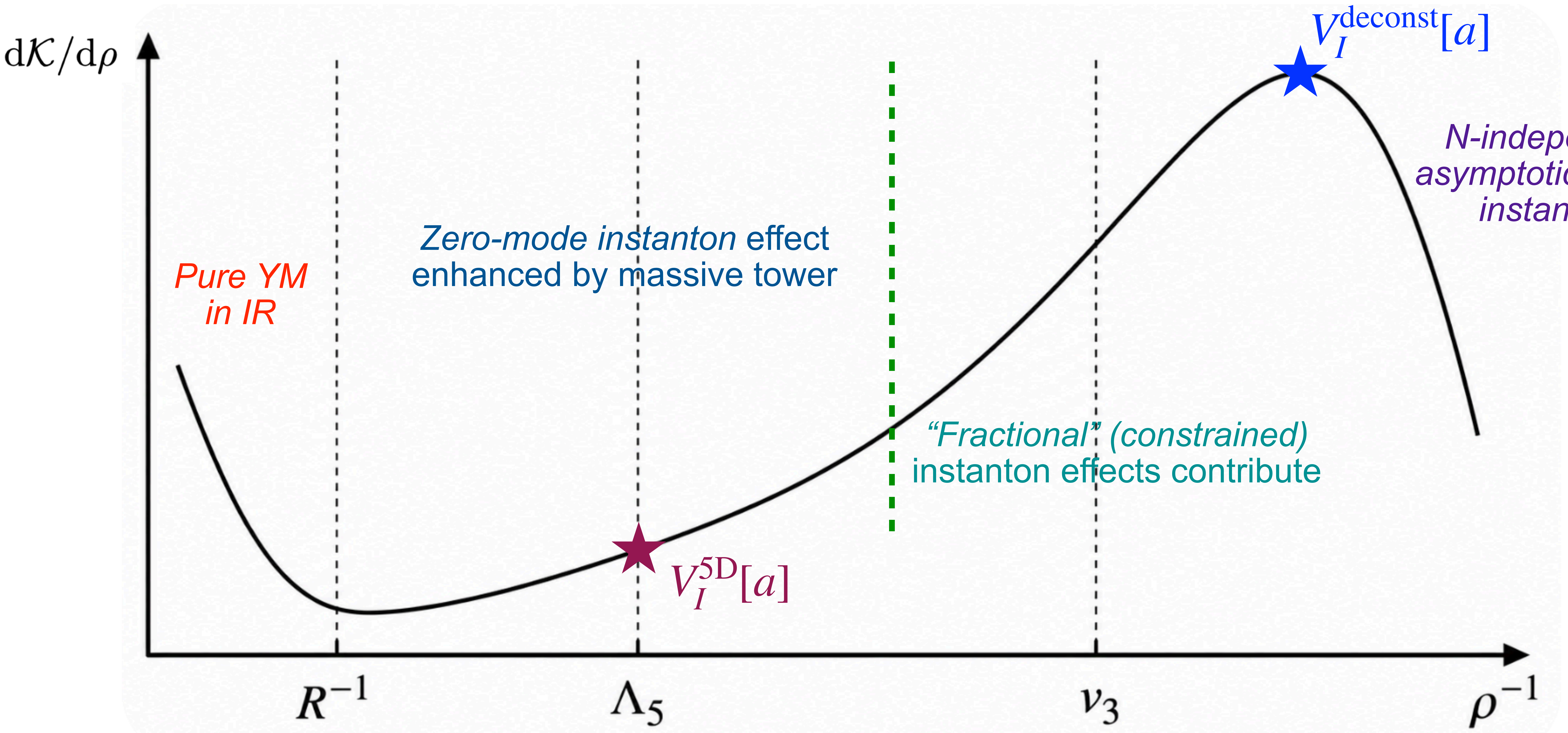

According to [Gherghetta, Khoze, Pomarol, Shirman (2020)] $V_I^{5\mathrm{D}}[a]$ *can exceed the ordinary QCD potential* $V_{\mathrm{QCD}}[a]$. **The deconstruction contribution could very likely be larger!**

Summary & Outlook

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- We construct a four-dimensional deconstruction of the extra-dim axion based on a moose of $U(1) \times SU(3)$ gauge theory.

- Axion-gluon coupling is reproduced by a **gauged Wess-Zumino-Witten term.**

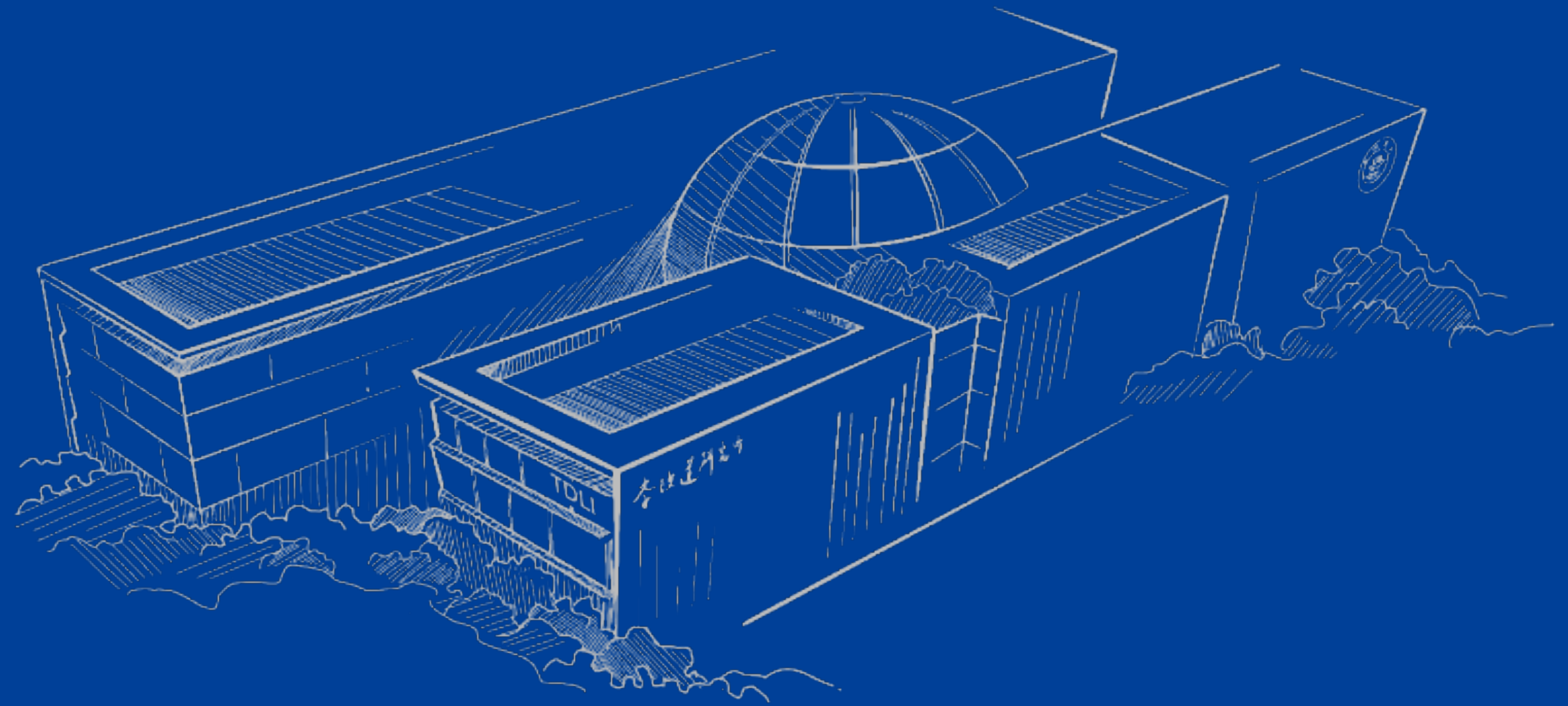
- ***"Fractional" instantons*** are **consistent with the 5D LEFT** due to the exponential suppression.

- The axion potential from the smaller instantons ***probing the deconstructed regime*** may be **much larger than the estimate based on the 5D theory**.

- Deconstruction opens up the window of the *post-inflationary* cosmology for extra-dim axion…

Thank you!

Backup



5D Orbifold Axion — Gauge Fields

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KK spectrum

One scalar zero mode: $A_{5,0}(x) = \dfrac{1}{\pi R}\displaystyle\int \mathrm{d}y\; A_5(x,y)\,,$

One gauge field zero mode: $G^a_{\mu,0}(x) = \dfrac{1}{\pi R}\displaystyle\int \mathrm{d}y\; G^a_\mu(x,y)\,,$

$$f_n^A = g_n^G = \sqrt{2}\sin\frac{ny}{R} \quad \text{for } n \geq 0\,,$$

$$g_0^A = f_0^G = 1\,, \quad g_n^A = f_n^G = \sqrt{2}\cos\frac{ny}{R} \quad \text{for } n \geq 1\,,$$

Towers of massive gauge bosons with masses $m_{V,n} = n/R$:

$$V_{\mu,n}(x) = \frac{1}{\pi R}\int \mathrm{d}y\; f_n^V(y) V_\mu(x,y)\,, \quad V_{5,n}(x) = \frac{1}{\pi R}\int \mathrm{d}y\; g_n^V(y) V_5(x,y)\,, \quad n \geq 1\,.$$

Stueckelberg mechanism

Define the *axion* — the phase of the Wilson line in the 5th direction

$$\theta_5(x) \equiv \int_0^{\pi R} \mathrm{d}y\, A_5(x,y) = A_{5,0}(x)\pi R$$

2π-periodicity

zero mode of A_5 component

5D Orbifold Axion — Gauge Fields

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Substitute the KK expansion

into the *field strength* (canonically normalized):

$$A_\mu(x,y) = \sum_{n=0}^{\infty} A_{\mu,n}(x) f_n^A(y)\,, \quad A_5(x,y) = \sum_{n=0}^{\infty} A_{5,n}(x) g_n^A(y)\,,$$

$$G_\mu(x,y) = \sum_{n=0}^{\infty} G_{\mu,n}(x) f_n^G(y)\,, \quad G_5(x,y) = \sum_{n=0}^{\infty} G_{5,n}(x) g_n^G(y)\,,$$

$$\mathcal{G}_{MN} \supset f_0 \left(\partial_\mu G_{0,\nu} - \partial_\nu G_{0,\mu} + i \underbrace{g_{5,c} f_0}_{\equiv\, g_{4,c}} \left[G_{0,\mu}, G_{0,\nu}\right]\right)$$

4D effective coupling: $\frac{1}{g_{4,c}^2} = \frac{\pi R}{g_{5,c}^2}$

into *kinetic term*:

Decay constant

$$-\int \mathrm{d}y \frac{1}{4g_5^2} \mathcal{F}_{MN}\mathcal{F}^{MN} \supset \left(\int_0^{\pi R} \mathrm{d}y\right) \frac{1}{2g_5^2} \partial_\mu A_{5,0} \partial^\mu A_{5,0} = \underbrace{\boxed{\frac{1}{2g_5^2 \pi R}}}_{\equiv \frac{1}{2} f_a^2} \partial_\mu \theta_5 \partial^\mu \theta_5$$

Canonically normalized field: $a_5 = f_a \theta_5 = \sqrt{\frac{1}{g_5^2 \pi R}}\,\theta_5$

Chern-Simons Term

$$S_5^{\rm CS} = -\int \mathrm{d}^5 x\, \frac{\kappa_{\rm CS}}{32\pi^2} \epsilon^{MNPQR} A_M\, \mathrm{tr}\left(\mathcal{G}_{NP}\mathcal{G}_{QR}\right)$$

Inserting the zero mode $\longrightarrow$

$$\mathcal{L}_{agg} = -\frac{\kappa_{\rm CS}}{16\pi^2}\frac{a_5(x)}{f_a}\,\mathrm{tr}\left(\mathcal{G}_{\mu\nu,0}(x)\widetilde{\mathcal{G}}_0^{\mu\nu}(x)\right)$$

Gauge invariant or not?

$$\boxed{A_M(x,y) \to A_M(x,y) - \partial_M \alpha_1(x,y)}$$

$$\delta_{U(1)} S_5^{\rm CS} = \frac{\kappa_{\rm CS}}{32\pi^2}\int \mathrm{d}^4 x \int_0^{\pi R} \mathrm{d}y\, \epsilon^{\mu\nu\rho\sigma}\partial_5\left[\alpha_1^{\rm re}(x,y)\mathrm{Tr}\left(\mathcal{G}_{\mu\nu}\mathcal{G}_{\rho\sigma}\right)\right] + \text{4D total derivatives}$$
$$= \kappa_{\rm CS}\int \mathrm{d}^4 x\left[\alpha_1^{\rm re}(x,y=\pi R)\mathcal{A}(y=\pi R) - \alpha_1^{\rm re}(x,y=0)\mathcal{A}(y=0)\right]$$

$$\mathcal{A} = \frac{1}{16\pi^2}\,\mathrm{tr}\left(\mathcal{G}_{\mu\nu}\widetilde{\mathcal{G}}^{\mu\nu}\right)$$

Chern-Simons Term

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$$= \kappa_{\mathrm{CS}} \int \mathrm{d}^4 x \left[\alpha_1^{\mathrm{re}}(x, y=\pi R)\mathcal{A}(y=\pi R) - \alpha_1^{\mathrm{re}}(x,y=0)\mathcal{A}(y=0)\right] = 0$$

$$\mathcal{A} = \frac{1}{16\pi^2}\mathrm{tr}\left(\mathcal{G}_{\mu\nu}\widetilde{\mathcal{G}}^{\mu\nu}\right)$$

Remnant gauge symmetry under BC:

$$A_\mu|_{0,\pi R} = 0 \longrightarrow \partial_\mu \alpha_1(x, y = 0, \pi R) = 0 \longrightarrow \alpha_1^{\mathrm{re}}(x,0) = \alpha_1^{\mathrm{re}}(x,\pi R) = 0$$

$U(1)$ reduced to **global symmetry** on boundaries

5D Orbifold Axion — Matter Fields

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For later discussion, we introduce a *fermion* and a *complex scalar* in the bulk.

	$U(1)$	$SU(3)$
$\Psi(x,y)$	$+1$	$\square$
$Q(x,y)$	$+1$	$\mathbb{1}$

Scalar

$$S_5^s = \int \mathrm{d}^4x \int_0^{\pi R} \mathrm{d}y \left([D_M Q(x,y)]^* D_M Q(x,y) - M_Q^2 |Q(x,y)|^2 \right)$$ with Neumann BC $\partial_y Q(x,y)|_{0,\pi R} = 0$

KK spectrum: $f_0^Q(y) = \sqrt{\frac{1}{\pi R}}\,, \quad f_n^Q(y) = \sqrt{\frac{2}{\pi R}} \cos\frac{ny}{R}\,, \quad n > 0\,, \qquad m_{Q,n}^2 = M_Q^2 + \frac{n^2}{R^2}$

Fermion $\Psi = \left(\eta_\alpha, \psi^{\dagger\dot{\alpha}}\right)^T$

$$S_5^f = \int \mathrm{d}^5x \left[\overline{\Psi}(x,y) i\Gamma^M D_M \Psi(x,y) - M_\Psi \overline{\Psi}(x,y)\Psi(x,y) \right]$$

with different BCs for different chiralities

$$\psi|_{0,\pi R} = 0$$

$$\partial_y \eta|_{0,\pi R} = 0$$

$$f_0^\psi = 0\,, \quad f_n^\psi = \sqrt{\frac{2}{\pi R}} \sin\frac{ny}{R} \quad \text{for } n \geq 1\,, \qquad m_{f,n}^2 = M_\Psi^2 + \frac{n^2}{R^2}$$

$$f_0^\eta = N_0 e^{-M_\Psi y}\,, \quad f_n^\eta = \sqrt{\frac{2}{\pi R}} \left(\frac{M_\Psi}{m_n} \sin\frac{ny}{R} - \frac{n}{m_n R} \cos\frac{ny}{R} \right) \quad \text{for } n \geq 1\,,$$

→ **Generating CS term**

Adachi, Lim, Maru (2022), Bonnefoy, Cox, etc (2021)

Deconstruction — Gauge Sector

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$SU(3)$ action $$S_D^{SU(3)} = \int \mathrm{d}^4x \left(-\frac{1}{2g_c^2} \sum_{j=1}^{N} \mathrm{tr}\left(\mathcal{G}_j^{\mu\nu}\mathcal{G}_{j,\mu\nu}\right) + \sum_{j=1}^{N-1} \mathrm{tr}\left(\left(D_\mu \Sigma_j\right)^\dagger D^\mu \Sigma_j\right) - V(\Sigma) \right)$$

Mass matrix obtained by

$$\Sigma_j \simeq v_3 \exp(i\pi_j/v_3)\,, \pi_j = \pi_j^a T^a$$

Replacing
$$\Delta_N^{\mathrm{D}} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$
with
$$\Delta_N^{\mathrm{N}} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix}$$

$$\mathcal{M}_{SU(3)}^2 = g_c^2 v_3^2 \Delta_N^{\mathrm{N}}$$

Spectrum

$$G_\mu^{(k)} = \frac{\sqrt{2}}{N} \sum_{j=1}^{N} \cos\left(\frac{(j-\frac{1}{2})k\pi}{N}\right) G_{j,\mu}\,, \qquad k = 1, \cdots, N-1\,, \qquad m_k^2 = 4g_c^2 v_3^2 \sin^2\left(\frac{k\pi}{2N}\right)$$

One massless gauge field — ***gluon***, corresponding to the remaining $SU(3)^{(0)}$

$$G_\mu^{(0)} = \frac{1}{N} \sum_{j=1}^{N} G_{j,\mu}$$ with coupling const. $$\frac{1}{g_{c,D}^2} = \frac{N}{g_c^2} \longleftrightarrow \frac{Na}{g_{5,c}^2} = \frac{\pi R}{g_{5,c}^2} = \frac{1}{g_{4,c}^2}$$

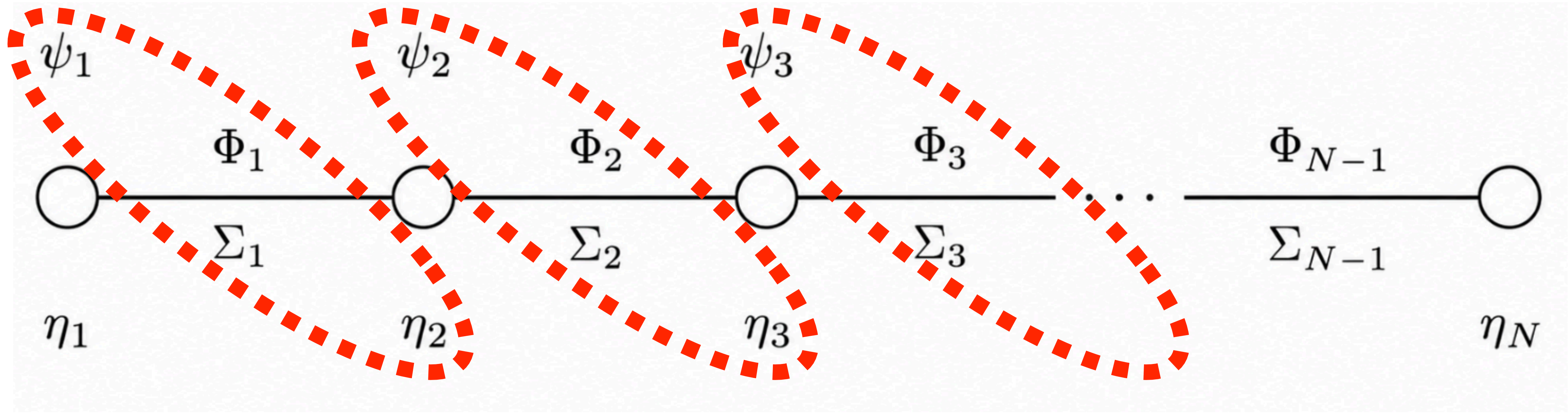
Deconstruction — Matter Sector

Discretizing fermion is non-trivial…

Nielson, Ninomiya (1981), Wilson (1974)

We consider the "Wilson fermion" approach

Hill, Leibovich (2002), Abe, Kobayashi, etc (2003)



Hopping terms

ψ_N removed according to the BC

$$S_D^f = \int \mathrm{d}^4x \Bigg[\sum_{j=1}^{N} \eta_j^\dagger i\bar{\sigma}^\mu D_\mu \eta_j + \sum_{j=1}^{N-1}\Big(\psi_j^\dagger i\bar{\sigma}^\mu D_\mu^* \psi_j - M_\Psi \left(\eta_j\psi_j + \psi_j^\dagger \eta_j^\dagger\right)\Big)$$

$$- \sum_{j=1}^{N-1}\left(\frac{\sqrt{2}g}{v_3}\psi_j \Phi_j \Sigma_j \eta_{j+1} - gv\eta_j\psi_j + \text{h.c.}\right)\Bigg]$$

Correct mass dimension

$$\boxed{a^{1/2}\psi(x,y_j) \longleftrightarrow \psi_j(x)\ , \quad a^{1/2}\eta(x,y_j) \longleftrightarrow \eta_j(x)}$$

$$-(gv)^2 \sum_{j=1}^{N-1}\left(\psi_j \frac{\tilde{\Phi}_j \tilde{\Sigma}_j \eta_{j+1} - \eta_j}{gv} + \text{h.c.}\right) \longleftrightarrow -\sum_{j=1}^{N-1} a\left(\psi(x,y_j) D_5^+ \eta(x,y_j) + \text{h.c.}\right)$$

Covariant derivatives defined with link variables

$$\tilde{\Phi}_j \equiv \frac{\sqrt{2}\Phi_j}{v} = e^{i\chi_j/v} \leftrightarrow e^{iaA_5},$$

$$\tilde{\Sigma}_j \equiv \frac{\Sigma_j}{v_3} = e^{i\pi_j/v_3} \leftrightarrow e^{iaG_5}$$

5D action reproduced!

Deconstruction — Matter Sector

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Hopping terms seems non-renormalizable…

$$\frac{\sqrt{2}g}{v_3}\psi_j\Phi_j\Sigma_j\eta_{j+1}$$

Introducing heavy scalar mediators S_j with mass M_S and *the same charge as the product of the two link fields*

	$U(1)_j$	$U(1)_{j+1}$	$SU(3)_j$	$SU(3)_{j+1}$
S_j	$+1$	-1	$\square$	$\overline{\square}$

$$\mathcal{L}_S \supset -y_S\psi_j S_j\eta_{j+1} - \mu_S S_j^\dagger \Sigma_j\Phi_j + \text{h.c.} \longrightarrow \mathcal{L}_{S,\text{eff}} \supset -\frac{y_S\mu_S}{M_S^2}\psi_j\Phi_j\Sigma_j\eta_{j+1} + \text{h.c.}$$

$$\frac{y_S\mu_S}{M_S^2} \simeq \frac{\sqrt{2}g}{v_3}$$

Mass spectrum

Zero mode: $\eta^{(0)} = \mathcal{N}_0\sum_{j=1}^{N}\left(-\frac{M'_\Psi}{gv}\right)^{j-1}\eta_j\,,\qquad |\mathcal{N}_0|^{-2} = \sum_{j=1}^{N}\left|\frac{M'_\Psi}{gv}\right|^{2(j-1)}\,,$

Massive Dirac fermions with mass squared $m_n^2 = M'^2_\Psi + (gv)^2 + 2M'_\Psi gv\cos\left(\frac{n\pi}{N}\right)$:

$$\eta^{(n)} = \sqrt{\frac{2}{N}}\sum_{j=1}^{N}\frac{1}{m_n}\left[M'_\Psi\sin\left(\frac{jn\pi}{N}\right) + gv\sin\left(\frac{(j-1)n\pi}{N}\right)\right]\eta_j\,, \tag{3.38}$$

$$\psi^{(n)} = \sqrt{\frac{2}{N}}\sum_{j=1}^{N-1}\sin\left(\frac{jn\pi}{N}\right)\psi_j\,,\qquad n = 1,\dots,N-1\,. \tag{3.39}$$

$$m_n^2 = M_\Psi^2 + 4[(gv)^2 - M_\Psi(gv)]\sin^2\left(\frac{n\pi}{2N}\right) \leftrightarrow M_\Psi^2 + \frac{n^2}{R^2}\,,$$

Match 5D KK spectrum well!

Deconstruction — Matter Sector

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$$S_D^s = \int \mathrm{d}^4x \left[\sum_{j=1}^{N} \left((D_\mu Q_j)^* D^\mu Q_j - M_Q^2 |Q_j|^2 \right) - g^2 \sum_{j=1}^{N-1} |\sqrt{2}\Phi_j Q_{j+1} - v Q_j|^2 \right]$$

$$= \int \mathrm{d}^4x \left[\sum_{j=1}^{N} \left((D_\mu Q_j)^* D^\mu Q_j - M_Q^2 |Q_j|^2 \right) - \underline{g^2 v^2 \sum_{j=1}^{N-1} |\tilde{\Phi}_j Q_{j+1} - Q_j|^2} \right] , \leftrightarrow |D_5^+ Q(x, y_j)|^2$$

Mass diagonalization gives

$$m_k^2 = M_Q^2 + \underline{4g^2 v^2 \sin^2\left(\frac{k\pi}{2N}\right)} , \leftrightarrow \frac{k^2}{R^2}$$

$$c_k = 1 \text{ for } k = 0 \text{ and } c_k = 2 \text{ for } k \geq 1$$

$$Q^{(k)} = \sqrt{\frac{c_k}{N}} \sum_{j=1}^{N} \cos\left(\frac{\left(j - \frac{1}{2}\right) k\pi}{N} \right) Q_j , \quad \text{with } k = 0, 1, \cdots, N-1 .$$

*Everything in 5D has been reproduced except **the Chern-Simons term**…*

Gauged Wess-Zumino-Witten Term

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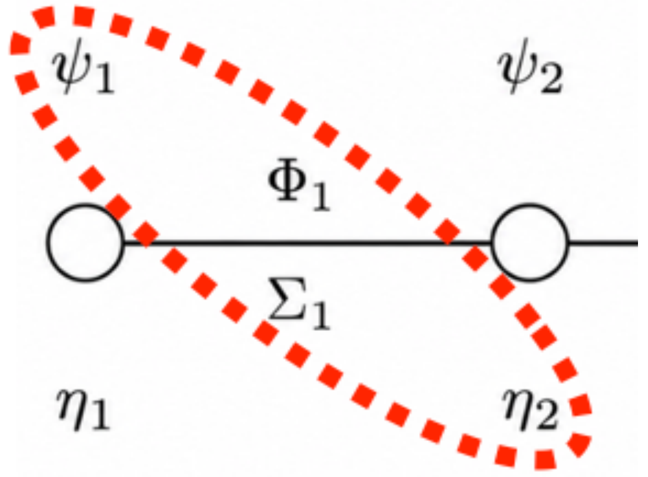
5D CS term can be generated by integrating out a bulk fermion $\Psi(x, y)$ charged under $U(1), SU(3)$.

Gauged WZW term can also be derived using bulk fermions $\Psi_i(x)$, guided by the **anomalies**.

$$\Psi_i = \left(\eta_i, \psi_i^\dagger\right)^T$$

One unit of deconstructed fermion action

$$\mathcal{L}_{\text{unit},i} = \eta_{i+1}^\dagger i\bar{\sigma}^\mu D_\mu \eta_{i+1} + \psi_i^\dagger i\bar{\sigma}^\mu D_\mu^* \psi_i - gv\left(\psi_i \tilde{\Phi}_i \tilde{\Sigma}_i \eta_{i+1} + \text{h.c.}\right)$$



Three types of anomalies exist:

$$\delta_1 S_{\text{unit},i} = \int_{R^4} \underbrace{-\frac{3}{32\pi^2}\left[\alpha_i \mathcal{F}_i^2 - \alpha_{i+1}\mathcal{F}_{i+1}^2\right]}_{U(1)^3} \underbrace{- \frac{1}{16\pi^2}\left[\alpha_i \operatorname{tr}\left(\mathcal{G}_i^2\right) - \alpha_{i+1} \operatorname{tr}\left(\mathcal{G}_{i+1}^2\right)\right]}_{U(1)\times SU(3)^2} ,$$

$$\underbrace{\delta_3 S_{\text{unit},i} = \int_{R^4} -\frac{1}{24\pi^2} \operatorname{tr}\left[\epsilon_i\left(\mathrm{d}G_i \mathrm{d}G_i + \frac{1}{2}\mathrm{d}(G_i^3)\right) - \epsilon_{i+1}\left(\mathrm{d}G_{i+1}\mathrm{d}G_{i+1} + \frac{1}{2}\mathrm{d}(G_{i+1}^3)\right)\right]}_{SU(3)^3}$$

[Manohar, Moore (1984)]

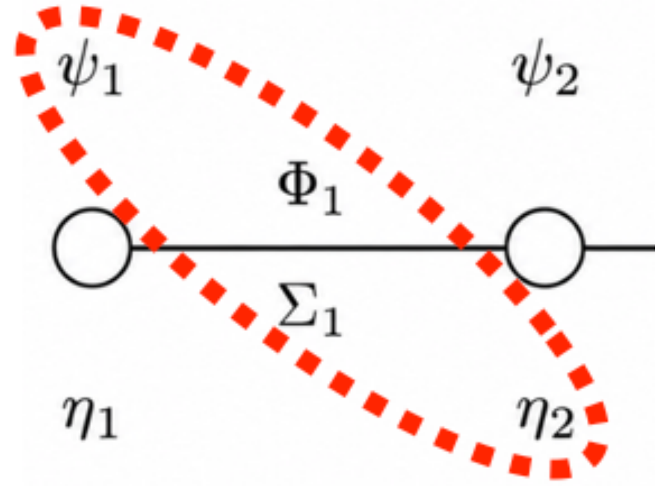
WZW term for mixed anomaly:

$$\mathcal{L}_{\text{WZW},i}^{\text{mix}} = \frac{1}{16\pi^2}\left[\left(A_i + A_{i+1} + \tilde{\Phi}_i^* \mathrm{d}\tilde{\Phi}_i\right) \operatorname{tr}\left(G_{i+1}\mathrm{d}G_{i+1} + \frac{2}{3}G_{i+1}^3\right) - \frac{1}{3}A_i \operatorname{tr}\left(\tilde{\Sigma}_i \mathrm{d}\tilde{\Sigma}_i^\dagger\right)^3 + \mathrm{d}A_i \operatorname{tr}\left(\tilde{\Sigma}_i^\dagger \mathrm{d}\tilde{\Sigma}_i G_{i+1} + G_i \tilde{\Sigma}_i G_{i+1}\tilde{\Sigma}_i^\dagger + G_i\tilde{\Sigma}_i \mathrm{d}\tilde{\Sigma}_i^\dagger\right) - \text{p.c.}\right] . \quad (4.6)$$

Gauged Wess-Zumino-Witten Term

In the limit $M'_\Psi = 0$, focus on one unit of hopping term:

$$\mathcal{L}_{\text{unit},i} = \eta^\dagger_{i+1} i\bar{\sigma}^\mu D_\mu \eta_{i+1} + \psi^\dagger_i i\bar{\sigma}^\mu D^*_\mu \psi_i - gv\left(\psi_i \tilde{\Phi}_i \tilde{\Sigma}_i \eta_{i+1} + \text{h.c.}\right)$$



gauge anomalies exist in the unit Lagrangian

Similar to the scenario considered in [Manohar, Moore (1984)], where the **SU(3) flavor symmetries are gauged** based on a phenomenological Lagrangian containing both quarks and pions.

$$SU(3)_{i,i+1} \longrightarrow SU(3)_{L,R} \qquad \mathrm{L}_\Sigma = \bar{\psi} i \not{\partial} \psi - m[\bar{\psi}_\ell \Sigma \psi_r + \text{p.c.}] + \cdots$$

In the low energy theory below the quark mass scale, the anomaly information should be manifest by an effective operator, called the **gauged Wess-Zumino-Witten (WZW) term**

By definition

$$\delta\tilde{\Gamma} = \frac{1}{24\pi^2}\int_{S^4} \omega^1_4(\varepsilon_r, A_r) - \omega^1_4(\varepsilon_\ell, A_\ell)$$

ω^1_4: integrated $SU(3)^3$ anomaly

$$\Gamma[\Sigma, A_r, A_\ell] = \underbrace{\frac{1}{240\pi^2}\int_{B_5(\Sigma)} \text{tr}\,(U^\dagger dU)^5}_{\text{ordinary WZW term}} + \frac{1}{48\pi^2}\int_{S^4} Z,$$

$$\boxed{\begin{aligned} -Z = {} & \text{tr}\,[A_\ell (d\Sigma\Sigma^\dagger)^3 - \text{p.c.}] + \text{tr}\,[(A_\ell dA_\ell + dA_\ell A_\ell + A_\ell^3)(\Sigma A_r \Sigma^\dagger + \Sigma\, d\Sigma^\dagger) - \text{p.c.}] \\ & + \text{tr}\,[d\Sigma\Sigma^\dagger\, dA_\ell \Sigma A_r \Sigma^\dagger - \text{p.c.}] + \tfrac{1}{2}\text{tr}\,(A_\ell\, d\Sigma\Sigma^\dagger A_\ell\, d\Sigma\Sigma^\dagger - \text{p.c.}) \\ & + \text{tr}\,[\Sigma A_r \Sigma^\dagger A_\ell\, d\Sigma\Sigma^\dagger\, d\Sigma\Sigma^\dagger - \text{p.c.}] \\ & - \text{tr}\,[A_r \Sigma^\dagger\, d\Sigma A_r \Sigma^\dagger A_\ell \Sigma - \text{p.c.}] - \tfrac{1}{2}\text{tr}\,(A_r \Sigma^\dagger A_\ell \Sigma A_r \Sigma^\dagger A_\ell \Sigma)\,. \end{aligned}}$$

Gauged Wess-Zumino-Witten Term

Our case is more complicated — $U(1)_i, U(1)_{i+1}, SU(3)_i, SU(3)_{i+1}$ are gauged.

$$\mathcal{L}_{\text{unit},i} = \eta_{i+1}^\dagger i\bar{\sigma}^\mu D_\mu \eta_{i+1} + \psi_i^\dagger i\bar{\sigma}^\mu D_\mu^* \psi_i - gv\left(\psi_i \tilde{\Phi}_i \tilde{\Sigma}_i \eta_{i+1} + \text{h.c.}\right)$$

Three types of anomalies exist:

$$\delta_1 S_{\text{unit},i} = \int_{R^4} \underbrace{-\frac{3}{32\pi^2}\left[\alpha_i \mathcal{F}_i^2 - \alpha_{i+1}\mathcal{F}_{i+1}^2\right]}_{U(1)^3} \underbrace{- \frac{1}{16\pi^2}\left[\alpha_i \operatorname{tr}\left(\mathcal{G}_i^2\right) - \alpha_{i+1}\operatorname{tr}\left(\mathcal{G}_{i+1}^2\right)\right]}_{U(1)\times SU(3)^2},$$

$$\delta_3 S_{\text{unit},i} = \int_{R^4} \underbrace{-\frac{1}{24\pi^2}\operatorname{tr}\left[\epsilon_i\left(\mathrm{d}G_i\mathrm{d}G_i + \frac{1}{2}\mathrm{d}(G_i^3)\right) - \epsilon_{i+1}\left(\mathrm{d}G_{i+1}\mathrm{d}G_{i+1} + \frac{1}{2}\mathrm{d}(G_{i+1}^3)\right)\right]}_{SU(3)^3}$$

WZW term for mixed anomaly:

$$\mathcal{L}_{\text{WZW},i}^{\text{mix}} = \frac{1}{16\pi^2}\left[\left(A_i + A_{i+1} + \tilde{\Phi}_i^* \mathrm{d}\tilde{\Phi}_i\right)\operatorname{tr}\left(G_{i+1}\mathrm{d}G_{i+1} + \frac{2}{3}G_{i+1}^3\right) - \frac{1}{3}A_i \operatorname{tr}\left(\tilde{\Sigma}_i \mathrm{d}\tilde{\Sigma}_i^\dagger\right)^3 \right.$$
$$\left. + \mathrm{d}A_i \operatorname{tr}\left(\tilde{\Sigma}_i^\dagger \mathrm{d}\tilde{\Sigma}_i G_{i+1} + G_i \tilde{\Sigma}_i G_{i+1}\tilde{\Sigma}_i^\dagger + G_i\tilde{\Sigma}_i \mathrm{d}\tilde{\Sigma}_i^\dagger\right) - \text{p.c.}\right]. \tag{4.6}$$

$U(1)$ transf.

$$\delta\mathcal{L}_{\text{WZW},i}^{\text{mix}} = \frac{1}{16\pi^2}\left[\mathrm{d}\alpha_i \operatorname{tr}\left(\omega_{3,i}\right) - \mathrm{d}\alpha_{i+1}\operatorname{tr}\left(\omega_{3,i+1}\right)\right] = \frac{1}{16\pi^2}\left[-\alpha_i \operatorname{tr}\left(\mathcal{G}_i^2\right) + \alpha_{i+1}\operatorname{tr}\left(\mathcal{G}_{i+1}^2\right)\right]$$

Instanton Effect — Zero-mode

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$$\underbrace{W_I = \int \mathrm{d}^4 x_0 \int \frac{\mathrm{d}\rho}{\rho^5} C_3 \left(\frac{2\pi}{\alpha_{c,D}}\right)^6}\, e^{-S^I_{\text{eff}}(\rho)}$$

Functional determinants

Collective coordinates from zero frequency mode of $\delta G_\mu^{(0)}$

Contributions to the effective action $-S^I_{\text{eff}}(R^{-1} < \rho^{-1} \lesssim v_3)$:

Classical action: $$-\frac{8\pi^2}{g_{c,D}(M)^2} = -\frac{8\pi^2}{g_{c,D}(R^{-1})^2} - \frac{11N_c}{3}\ln(MR) - \frac{7N_c}{2}\sum_n \ln(M/m_n)$$

$\delta G_\mu^{(0)}, \delta c^{(0)}$ →

$$4N_c \ln(M\rho) \quad - \quad \frac{N_c}{3}\ln(M\rho) = \underbrace{\frac{11N_c}{3}}_{b_{\text{YM}}}\ln(M\rho)$$

Zero freq. mode of $\delta G_\mu^{(0)}$

Positive freq. mode of $\delta G_\mu^{(0)}$ & $\delta c^{(0)}$

$\delta G_\mu^{(n)}, \delta\pi^{(n)}, \delta c^{(n)}$ →

Light $m_n < \rho^{-1}$ $$4N_c \ln(M/m_n) \quad - \quad 3\times\frac{N_c}{6}\ln(M\rho)$$

Would-be zero freq. mode of $\delta G_\mu^{(n)}$

Positive freq. mode of $\delta G_\mu^{(n)}, \delta c^{(n)}, \delta\pi^{(n)}$

Heavy $m_n > \rho^{-1}$ $$4N_c \ln(M/m_n) \quad - \quad 3\times\frac{N_c}{6}\ln(M/m_n) = \frac{7N_c}{2}\ln(M/m_n)$$

Instanton Effect — Zero-mode

$$e^{-S^I_{\rm eff}(R^{-1}<\rho^{-1}\lesssim v_3)} = \exp\left(-\frac{8\pi^2}{g^2_{c,D}(1/R)} + 11\ln\frac{\rho}{R} - \frac{3}{2}\underline{\sum_{n=1}^{K(\rho)}\ln(m_n\rho)}\right)$$

$K(\rho)$: # of modes lighter than ρ^{-1}

Massive (KK) tower enhancement

Especially for 5D regime $R^{-1} \ll \rho^{-1} < \Lambda_5$, $m_n = n/R$, $K(\rho) \simeq R/\rho$

$$\sum_{n=1}^{K(\rho)}\ln(m_n\rho) \simeq \sum_{n=1}^{K(\rho)}\ln(\frac{n}{R}\rho) = \cancel{K(\rho)\ln\frac{\rho}{R}} + \underline{\ln K(\rho)!}$$

Stirling formula for large $K(\rho)$

$$\simeq \cancel{K(\rho)\ln K(\rho)} - K(\rho) + \frac{1}{2}\ln(\pi K(\rho))$$

$$\longrightarrow \quad e^{-S^I_{\rm eff}(R^{-1}\ll\rho^{-1}\lesssim \Lambda_5)} = \exp\left(-\frac{8\pi^2}{g^2_{c,D}(1/R)} + \underline{\frac{3}{2}\frac{R}{\rho}} + \frac{47}{4}\ln\frac{\rho}{R}\right)$$

Power-law enhancement

Resulting axion potential

$$V_I[a] = -\frac{1}{\mathcal{V}_4}\ln Z[a] = -2\mathcal{K}\cos\left(\frac{a}{f_a}\right)$$

$$\mathcal{K} \equiv \int\frac{d\rho}{\rho^5}C_3\left(\frac{2\pi}{\alpha_{c,D}}\right)^6 e^{-S^I_{\rm eff}(\rho)}$$

with 5D region amplitude peaked at $\rho \simeq \Lambda_5^{-1}$

$$\mathcal{K}_5 \simeq C_3\left(\frac{2\pi}{\alpha_{c,D}(\Lambda_5)}\right)^6 \Lambda_5^4 e^{-S^{I,5}_{\rm eff}(\Lambda_5^{-1})}$$