Quantisation of constrained systems

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Nonrelativistic harmonic oscillator

Lagrange function of harmonic oscillator:

$$L(x, \dot{x}) = T - V = \frac{m\dot{x}^2}{2} - \frac{1}{2}m\omega^2 x^2$$

Euler–Lagrange equation of motion:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \quad \Longrightarrow \quad \ddot{x} + \omega^2 x = 0$$

Momentum and Hamilton function:

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad H = p\dot{x} - L = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

Canonical (Poisson) bracket:

$$\boxed{\{p, x\} = 1}$$

$$\{AB\} = \frac{\partial A}{\partial p}\frac{\partial B}{\partial x} - \frac{\partial A}{\partial x}\frac{\partial B}{\partial p}$$

Canonical quantisation of harmonic oscillator

Canonically conjugated coordinate and momentum become operators:

$$p \to \hat{p} \qquad x \to \hat{x}$$

Canonical (Poisson) bracket turns to quantum commutator:

$$\{p, x\} = 1 \quad \Longrightarrow \quad [\hat{p}, \hat{x}] = \hat{p}\hat{x} - \hat{x}\hat{p} = -i$$

Hamilton function turns to operator in Hilbert space of wave functions:

$$\hat{H}\psi_n(x) = E_n \psi_n(x)$$

Spectrum of harmonic oscillator:

$$E_n = \omega \left(n + \frac{1}{2} \right) \qquad n = 0, 1, 2, \ldots$$

Eigenfunctions of harmonic oscillator:

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi} \right)^{1/4} \exp\left(-\frac{m\omega x^2}{2} \right) H_n\left(\sqrt{m\omega} x \right)$$

Harmonic oscillator in Holomorphic representation

Equation of motion:

$$\ddot{x} + \omega^2 x = 0$$

General solution (coordinate is real $\Longrightarrow x^* = x$) and canonical momentum:

$$x(t) = \frac{1}{\sqrt{2m\omega}}(ae^{-i\omega t} + a^* e^{i\omega t}) \equiv \frac{1}{\sqrt{2m\omega}}(a(t) + a^*(t))$$

$$p(t) = m\dot{x}(t) = -i\sqrt{\frac{m\omega}{2}}(ae^{-i\omega t} - a^* e^{i\omega t}) = -i\sqrt{\frac{m\omega}{2}}(a(t) - a^*(t))$$

Proceed from x and p to a and a^*:

$$a(t) = \sqrt{\frac{m\omega}{2}}x + i\frac{p}{\sqrt{2m\omega}} \qquad a^*(t) = \sqrt{\frac{m\omega}{2}}x - i\frac{p}{\sqrt{2m\omega}}$$

$$H = \omega\Big[\zeta a^* a + (1-\zeta) a a^*\Big] \text{ with any } 0 \leqslant \zeta \leqslant 1$$

Quantisation:

$$\{a, a^*\} = 1 \quad \Longrightarrow \quad [a a^\dagger] = 1$$

Then only $\zeta = \frac{1}{2}$ gives the quantum spectrum consistent with experiment

What is constrained system?

How to restrict the motion of a 3D harmonic oscillator with the Hamiltonian

$$H = \frac{\boldsymbol{p}^2}{2m} + \frac{1}{2}m\omega^2\boldsymbol{r}^2$$

to move in the 2D plane (x, y)?

Way 1: Solve the equations of motion

$$\ddot{\boldsymbol{r}} + \omega^2\boldsymbol{r} = 0 \quad \Longrightarrow \quad \boldsymbol{r} = \boldsymbol{A}\sin(\omega t) + \boldsymbol{B}\cos(\omega t)$$

and then set $A_z = B_z = 0$

Way 2: Impose a constraint $\varphi_1 = z \approx 0$ on the system and ensure its conservation:

$$\varphi_2 = \dot{\varphi}_1 = \frac{\partial\varphi_1}{\partial t} + \{\varphi_1, H\} = \frac{\partial\varphi_1}{\partial\boldsymbol{p}}\frac{\partial H}{\partial\boldsymbol{r}} - \frac{\partial\varphi_1}{\partial\boldsymbol{r}}\frac{\partial H}{\partial\boldsymbol{p}} = -\frac{p_z}{m} \approx 0$$

Since $\varphi_3 \propto \varphi_1 \approx 0$, the system of constraints is closed and $\{\varphi_1, \varphi_2\} \neq 0$

$\Longrightarrow$ use φ_1 and φ_2 to get rid of unphysical variables z and p_z:

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2)$$

Free relativistic particle: Different forms of dynamics

Consider a free relativistic particle described by the action

$$S = \int_{\tau_i}^{\tau_f} d\tau \mathcal{L} \qquad \mathcal{L} = -m\sqrt{\dot{x}^2} \qquad \dot{x}_\mu = \frac{\partial x_\mu}{\partial \tau}$$

Let us proceed from Lagrangian to Hamiltonian:

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = -m\frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}} \quad \Longrightarrow \quad (p\dot{x}) = \mathcal{L} \quad \Longrightarrow \quad \mathcal{H} = (p\dot{x}) - \mathcal{L} = 0$$

On the other hand, dynamics of the particle is constrained by mass shell condition

$$\varphi = p^2 - m^2 \approx 0$$

Quantisation of the system (Klein–Gordon equation for scalar particle):

$$\left(\Box + m^2\right)\Psi = 0$$

Reparametrisation invariance: the action is invariant under time redifinition

$$\tau \to f(\tau) \qquad \dot{f}(\tau) > 0 \qquad f(\tau_i) = \tau_i \qquad f(\tau_f) = \tau_f$$

Alternative approach to quantisation — fix the freedom by choosing time τ:

$$\tau = x_0 \quad \Longrightarrow \quad \mathcal{L} = -m\sqrt{1 - \dot{\boldsymbol{r}}^2} \quad \Longrightarrow \quad \mathcal{H} = \sqrt{\boldsymbol{p}^2 + m^2}$$

Some definitions and ideas (a la Dirac)

- Constraints imposed on the system are called primary constraints
- Constraints that guarantee conservation in time of the primary constrains are called secondary constraints
- Constraints that commute with each other (their Poisson bracket vanishes) are called constraints of the first class
- The matrix $C_{ab} = \{\varphi_a, \varphi_b\}$ composed of the Poisson brackets of constrains with each other is called Hessian matrix
- Constraints such that $\det(C) \neq 0$ are called constraints of the second class
- Constraints of the second class can be worked out by removing redundant variables from the system
- Primary constraints of the first class imply symmetry/invariance of the system
- Constraints of the first class are imposed as weak (operator) equations for the wave function after quantisation
- Alternatively, additional primary constraints can be imposed on the system to fix constraints of the first class to become constraints of the second class (gauge fixing)

Electromagnetic field as constrained system

Canonical momentum of electromagnetic field ($\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2$)

$$\Pi^\mu(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu(x)} = F^{\mu 0}(x) \implies \varphi_1 = \Pi^0 \approx 0 \qquad \boldsymbol{\Pi}(x) = -\dot{\boldsymbol{A}} - \nabla A_0 = \boldsymbol{E}$$

Hamiltonian takes the form ($\boldsymbol{H} = [\nabla \times \boldsymbol{A}]$)

$$\mathcal{H} = \Pi^\mu \dot{A}_\mu - \mathcal{L} = \int d^3x \left(\frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{H}^2) + A_0(\rho - \nabla \boldsymbol{E}) - \boldsymbol{j}\boldsymbol{A}\right)$$

Hamiltonian equations of motion

$$\varphi_2 = \dot{\Pi}_0(x) = \{\Pi_0, \mathcal{H}\} = -\frac{\delta \mathcal{H}}{\delta A_0} \approx 0 \implies \nabla \boldsymbol{E} = \rho \quad \text{(Gauss law)}$$

$$\dot{\boldsymbol{A}} = \{\boldsymbol{A}, \mathcal{H}\} = -\frac{\delta \mathcal{H}}{\delta \boldsymbol{E}} = -\boldsymbol{E} - \nabla A_0 \approx 0 \implies \boldsymbol{E} = -\dot{\boldsymbol{A}} - \nabla A_0$$

$$\dot{\boldsymbol{\Pi}} = \{\boldsymbol{\Pi}, \mathcal{H}\} = \frac{\delta \mathcal{H}}{\delta \boldsymbol{A}} = [\nabla \times \boldsymbol{H}] - \boldsymbol{j} \approx 0 \implies \text{rot}\boldsymbol{H} - \dot{\boldsymbol{E}} = \boldsymbol{j}$$

Gauge invariance

$$\{\varphi_1 \varphi_2\} \approx 0$$

Free relativistic particle: Einbein field formalism

Modify the Lagrange function and introduce einbein field μ (with no velocity $\dot{\mu}$)

$$\mathcal{L} = -\frac{m^2}{2\mu} - \frac{\mu \dot{x}^2}{2} \quad \underset{\mu \to \mu_{\text{ext}}}{\Longrightarrow} \quad \mathcal{L} = -m\sqrt{\dot{x}^2}$$

Evaluate canonical momenta for coordinates x_μ and μ

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -\mu \dot{x}_\mu \qquad \pi = \frac{\partial L}{\partial \dot{\mu}} = 0 \quad \Longrightarrow \quad \varphi_1 = \pi \approx 0$$

The Hamiltonian takes the form

$$\mathcal{H} = -\frac{1}{2\mu}(p^2 - m^2) + \lambda \varphi_1 \quad \Longrightarrow \quad \varphi_2 = \{\varphi_1, \mathcal{H}\} = \frac{1}{2\mu^2}(p^2 - m^2) \approx 0$$

Quantisation follows the standard lines

$$[\hat{p}_\mu \hat{x}_\nu] = -i g_{\mu\nu} \qquad \hat{p}_\mu = -i\frac{\partial}{\partial x^\mu} \qquad [\hat{\pi}\hat{\mu}] = -1 \qquad \hat{\pi} = \frac{\partial}{\partial \mu}$$

To arrive at

$$(\Box + m^2)\Psi = 0 \qquad \frac{\partial \Psi}{\partial \mu} = 0$$

Free relativistic particle: Gauge fixing

To fix laboratory gauge impose an extra constraint

$$\varphi_3 = x_0 + \tau \quad \underset{x_0 \to x_0' = x_0 + \tau}{\Longrightarrow} \quad \varphi_3 = x_0'$$

The partition function for this canonical transformation takes the form

$$F(x, p', \tau) = p'_\mu (x^\mu + \tau g^{\mu 0})$$

and the modified Hamilton function becomes

$$\mathcal{H}' = \mathcal{H} + \frac{\partial F}{\partial \tau} = \mathcal{H} + p_0'$$

On the constraints surface

$$p'^2 \approx m^2 \quad \Longrightarrow \quad p_0' = \sqrt{\boldsymbol{p}^2 + m^2} \quad \& \quad \mathcal{H} \approx 0$$

Therefore, after fixing the gauge of the laboratory time we have

$$\mathcal{H}' = \sqrt{\boldsymbol{p}^2 + m^2}$$

Two noninteracting relativistic particles

Start from the Lagrange function and introduce two einbeins

$$\mathcal{L} = -m_1\sqrt{\dot{x}_1^2} - m_2\sqrt{\dot{x}_1^2} \quad \Longrightarrow \quad -\frac{\mu_1 \dot{x}_1^2}{2} - \frac{m_1^2}{2{\mu_1}^2} - \frac{\mu_2 \dot{x}_2^2}{2} - \frac{m_2^2}{2{\mu_2}^2}$$

Separate the centre-of-mass motion (formal nonrelativistic relations apply!)

$$x_\mu = x_{1\mu} - x_{2\mu} \qquad X_\mu = \zeta x_{1\mu} + (1-\zeta)x_{2\mu} \qquad \zeta = \frac{\mu_1}{\mu_1+\mu_2} \qquad M = \mu_1 + \mu_2$$

In terms of the new variables the Lagrange function becomes

$$\mathcal{L} = -\frac{m_1^2}{2M\zeta} - \frac{m_2^2}{2M(1-\zeta)} - \frac{1}{2}M\left(\dot{X} - \dot{\zeta}x\right)^2 - \frac{1}{2}M\zeta(1-\zeta)\dot{x}^2$$

Then we define the four canonically conjugated momenta

$$P_\mu = \frac{\partial L}{\partial \dot{X}_\mu} \qquad p_\mu = \frac{\partial L}{\partial \dot{x}_\mu} \qquad \Pi = \frac{\partial L}{\partial \dot{M}} \qquad \pi = \frac{\partial L}{\partial \dot{\zeta}}$$

and proceed to the Hamilton function

$$\mathcal{H} = \frac{m_1^2 - p^2}{2M\zeta} + \frac{m_2^2 - p^2}{2M(1-\zeta)} - \frac{P^2}{2M} + \Lambda\varphi_1 + \lambda\varphi_2 \qquad \varphi_1 = \Pi \qquad \varphi_2 = \pi + (Px)$$

Two noninteracting relativistic particles

The Poisson bracket is defined as

$$\{AB\} = \frac{\partial A}{\partial P_\mu}\frac{\partial B}{\partial X_\mu} - \frac{\partial A}{\partial X_\mu}\frac{\partial B}{\partial P_\mu} + \frac{\partial A}{\partial p_\mu}\frac{\partial B}{\partial x_\mu} - \frac{\partial A}{\partial x_\mu}\frac{\partial B}{\partial p_\mu} + \frac{\partial A}{\partial \Pi}\frac{\partial B}{\partial M} - \frac{\partial A}{\partial M}\frac{\partial B}{\partial \Pi} + \frac{\partial A}{\partial \pi}\frac{\partial B}{\partial \mu} - \frac{\partial A}{\partial \mu}\frac{\partial B}{\partial \pi}$$

and the secondary constraints take the form

$$\varphi_3 = \{\varphi_1 H\} = -\frac{\varepsilon_1^2}{2M^2\zeta} - \frac{\varepsilon_2^2}{2M^2(1-\zeta)} + \frac{P^2}{2M^2}$$

$$\varphi_4 = \{\varphi_2 H\} = -\frac{\varepsilon_1^2}{2M\zeta^2} + \frac{\varepsilon_2^2}{2M(1-\zeta)^2} + \frac{(pP)}{M\zeta(1-\zeta)}$$

$$\varepsilon_1^2 = m_1^2 - p^2 \qquad \varepsilon_2^2 = m_2^2 - p^2$$

Constraints $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ are first class constraints (2 primary and 2 secondary)

$\Longrightarrow$ There are two "guage" freedoms: for c.m. motion & for relative motion

Time-like gauge

Two fix the two gauges, impose two extra constraints

$$\varphi_5 = X_0 + \tau \qquad \varphi_6 = x_0$$

and "commute" them with the Hamilton function for secondary constraints

$$\varphi_7 = \{\varphi_5 H\} = \frac{P_0}{M} - 1 \qquad \varphi_8 = \{\varphi_6 H\} = \frac{p_0}{M\zeta(1-\zeta)}$$

The Hamilton function becomes simply

$$H = M = P_0$$

and the canonical bracket has to be redefined: Poisson bracket $\Longrightarrow$ Dirac bracket:

$$\{AB\}^* = \{AB\} - \sum_{a,b} \{A\varphi_a\} C^{-1}_{ab} \{\varphi_b B\}$$

where C^{-1} is the inverse matrix with respect to $(a, b = 1..8)$

$$C_{ab} = \{\varphi_a \varphi_b\}$$

Time-like gauge

The Hamiltonian takes the form

$$\mathcal{H} = \sqrt{\boldsymbol{P}^2 + \mathcal{E}_1^2 + \mathcal{E}_2^2 + 2\sqrt{\mathcal{E}_1^2\mathcal{E}_2^2 + (\boldsymbol{pP})^2}}$$

where $\mathcal{E}_i = \sqrt{\boldsymbol{p}^2 + m_i^2}$ $(i = 1, 2)$ and the set of canonical (Dirac) brackets reads

$$\{P_i X_k\}^* = \delta_{ik}$$

$$\{X_i p_k\}^* = \frac{p_i P_k}{M^2}$$

$$\{X_i X_k\}^* = \frac{x_i p_k - x_k p_i}{M^2}$$

$$\{X_i x_k\}^* = \frac{x_i P_k}{M^2} + \frac{x_i p_k}{M^2}\left(\frac{1-\zeta}{\zeta} - \frac{\zeta}{1-\zeta}\right)$$

$$\{p_i x_k\}^* = \delta_{ik} - \frac{P_i P_k}{M^2} - \frac{P_i p_k}{M^2}\left(\frac{1-\zeta}{\zeta} - \frac{\zeta}{1-\zeta}\right)$$

$$\{P_i P_k\}^* = \{P_i p_k\}^* = \{P_i x_k\}^* = \{p_i p_k\}^* = \{x_i x_k\}^* = 0$$

Newton–Wigner coordinate

Let us define the Newton–Wigner coordinate

$$Q_i = X_i - \frac{S_{ik}P_k}{E(M+E)} \qquad E = \sqrt{M^2 - \boldsymbol{P}^2} \qquad S_{ik} = x_i p_k - x_k p_i$$

and the proper internal variables

$$\boldsymbol{k} = \boldsymbol{p} + \frac{(\boldsymbol{pP})\boldsymbol{P}}{E(M+E)} \qquad \boldsymbol{r} = \boldsymbol{x} + \frac{(\boldsymbol{xP})\boldsymbol{P}}{E(M+E)} + \frac{(\boldsymbol{xP})\boldsymbol{k}}{E\omega_1\omega_2}\left(\omega_1 - \omega_2 - \frac{(\boldsymbol{kP})}{M}\right)$$

where

$$M = \sqrt{\boldsymbol{P}^2 + (\omega_1+\omega_2)^2} \qquad \omega_1 = \sqrt{m_1^2 + \boldsymbol{k}^2} \qquad \omega_2 = \sqrt{m_2^2 + \boldsymbol{k}^2}$$

with the canonical Dirac brackets:

$$\{k_i r_k\}^* = \delta_{ik} \qquad \{k_i k_k\}^* = \{r_i r_k\}^* = \{k_i Q_k\}^* = \{r_i Q_k\}^* = 0$$

and the Hamilton function takes the desired form

$$\mathcal{H} = M = \sqrt{\boldsymbol{P}^2 + \left(\sqrt{m_1^2 + \boldsymbol{k}^2} + \sqrt{m_2^2 + \boldsymbol{k}^2}\right)^2}$$

Proper-time gauge

Let us fix one gauge freedom of two imposing Lorenz-covariant constraint φ_5 that gives rise to a covariant secondary constraint φ_6:

$$\varphi_5 = (Px) \approx 0 \quad \Longrightarrow \quad \varphi_6 = (Pp) \approx 0$$

After excluding 6 second-class constraints, Newton–Wigner coordinate is defined as

$$Z_\mu = X_\mu + \frac{1}{2} S_{ij} \Gamma_{ij\mu} \qquad \{Z_\mu Z_\nu\}^* = 0 \qquad \{P_\mu Z_\nu\}^* = g_{\mu\nu}$$

with $S_{ij} = e_{i\mu} e_{j\nu} S_{\mu\nu}$ and a tetrade of vectors and Christoffel symbols introduced as

$$e_{0\mu} = \frac{P_\mu}{\sqrt{P^2}} \quad e_{i\mu} e_{j\mu} = -\delta_{ij} \quad e_{0\mu} e_{j\mu} = 0 \quad \Gamma_{ij\alpha} = e_{i\mu} \frac{\partial}{\partial P_\alpha} e_{j\mu} \quad \Gamma_{0j\alpha} = e_{0\mu} \frac{\partial}{\partial P_\alpha} e_{j\mu}$$

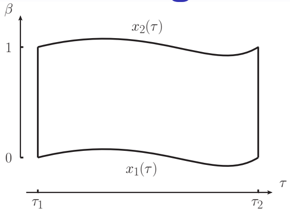
Then the coordinate and momentum of relative motion can be defined as

$$x_i = -e_{i\mu} x^\mu \qquad p_i = -e_{i\mu} p^\mu$$

and the system is quantised employing the remaining primary first-class constraint

$$\left(\hat{P}^2 - \mathcal{M}^2(\hat{p}_i)\right) \Psi = 0 \qquad \mathcal{M}^2 = \left(\sqrt{m_1^2 + p_i^2} + \sqrt{m_2^2 + p_i^2}\right)^2$$

Straight-line string: Lagrange function



$$S_{\text{string}} = -\sigma \int_{\tau_1}^{\tau_2} d\tau \int_0^1 d\beta \sqrt{(\dot{w}w')^2 - \dot{w}^2 w'^2}$$

$$w_\mu(\beta,\tau) = (1-\beta)x_{1\mu}(\tau) + \beta x_{2\mu}(\tau)$$

$$\dot{w}_\mu = \frac{\partial w_\mu}{\partial \tau} \qquad w'_\mu = \frac{\partial w_\mu}{\partial \beta} = x_{2\mu} - x_{1\mu} \equiv x_\mu$$

Continuous einbein field $\nu(\beta)$ is introduced to get rid of the square root

$$L = -\frac{1}{2}\int_0^1 d\beta \nu(\beta)\left(\dot{w}^2 - \frac{(\dot{w}x)^2}{x^2}\right) + \frac{1}{2}\int_0^1 d\beta \frac{\sigma^2 x^2}{\nu(\beta)}$$

with canonical momenta ($X_\mu = \zeta x_{1\mu} + (1-\zeta)x_{2\mu}$) with $\zeta = \int_0^1 d\beta \nu\beta / \int_0^1 d\beta\nu$)

$$P_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = -M\left(\dot{X}_\mu - \frac{x_\mu(x\dot{X})}{x^2}\right) \quad \Longrightarrow \quad \varphi_1 = (Px) \approx 0$$

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -m\left(\dot{x}_\mu - \frac{x_\mu(x\dot{x})}{x^2}\right) \quad \Longrightarrow \quad \varphi_2 = (px) \approx 0$$

$$\kappa(\beta) = \frac{\delta L}{\delta \dot{\nu}(\beta)} = 0 \quad \Longrightarrow \quad \varphi_3(\beta) = \kappa(\beta) \approx 0$$

Straight-line string: Hamilton function

The Hamilton function takes the form

$$\mathcal{H} = \underbrace{-\frac{P^2}{2M} - \frac{p^2}{2m} - \frac{kx^2}{2}}_{\mathcal{H}_0} + \Lambda \underbrace{(Px)}_{\varphi_1} + \lambda \underbrace{(px)}_{\varphi_2} + \int_0^1 d\beta e(\beta) \underbrace{\kappa(\beta)}_{\varphi_3(\beta)}$$

$$M = \int_0^1 d\beta \nu \qquad m = \int_0^1 d\beta \nu (\beta - \zeta)^2 \qquad k = \int_0^1 \frac{\sigma^2}{\nu}$$

The Poisson bracket is defined as

$$\{AB\} = \frac{\partial A}{\partial P_\mu}\frac{\partial B}{\partial X_\mu} - \frac{\partial A}{\partial X_\mu}\frac{\partial B}{\partial P_\mu} + \frac{\partial A}{\partial p_\mu}\frac{\partial B}{\partial x_\mu} - \frac{\partial A}{\partial x_\mu}\frac{\partial B}{\partial p_\mu} + \int_0^1 d\beta \left(\frac{\delta A}{\delta \kappa(\beta)}\frac{\delta B}{\delta \nu(\beta)} - \frac{\delta A}{\delta \nu(\beta)}\frac{\delta B}{\delta \kappa(\beta)} \right)$$

Primary constraints lead to secondary ones and finally Lagrange multipliers get fixed

$$\left.\begin{array}{lllll} \varphi_1 \approx 0 & \Longrightarrow & \varphi_4 = \{\varphi_1, \mathcal{H}\} \approx 0 & \Longrightarrow & \{\varphi_4, \mathcal{H}\} \approx 0 \\ \varphi_2 \approx 0 & \Longrightarrow & \varphi_5 = \{\varphi_2, \mathcal{H}\} \approx 0 & \Longrightarrow & \{\varphi_5, \mathcal{H}\} \approx 0 \\ \varphi_3 \approx 0 & \Longrightarrow & \varphi_6 = \{\varphi_3, \mathcal{H}\} \approx 0 & \Longrightarrow & \{\varphi_6, \mathcal{H}\} \approx 0 \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \Lambda = 0 \\ \lambda = 0 \\ e(\beta) = e_0 \nu(\beta) \end{array}\right.$$

The remaining pair of first-class constraints is

$$\Phi_1 = \int_0^1 d\beta \nu(\beta) \varphi_3(\beta) \qquad \Phi_2 = \int_0^1 d\beta \nu(\beta) \varphi_6(\beta) \approx -\mathcal{H}_0$$

Straight-line string: Quantisation and spectrum

We introduce the standard Newton–Wigner coordinate

$$Z_\mu = X_\mu + \frac{1}{2} S_{ij} \Gamma_{ij\mu}$$

where

$$n_i = -e_{i\mu} \frac{x_\mu}{\sqrt{-x^2}} \quad S_{ij} = e_{i\mu} e_{j\nu} S_{\mu\nu} \quad S_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \quad L^2 = \frac{1}{2} S_{ik} S_{ik}$$

and tetrade of vectors and Christoffel symbols are defined as before

$$e_{0\mu} = \frac{P_\mu}{\sqrt{P^2}} \quad e_{i\mu} e_{j\mu} = -\delta_{ij} \quad e_{0\mu} e_{j\mu} = 0 \quad \Gamma_{ij\alpha} = e_{i\mu} \frac{\partial}{\partial P_\alpha} e_{j\mu} \quad \Gamma_{0j\alpha} = e_{0\mu} \frac{\partial}{\partial P_\alpha} e_{j\mu}$$

Employing second-class constraints to exclude redundant degrees of freedom we find

$$\nu(\beta) = \frac{N}{\sqrt{1-(2\beta-1)^2}} \quad M = 8m = \frac{\pi N}{2} \quad k = \frac{\pi \sigma^2}{4N} \quad x^2 = -\frac{8L}{\pi\sigma} \quad p^2 = -\frac{\pi \sigma L}{8}$$

with arbitrary N. After quantisation we get for bound state equation and spectrum

$$(\hat{P}^2 - 2\pi\sigma \hat{L})\Psi = 0 \quad \Longrightarrow \quad \boxed{\mathcal{M}_L^2 = 2\pi\sigma\sqrt{L(L+1)}}$$

Generalisation to fermion in external field

Spin-$\frac{1}{2}$ fermion in external electromagnetic field is described by Lagrange function

$$\mathcal{L} = -\frac{\mu}{2}\dot{x}_\nu[\dot{x}^\nu - i\chi\psi^\nu] - \frac{i}{2}\psi_\nu\dot{\psi}^\nu - \frac{m^2}{2\mu} + \frac{i}{2}[\psi_5\dot{\psi}_5 + m\chi\psi_5] - eA_\mu\dot{x}^\mu + \frac{ie}{2\mu}\psi_\mu\psi_\nu F^{\mu\nu}$$

where ψ_μ describes the spin and μ and χ are einbeins (ψ_μ and χ are Grassmannian)

Balachandran, Salomonson, Skagerstam,Winnberg'1977

Two gauge-like symmetries of the action:

- Reparametrisation invariance

$$\tau \to f(\tau) \qquad \mu \to \frac{\mu}{\dot{f}(\tau)} \qquad \chi \to \frac{\chi}{\dot{f}(\tau)}$$

- Supergauge transformations (α is Grassmannian)

$$\delta x_\nu = i\alpha\psi_\nu \quad \delta\psi_\nu = -\alpha\mu(\dot{x}_\nu - \frac{i}{2}\chi\psi_\nu) \quad \delta\mu = i\alpha\mu^2\chi \quad \delta\chi = 2\dot{\alpha} \quad \delta\psi_5 = m\alpha$$

Gitman,Tyutin'1990; Grigoryan,Grigoryan'1991

Quantisation with first-class constraints

Quantisation proceeds via setting

$$\hat{p}_\mu = i\frac{\partial}{\partial x_\mu} \qquad \psi_\mu = \frac{1}{\sqrt{2}}\gamma_5\gamma_\mu \qquad \psi_5 = \frac{1}{\sqrt{2}}\gamma_5$$

Primary constraints $\Phi_1 = \pi \approx 0$ and $\Phi_2 = \pi_\chi \approx 0$ give rise to secondary first-class constraints Φ_{KG} and Φ_D that, after quantisation, transform into equations

$$\hat{\Phi}_D\Psi = \gamma_5\Big(\gamma(p+eA) - m\Big)\Psi = 0$$

$$\hat{\Phi}_{KG}\Psi = \left((p+eA)^2 - m^2 - \frac{ie}{2}\sigma_{\mu\nu}F_{\mu\nu}\right)\Psi = 0$$

These are the famous Dirac equation and so-called "Klein–Gordon with spin"

Quantum albebra of constraints is closed:

$$[\hat{\Phi}_{KG}, \hat{\Phi}_{KG}]_- = 0 \qquad [\hat{\Phi}_{KG}, \hat{\Phi}_D]_- = 0 \qquad [\hat{\Phi}_D, \hat{\Phi}_D]_+ = -\hat{\Phi}_{KG}$$

Quantisation after "laboratory gauge" fixing

Fix "gauge" freedom imposing additional primary constraint

$$\tau = x_0$$

Then, after some algebra, the Hamilton function and Dirac constraint take the form

$$H = \mu_0 + eA_0 + \frac{ie}{\mu_0} F_{0i}\psi_0\psi_i \qquad \varphi_D = \mu_0\psi_0 - (\boldsymbol{p} - e\boldsymbol{A})\boldsymbol{\psi} + m\psi_5$$

where the einbein field takes the form (solution with $\mu_0 < 0$ thrown away!)

$$\mu_0 = \sqrt{(\boldsymbol{p} - e\boldsymbol{A})^2 + m^2 - ieF_{ik}\psi_i\psi_k}$$

However, the quantum algebra is not closed:

$$[\hat{H}, \hat{\varphi}_D]_- \neq 0$$

since the Grassmann algebra for ψ's

$$\psi_\mu\psi_\nu = -\psi_\nu\psi_\mu \qquad \psi_5\psi_\mu = -\psi_\mu\psi_5 \qquad \psi_5\psi_5 = 0$$

does not translate into similar algebra for Dirac matrices

Kalashnikova,AN'1999

The physical origins of the problem

Start from Dirac equation in external field (for simplicity, $\boldsymbol{A} = 0$) and perform its Foldy–Wounthuysen transformation to order $1/m^2$

$$H = \frac{p^2}{2m} - \frac{p^4}{8m^3} + eA_0 - \underbrace{\frac{e}{4m^2}\boldsymbol{\sigma}[\boldsymbol{E}\boldsymbol{p}]}_{\text{Spin-orbit term}} - \underbrace{\frac{e}{8m^2}\mathrm{div}\boldsymbol{E}}_{\text{Darwin term}}$$

where

$$V_{\mathrm{Darwin}} = -\frac{e}{8m^2}\mathrm{div}\boldsymbol{E} \propto \Delta A_0 \propto \rho_{\mathrm{ext}}(\boldsymbol{x})$$

and, in central field $A_0(r)$,

$$\boldsymbol{E} = -\frac{\boldsymbol{r}}{r}\frac{dA_0}{dr}$$

- Quantisation with first class constraints $\Longrightarrow$ Darwin term is present
- Quantisation after laboratory time fixing $\Longrightarrow$ Darwin is missing

Conclusion: Zitterbewegung (motion with $\mu_0 < 0$) lost after fixing laboratory time $\Longrightarrow$ no problem for free particles but potential danger for interacting particles

Summary: How to quantise constrained system

- Start from Lagrange function
- Evaluate canonical momenta
- Establish primary constraints
- Build Hamilton function
- Add primary constraints to Hamilton function with Lagrange multiplies
- Commute primary constraints with Hamilton function for secondary constraints
- If the process truncates at some step proceed to the next stage (give up if not)
- Commute (build Poisson brackets) all constraints with each other
- Identify first- and second-class constraints
- Choose whether or not to fix the first-class constraints
- Impose additional constraints to fix the chosen first-class constraints (Caution!)
- Establish the corresponding secondary constraints
- Get rid of second-class constraints by eliminating unphysical degrees of freedom
- Build Hessian matrix and Dirac brackets for all physical degrees of freedom
- Bring Dirac brackets to canonical form by defining the proper coordinates
- Express everything through the physical variables
- Quantise system employing Hamilton function and remaining first-class constraints
- Check if the resulting theory describes Nature — if yes, enjoy your theory!

Conclusions

- Quantisation of a classical system is an Art
    - Any quantum theory possesses the only classical limit
    - The same classical theory may correspond to an infinite number of quantum theories (operators ordering, gauge fixing, etc)
- Presence of constraints makes quantisation an Art^2
- General principles for quantisation of constrained systems can still be formulated that work in practical applications and lead to meaningful theories

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Forms of Relativistic Dynamics

P. A. M. DIRAC
St. John's College, Cambridge, England

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Thank you for your attention!

Possible choices for tetrades

Orthogonality and completeness conditions

$$e_{\mu\nu}e_{\mu'\nu} = g_{\mu\mu'} \qquad e_{\mu\nu}e_{\mu\nu'} = g_{\nu\nu'}$$

where μ is serial number of tetrade in the set and ν is its Lorentz component

- Simplest choice

$$e_{\mu\nu} = g_{\mu\nu}$$

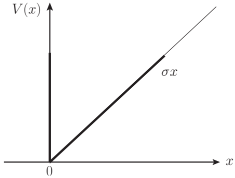
- Convenient choice for co-moving frame

$$e_{0\mu} = \frac{P_\mu}{\sqrt{P^2}} \qquad e_{i0} = \frac{P_i}{\sqrt{P^2}} \qquad e_{ik} = \delta_{ik} + \frac{P_i P_k}{\sqrt{P^2}(P_0 + \sqrt{P^2})}$$

Einbein field as variational parameter

One-dimensional linear potential

$$V(x) = \begin{cases} \sigma x, & x \geqslant 0 \\ \infty, & x < 0 \end{cases}$$



Schrödinger equation and wave function

$$\left(\frac{p^2}{2m} + V(x)\right)\psi = E\psi \qquad \psi_n(x) = \mathcal{N}_n Ai(y) \qquad y = (2m\sigma)^{1/3}(x - E_n/\sigma)$$

Quantisation condition

$$\psi_n(x=0) = 0 \quad \Longrightarrow \quad E_n = -\frac{\sigma^{2/3}}{(2m)^{1/3}} a_n \ \text{ with } \ Ai(a_n) = 0$$

Approximate formula for zeros of Airy function

$$a_n \approx -\underbrace{\left(\frac{3\pi}{2}\right)^{2/3}}_{\approx 2.81}\left(n - \frac{1}{4}\right)^{2/3} \qquad n = 1, 2, \ldots$$

Einbein field as variational parameter

One

In einbein field formalism:

$$\sigma x \to \frac{\nu}{2} + \frac{1}{2} m\omega^2 x^2 \qquad \omega = \frac{\sigma}{\sqrt{m\nu}}$$

$$E_n = \frac{\nu}{2} + \omega \left[\{2(n-1)+1\} + \frac{1}{2} \right] \quad \Longrightarrow \quad -\frac{\sigma^{2/3}}{(2m)^{1/3}} a_n^{\text{ein}}$$

$$a_n^{\text{ein}} = -3 \left(n - \frac{1}{4} \right)^{2/3} \qquad n = 1, 2, \ldots$$

Sch

$$\left(\frac{p}{2\ldots} \right. \quad \ldots - E_n/\sigma)$$

Qua

$$\psi_n(x=0) = 0 \quad \Longrightarrow \quad E_n = -\frac{\ldots}{(2m)^{1/3}} a_n \quad \text{with} \quad Ai(a_n) = 0$$

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