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Lineshape of $a_1(1260)$ from heavy-lepton decay $\tau \to \upsilon 3\pi$

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Outline

Motivation

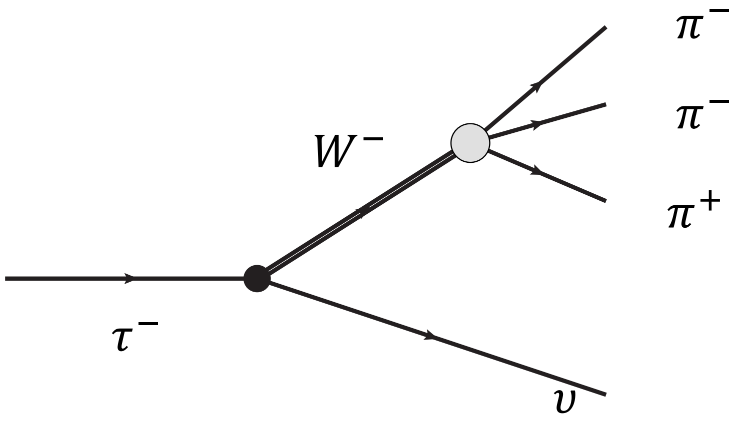
- ALEPH and CLEO data on $\tau \to \upsilon 3\pi$
- $\tau \to \upsilon 3\pi$ offers a clean view of $a_1(1260)$ production
- $a_1(1420)$ triangle singularity in $a_1(1260) \to \bar{K}K^* \to f_0\pi \to 3\pi$

Dispersive framework for three-pion decays

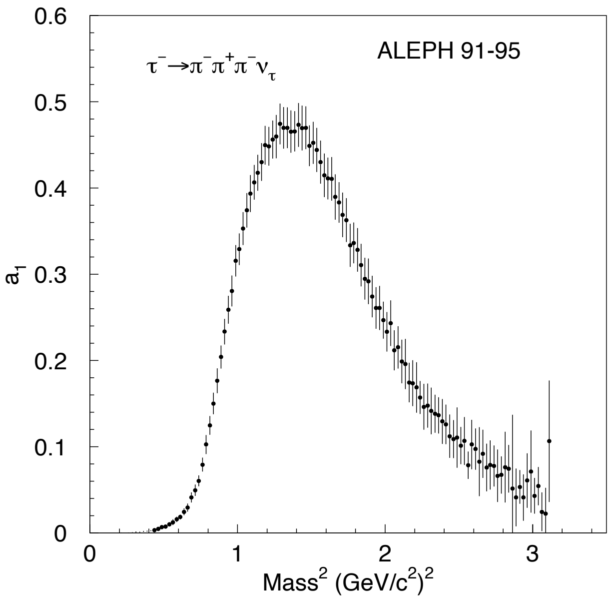
- construction by unitarity and analyticity
- phase-shifts with good accuracy available
- integral equations for $a_1 \to 3\pi$

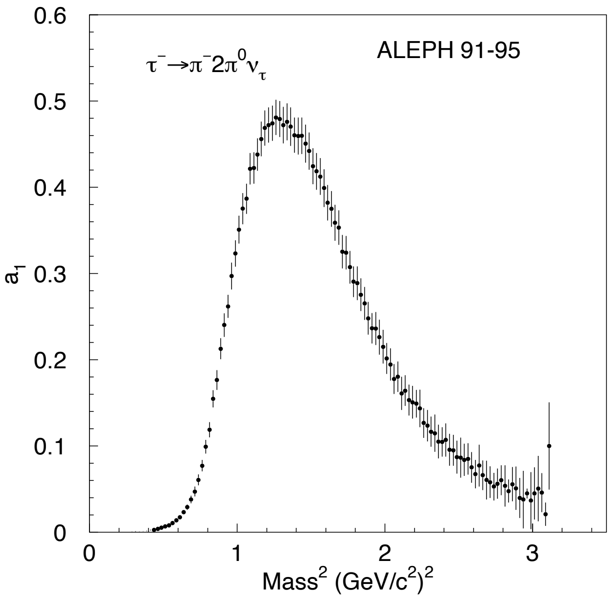
Numerical calculation

$\tau \to \upsilon 3\pi$ decay



ALEPH data on $\tau \to \upsilon 3\pi$





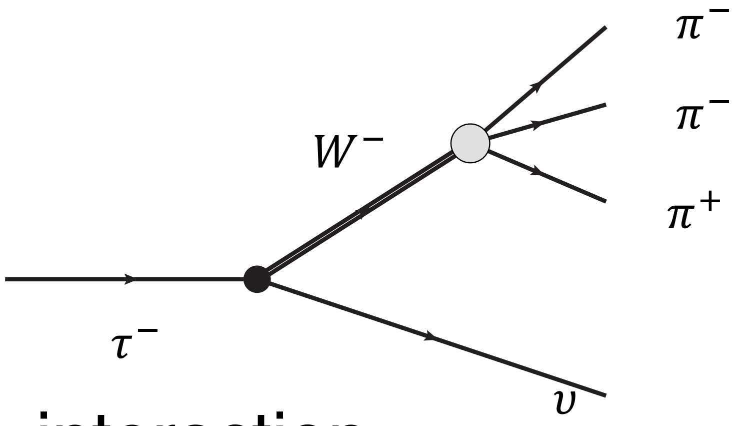
- accurate measurements of the mass distribution
- component $J^{PC}=0^{-+}$ is suppressed by the PCAC
- the $J^{PC}=1^{++}$ is dominant

The spectral function for the 3π final state

ALEPH Collaboration, Phys. Rept. 421, 191(2005).

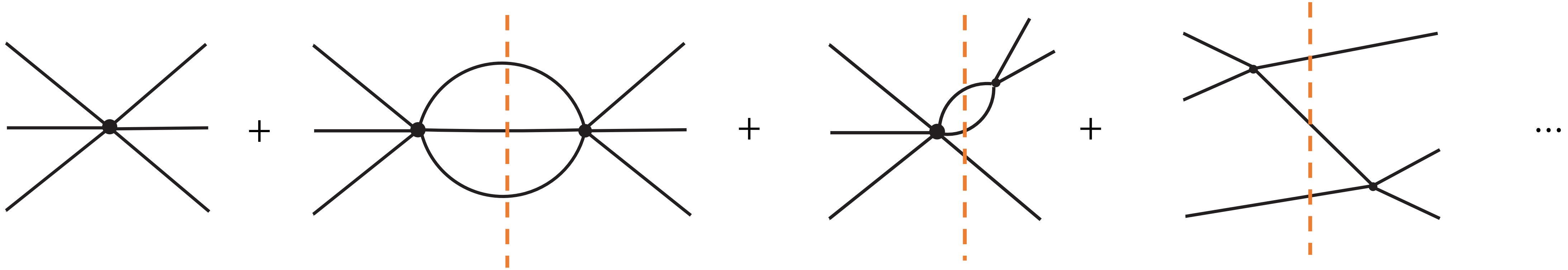
J. H. Kuhn and E. Mirkes, Z. Phys. C 56, 661 (1992)

$\tau \to \upsilon 3\pi$ decay



Final-state interactions in hadronic three-body decays play essential role

- three-body unitary framework used to describe 3π interaction



Daniel Sadasivan et al., Phys. Rev. D 105, 054020 (2022)

- $a_1(1260)$ generated from 3π interaction

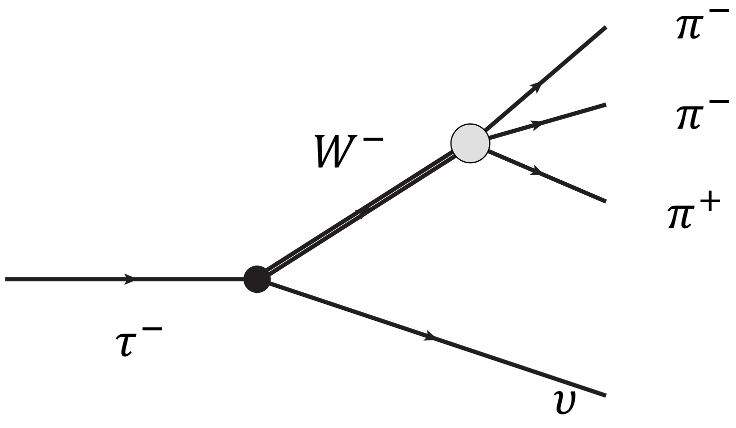
- approximate three-body unitary

JPAC Collaboration, Phys. Rev. D 98, 096021 (2018)

$\tau \to \upsilon 3\pi$ decay

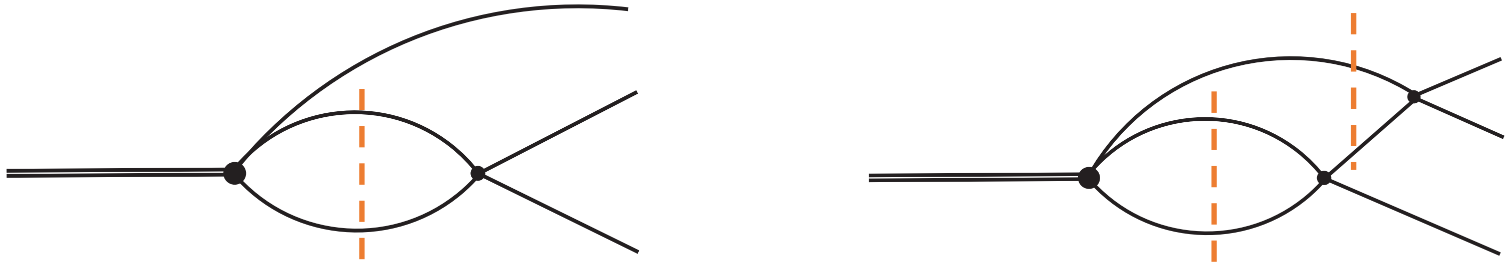
- model of this work $\tau \to \upsilon 3\pi$ decay

$$\frac{d\Gamma}{d\sigma} = \text{cons.}\,(m_\tau^2 - \sigma)^2 \sum_\lambda \left| \frac{A_\lambda(\sigma)}{\sigma - m_{a_1}^{(0)2} + \Sigma(\sigma)} \right|^2 d\Phi^3$$

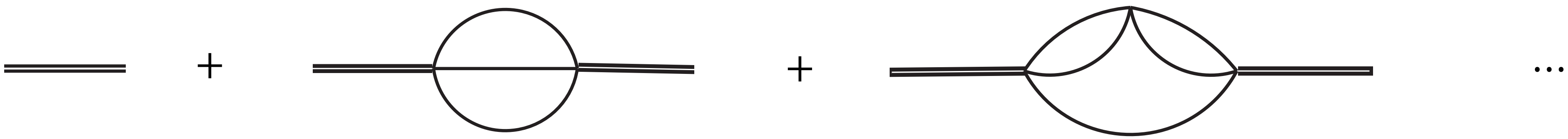


- unitarity relation of $a_1 \to 3\pi$

N. Khuri, S. Treiman, Phys. Rev. 119, 1115 (1960)

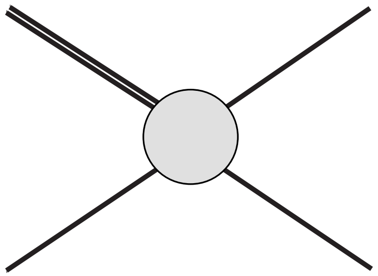


- self-energy $\Sigma(\sigma)$



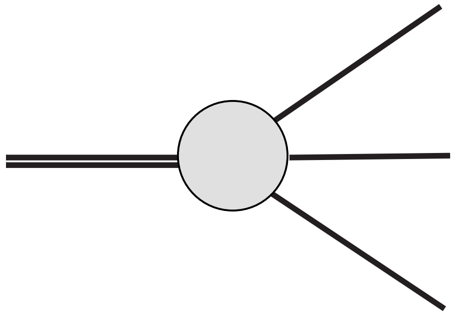
Three-body decay $a_1 \to 3\pi$

scattering region



$\sigma < 3m_\pi$
$s \in [\,(\sigma + m_\pi)^2\,, \infty]$

continuation in
σ and s



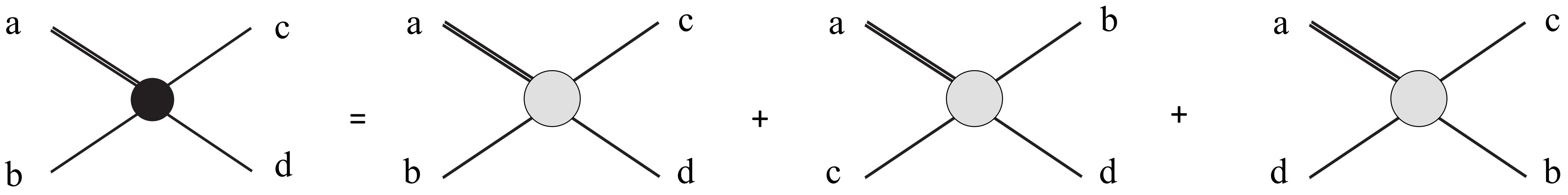
$\sigma\ > 3m_\pi$
$s \in [\,4m_\pi^2\,, (\sigma - m_\pi)^2]$

decay region

σ is three-body
invariant mass

dispersive treatment

Cartesian isospin indices



s-channel

t-channel

u-channel

amplitude can be decomposed as

$$A_\lambda^{(s)abcd}(s,t,u) = \bar{A}_\lambda^{(s)abcd}(s,t,u) + \Sigma_{\lambda'}(-1)^{\lambda'}\, d^J_{\lambda'\lambda}(\omega_t)\bar{A}_{\lambda'}^{(t)acbd}(t,s,u) + \Sigma_{\lambda'}(-1)^{\lambda'}\, d^J_{\lambda'\lambda}(\omega_u)\bar{A}_{\lambda'}^{(u)adbc}(u,t,s)$$

From unitarity to integral equations

$$\bar{A}_{\lambda}^{(I)}(s,t,u) = \Sigma_j\,(2j+1)d_{\lambda 0}^{j}(\theta_s)a_{j\lambda}^{(I)}(s)$$

$$T^{(I)}(s,\cos\theta_s) = 32\pi\sum_{l=0}^{\infty}(2l+1)\,P_l(\cos\theta_s)\tau_l^{(I)}\,(s)$$

- unitarity relation

$$A_{\lambda}^{(I)}(s,t,u) = \bar{A}_{\lambda}^{(I)}(s,t,u) + \Sigma_{\lambda'}(-1)^{\lambda'}d_{\lambda'\lambda}^{J}(\omega_t)\bar{A}_{\lambda'}^{(I')}(t,s,u)\tfrac{1}{2}C_{II'} + \Sigma_{\lambda'}(-1)^{\lambda'}d_{\lambda'\lambda}^{J}(\omega_u)\bar{A}_{\lambda'}^{(I')}(u,t,s)\tfrac{1}{2}C_{II'}$$

$$\text{Disc}\,A_{\lambda}^{(I)}(s,t,u) = \frac{\rho(s)}{64\pi^2}\int d\Omega'\;T^{*(I)}(s,t'',u'')\times A_{\lambda}^{(I)}(s,t',u')$$

$$\text{Disc}\,a_{j\lambda}^{(I)}(s) = \rho(s)\tau_j^{(I)}(s)\left\{\underbrace{a_{j\lambda}^{(I)}(s)}_{\text{right-hand cut}} + \underbrace{\bar{\bar{a}}_{j\lambda}^{(I)}(s)}_{\text{left-hand cut}}\right\}$$

JPAC, Phys. Rev. D 101, 054018 (2020)

- kinematic singularities arising from the kinematics (spinor)

kinematic singularities free (KSF) amplitude

$$a_{j\lambda}^{(I)}(s) = \frac{(\lambda_X^{1/2}(s))^{j-1}(\lambda_\pi^{1/2}(s))^{j}}{s^{|\lambda|/2}}\,\hat{a}_{j\lambda}^{(I)}(s)$$

Källén or triangle function

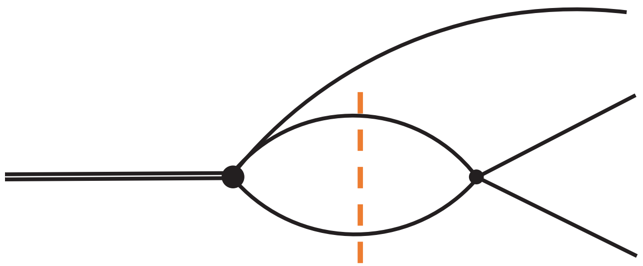
$$\lambda_M^{1/2}(s) = \lambda(s,M^2,m_\pi^2)$$

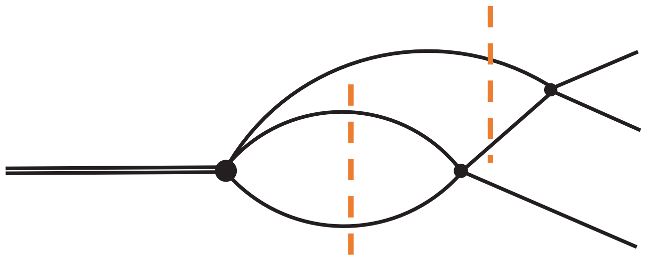
$$\lambda_\pi^{1/2}(s) = \lambda(s,m_\pi^2,m_\pi^2)$$

From unitarity to integral equations

- unitarity relation

$$\text{Disc}\,\hat{a}^{(I)}_{j\lambda}(s) = \rho(s)\tau^{(I)}_j(s)\left\{\underbrace{\hat{a}^{(I)}_{j\lambda}(s)}_{\text{right-hand cut}} + \underbrace{\tilde{a}^{(I)}_{j\lambda}(s)}_{\text{left-hand cut}}\right\}$$





- homogeneity contribution

$$\hat{a}^{(I)}_{j\lambda}(s) = P(s)\Omega(s) = P(s)\exp\left[\frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{\delta^I_j(s')}{s'(s'-s-i\varepsilon)}\,\mathrm{d}s'\right]$$

R. Omnès, Nuovo Cim. 8, 316 (1958)

Integral equations

A. V. Anisovich and H. Leutwyler, Phys. Lett. B 375, 335 (1996)

- unitarity relation

$$\text{Disc}\ \hat{a}_{j\lambda}^{(I)}(s) = \rho(s)\tau_j^{(I)}(s)\left\{\hat{a}_{j\lambda}^{(I)}(s) + \tilde{a}_{j\lambda}^{(I)}(s)\right\}$$

It does not determine the amplitude uniquely

$$\hat{a}_{j\lambda}^{(I)}(s) = P(s) + \frac{s^n}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^n}\frac{\sin\delta_j^I(s')e^{-i\delta_j^I(s')}}{(s'-s-i\varepsilon)}\left\{\hat{a}_{j\lambda}^{(I)}(s) + \tilde{a}_{j\lambda}^{(I)}(s)\right\}$$

$$\hat{a}_{j\lambda}^{(I)}(s) = \Omega_j^I(s)\left\{P(s) + \frac{s^n}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^n}\frac{\sin\delta_j^I(s')}{|\Omega_j^I|(s'-s-i\varepsilon)}\tilde{a}_{j\lambda}^{(I)}(s')\right\}$$

Singularities

- inhomogeneities $\tilde{a}^{(I)}_{j\lambda}(s)$

$$C_{II'} = \frac{1}{(2I + 1)} \Sigma_{a,b,c,c} P^{I}_{a,b,c,d} P^{I'}_{a,b,c,d}$$

$$d^{j}_{\lambda 0}(\theta) = (\sin\theta)^{|\lambda|} \hat{d}^{j}_{\lambda\, 0}(\theta)$$

$$\tilde{a}^{(I)}_{j\lambda}(s) = (-1)^{\lambda} \sum_{I',\lambda',j'} (2j' + 1)\, C_{II'} \int d\cos\theta'\ \hat{C}^{jj'}_{II'}(s,\theta')\ \hat{d}^{j}_{\lambda\, 0}(\theta')\hat{d}^{j'}_{\lambda' 0}(\theta'_t)\hat{a}^{(I')}_{j'\lambda'}(t')$$

$\cos\theta' \to s'$

$$\tilde{a}^{(I)}_{j\lambda}(s) = (-1)^{\lambda} \sum_{I',\lambda',j'} (2j' + 1)\, C_{II'}\ \underbrace{\frac{2s}{\lambda_M^{n/2}(s)\lambda_\pi^{n/2}(s)}}_{}\int d s'\ K(s,s')\ \hat{a}^{(I')}_{j'\lambda'}(s')$$

$n = 1,3,5$

singularities

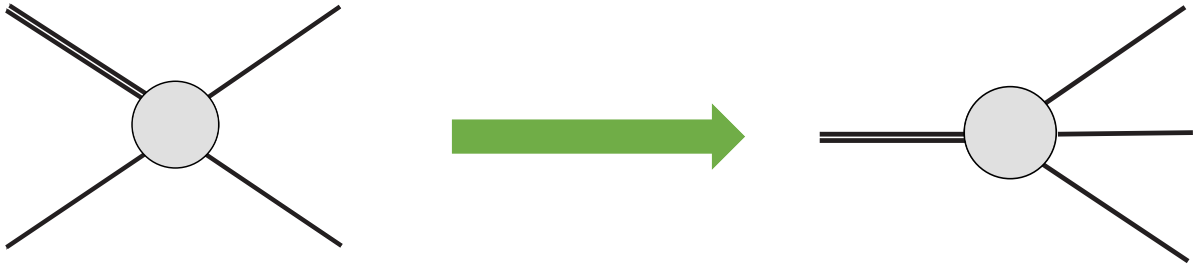
$$\hat{C}^{jj'}_{II'}(s,\theta) = (\sin^2\theta)^{|\lambda|} K^{-1}_{j\lambda}(s,\theta)\ d^{1}_{\lambda'\lambda}(\omega_t(s,\theta)) K_{j'\lambda'}(t(s,\theta),\theta_t(s,\theta))$$

$$K_{j\lambda}(s,\theta) = (s^{-1/2}\sin\theta)^{|\lambda|}\ (\lambda_X^{1/2}(s))^{|j-1|}(\lambda_\pi^{1/2}(s))^{j}$$

$$s' = (M^2 + 3m_\pi^2 - s)/2\ +\ \lambda_M^{1/2}(s)\lambda_\pi^{1/2}(s)\,/(2s)\ \cos\theta'$$

Decay region

- continuation into the decay region

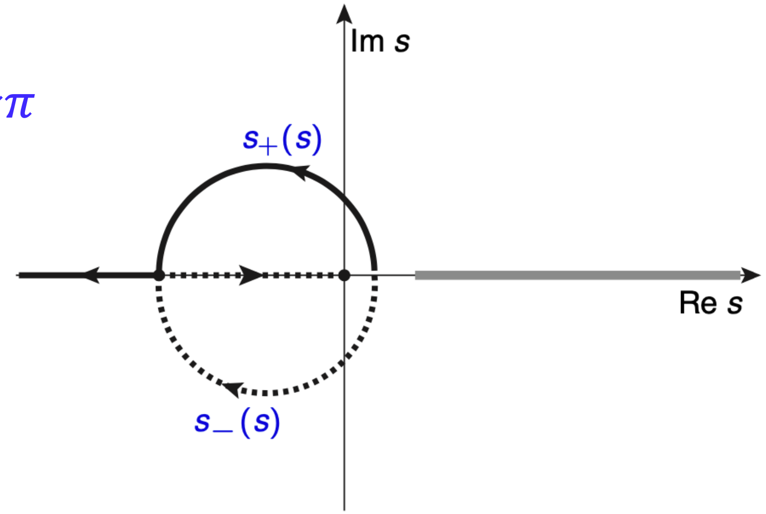


- $\sigma \to \sigma + i\,\delta$

J. B. Bronzan and C. Kacser, Phys. Rev. 132, 2703 (1963)

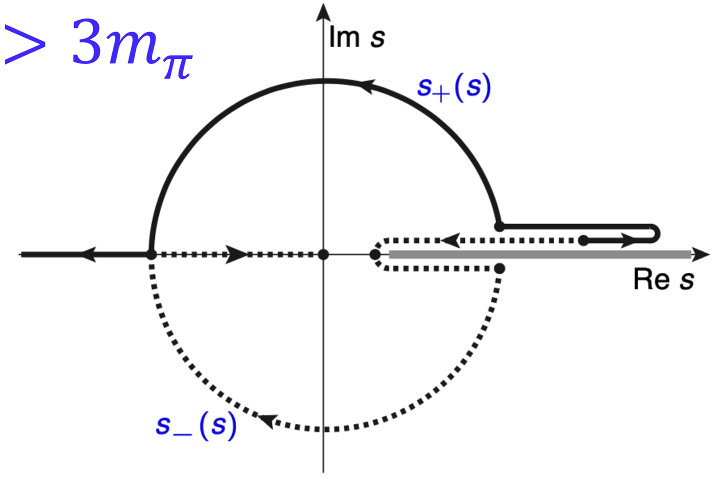
- path deformation needed

$\sigma < 3m_\pi$



Integration does not run over the cut

$\sigma > 3m_\pi$



Integration does run over the cut

Amplitude structure

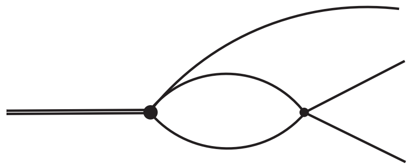
$a_1^- \to \pi^-(p_1)\,\pi^-(p_2)\,\pi^+(p_3)$

- Decay amplitude $A_\lambda\,(s,t,u)$

Only $I=1,\ J=1$ $\pi\pi$ scattering constrained

$$A_\lambda\,(s,t,u) = \bar{A}_\lambda\,(s,t,u) + \Sigma_{\lambda'}(-1)^{\lambda'}\, d^J_{\lambda'\lambda}(\omega_t)\bar{A}_\lambda\,(t,s,u)$$

$$\bar{A}_\lambda\,(s,t,u) = \Sigma_j\,(2j+1)d^j_{\lambda 0}(\theta_s)a_{j\lambda}(s) \qquad a_{j\lambda}(s) = \frac{(\lambda_X^{1/2}(s))^{j-1}(\lambda_\pi^{1/2}(s))^j}{s^{|\lambda|/2}}\,\hat{a}_{j\lambda}(s)$$



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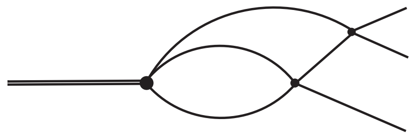
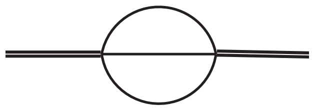


$$\hat{a}_{j\lambda}(s) = \Omega(s)\left\{P(s) + \frac{s^n}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^n}\,\frac{\sin\delta_j^I\,(s')}{|\Omega|(s' - s - i\,\varepsilon)}\,\tilde{a}_{j\lambda}^{(I)}(s')\right\}$$

- Self-energy $\Sigma(\sigma)$

+



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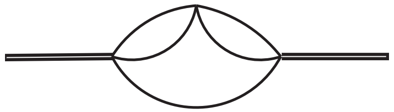


$$\mathrm{Im}\Sigma(\sigma) \sim \int \Sigma_\lambda |A_\lambda(s,t,u)|^2\,ds dt \qquad \Sigma(\sigma) \sim \frac{\sigma}{\pi}\int_{9m_\pi^2}^{\infty}\frac{\mathrm{Im}\Sigma(\sigma')}{\sigma'(\sigma' - \sigma - i\varepsilon)}\,d\sigma$$

Real s,t,u and σ

$$\mathrm{Im}\Sigma(\sigma) \sim \int \Sigma_\lambda A_\lambda(s,t,u) * A_\lambda^{(\mathrm{unphy})}\,(s,t,u)\,ds dt$$

Complex s,t,u and σ

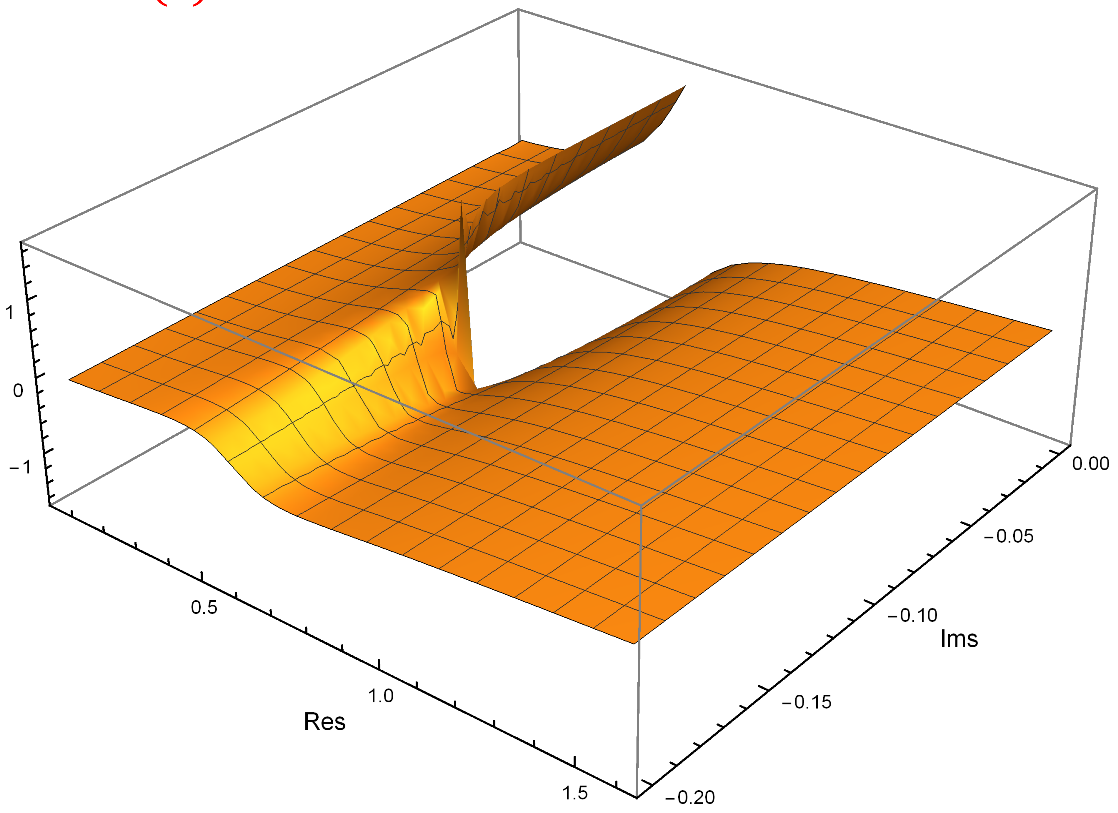
- Analytic structure of $\delta^{I=1}(s)$

$$\mathrm{Tan}\delta^{I=1}(s) = \sqrt{\frac{s-4m_\pi^2}{s}}\left(\frac{s-4m_\pi^2}{4m_\pi^2}\right)\left(a + b\left(\frac{s-4m_\pi^2}{4m_\pi^2}\right)\right)$$

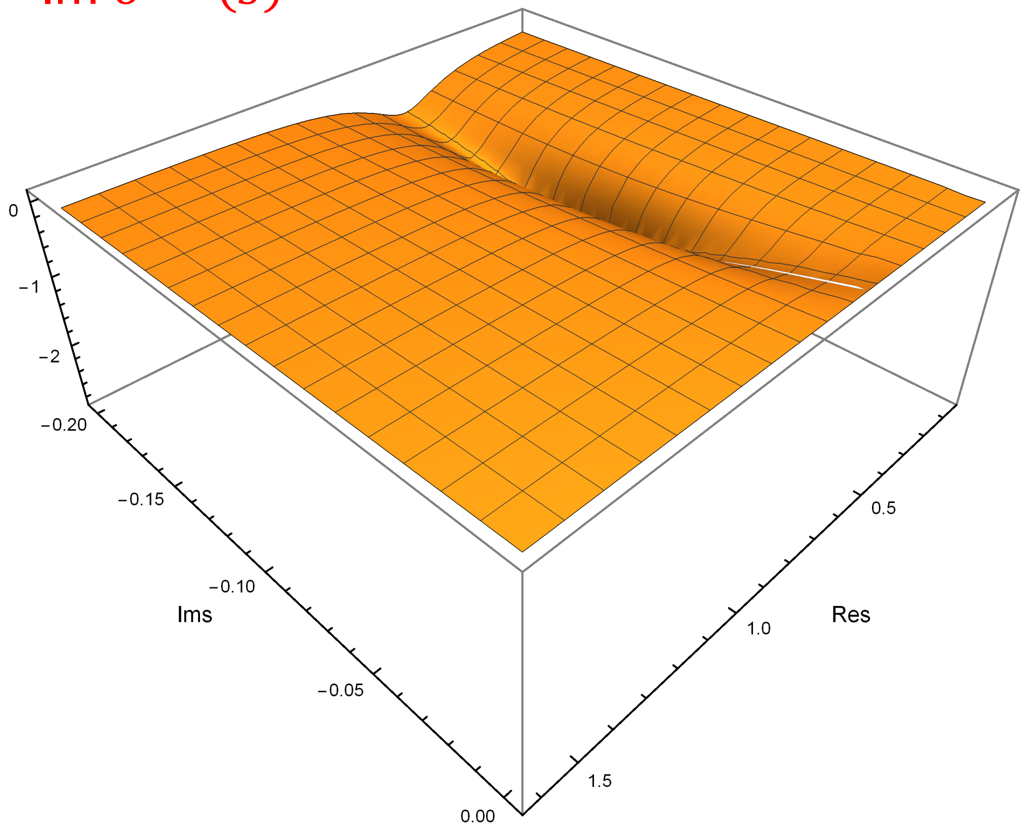
A. Schenk, Nucl.Phys.B 363, 97 (1991)
G. Colangelo, J. Gasser and H. Leutwyler, Nucl.Phys.B 603, 125 (2001)

$$\delta^{I=1}(s) = \frac{1}{2i}\log\frac{1+i\,\mathrm{Tan}\delta^{I=1}(s)}{1-i\,\mathrm{Tan}\delta^{I=1}(s)}$$

Re $\delta^{I=1}(s)$



Im $\delta^{I=1}(s)$



Branch point: pole of ρ meson

- Analytic structure of omnès function $\Omega(s)$

First Riemann sheet $\Omega(s) = \exp[\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\delta_j^I(s')}{s'(s'-s-i\varepsilon)} \mathrm{d}s']$

Second Riemann sheet $\Omega^{(\mathrm{II})}(s) = \exp[\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\delta_j^I(s')}{s'(s'-s-i\varepsilon)} \mathrm{d}s' + i\delta^{I=1}(s)]$

- Analytic structure of singularities free amplitude $\hat{a}_{j\lambda}(s)$

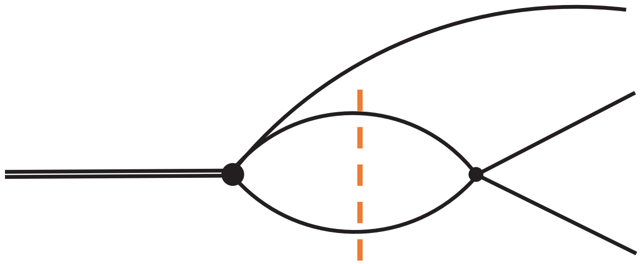
First Riemann sheet

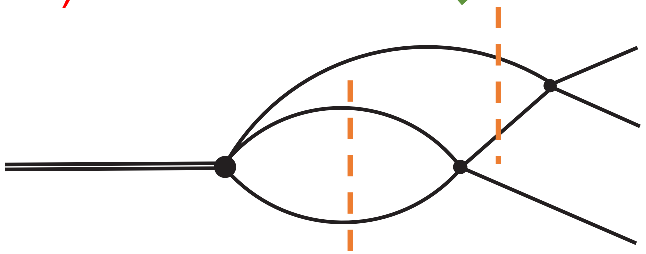
$$\hat{a}_{j\lambda}(s) = \Omega(s)\left\{P(s) + \frac{s^n}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^n}\frac{e^{-i\delta^{I=1}(s)}\sin\delta_j^I(s')}{\Omega(s')(s'-s-i\varepsilon)}\tilde{a}_{j\lambda}^{(I)}(s')\right\}$$

Across two-body cut

$$\hat{a}_{j\lambda}^{(II)}(s) = \Omega^{(\mathrm{II})}(s)\left\{P(s) + \frac{s^n}{\pi}\int_{C}\frac{ds'}{s'^n}\frac{e^{-i\delta^{I=1}(s)}\sin\delta_j^I(s')}{\Omega(s')(s'-s-i\varepsilon)}\tilde{a}_{j\lambda}^{(I)}(s')\right\}$$

Contour deformation (Pick up a residue)





- Three-body and complex two-body cut

$$\mathrm{Im}\Sigma(\sigma) \sim \int \Sigma_\lambda A_\lambda(s,t,u) * A_\lambda^{(\mathrm{unphy})}(s,t,u)\, ds dt \qquad \Sigma(\sigma) \sim \frac{\sigma}{\pi}\int_{9m_\pi^2}^{\infty} \frac{\mathrm{Im}\Sigma(\sigma')}{\sigma'(\sigma' - \sigma - i\varepsilon)} d\sigma$$

Complex s, t, u and σ

$$\Omega^{(\mathrm{II})}(s) = \exp[\frac{s}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\delta_j^I(s')}{s'(s' - s - i\varepsilon)} ds' + i\delta^{I=1}(s)]$$

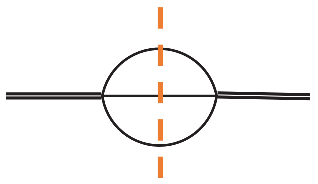
Branch point

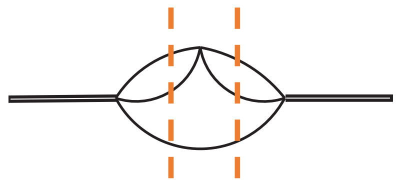
Zero

$$\hat{a}_{j\lambda}^{(II)}(s) = \Omega^{(\mathrm{II})}(s)\left\{P(s) + \frac{s^n}{\pi}\int_C \frac{ds'}{s'^n} \frac{e^{-i\delta^{I=1}(s)} \sin\delta_j^I(s')}{\Omega(s')(s' - s - i\,\varepsilon)} \tilde{a}_{j\lambda}^{(I)}(s')\right\}$$

Zero

$s \in [2m_\pi^2, (\sqrt{\sigma} - m_\pi)^2]$





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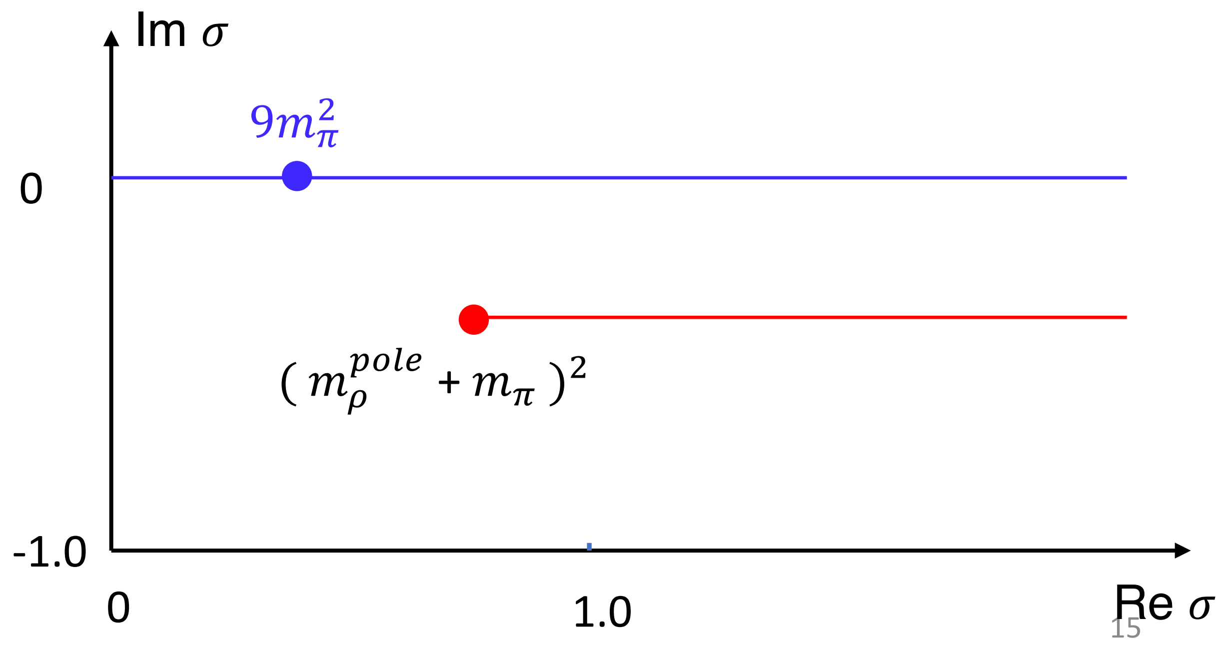


Phase-shift input

R. García-Martín et al., Phys. Rev. D 83, 074004 (2011)

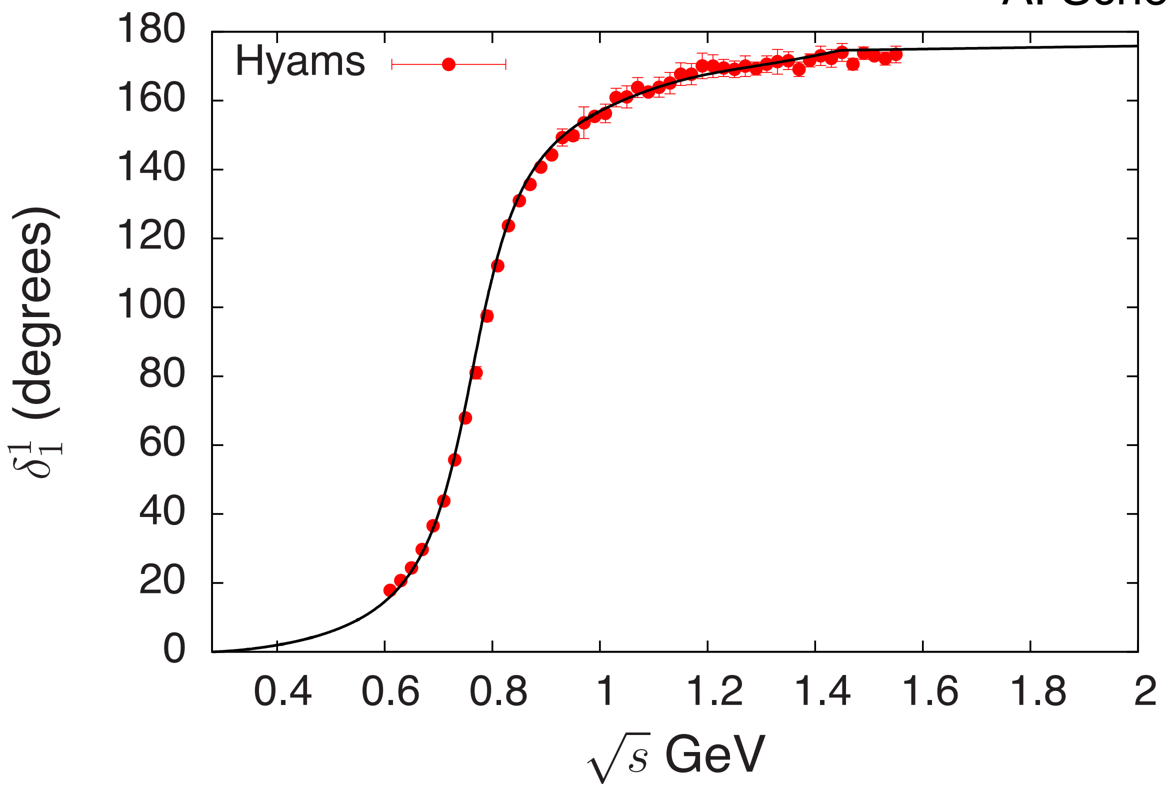
P-wave, $I = 1$

$\pi\pi$ scattering constrained by analyticity and unitarity

Roy equations = partial-wave dispersion relations + crossing symmetry + unitarity

Schenk parameterization

A. Schenk, Nucl.Phys.B 363, 97 (1991)

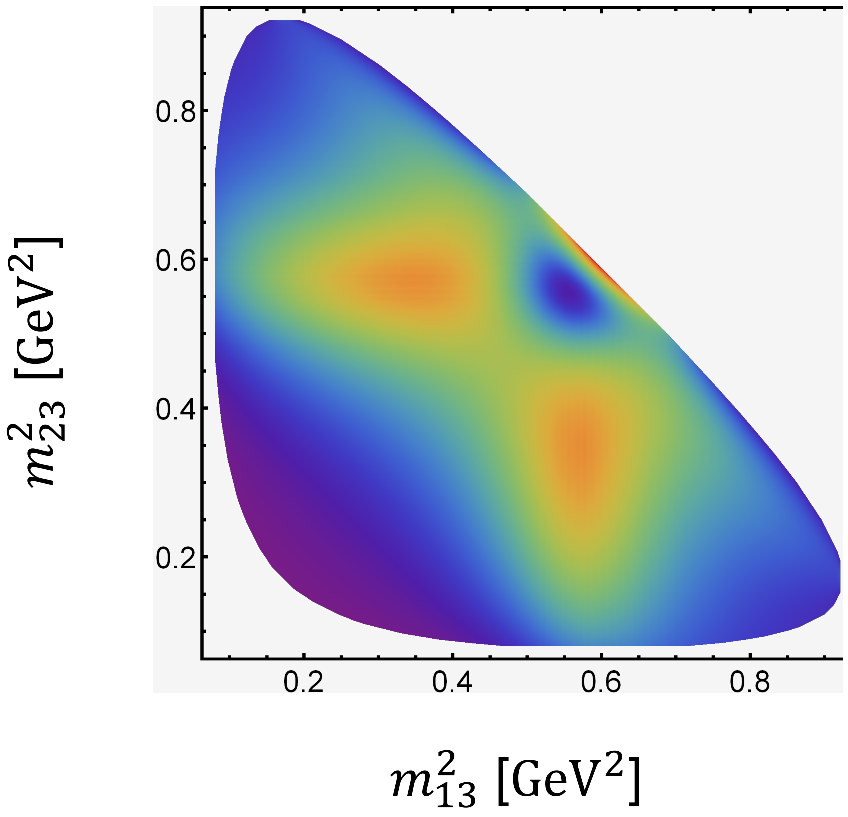


B. Hyams et al., Nucl.Phys.B 64, 134 (1973)

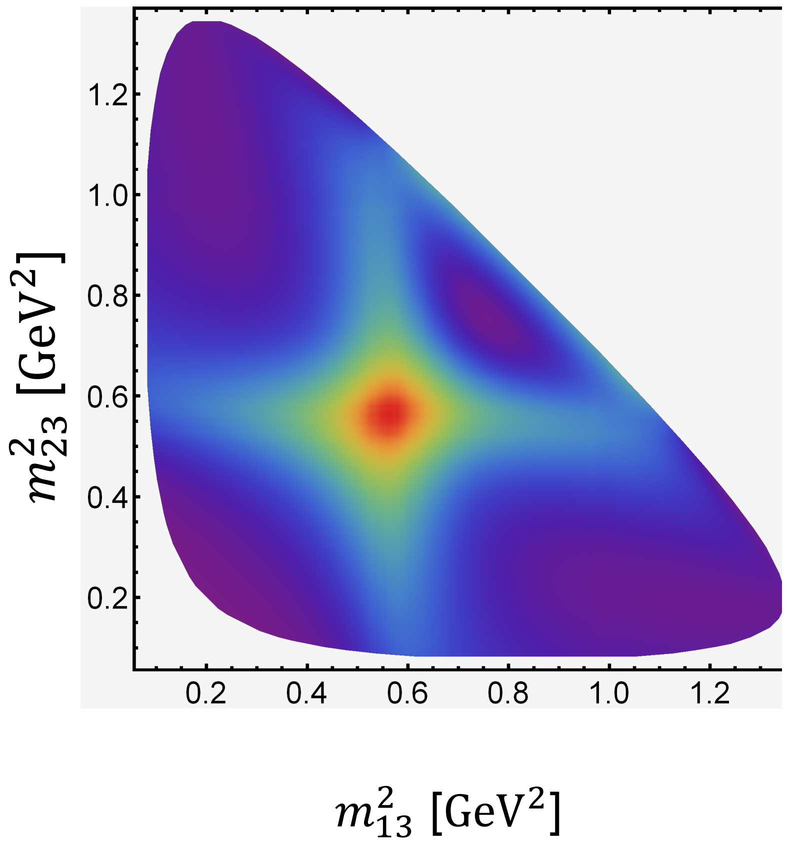
Numerical results

Dalitz plots

$\boldsymbol{a_1^- \to \pi^-(p_1)\,\pi^-(p_2)\,\pi^+(p_3)}$



$M = 1.1$ GeV



$M = 1.3$ GeV

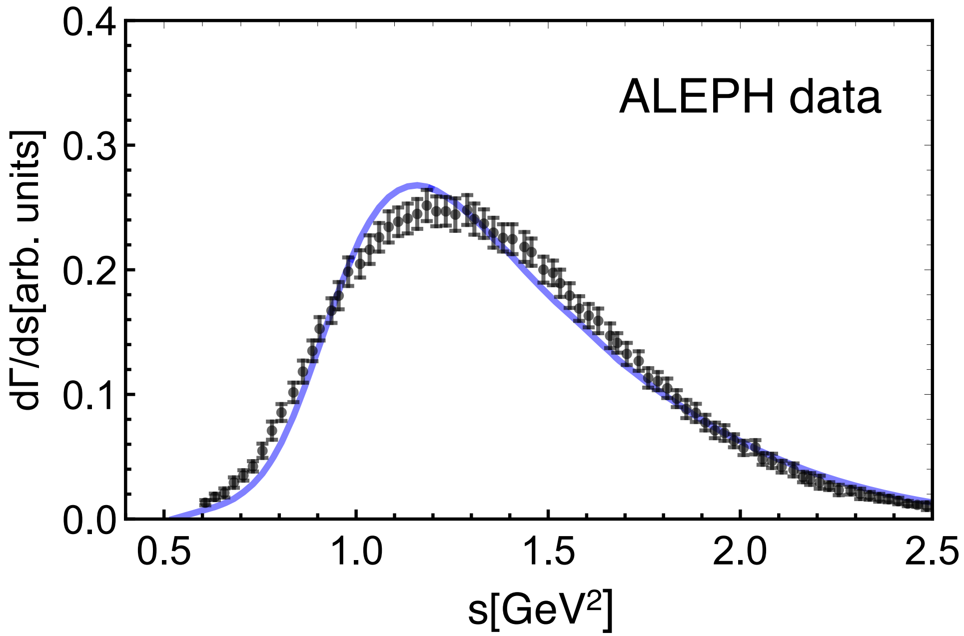
Dalitz plot shows ρ-resonance bands

Numerical results

Fit A

$$\frac{d\Gamma}{d\sigma} = \text{cons.}\,(m_\tau^2 - \sigma)^2 \sum_\lambda \left| \frac{A_\lambda(\sigma)}{\sigma - m_{a_1}^2 + i\mathrm{Im}\Sigma(\sigma)} \right|^2 d\Phi^3$$

$$\mathrm{Im}\Sigma(\sigma) = \frac{1}{2}\frac{1}{3}\int \Sigma_\lambda |A_\lambda(s,t,u)|^2 \; d\Phi^3$$



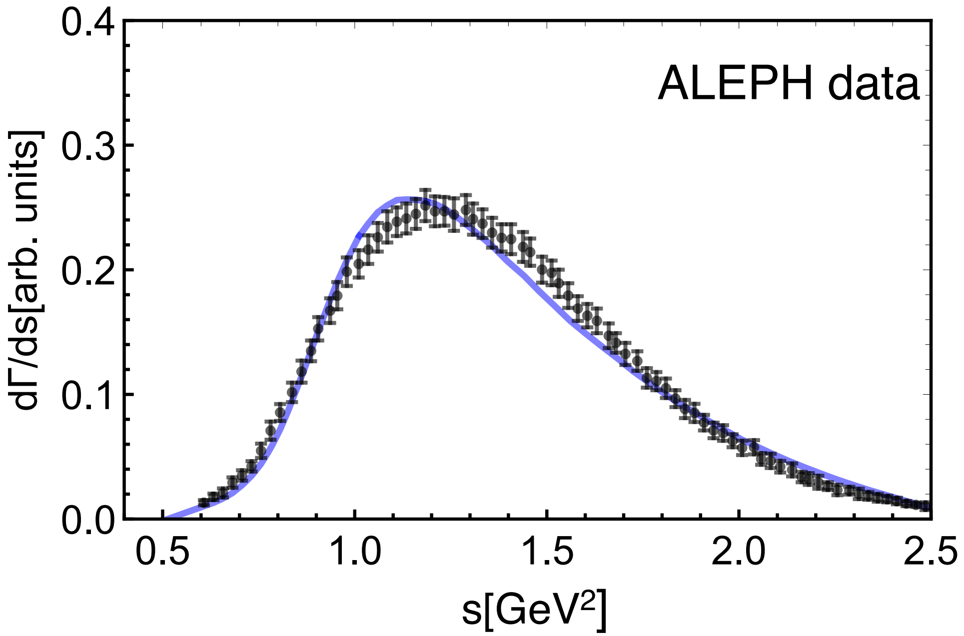
χ^2/d.o.f =220.7/(77-3)

m_{a_1}= 1.253 GeV

Fit B

$$\frac{d\Gamma}{d\sigma} = \text{cons.}\,(m_\tau^2 - \sigma)^2 \sum_\lambda \left| \frac{A_\lambda(\sigma)}{S - m_{a_1}^{(0)2} + \Sigma(\sigma)} \right|^2 d\Phi^3$$

$$\Sigma(\sigma) = \frac{\sigma}{\pi}\int_{9m_\pi^2}^{\infty} \frac{\mathrm{Im}\Sigma(\sigma')}{\sigma'(\sigma' - \sigma - i\varepsilon)} d\sigma'$$



χ^2/d.o.f =143.3/(77-3)

$m_{a_1}^{(0)}$= 3.24 GeV

Summary

- Only P-wave $\pi\pi$ scattering is included
- The 3π mass distribution was fitted
- The pole position will be explored
- $a_1(1420)$ triangle singularity in $a_1(1260) \to \bar{K}K^* \to f_0\pi \to 3\pi$ will be studied

Thank you !