

两体长程相互作用的有限体积量子化条件



Modified Lüscher formula for long-range interaction

强子物理论坛 2025年10月31日

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Based on [\[JHEP 05 \(2024\) 168\]](#) [\[arXiv: 2507.18399\]](#)

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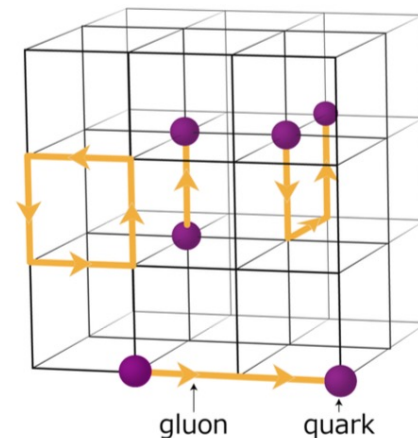
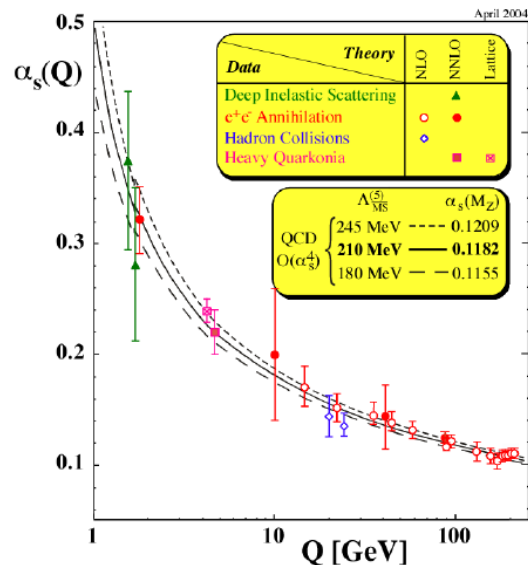
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01 Motivation

Lattice QCD

- Lattice QCD – important un-perturbative method
- Discretization of Euclidean space-time
 - Lattice spacing – continue limit
 - Lattice size – finite volume analysis
 - Pion (quark) mass – chiral limit extrapolation
 - Euclidean space-time – imaginary time
- Path integral
 - Correlation function on lattice
 - Match to amplitude in real world



$$\langle \mathcal{O}(x) \mathcal{O}^\dagger(0) \rangle = \frac{1}{Z} \int \mathcal{D}\bar{q} \mathcal{D}q \mathcal{D}A \mathcal{O}(x) \mathcal{O}^\dagger(0) e^{-S[\bar{q}, q, A]} \rightarrow \text{M.C. integral}$$

$$\langle k_1, k_2; \text{out} | j(0) | 0 \rangle = \Phi(s, P_n, L) \langle P_n | j(0) | 0 \rangle$$

Decay matrix element

Finite volume correction
[arXiv:1205.4642]

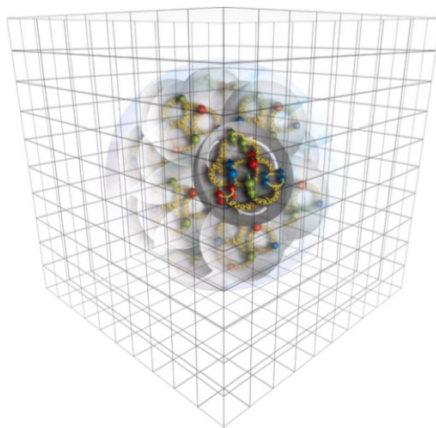
Finite volume matrix element

01 Motivation

Lattice QCD in hadron spectroscopy

➤ Finite volume energy levels

- Build up operator for hadron system
- Extract out discrete energy levels in a finite volume
- Lattice spectrum (finite volume spectrum)



$$\langle \mathcal{O}(\tau) \mathcal{O}^\dagger(0) \rangle = \sum_n c_n e^{-E_n \tau}$$

Operator for hadron system

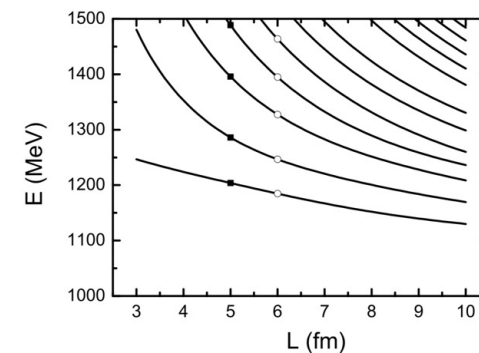
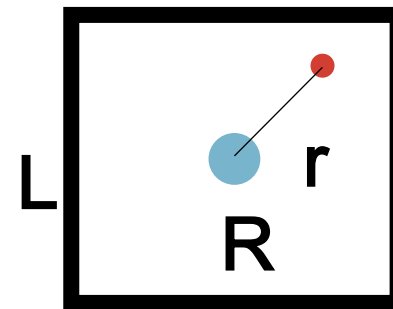
Finite volume energy level

➤ One-hadron system

- Stable hadron: inf. vol. limit of energy level $\rightarrow$ hadron mass
- Finite volume correction is **exponentially suppressed**

➤ Two-hadron system

- Bound states: inf. vol. limit of energy level $\rightarrow$ binding energy B (Finite volume correction $e^{-\kappa L}$)
- Scattering states: inf. vol. limit of energy level is **NOTHING** but threshold (No-go theorem)
- Asymptotic state touches the boundary $\rightarrow$ Finite volume correction is power-law $\rightarrow$ Finite volume analysis is relevant.



01 Motivation

Lüscher formula

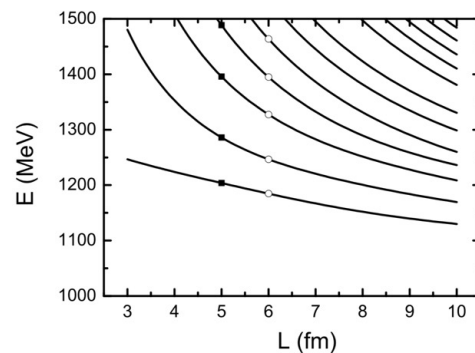
➤ Lüscher formula [M. Lüscher, NPB 354 (1991) 531]

- Energy level → phase shift
- Interaction is **local**: phase shift in the infinite volume and energy level in a finite volume are generated by **the same potential**.
- ρ meson in $\pi\pi$ scattering

➤ EFT and LQCD

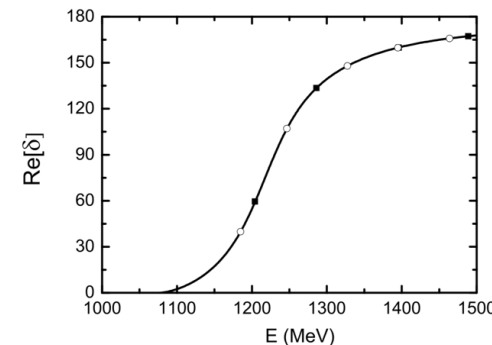
- In low energy region
- LQCD as finite volume QCD = EFT in a finite volume
- Lattice spectrum → interaction at hadron level (LEC)
- Quantization condition: the equation that determines energy levels in a finite volume via dynamical interaction in the infinite volume.
- Two-body EFT → model-indept. quantization condition

$$\det \left(\frac{q_0^{2\ell+1} \cot \delta_\ell}{4\pi} \delta_{\ell m, \ell' m'} - Z_{\ell m, \ell' m'}(q_0^2) \right) = 0 \quad \text{Lüscher zeta function} \quad Z_{\ell m, \ell' m'}(q_0^2) = \frac{4\pi}{L^3} \sum_{\mathbf{p}} \frac{\mathcal{Y}_{\ell m}^*(\mathbf{p}) \mathcal{Y}_{\ell' m'}(\mathbf{p})}{p^2 - q_0^2}$$

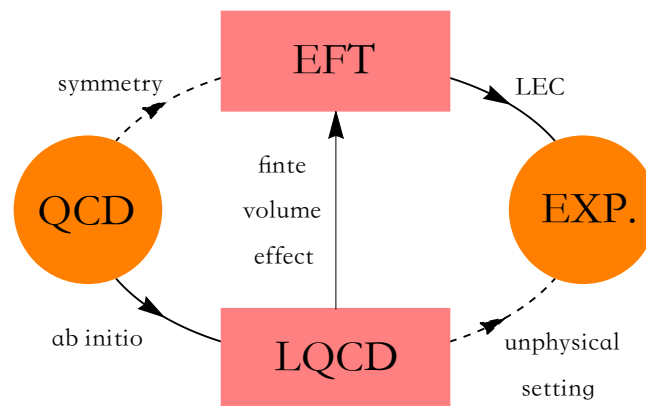


$$q_0 \cot \delta_0 = \frac{2}{\sqrt{\pi} L} Z_{00}(\eta^2)$$

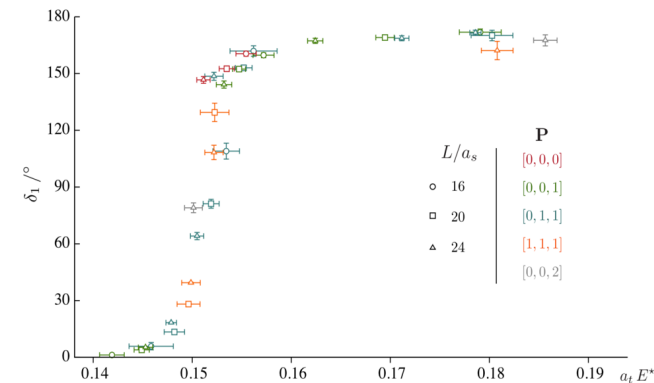
$$\sqrt{mE} = q_0 = \frac{2\pi}{L} \eta$$



[J.-J. Wu et al., PRD 95,114507(2017)]



[Hammer, JYP and Rusetsky, 2017]



[arXiv:1809.07350]

01 Motivation

Finite volume LS equation

➤ LS equation

- Solve inf. vol. LS equation → match to on-shell amplitude
- Solve fin. vol. LS equation with **the same short-range kernel**
- Pole position of fin. vol. amplitude = Quantization condition

➤ Hamiltonian EFT [J.-J. Wu et al.]

- Formulate the interaction
- Hamiltonian in plane-wave basis
- Cubic projection

Infinite volume

$$T_\ell(p, q; q_0^2) = V_\ell(p, q) + \int \frac{d^3 k k^{2\ell}}{(2\pi)^3} \frac{V_\ell(p, k) T_\ell(k, q; q_0^2)}{k^2 - q_0^2 - i\epsilon}$$

$$T_\ell(q_0^2) = \frac{1}{R_\ell^{-1}(q_0^2) - \langle G_\ell(q_0^2) \rangle} \xrightarrow{\text{match to}} \frac{4\pi}{q_0^{2\ell+1} \cot \delta_\ell - i q_0^{2\ell+1}}$$

$$R_\ell^{-1}(q_0^2) \xrightarrow{\text{renormalization}} \frac{q_0^{2\ell+1} \cot \delta_\ell}{4\pi}$$

Finite volume

$$T_{\ell m, \ell' m'}^{(L)}(p, q; q_0^2) = V_\ell(p, q) + \frac{4\pi}{L^3} \sum_{j\nu} \sum_{\mathbf{k}} \frac{V_\ell(p, k) \mathcal{Y}_{\ell m}^*(\mathbf{k}) \mathcal{Y}_{j\nu}(\mathbf{k}) T_{j\nu, \ell' m'}^{(L)}(k, q; q_0^2)}{k^2 - q_0^2}$$

$$\left[T^{(L)} \right]^{-1} = R^{-1} - Z \xrightarrow{\text{renormalization}} \frac{q_0^{2\ell+1} \cot \delta_\ell}{4\pi} \delta_{\ell m, \ell' m'} - Z_{\ell m, \ell' m'}(q_0^2)$$

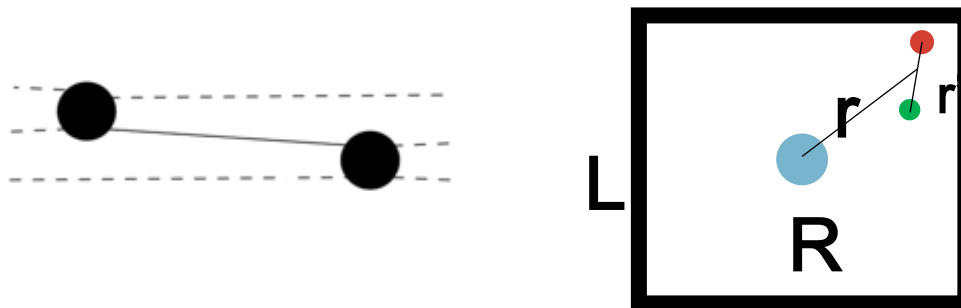
$$\det \left(\frac{q_0^{2\ell+1} \cot \delta_\ell}{4\pi} \delta_{\ell m, \ell' m'} - Z_{\ell m, \ell' m'}(q_0^2) \right) = 0$$

01 Motivation

Finite volume analysis of 3-body

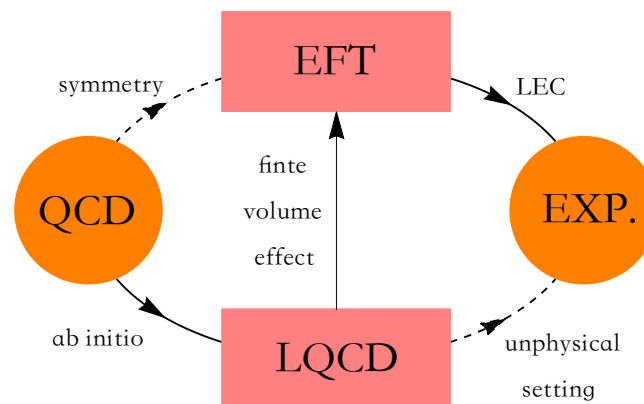
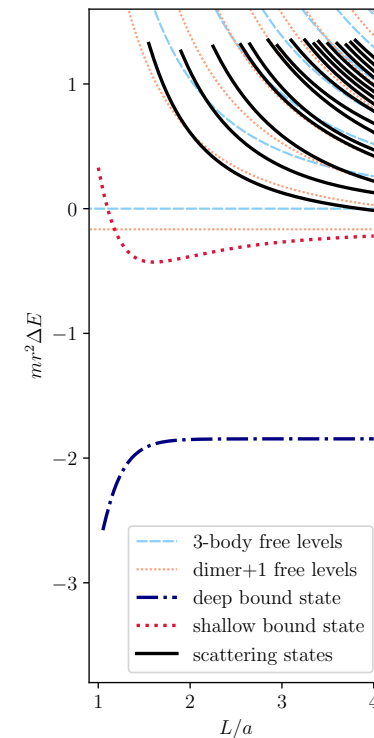
➤ Three-hadron system

- Physical singularity – 3-body cut
- Complicate kinematics
- Parameterization of on-shell amplitude

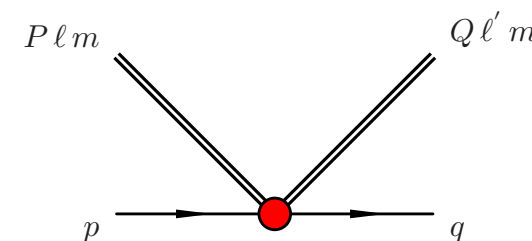


➤ Three-body quantization condition

- Energy level $\rightarrow$ quantization condition $\rightarrow$ 3-body force
- Relativistic Field Theory
[Hansen, Sharpe \[PRD90,116003,2014\], \[PRD92,114509,2015\]](#)
- Non-Relativistic Effect Field Theory
[Hammer, JYP, Rusetsky \[JHEP09, 109,2017\], \[JHEP10,115,2017\]](#)
- Finite Volume Unitarity
[Mai, Döring \[EPJA53, 240, 2017\], \[PRL, 122, 062503, 2019\]](#)



[Hammer, JYP and Rusetsky, 2017]



01 Motivation

Inclusion of the long-range forces

➤ Interpretation of $T_{cc}^+(3875)$

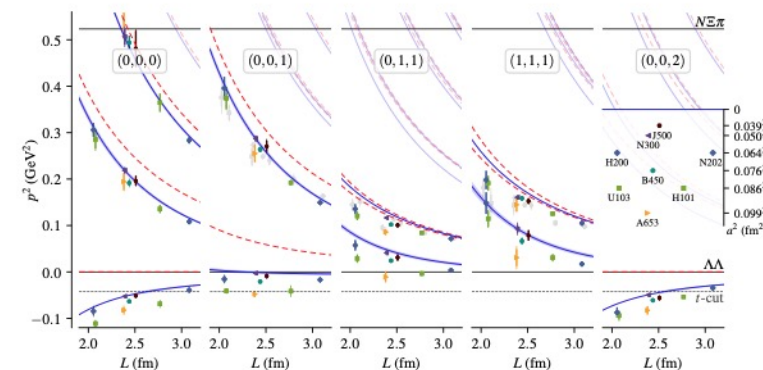
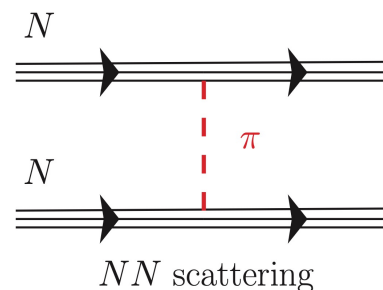
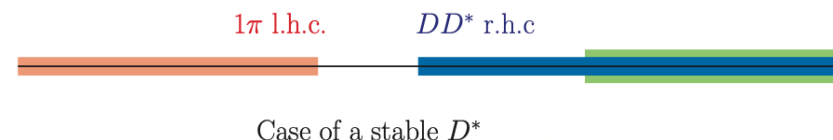
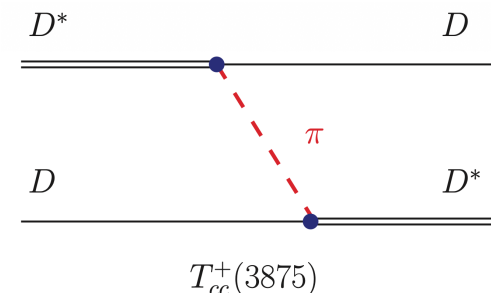
- Left-hand cut extremely close to threshold
- Affects the extraction of the phase shifts and the parameters of the $T_{cc}^+(3875)$ from lattice data

➤ NN scattering

- Left-hand cut: $-\infty < s \leq \underbrace{(2m_N)^2 - M_\pi^2}_{(1875\text{MeV})^2}$ right-hand cut: $\underbrace{(2m_N)^2}_{(1880\text{MeV})^2} \leq s < +\infty$
- Phase shift **real** below left-handed branch point?

➤ Finite volume problem of long-range force

- Left-hand cut close to threshold: Energy levels below the left-handed branch point cannot be used
- Slowly converging partial-wave expansion: Expecting strong admixture of higher partial-waves in the quantization condition
- Exponentially suppressed corrections still sizable



J.R. Green *et al.*, PRL 127(24) (2021) 242003

01 Motivation

Schemes to including long-range force

➤ Plane-wave basis

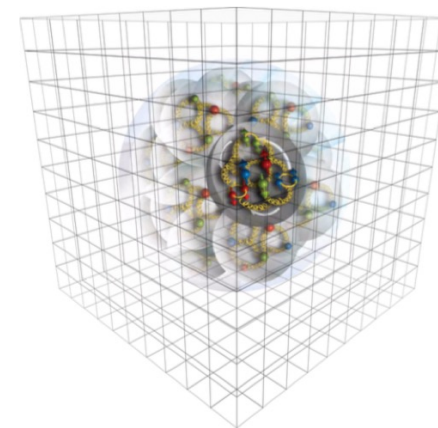
- Describe the system in terms of the parameters of the effective Lagrangian
- Work in the plane-wave basis
- For the NN scattering, it was shown that, at the physical quark masses, the partial-wave mixing is sizable [Meng & Epelbaum, 2021]
- A consistent fit of the DD^* scattering phases to lattice data in the left-hand cut region has been performed [Meng et al. 2023]

➤ Alternative approaches

- Splitting long- and short-range interactions [Hansen & Raposo, 2023]
- Using Lüscher equation plus EFT with long-range force in the infinite volume [Collins et al., 2024]
- Other contributions [Lyu et al., 2023] [Du, Guo & Wu, 2024] [Dawid et al., 2025] etc.

➤ Our scheme

- Modified Lüscher formula [Bubna, Hammer, Müller, JYP, Rusetsky & Jia-Jun Wu, JHEP 05 (2024) 168]
[Bubna, Hammer, Boid, JYP, Rusetsky & Jia-Jun Wu, arXiv:2507.18399]
- Finite volume version of the modified effective range expansion [van Haeringen & Kok, 1982]



$$V(r) = \underbrace{V_L(r)}_{\text{known, local}} + \underbrace{V_S(r)}_{\text{unknown}}$$

$$K_\ell^M(q^2) = M_\ell(q) + \frac{q^{2\ell+1}}{|f_\ell(q)|^2} (\cot(\delta_\ell(q) - \sigma_\ell(q)) - i) = -\frac{1}{\tilde{a}_\ell} + \frac{1}{2} \tilde{r}_\ell q^2 + O(q^4)$$

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02 Modified Lüscher formula (infinite volume part)

Decomposition of interaction

➤ Decomposition of potential

$$V = V_L + V_S$$

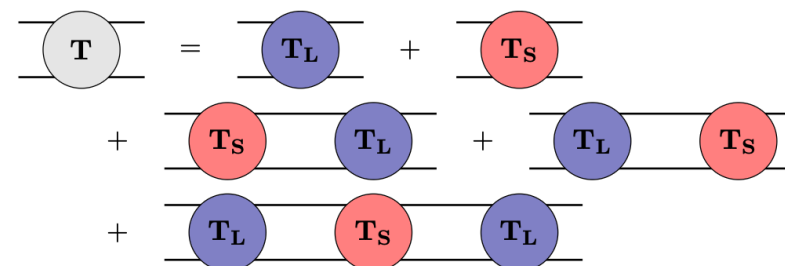
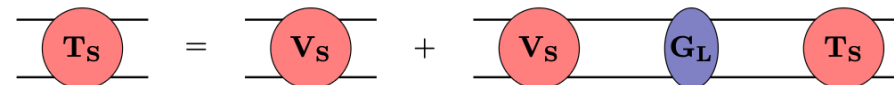
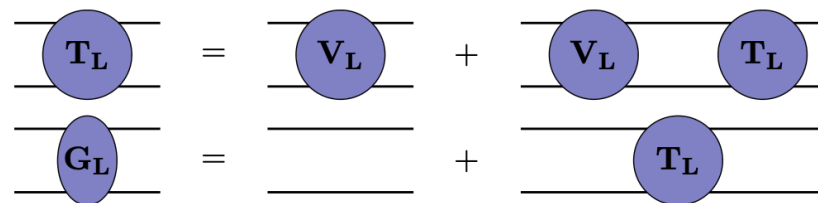
- V_L : long-range potential which is known from, e.g., χ PT
- V_S : short-range potential which is unknown, but **local**

➤ Amplitudes and full propagators

- T_L : long-range amplitude which is established only by V_L
- G_L : long-range propagator
- T_S : effective short-range amplitude which is established by both long- and short-range interaction

➤ Decomposition of T -matrix

$$T = T_L + (1 + T_L G) T_S (G T_L + 1)$$



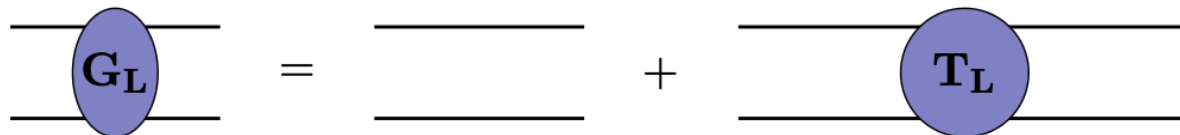
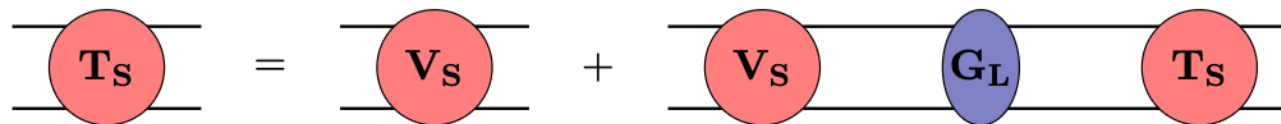
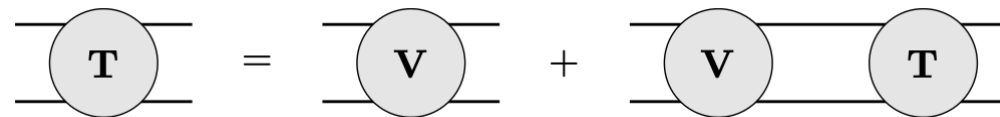
02 Modified Lüscher formula (infinite volume part)

Pole structure

➤ Pole position

$$T = T_L + (1 + T_L G) T_S (G T_L + 1)$$

- Poles of T are equivalent to T_S
- Free energy (pole of G)
- Long-range eigen-energy (pole of T_L)
- Eigen-energy (pole of T)



➤ Effective short-range LS equation

$$T_S = V_S + V_S G_L T_S$$

- Solve effective equation for T_S instead of total LS equation for T

$$G_L = G + G T_L G$$

02 Modified Lüscher formula (infinite volume part)

Effective short-range amplitude

➤ Effective short-range LS equation

$$T_S = V_S + V_S G_L T_S$$

➤ Effective short-range amplitude

$$\langle \psi^{(+)}(q_0^2), \ell m | T_S | \psi^{(+)}(q_0^2), \ell m \rangle = \frac{1}{q_0 \cot(\delta_\ell - \sigma_\ell) - i q_0}.$$

- $|\psi^{(+)}(q_0^2), \ell m\rangle$: Partial-wave long-range basis
- δ_ℓ : phase shift of total interaction (need to determine)
- σ_ℓ : phase shift of long interaction (known)

$$\langle q_0, \ell m | \Omega_L^{(-)\dagger} T_S \Omega_L^{(+)} | q_0, \ell m \rangle = \langle \psi^{(-)}(q_0), \ell m | T_S | \psi^{(+)}(q_0), \ell m \rangle = \frac{e^{2i\sigma_\ell}}{q_0 \cot \tilde{\delta}_\ell - i q_0}.$$

$$T = T_L + \Omega_L^{(-)\dagger} T_S \Omega_L^{(+)}.$$

$$\tilde{\delta}_\ell = \delta_\ell - \sigma_\ell.$$

$$\begin{aligned} \langle q_0, \ell m | T | q_0, \ell m \rangle &= \frac{1}{q_0 \cot \delta_\ell - i q_0}, \\ \langle q_0, \ell m | T_L | q_0, \ell m \rangle &= \frac{1}{q_0 \cot \sigma_\ell - i q_0}. \end{aligned}$$

02 Modified Lüscher formula (infinite volume part)

Effective short-range LS equation

➤ Effective LS equation in long-range bases

$$T_S = V_S + V_S G_L T_S$$

$$B(\mathbf{p}, \mathbf{q}; q_0^2 + i\epsilon) = \tilde{V}_S(\mathbf{p}, \mathbf{q}) + \int \frac{d^3 k}{(2\pi)^3} \tilde{V}_S(\mathbf{p}, \mathbf{k}) \frac{1}{k^2 - q_0^2 - i\epsilon} B(\mathbf{k}, \mathbf{q}; q_0^2 + i\epsilon)$$

- Effective amplitude $B(\mathbf{p}, \mathbf{q}; q_0^2 + i\epsilon) = \langle \psi_{\mathbf{p}}^{(+)} | T_S(q_0^2 + i\epsilon) | \psi_{\mathbf{q}}^{(+)} \rangle$

- Effective short-range potential $\tilde{V}_S(\mathbf{p}, \mathbf{q}) = \langle \psi_{\mathbf{p}}^{(+)} | V_S | \psi_{\mathbf{q}}^{(+)} \rangle$

- Long-range propagator

Long-range propagator's spectral representation $G_L(q_0^2 + i\epsilon) = \int \frac{d^3 p}{(2\pi)^3} \frac{|\psi_{\mathbf{p}}^{(\pm)}\rangle \langle \psi_{\mathbf{p}}^{(\pm)}|}{p^2 - q_0^2 - i\epsilon} \longrightarrow \langle \psi_{\mathbf{p}}^{(\pm)} | G_L(q_0^2 + i\epsilon) | \psi_{\mathbf{k}}^{(\pm)} \rangle = \frac{(2\pi)^3 \delta(\mathbf{p} - \mathbf{k})}{k^2 - q_0^2 - i\epsilon}$

02 Modified Lüscher formula (infinite volume part)

Separation of effective short-range potential

$$B(\mathbf{p}, \mathbf{q}; q_0^2 + i\epsilon) = \tilde{V}_S(\mathbf{p}, \mathbf{q}) + \int \frac{d^3k}{(2\pi)^3} \tilde{V}_S(\mathbf{p}, \mathbf{k}) \frac{1}{k^2 - q_0^2 - i\epsilon} B(\mathbf{k}, \mathbf{q}; q_0^2 + i\epsilon)$$

➤ Effective short-range potential in long-range bases

$$\tilde{V}_S(\mathbf{p}, \mathbf{q}) = \langle \psi_{\mathbf{p}}^{(+)} | V_S | \psi_{\mathbf{q}}^{(+)} \rangle$$

Low-energy constants

- Polynomial representation of local potential

$$\langle \mathbf{p} | V_S | \mathbf{q} \rangle = 4\pi \sum_{\ell m} \mathcal{Y}_{\ell m}(\mathbf{p}) \left[\sum_{a,b} C_{\ell}^{ab} (p^2)^a (q^2)^b \right] \mathcal{Y}_{\ell m}^*(\mathbf{q})$$

- Potential in long-range bases

- Separation of long- and short-range interaction

$$\tilde{V}_S^{(\ell)}(k, k') = \sum_{a,b} C_{\ell}^{ab} \underbrace{(A_{\ell}^a(k))^*}_{\text{UV integral of long-range wave-function}} \underbrace{A_{\ell}^b(k')}_{\text{UV integral of long-range wave-function}}$$

$$A_{\ell}^a(k) = \int \frac{d^3p}{(2\pi)^3} (p^2)^a p^{\ell} \psi_{\ell}^{(+)}(p; k)$$

$$\tilde{V}_S^{(\ell)}(p, q) = \underbrace{\frac{(pq)^{\ell}}{[(2\ell + 1)!!]^2 f_{\ell}^*(p) f_{\ell}(q)}}_{\text{Long-range contribution in effective potential}} \underbrace{\bar{V}_S^{(\ell)}(p, q)}_{\text{Short-range kernel}}$$

Short-range kernel

Long-range contribution in effective potential

$$\bar{V}_S^{(\ell)}(p, q) = \sum_{a,b} C_{\ell}^{ab} \mathfrak{p}_a(p^2) \mathfrak{p}_b(q^2)$$

Separation of UV integral $A_{\ell}^a(k) = \underbrace{v_{\ell}^{(+)}(k)}_{\text{Long-range Jost function}} \underbrace{\mathfrak{p}_a(k^2)}_{\text{Local polynomials}}$

$$v_{\ell}^{(+)}(k) = \frac{1}{4\pi} \int \frac{d^3p}{(2\pi)^3} p^{\ell} \psi_{\ell}^{(+)}(p; k) = \frac{k^{\ell}}{(2\ell + 1)! f_{\ell}(k)}$$

Long-range Jost function

02 Modified Lüscher formula (infinite volume part)

Solve effective LS equation in the infinite volume

➤ Threshold expansion

$$B_\ell(p, q; q_0^2 + i\epsilon) = \tilde{V}_S^{(\ell)}(p, q) + \int \frac{d^3k}{(2\pi)^3} \tilde{V}_S^{(\ell)}(p, k) \frac{1}{k^2 - q_0^2 - i\epsilon} B_\ell(k, q; q_0^2 + i\epsilon).$$

- Equation driven by short-range kernel $\tilde{V}_S$

$$\tilde{V}_S^{(\ell)}(p, q) = \frac{(pq)^\ell}{[(2\ell + 1)!!]^2 f_\ell^*(p) f_\ell(q)} \tilde{V}_S^{(\ell)}(p, q) \quad B_\ell(p, q; q_0^2 + i\epsilon) = \frac{(pq)^\ell}{[(2\ell + 1)!!]^2 f_\ell^*(p) f_\ell(q)} \bar{B}_\ell(p, q; q_0^2 + i\epsilon)$$

$$\bar{B}_\ell(p, q; q_0^2 + i\epsilon) = \tilde{V}_S^{(\ell)}(p, q) + \int \frac{d^3k}{(2\pi)^3} \frac{k^{2\ell}}{[(2\ell + 1)!!]^2 |f_\ell(k)|^2} \frac{\tilde{V}_S^{(\ell)}(p, k) \bar{B}_\ell(k, q; q_0^2 + i\epsilon)}{k^2 - q_0^2 - i\epsilon}$$

- Singularity-free R -equation

$$R_\ell(p, q; q_0^2) = \tilde{V}_S^{(\ell)}(p, q) + \int \frac{d^3k}{(2\pi)^3} \frac{k^{2\ell}}{[(2\ell + 1)!!]^2 |f_\ell(k)|^2} \frac{\tilde{V}_S^{(\ell)}(p, k) R_\ell(k, q; q_0^2) - \tilde{V}_S^{(\ell)}(p, q_0) R_\ell(k, q_0; q_0^2)}{k^2 - q_0^2}$$

Singularity-free

- On-shell effective amplitude

$$B_\ell(q_0^2 + i\epsilon) = \frac{q_0^{2\ell}}{[(2\ell + 1)!!]^2 |f_\ell(q_0)|^2} \frac{1}{R_\ell^{-1}(q_0, q_0; q_0^2) - \langle G_L^{(\ell)}(q_0^2 + i\epsilon) \rangle}$$

Loop integral $\langle G_L^{(\ell)}(q_0^2 + i\epsilon) \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{[(2\ell + 1)!!]^2 |f_\ell(k)|^2} \frac{k^{2\ell}}{k^2 - q_0^2 - i\epsilon}$

02 Modified Lüscher formula (infinite volume part)

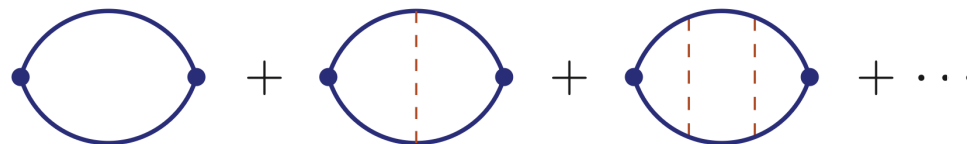
Renormalization in the infinite volume

➤ Match equation in the infinite volume

$$B_\ell(q_0^2 + i\epsilon) = \frac{q_0^{2\ell}}{[(2\ell + 1)!!]^2 |f_\ell(q_0)|^2} \frac{1}{R_\ell^{-1}(q_0, q_0; q_0^2) - \langle G_L^{(\ell)}(q_0^2 + i\epsilon) \rangle} \quad \longleftrightarrow \quad B_\ell(q_0^2 + i\epsilon) = \frac{4\pi q_0^{2\ell}}{q_0^{2\ell+1} \cot(\delta_\ell - \sigma_\ell) - i q_0^{2\ell+1}}$$

• Renormalization of loop integral

$$\langle G_L^{(\ell)}(q_0^2 + i\epsilon) \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{[(2\ell + 1)!!]^2 |f_\ell(k)|^2} \frac{k^{2\ell}}{k^2 - q_0^2 - i\epsilon}$$



$$\langle G_L^{(\ell)}(q_0^2 + i\epsilon) \rangle = \frac{M_\ell(q_0) + \mathcal{P}_\ell(q_0^2)}{4\pi[(2\ell + 1)!!]^2} \quad \text{Counter-terms} \quad M_\ell(q) = \frac{(-iq)^\ell}{2^\ell \ell!} \frac{1}{f_\ell(q)} \lim_{r \rightarrow 0} \frac{d^{2\ell+1}}{dr^{2\ell+1}} [r^\ell f_\ell(q, r)] \quad f_\ell(q) = \frac{(-iq)^\ell}{(2\ell - 1)!!} \lim_{r \rightarrow 0} [r^\ell f_\ell(q, r)]$$

• Match equation and modified K -matrix [Phys. Rev. A 26 (1982) 1218]

Renormalization
condition

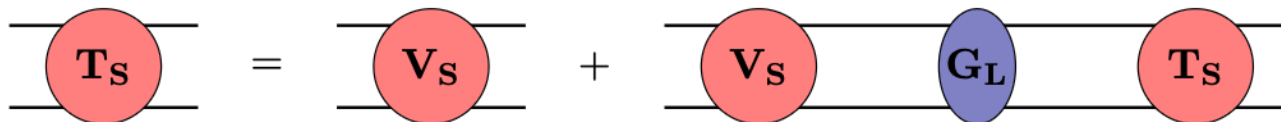
$$4\pi[(2\ell + 1)!!]^2 R_\ell^{-1}(q_0, q_0; q_0^2) = K_\ell^M(q_0^2) + \mathcal{P}_\ell(q_0^2)$$

Modified K -matrix

$$K_\ell^M(q_0^2) = M_\ell(q_0) + \frac{q_0^{2\ell+1} \cot(\delta_\ell - \sigma_\ell) - i q_0^{2\ell+1}}{|f_\ell(q_0)|^2}$$

02 Modified Lüscher formula

Finite volume effective LS equation



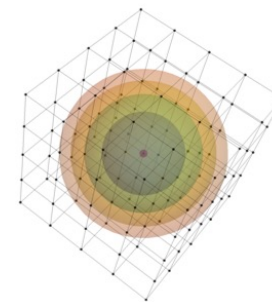
Equation in the infinite volume

$$B(\mathbf{p}, \mathbf{q}; q_0^2 + i\epsilon) = \tilde{V}_S(\mathbf{p}, \mathbf{q}) + \int \frac{d^3k}{(2\pi)^3} \tilde{V}_S(\mathbf{p}, \mathbf{k}) \frac{1}{k^2 - q_0^2 - i\epsilon} B(\mathbf{k}, \mathbf{q}; q_0^2 + i\epsilon)$$



Equation in a finite volume

?



➤ Finite volume basis

- Discrete long-range bases
- Finite volume effective LS equation in discrete long-range bases

$$B(\mathbf{p}, \mathbf{q}; q_0^2 + i\epsilon) = \langle \psi_{\mathbf{p}}^{(+)} | T_S(q_0^2 + i\epsilon) | \psi_{\mathbf{q}}^{(+)} \rangle$$

$$\tilde{V}_S(\mathbf{p}, \mathbf{q}) = \langle \psi_{\mathbf{p}}^{(+)} | V_S | \psi_{\mathbf{q}}^{(+)} \rangle$$

$$|\psi_{\mathbf{n}}\rangle = \frac{1}{L^3} \sum_{\mathbf{p}} \psi_{\mathbf{n}}(\mathbf{p}) |\mathbf{p}\rangle$$

$$\langle \psi_{\mathbf{n}} | \psi_{\mathbf{n}'} \rangle = \delta_{\mathbf{n}\mathbf{n}'},$$

$$\sum_{\mathbf{n}} |\psi_{\mathbf{n}}\rangle \langle \psi_{\mathbf{n}}| = 1.$$

$$B(\mathbf{n}, \mathbf{n}'; q_0^2) = \langle \psi_{\mathbf{n}} | T_S(q_0^2) | \psi_{\mathbf{n}'} \rangle,$$

$$\tilde{V}_S(\mathbf{n}, \mathbf{n}') = \langle \psi_{\mathbf{n}} | V_S | \psi_{\mathbf{n}'} \rangle.$$

Equation in a finite volume

$$B(\mathbf{n}, \mathbf{n}'; q_0^2) = \tilde{V}_S(\mathbf{n}, \mathbf{n}') + \sum_{\mathbf{k}} \tilde{V}_S(\mathbf{n}, \mathbf{k}) \frac{1}{q_{\mathbf{k}}^2 - q_0^2} B(\mathbf{k}, \mathbf{n}'; q_0^2)$$

02 Modified Lüscher formula

Separation of effective short-range potential in a finite volume

$$B(\mathbf{n}, \mathbf{n}'; q_0^2) = \tilde{V}_S(\mathbf{n}, \mathbf{n}') + \sum_{\mathbf{k}} \tilde{V}_S(\mathbf{n}, \mathbf{k}) \frac{1}{q_{\mathbf{k}}^2 - q_0^2} B(\mathbf{k}, \mathbf{n}'; q_0^2)$$

➤ Finite volume effective short-range potential in long-range bases

$$\tilde{V}_S(\mathbf{n}, \mathbf{n}') = \langle \psi_{\mathbf{n}} | V_S | \psi_{\mathbf{n}'} \rangle$$

- Polynomial representation of local potential is the same in both inf. and fin. volume.

$$\langle \mathbf{p} | V_S | \mathbf{q} \rangle = 4\pi \sum_{\ell m} \mathcal{Y}_{\ell m}(\mathbf{p}) \left[\sum_{a,b} C_{\ell}^{ab} (p^2)^a (q^2)^b \right] \mathcal{Y}_{\ell m}^*(\mathbf{q})$$

- Potential in long-range bases

$$\tilde{V}_S(\mathbf{n}, \mathbf{n}') = 4\pi \sum_{\ell m} \sum_{a,b} C_{\ell}^{ab} (\mathcal{A}_{\ell m}^a(\mathbf{n}))^* \mathcal{A}_{\ell m}^b(\mathbf{n}')$$

UV summation of long-range wave-funct $\mathcal{A}_{\ell}^a(\mathbf{n}) = \frac{1}{L^3} \sum_{\mathbf{p}} (p^2)^a \mathcal{Y}_{\ell m}^*(\mathbf{p}) \psi_{\mathbf{n}}(\mathbf{p})$

- Separation of long- and short-range interaction

$$\tilde{V}_S(\mathbf{n}, \mathbf{n}') = 4\pi \sum_{\ell m} \underbrace{v_{\ell m}^*(\mathbf{n})}_{\text{Long-range contribution in effective potential}} \underbrace{\bar{V}_S^{(\ell)}(q_{\mathbf{n}}, q_{\mathbf{n}'})}_{\text{Short-range kernel NOT affected by finite vol.}} \underbrace{v_{\ell m}(\mathbf{n}')}_{\text{Long-range contribution in effective potential}}$$

Long-range contribution in effective potential

$$\bar{V}_S^{(\ell)}(p, q) = \sum_{a,b} C_{\ell}^{ab} \mathbf{p}_a(p^2) \mathbf{p}_b(q^2)$$

Short-range kernel NOT affected by finite vol.

Separation of UV summation $\mathcal{A}_{\ell m}^a(\mathbf{n}) = \underbrace{v_{\ell m}(\mathbf{n})}_{\text{Long-range contribution in effective potential}} \underbrace{\mathbf{p}_a(q_{\mathbf{n}}^2)}_{\text{Local polynomials: UV limits of } V_L \text{ and its derivative are NOT affected by finite vol.}}$

$$v_{\ell m}(\mathbf{n}) = \frac{1}{L^3} \sum_{\mathbf{p}} \mathcal{Y}_{\ell m}^*(\mathbf{p}) \psi_{\mathbf{n}}(\mathbf{p})$$

Local polynomials:
UV limits of V_L and its derivative
are NOT affected by finite vol.

02 Modified Lüscher formula

Solve effective LS equation in a finite volume

➤ “Threshold expansion” in a finite volume

- Finite volume effective LS equation in long-range bases

- Equation driven by short-range kernel $\bar{V}_S$

$$\bar{B}_{\ell m, \ell' m'}(q_{\mathbf{n}}, q_{\mathbf{n}'}; q_0^2) = \bar{V}_S^{(\ell)}(q_{\mathbf{n}}, q_{\mathbf{n}'}; q_0^2) \delta_{\ell m, \ell' m'} + 4\pi \sum_{j\nu} \sum_{\mathbf{k}} \bar{V}_S^{(\ell)}(q_{\mathbf{n}}, q_{\mathbf{k}}) \frac{v_{\ell m}(\mathbf{k}) v_{j\nu}^*(\mathbf{k})}{q_{\mathbf{k}}^2 - q_0^2} \bar{B}_{j\nu, \ell' m'}(q_{\mathbf{k}}, q_{\mathbf{n}'}; q_0^2).$$

- Singularity-free R -equation

$$R_{\ell m, \ell' m'}(q_{\mathbf{n}}, q_{\mathbf{n}'}; q_0^2) = \bar{V}_S^{(\ell)}(q_{\mathbf{n}}, q_{\mathbf{n}'}; q_0^2) \delta_{\ell m, \ell' m'} + 4\pi \sum_{j\nu} \sum_{\mathbf{k}} v_{\ell m}(\mathbf{k}) v_{j\nu}^*(\mathbf{k}) \times \frac{\bar{V}_S^{(\ell)}(q_{\mathbf{n}}, q_{\mathbf{k}}) R_{j\nu, \ell' m'}(q_{\mathbf{k}}, q_{\mathbf{n}'}; q_0^2) - \bar{V}_S^{(\ell)}(q_{\mathbf{n}}, q_0) R_{j\nu, \ell' m'}(q_0, q_{\mathbf{n}'}; q_0^2)}{q_{\mathbf{k}}^2 - q_0^2}$$

Singularity-free

$$B(\mathbf{n}, \mathbf{n}'; q_0^2) = \tilde{V}_S(\mathbf{n}, \mathbf{n}') + \sum_{\mathbf{k}} \tilde{V}_S(\mathbf{n}, \mathbf{k}) \frac{1}{q_{\mathbf{k}}^2 - q_0^2} B(\mathbf{k}, \mathbf{n}'; q_0^2)$$

$$\tilde{V}_S(\mathbf{n}, \mathbf{n}') = 4\pi \sum_{\ell m} \underline{v_{\ell m}^*(\mathbf{n})} \underline{\bar{V}_S^{(\ell)}(q_{\mathbf{n}}, q_{\mathbf{n}'}; q_0^2)} \underline{v_{\ell m}(\mathbf{n}')} \quad \leftarrow$$

$$B(\mathbf{n}, \mathbf{n}'; q_0^2) = 4\pi \sum_{\ell m, \ell' m'} \underline{v_{\ell m}^*(\mathbf{n})} \underline{\bar{B}_{\ell m, \ell' m'}(q_{\mathbf{n}}, q_{\mathbf{n}'}; q_0^2)} \underline{v_{\ell' m'}(\mathbf{n}')} \quad \leftarrow$$

- “On-shell” amplitude

$$\bar{B}_{\ell m, \ell' m'}(q_0, q_0; q_0^2) = R_{\ell m, \ell' m'}(q_0, q_0; q_0^2) + \sum_{j\nu, j'\nu'} R_{\ell m, j\nu}(q_0, q_0; q_0^2) H_{j\nu, j'\nu'}(q_0^2) \bar{B}_{j'\nu', \ell' m'}(q_0, q_0; q_0^2)$$

Modified zeta-function $H_{\ell m, \ell' m'}(q_0^2) = 4\pi \sum_{\mathbf{n}} \frac{v_{\ell m}(\mathbf{n}) v_{\ell' m'}^*(\mathbf{n})}{q_{\mathbf{n}}^2 - q_0^2}$

02 Modified Lüscher formula

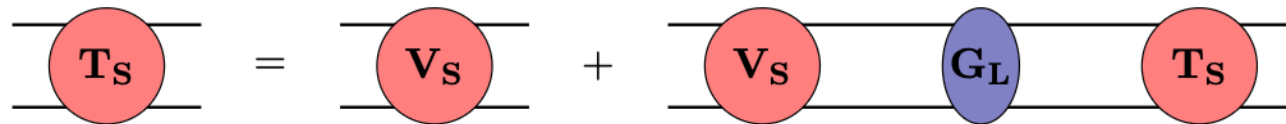
Modified Lüscher formula

➤ Determination of finite volume energy levels

- Finite volume energy levels are determined by poles of T
- T 's poles are equivalent to effective short-range part T_S
- T_S can be formulated by effective amplitude $\bar{B}$
- On-shell effective amplitude $\bar{B}$

$$\bar{B} = (R^{-1} - H)^{-1}$$

$$\det(R^{-1} - H) = 0$$



$$\bar{B}_{\ell m, \ell' m'}(q_{\mathbf{n}}, q_{\mathbf{n}'}; q_0^2) = \bar{V}_S^{(\ell)}(q_{\mathbf{n}}, q_{\mathbf{n}'}; q_0^2) \delta_{\ell m, \ell' m'}$$

$$+ 4\pi \sum_{j\nu} \sum_{\mathbf{k}} \bar{V}_S^{(\ell)}(q_{\mathbf{n}}, q_{\mathbf{k}}; q_0^2) \frac{v_{\ell m}(\mathbf{k}) v_{j\nu}^*(\mathbf{k})}{q_{\mathbf{k}}^2 - q_0^2} \bar{B}_{j\nu, \ell' m'}(q_{\mathbf{k}}, q_{\mathbf{n}'}; q_0^2).$$

$$\bar{B}_{\ell m, \ell' m'}(q_0, q_0; q_0^2) = R_{\ell m, \ell' m'}(q_0, q_0; q_0^2)$$

$$+ \sum_{j\nu, j'\nu'} R_{\ell m, j\nu}(q_0, q_0; q_0^2) H_{j\nu, j'\nu'}(q_0^2) \bar{B}_{j'\nu', \ell' m'}(q_0, q_0; q_0^2)$$

➤ Quantization condition

$$\det(R^{-1} - H) = 0$$

- Finite volume correction of R -function is exponentially suppressed
- Long-range interaction is encoded in modified zeta function H

$$H_{\ell m, \ell' m'}(q_0^2) = 4\pi \sum_{\mathbf{n}} \frac{v_{\ell m}(\mathbf{n}) v_{\ell' m'}^*(\mathbf{n})}{q_{\mathbf{n}}^2 - q_0^2} = \frac{4\pi}{L^6} \sum_{\mathbf{p}, \mathbf{q}} \mathcal{Y}_{\ell m}^*(\mathbf{p}) \langle \mathbf{p} | G_L(q_0^2) | \mathbf{q} \rangle \mathcal{Y}_{\ell' m'}(\mathbf{q})$$

Singularity-free $\longrightarrow$ Exponentially suppression

$$R_{\ell m, \ell' m'} = R_{\ell} \delta_{\ell m, \ell' m'} + O(e^{-E_{\text{th}} L}) \longleftarrow \begin{cases} \frac{1}{L^3} \sum_{\mathbf{k}} \cdots = \int \frac{d^3 k}{(2\pi)^3} \cdots + O(e^{-E_{\text{th}} L}) \\ v_{\ell m}(\mathbf{k}) = v_{\ell}^{(+)}(k) \mathcal{Y}_{\ell m}(\mathbf{k}) + O(e^{-E_{\text{th}} L}) \end{cases}$$

02 Modified Lüscher formula

Renormalization in a finite volume

➤ Regularization in quantization condition

$$\det \mathcal{A}(q_0^2) = 0$$

quantization matrix

$$\mathcal{A}_{\ell m, \ell' m'}(q_0^2) = R_\ell^{-1}(q_0, q_0; q_0^2) \delta_{\ell m, \ell' m'} - H_{\ell m, \ell' m'}(q_0^2)$$

- Renormalization of R -function in the finite volume

Renormalization condition

$$4\pi[(2\ell + 1)!!]^2 R_\ell^{-1}(q_0, q_0; q_0^2) = K_\ell^M(q_0^2) + \mathcal{P}_\ell(q_0^2)$$

Counter-terms

Modified K-matrix

$$K_\ell^M(q_0^2) = M_\ell(q_0) + \frac{q_0^{2\ell+1} \cot(\delta_\ell - \sigma_\ell) - i q_0^{2\ell+1}}{|f_\ell(q_0)|^2}$$

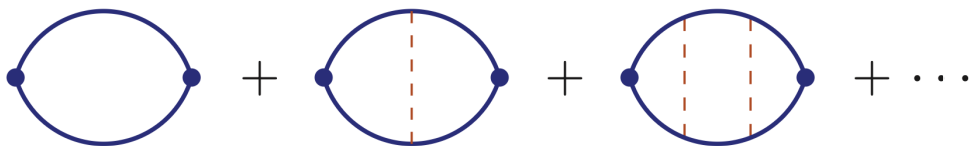
- Regularization of modified zeta function H in a finite volume

$$H_{\ell m, \ell' m'}(q_0^2) = \frac{4\pi}{L^6} \sum_{\mathbf{p}, \mathbf{q}} \mathcal{Y}_{\ell m}^*(\mathbf{p}) \langle \mathbf{p} | G_L(q_0^2) | \mathbf{q} \rangle \mathcal{Y}_{\ell' m'}(\mathbf{q})$$

Infinite vol. limit

$$H_{\ell m, \ell' m'}^\infty(q_0^2) = 4\pi \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \mathcal{Y}_{\ell m}^*(\mathbf{p}) \langle \mathbf{p} | G_L(q_0^2 + i\epsilon) | \mathbf{q} \rangle \mathcal{Y}_{\ell' m'}(\mathbf{q})$$

$$= \delta_{\ell m, \ell' m'} \langle G_L^{(\ell)}(q_0^2 + i\epsilon) \rangle,$$



Regularization in the infinite volume

$$\langle G_L^{(\ell)}(q_0^2 + i\epsilon) \rangle = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{[(2\ell + 1)!!]^2 |f_\ell(k)|^2} \frac{k^{2\ell}}{k^2 - q_0^2 - i\epsilon} = \frac{M_\ell(q_0) + \mathcal{P}_\ell(q_0^2)}{4\pi[(2\ell + 1)!!]^2}$$

Counter-terms

sum-integral difference $\Delta H_{\ell m, \ell' m'}(q_0^2) = H_{\ell m, \ell' m'}(q_0^2) - \Re H_{\ell m, \ell' m'}^\infty(q_0^2)$

Regularization $H_{\ell m, \ell' m'}(q_0^2) = \Delta H_{\ell m, \ell' m'}(q_0^2) + \delta_{\ell m, \ell' m'} \frac{\Re M_\ell(q_0) + \mathcal{P}_\ell(q_0^2)}{4\pi[(2\ell + 1)!!]^2}$

02 Modified Lüscher formula

Modified Lüscher formula

➤ Modified Lüscher formula

- Regularized quantization matrix

$$\det \mathcal{A}(q_0^2) = 0$$

quantization matrix

$$R_\ell^{-1}(q_0, q_0; q_0^2) = \frac{K_\ell^M(q_0^2) + \mathcal{P}_\ell(q_0^2)}{4\pi[(2\ell+1)!!]^2}$$

$$H_{\ell m, \ell' m'}(q_0^2) = \Delta H_{\ell m, \ell' m'}(q_0^2) + \delta_{\ell m, \ell' m'} \frac{\Re M_\ell(q_0) + \mathcal{P}_\ell(q_0^2)}{4\pi[(2\ell+1)!!]^2}$$

Conter-term
cancellation

$$\mathcal{A}_{\ell m, \ell' m'}(q_0^2) = R_\ell^{-1}(q_0, q_0; q_0^2) \delta_{\ell m, \ell' m'} - H_{\ell m, \ell' m'}(q_0^2)$$

$$\mathcal{A}_{\ell m, \ell' m'}(q_0^2) = \frac{K_\ell^M(q_0^2) - \Re M_\ell(q_0)}{4\pi[(2\ell+1)!!]^2} \delta_{\ell m, \ell' m'} - \Delta H_{\ell m, \ell' m'}(q_0^2)$$

Physical observable
including both long- and
short-range information

Infinite (Finite) volume function
determined by **only** long-range
interaction

- Advantage of modified Lüscher formula

$$q_0^{2\ell+1} \cot \delta_\ell = -\frac{1}{a_\ell} + \frac{1}{2} r_\ell q_0^2 + O(q_0^4) \quad (\text{poor convergence for long-range force})$$

$$K_\ell^M(q_0^2) = M_\ell(q_0) + \frac{q_0^{2\ell+1} \cot(\delta_\ell - \sigma_\ell) - i q_0^{2\ell+1}}{|f_\ell(q_0)|^2} = -\frac{1}{\tilde{a}_\ell} + \frac{1}{2} \tilde{r}_\ell q_0^2 + O(q_0^4) \quad (\text{much larger convergence radius})$$

- Reproduce of standard Lüscher formula

Long-range interaction disappears

$$\sigma_\ell = 0 \quad [(2\ell+1)!!]^2 |f_\ell(q_0)|^2 = 1$$

$$K_\ell^M(q_0^2) \rightarrow [(2\ell+1)!!]^2 q_0^{2\ell+1} \cot \delta_\ell$$

$$\Re M_\ell(q_0) \rightarrow 0$$

$$\begin{aligned} \Delta H_{\ell m, \ell' m'}(q_0^2) &\rightarrow \frac{4\pi}{L^6} \sum_{\mathbf{p}, \mathbf{q}} \mathcal{Y}_{\ell m}^*(\mathbf{p}) \langle \mathbf{p} | G(q_0^2) | \mathbf{q} \rangle \mathcal{Y}_{\ell' m'}(\mathbf{q}) \\ &\quad - 4\pi \Re \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \mathcal{Y}_{\ell m}^*(\mathbf{p}) \langle \mathbf{p} | G(q_0^2 + i\epsilon) | \mathbf{q} \rangle \mathcal{Y}_{\ell' m'}(\mathbf{q}) \\ &= \left(\frac{1}{L^3} \sum_{\mathbf{k}} -\text{PV} \int \frac{d^3 k}{(2\pi)^3} \right) \frac{4\pi \mathcal{Y}_{\ell m}^*(\mathbf{k}) \mathcal{Y}_{\ell' m'}(\mathbf{k})}{k^2 - q_0^2}. \end{aligned}$$

Quantization matrix in standard Lüscher formula

$$\frac{q_0^{2\ell+1} \cot \delta_\ell}{4\pi} \delta_{\ell m, \ell' m'} - \frac{1}{L^3} \sum_{\mathbf{k}} \frac{4\pi \mathcal{Y}_{\ell m}^*(\mathbf{k}) \mathcal{Y}_{\ell' m'}(\mathbf{k})}{k^2 - q_0^2}$$

01 Motivation

02 Modified Lüscher formula

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03 Zeta function

04 Numerical result

05 Conclusion and outlook

03 Zeta function

Definition of zeta function

> Definition

- H function is regularized by sum-integral difference in our formalism

$$\Delta H_{\ell m, \ell' m'}(q_0^2) = H_{\ell m, \ell' m'}(q_0^2) - \Re H_{\ell m, \ell' m'}^\infty(q_0^2) \quad \left\{ \begin{array}{l} H_{\ell m, \ell' m'}(q_0^2) = \frac{4\pi}{L^6} \sum_{\mathbf{p}, \mathbf{q}} \mathcal{Y}_{\ell m}^*(\mathbf{p}) \langle \mathbf{p} | G_L(q_0^2) | \mathbf{q} \rangle \mathcal{Y}_{\ell' m'}(\mathbf{q}), \\ H_{\ell m, \ell' m'}^\infty(q_0^2) = 4\pi \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \mathcal{Y}_{\ell m}^*(\mathbf{p}) \langle \mathbf{p} | G_L(q_0^2 + i\epsilon) | \mathbf{q} \rangle \mathcal{Y}_{\ell' m'}(\mathbf{q}). \end{array} \right.$$

$$\text{Diagram: } \text{---} \bigcirc \text{G}_L \text{---} = \text{---} + \text{---} \bigcirc \text{T}_L \text{---}$$

$$G_L = G + G T_L G$$

- Finite volume H function is composed of free part and long-range part

$$\begin{aligned} H_{\ell m, \ell' m'}(q_0^2) &= \frac{4\pi}{L^6} \sum_{\mathbf{p}, \mathbf{q}} \mathcal{Y}_{\ell m}^*(\mathbf{p}) \langle \mathbf{p} | G(q_0^2) | \mathbf{q} \rangle \mathcal{Y}_{\ell' m'}(\mathbf{q}) \xrightarrow{\text{Free part}} H_{\ell m, \ell' m'}^{(0)}(q_0^2) = \frac{4\pi}{L^3} \sum_{\mathbf{p}} \frac{\mathcal{Y}_{\ell m}^*(\mathbf{p}) \mathcal{Y}_{\ell' m'}(\mathbf{p})}{p^2 - q_0^2} \quad (\text{Standard zeta function}) \\ &+ \frac{4\pi}{L^6} \sum_{\mathbf{p}, \mathbf{q}} \mathcal{Y}_{\ell m}^*(\mathbf{p}) \langle \mathbf{p} | G(q_0^2) T_L(q_0^2) G(q_0^2) | \mathbf{q} \rangle \mathcal{Y}_{\ell' m'}(\mathbf{q}) \longrightarrow H_{\ell m, \ell' m'}(q_0^2) - H_{\ell m, \ell' m'}^{(0)}(q_0^2) = \frac{4\pi}{L^6} \sum_{\mathbf{p}, \mathbf{q}} \frac{\mathcal{Y}_{\ell m}^*(\mathbf{p})}{p^2 - q_0^2} T_L(\mathbf{p}, \mathbf{q}; q_0^2) \frac{\mathcal{Y}_{\ell' m'}(\mathbf{q})}{q^2 - q_0^2} \end{aligned}$$

- Finite volume long-range LS equation

$$\text{Diagram: } \text{---} \bigcirc \text{T}_L \text{---} = \text{---} \bigcirc \text{V}_L \text{---} + \text{---} \bigcirc \text{V}_L \text{---} \bigcirc \text{T}_L \text{---}$$

$$T_L(\mathbf{p}, \mathbf{q}; q_0^2) = V_L(\mathbf{p}, \mathbf{q}) + \frac{1}{L^3} \sum_{\mathbf{k}} V_L(\mathbf{p}, \mathbf{k}) G(k; q_0^2) T_L(\mathbf{k}, \mathbf{q}; q_0^2)$$

03 Zeta function

UV property of zeta function

> Definition

- ΔH is also composed of free part and long-range part

$$\Delta H_{\ell m, \ell' m'}(q_0^2) = \Delta H_{\ell m, \ell' m'}^{(0)}(q_0^2) - \Delta H'_{\ell m, \ell' m'}(q_0^2) \quad \left\{ \begin{array}{l} \Delta H_{\ell m, \ell' m'}^{(0)}(q_0^2) = \left(\frac{1}{L^3} \sum_{\mathbf{p}} -\text{PV} \int \frac{d^3 p}{(2\pi)^3} \right) \frac{4\pi \mathcal{Y}_{\ell m}^*(\mathbf{p}) \mathcal{Y}_{\ell' m'}(\mathbf{p})}{p^2 - q_0^2} \quad (\text{Standard zeta function}) \\ \Delta H'_{\ell m, \ell' m'}(q_0^2) = \frac{4\pi}{L^6} \sum_{\mathbf{p}, \mathbf{q}} \frac{\mathcal{Y}_{\ell m}^*(\mathbf{p})}{p^2 - q_0^2} T_L(\mathbf{p}, \mathbf{q}; q_0^2) \frac{\mathcal{Y}_{\ell' m'}(\mathbf{q})}{q^2 - q_0^2} - 4\pi \Re \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \frac{\mathcal{Y}_{\ell m}^*(\mathbf{p})}{p^2 - q_0^2} T_L^\infty(\mathbf{p}, \mathbf{q}; q_0^2) \frac{\mathcal{Y}_{\ell' m'}(\mathbf{q})}{q^2 - q_0^2} \end{array} \right.$$

- Free part is regularized directly in all methods to calculate standard zeta function

> UV property

- Direct subtraction in long-range part $\Delta H'$ is numerically unstable

$$\underbrace{\Delta H'}_{\text{UV div.-free but numerically unstable}} = \underbrace{\sum GT_L G}_{\text{UV div.}} - \underbrace{\int GT_L^\infty G}_{\text{UV div.}} \quad \left\{ \begin{array}{l} T_L = V_L + \sum V_L G V_L + \dots \longrightarrow H' = \underbrace{\sum GV_L G}_{\text{UV div.}} + \underbrace{\sum GV_L G V_L G}_{\text{UV div.}} + \dots \\ T_L^\infty = V_L + \int V_L G V_L + \dots \longrightarrow H'^\infty = \underbrace{\int GV_L G}_{\text{UV div.}} + \underbrace{\int GV_L G V_L G}_{\text{UV div.}} + \dots \end{array} \right.$$

- One-by-one Subtraction is not feasible in practice for un-separable V_L

$$\Delta H' = \underbrace{(\sum - \int) GV_L G}_{\text{UV div.-free}} + \underbrace{(\sum - \int) GV_L G V_L G}_{\text{UV div.-free}} + \dots$$

03 Zeta function

Fast convergent method for long-range part

➤ Decomposition of propagator

- G : singular and UV slowly convergent
- G_1 : regular but UV slowly convergent
- ΔG : singular but UV fast convergent
- Decomposition is μ -dept. but result is μ -indept.

$$G = G_1 + \Delta G \quad \left\{ \begin{array}{l} G(p) = \frac{1}{p^2 - q_0^2}; \\ G_1(p) = \frac{1}{p^2 + \mu^2} + \frac{\mu^2 + q_0^2}{(p^2 + \mu^2)^2} + \frac{(\mu^2 + q_0^2)^2}{(p^2 + \mu^2)^3} + \cdots + \frac{(\mu^2 + q_0^2)^{n-1}}{(p^2 + \mu^2)^n}; \\ \Delta G(p) = \frac{1}{p^2 - q_0^2} \frac{(\mu^2 + q_0^2)^n}{(p^2 + \mu^2)^n}. \end{array} \right.$$

➤ Decomposition of long-range propagator

- G_μ : regular
- G_f : UV fast convergent

regular $T_1 = V_L + V_L G_1 T_1$

$\tilde{G} = \Delta G + \Delta G T_L \Delta G$

$T_L = V_L + V_L G T_L$

$G_L = \underbrace{G_1 + G_1 T_1 G_1}_{G_\mu} + \underbrace{(1 + G_1 T_1) \tilde{G} (T_1 G_1 + 1)}_{G_f}$

03 Zeta function

Fast convergent method for long-range part

➤ Decomposition of zeta function

$$G_L = \underbrace{G_1 + G_1 T_1 G_1}_{G_\mu} + \underbrace{(1 + G_1 T_1) \tilde{G} (T_1 G_1 + 1)}_{G_f}$$

- G_μ is **regular** $\rightarrow \Delta H^{(\mu)}$ is **exponentially suppressed**

$$\Delta H^{(\mu)} = H^{(\mu)} - H^{(\mu)\infty} = O(e^{-E_{\text{th}} L})$$

- G_f is **UV fast convergent** $\rightarrow H^{(f)}$ and $H^{(f)\infty}$ are both **UV convergent**

$$\begin{aligned} H_{\ell m, \ell' m'} &= \frac{4\pi}{L^6} \sum_{\mathbf{p}, \mathbf{q}} \mathcal{Y}_{\ell m}^*(\mathbf{p}) G_L(\mathbf{p}, \mathbf{q}) \mathcal{Y}_{\ell' m'}(\mathbf{q}) \\ &= \underbrace{\frac{4\pi}{L^6} \sum_{\mathbf{p}, \mathbf{q}} \mathcal{Y}_{\ell m}^*(\mathbf{p}) G_\mu(\mathbf{p}, \mathbf{q}) \mathcal{Y}_{\ell' m'}(\mathbf{q})}_{H_{\ell m, \ell' m'}^{(\mu)}} + \underbrace{\frac{4\pi}{L^6} \sum_{\mathbf{p}, \mathbf{q}} \mathcal{Y}_{\ell m}^*(\mathbf{p}) G_f(\mathbf{p}, \mathbf{q}) \mathcal{Y}_{\ell' m'}(\mathbf{q})}_{H_{\ell m, \ell' m'}^{(f)}} \end{aligned}$$

$$\begin{aligned} \Delta H_{\ell m, \ell' m'} &= \boxed{\frac{4\pi}{L^6} \sum_{\mathbf{p}, \mathbf{q}} \mathcal{Y}_{\ell m}^*(\mathbf{p}) G_f(\mathbf{p}, \mathbf{q}) \mathcal{Y}_{\ell' m'}(\mathbf{q})} \quad \text{UV conv.} \\ &\quad - \boxed{4\pi \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \mathcal{Y}_{\ell m}^*(\mathbf{p}) G_f^\infty(\mathbf{p}, \mathbf{q}) \mathcal{Y}_{\ell' m'}(\mathbf{q})} \quad \text{UV conv.} \end{aligned}$$

$$\begin{aligned} \Delta H^{(\mu)} &= H^{(\mu)} - H^{(\mu)\infty} = O(e^{-E_{\text{th}} L}) & \Delta H^{(f)} &= H^{(f)} - H^{(f)\infty} \\ \Delta H &= \Delta H^{(f)} + O(e^{-E_{\text{th}} L}) \end{aligned}$$

- $H^{(f)}$ and $H^{(f)\infty}$ can be calculated separately $\left\{ \begin{aligned} H_{\ell m, \ell' m'}^{(f)} &= \frac{4\pi}{L^6} \sum_{\mathbf{k}, \mathbf{k}'} \mathcal{Y}_{\ell m}^*(\mathbf{k}) G_f(\mathbf{k}, \mathbf{k}') \mathcal{Y}_{\ell' m'}(\mathbf{k}') \\ H_{\ell m, \ell' m'}^{(f)\infty} &= \frac{4\pi}{L^6} \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 k'}{(2\pi)^3} \mathcal{Y}_{\ell m}^*(\mathbf{k}) G_f^\infty(\mathbf{k}, \mathbf{k}') \mathcal{Y}_{\ell' m'}(\mathbf{k}') \end{aligned} \right.$

03 Zeta function

Fast convergent method for long-range part

➤ Wave-function in modified zeta function

$$G_f = (1 + G_1 T_1) \tilde{G} (T_1 G_1 + 1)$$

$$= \Psi^* \tilde{G} \Psi$$

$$H_{\ell m, \ell' m'}^{(f)} = \frac{4\pi}{L^6} \sum_{\mathbf{k}, \mathbf{k}'} \mathcal{Y}_{\ell m}^*(\mathbf{k}) G_f(\mathbf{k}, \mathbf{k}') \mathcal{Y}_{\ell' m'}(\mathbf{k}')$$

$$= \frac{4\pi}{L^6} \sum_{\mathbf{p}, \mathbf{p}'} \Psi_{\ell m}^*(\mathbf{p}) \tilde{G}(\mathbf{p}, \mathbf{p}') \Psi_{\ell' m'}(\mathbf{p}')$$

$$H_{\ell m, \ell' m'}^{(f)\infty} = 4\pi \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 k'}{(2\pi)^3} \mathcal{Y}_{\ell m}^*(\mathbf{k}) G_f^\infty(\mathbf{k}, \mathbf{k}') \mathcal{Y}_{\ell' m'}(\mathbf{k}')$$

$$= 4\pi \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \Psi_{\ell m}^{\infty*}(\mathbf{p}) \tilde{G}^\infty(\mathbf{p}, \mathbf{p}') \Psi_{\ell' m'}^\infty(\mathbf{p}')$$

$$\tilde{G}(\mathbf{p}, \mathbf{p}') = \Delta G(p) L^3 \delta_{\mathbf{p}\mathbf{p}'} + \Delta G(p) T_L(\mathbf{p}, \mathbf{p}') \Delta G(p')$$

$$\tilde{G}^\infty(\mathbf{p}, \mathbf{q}) = \Delta G(p) (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}) + \Delta G(p) T_L^\infty(\mathbf{p}, \mathbf{q}) \Delta G(q)$$

$$\left. \begin{aligned} \Psi_{\ell m}^*(\mathbf{p}) &= \frac{1}{L^3} \sum_{\mathbf{k}} \mathcal{Y}_{\ell m}^*(\mathbf{k}) (L^3 \delta_{\mathbf{k}\mathbf{p}} + G_1(k) T_1(\mathbf{k}, \mathbf{p})); \\ \Psi_{\ell' m'}(\mathbf{p}') &= \frac{1}{L^3} \sum_{\mathbf{k}'} (T_1(\mathbf{p}', \mathbf{k}') G_1(k') + L^3 \delta_{\mathbf{p}'\mathbf{k}'}) \mathcal{Y}_{\ell' m'}(\mathbf{k}'). \end{aligned} \right\} \begin{array}{l} \xrightarrow{\text{Finite vol. correction is}} \\ \text{exponentially suppressed} \end{array} \left\{ \begin{aligned} \Psi_{\ell m}^{\infty*}(\mathbf{p}) &= \int \frac{d^3 k}{(2\pi)^3} \mathcal{Y}_{\ell m}^*(\mathbf{k}) ((2\pi)^3 \delta(\mathbf{k} - \mathbf{p}) + G_1(k) T_1^\infty(\mathbf{k}, \mathbf{p})); \\ \Psi_{\ell' m'}^\infty(\mathbf{p}') &= \int \frac{d^3 q'}{(2\pi)^3} (T_1^\infty(\mathbf{p}', \mathbf{q}') G_1(q') + (2\pi)^3 \delta(\mathbf{p}' - \mathbf{q}')) \mathcal{Y}_{\ell' m'}(\mathbf{q}'). \end{aligned} \right.$$

- Finite volume wave-function can be replaced by infinite volume one

03 Zeta function

Symmetric projection

➤ Partial-wave projection

- Rotational symmetry allows partial-wave projection in the infinite volume

$$T_1^\infty(\mathbf{p}, \mathbf{q}) = 4\pi \sum_{\ell m} \mathcal{Y}_{\ell m}(\mathbf{p}) T_{1,\ell}^\infty(p, q) \mathcal{Y}_{\ell m}^*(\mathbf{q})$$

$$\Psi_{\ell m}^{\infty*}(\mathbf{p}) = \Psi_\ell^{\infty*}(p) \mathcal{Y}_{\ell m}^*(\mathbf{p});$$

$$\Psi_{\ell m}^\infty(\mathbf{q}) = \mathcal{Y}_{\ell m}(\mathbf{q}) \Psi_\ell^\infty(q).$$

$$\Psi_\ell^{\infty*}(p) = 1 + \int \frac{d^3 k}{(2\pi)^3} k^{2\ell} G_1(k) T_{1,\ell}^\infty(k, p);$$

$$\Psi_\ell^\infty(q) = \int \frac{d^3 k}{(2\pi)^3} T_{1,\ell}^\infty(q, k) G_1(k) k^{2\ell} + 1.$$

$$T_L^\infty(\mathbf{p}, \mathbf{q}) = 4\pi \sum_{\ell m} \mathcal{Y}_{\ell m}(\mathbf{p}) T_{L,\ell}^\infty(p, q) \mathcal{Y}_{\ell m}^*(\mathbf{q})$$

$$\tilde{G}^\infty(\mathbf{p}, \mathbf{q}) = 4\pi \sum_{\ell m} \mathcal{Y}_{\ell m}(\mathbf{p}) \tilde{G}_\ell^\infty(p, q) \mathcal{Y}_{\ell m}^*(\mathbf{q})$$

$$\tilde{G}_\ell^\infty(p, q) = \Delta G(p) \frac{(2\pi)^3 \delta(p - q)}{4\pi p^2} + \Delta G(p) T_{L,\ell}^\infty(p, q) \Delta G(q)$$

- Infinite volume H $H_{\ell m, \ell' m'}^{(f)\infty} = \delta_{\ell m, \ell' m'} H_\ell^{(f)\infty}$ with $H_\ell^{(f)\infty} = \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \Psi_\ell^{\infty*}(p) p^{2\ell} \tilde{G}_\ell^\infty(p, q) q^{2\ell} \Psi_\ell^\infty(q)$

- Finite volume H $H_{\ell m, \ell' m'}^{(f)} = \frac{4\pi}{L^6} \sum_{\mathbf{p}, \mathbf{q}} \Psi_\ell^{\infty*}(p) \mathcal{Y}_{\ell m}^*(\mathbf{p}) \tilde{G}(\mathbf{p}, \mathbf{q}) \mathcal{Y}_{\ell' m'}(\mathbf{q}) \Psi_{\ell'}^\infty(q)$

- Zeta function ΔH $\Delta H_{\ell m, \ell' m'} = H_{\ell m, \ell' m'}^{(f)} - \delta_{\ell m, \ell' m'} H_\ell^{(f)\infty}$

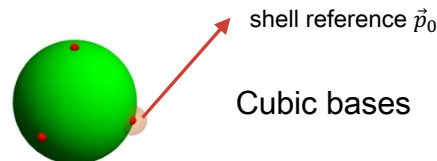
NOT partial-wave expanded

03 Zeta function

Symmetric projection

➤ Cubic projection

- rotational symmetry → O_h group (rest frame) or little group (moving frame)
- 3-momentum state characterized by **length** + **direction** → **shell** + **projection irreps**.
- finite volume shell: r -shell = $\{\mathbf{p} | \mathbf{p} = g\mathbf{p}_0^{(r)}, g \in \mathcal{G}\}$



Cubic bases $v_{\beta,\alpha}^{(\Gamma)}(\mathbf{p}) = \sqrt{\frac{s_\Gamma}{\vartheta_r}} T_{\alpha\beta}^{(\Gamma)}(g)$, with $\mathbf{p} = g\mathbf{p}_0^{(r)}$

$$\tilde{G}(\mathbf{p}, \mathbf{q}) = L^3 \sum_{\Gamma, \alpha} \sum_{\beta, \beta'} v_{\beta, \alpha}^{(\Gamma)}(\mathbf{p}) v_{\beta', \alpha}^{(\Gamma)*}(\mathbf{q}) \tilde{G}_{\beta\beta'}^{(\Gamma)}(r, r')$$

$$\tilde{G}_{\beta\beta'}^{(\Gamma)}(r, r') = \Delta G(p) \delta_{\beta\beta'} \delta_{rr'} + \Delta G(p) T_{L, \beta\beta'}^{(\Gamma)}(r, r') \Delta G(q)$$

$$T_L(\mathbf{p}, \mathbf{q}) = L^3 \sum_{\Gamma, \alpha} \sum_{\beta, \beta'} v_{\beta, \alpha}^{(\Gamma)}(\mathbf{p}) v_{\beta', \alpha}^{(\Gamma)*}(\mathbf{q}) T_{L, \beta\beta'}^{(\Gamma)}(r, r')$$

$$T_{L, \beta\beta'}^{(\Gamma)}(r, r') = V_{L, \beta\beta'}^{(\Gamma)}(r, r') + \frac{1}{L^3} \sum_s \sum_\gamma V_{L, \beta\gamma}^{(\Gamma)}(r, s) G(p_s) T_{L, \gamma\beta'}^{(\Gamma)}(s, r')$$

$$\text{Projected potential } V_{L, \beta\beta'}^{(\Gamma)}(r, r') = \frac{\sqrt{\vartheta_r \vartheta_{r'}}}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} T_{\beta\beta'}^{(\Gamma)}(g) \langle \mathbf{p}_0^{(r)} | V_L | g\mathbf{q}_0^{(r')} \rangle$$

- Cubic harmonics $\mathcal{Y}_{\ell t, \alpha}^{(\Gamma)} = \sum_m c_{\Gamma \ell t, \alpha}^m \mathcal{Y}_{\ell m}$

$$\Delta H_{\ell t, \ell' t'}^{(\Gamma)} = \sum_{m, m'} c_{\Gamma \ell t, \alpha}^{m*} \Delta H_{\ell m, \ell' m'} \mathcal{Y}_{\ell' t', \alpha}^{m'} = H_{\ell t, \ell' t'}^{(\Gamma, f)} - \delta_{\ell t, \ell' t'} H_\ell^{(f) \infty}$$

$$H_{\ell t, \ell' t'}^{(\Gamma, f)} = \frac{4\pi}{L^3} \sum_{r, r'} \sum_{\beta, \beta'} \tilde{\Psi}_{\ell t, \beta}^*(r) \tilde{G}_{\beta\beta'}^{(\Gamma)}(r, r') \tilde{\Psi}_{\ell' t', \beta'}(r') \quad \text{with} \quad \tilde{\Psi}_{\ell t, \beta}(r) = \sqrt{\frac{\vartheta_r}{s_\Gamma}} \mathcal{Y}_{\ell t, \beta}^{(\Gamma)}(\mathbf{p}_0^{(r)}) \Psi_\ell^\infty(p)$$

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04 Numerical result

OPE potential and energy levels

➤ Decomposition of potential

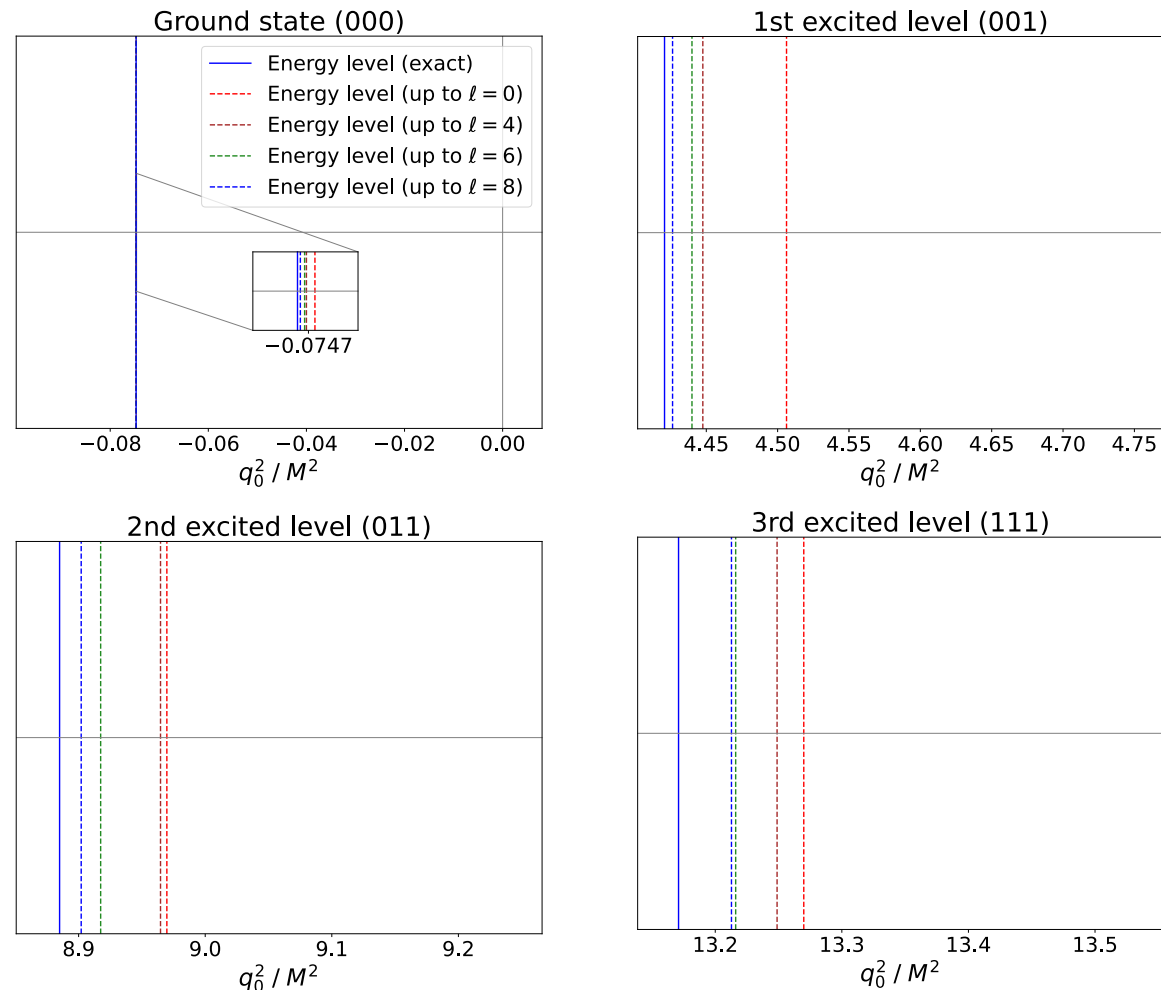
$$V(\mathbf{p}, \mathbf{q}) = \underbrace{\frac{4\pi g}{M^2 + (\mathbf{p} - \mathbf{q})^2}}_{V_L} + \underbrace{\frac{4\pi g_S}{M_S^2 + (\mathbf{p} - \mathbf{q})^2}}_{V_S}$$

- One-pion-exchange potential
- Everything scaled by pion mass M
- Long-range coupling (attractive) $g = 0.489M$
- Short-range coupling (repulsive) $g_S = -1.34M$
- Two sets of short-range interaction $M_S/M = 2, 10$

➤ Energy levels in a finite volume

- Lattice size $ML = 3$
- Ground state is below threshold
- Partial-wave mixing in the excited states

$$M_S/M = 10 \quad ML = 3$$



04 Numerical result

Modified zeta functions

➤ Definition of zeta functions

- ΔH : modified zeta function

$$\Delta H_{\ell t, \ell' t'}^{(\Gamma)}(q_0^2) = \frac{4\pi}{L^6} \sum_{\mathbf{p}, \mathbf{q}} \mathcal{Y}_{\ell t, \alpha}^{(\Gamma)*}(\mathbf{p}) G_L(\mathbf{p}, \mathbf{q}; q_0^2) \mathcal{Y}_{\ell' t', \alpha}^{(\Gamma)}(\mathbf{q}) - 4\pi \Re \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \mathcal{Y}_{\ell t, \alpha}^{(\Gamma)*}(\mathbf{p}) G_L^\infty(\mathbf{p}, \mathbf{q}; q_0^2) \mathcal{Y}_{\ell' t', \alpha}^{(\Gamma)}(\mathbf{q})$$

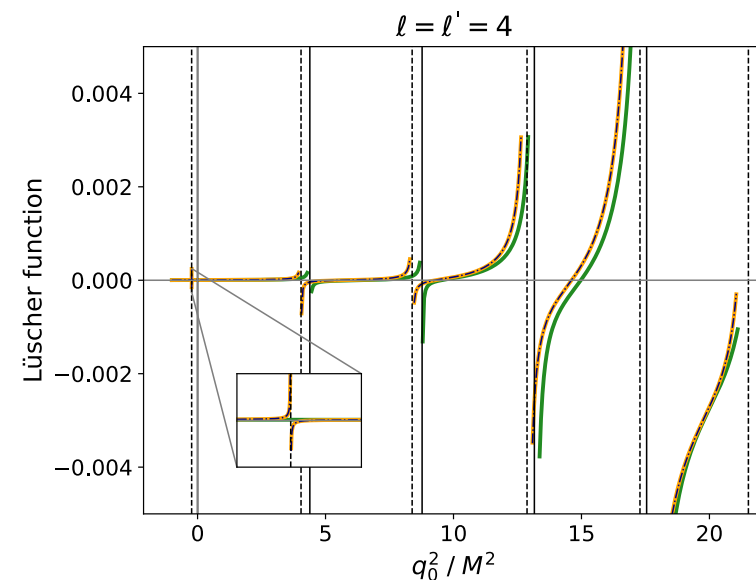
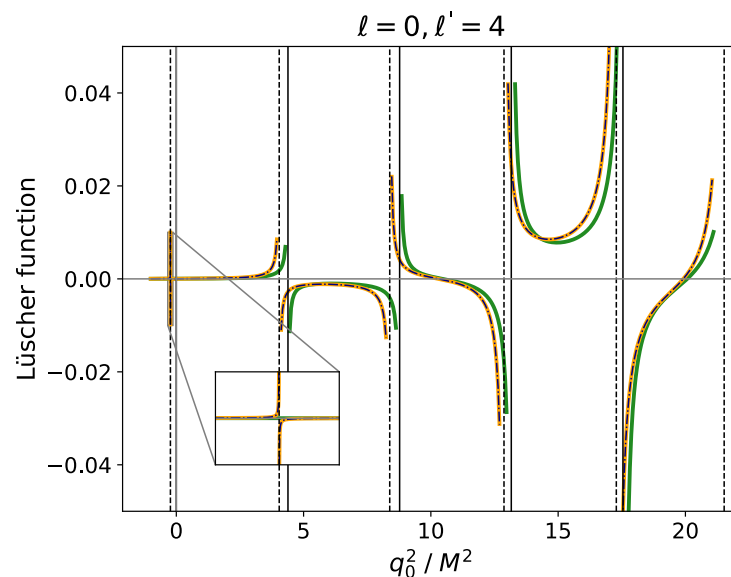
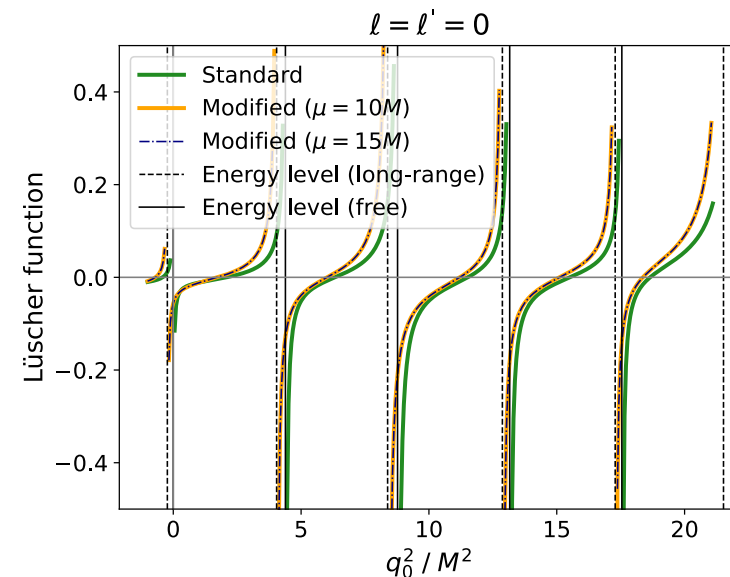
- Z : standard zeta function

$$Z_{\ell t, \ell' t'}^{(\Gamma)}(q_0^2) = 4\pi \left(\frac{1}{L^3} \sum_{\mathbf{p}} -\text{PV} \int \frac{d^3 p}{(2\pi)^3} \right) \frac{\mathcal{Y}_{\ell t, \alpha}^{(\Gamma)*}(\mathbf{p}) \mathcal{Y}_{\ell' t', \alpha}^{(\Gamma)}(\mathbf{p})}{p^2 - q_0^2}$$

- CMS A_1^+ -irreps (implicity index t, t' and bases index are trivial)

➤ Property of zeta functions

- Modified zeta function is singular at long-range energy level
- Standard zeta function is singular at free energy level
- Modified zeta function is μ -indept.



04 Numerical result

Modified Lüscher formula

➤ Quantization condition $\det \mathcal{A}^{(\Gamma)}(q_0^2) = 0$

- Modified quantization matrix

$$\mathcal{A}_{\ell t, \ell' t'}^{(\Gamma)}(q_0^2) = \frac{K_{\ell}^M(q_0^2) - \Re M_{\ell}(q_0)}{4\pi[(2\ell+1)!!]^2} \delta_{\ell t, \ell' t'} - \Delta H_{\ell t, \ell' t'}^{(\Gamma)}(q_0^2)$$

- Standard quantization matrix

$$\mathcal{A}_{\ell t, \ell' t'}^{(\Gamma)}(q_0^2) = \frac{q_0^{2\ell+1} \cot \delta_{\ell}}{4\pi} \delta_{\ell t, \ell' t'} - Z_{\ell t, \ell' t'}^{(\Gamma)}(q_0^2)$$

- A_1^+ -irreps truncated at S -wave

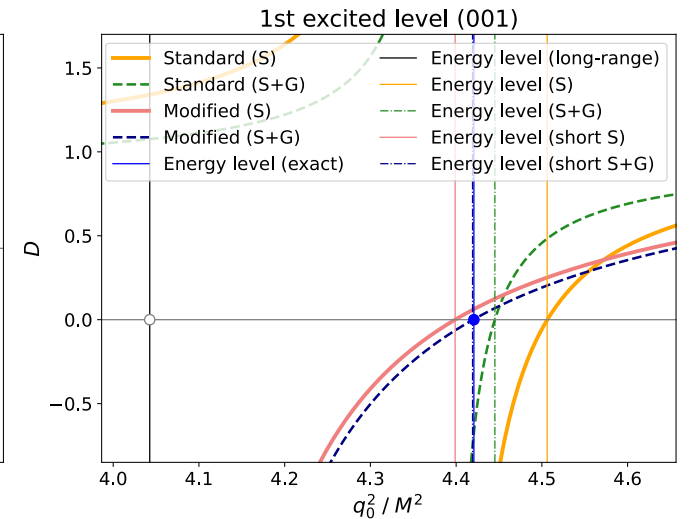
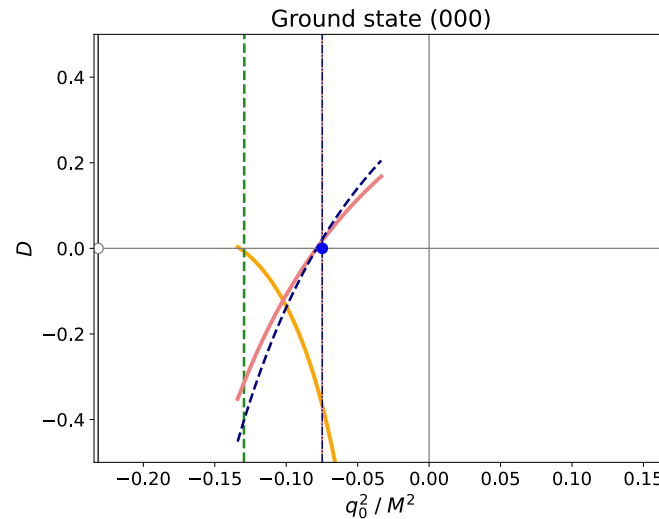
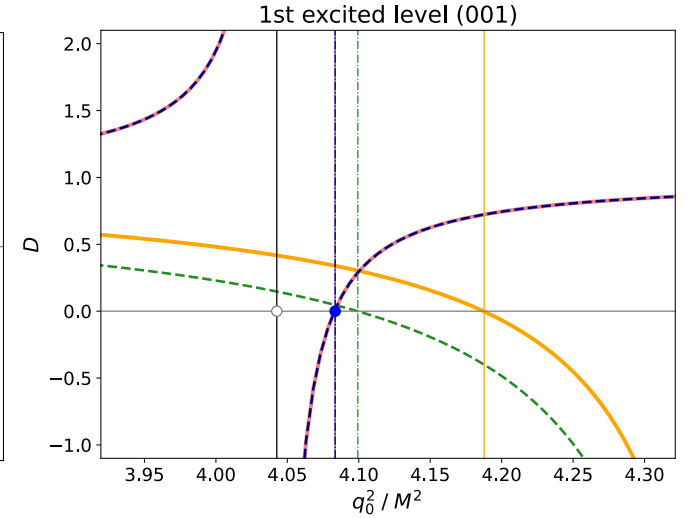
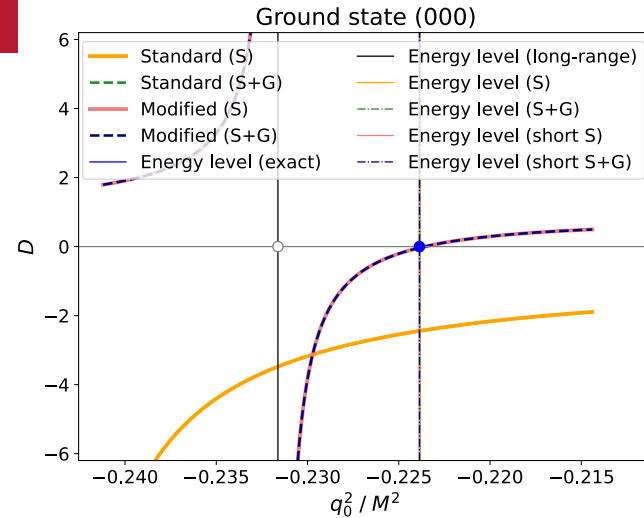
$$\text{Modified } \begin{cases} D_S^{(A_1^+)} = \mathcal{K}_S - \Delta H_S^{(A_1^+)} \\ \Delta H_S^{(A_1^+)}(q_0^2) = \Delta H_{00}^{(A_1^+)}(q_0^2) \end{cases} \quad \left\{ \begin{array}{l} \mathcal{K}_S(q_0^2) = \frac{q_0 \cot(\delta_0 - \sigma_0)}{4\pi|f_0(q_0)|^2} \\ \Delta H_{00}^{(A_1^+)}(q_0^2) = \Delta H_{00}^{(A_1^+)}(q_0^2) \end{array} \right.$$

$$\text{Standard } D_S^{(A_1^+)} = \frac{q_0 \cot \delta_0}{4\pi} - Z_S^{(A_1^+)} \quad \text{with } Z_S^{(A_1^+)}(q_0^2) = Z_{00}^{(A_1^+)}(q_0^2)$$

- A_1^+ -irreps truncated at G -wave

$$\text{Modified } \begin{cases} D_{S+G}^{(A_1^+)} = \mathcal{K}_S - \Delta H_{S+G}^{(A_1^+)} \\ \Delta H_{S+G}^{(A_1^+)}(q_0^2) = \Delta H_{00}^{(A_1^+)}(q_0^2) + \Delta H_{44}^{(A_1^+)}(q_0^2) \frac{\mathcal{K}_S}{\mathcal{K}_G} - \frac{\Delta H_{00}^{(A_1^+)} \Delta H_{44}^{(A_1^+)} - (\Delta H_{04}^{(A_1^+)})^2}{\mathcal{K}_G} \end{cases} \quad \text{with } \mathcal{K}_G(q_0^2) = \frac{q_0^9 \cot(\delta_4 - \sigma_4)}{4\pi[9!!]^2|f_4(q_0)|^2}$$

$$\text{Standard } D_{S+G}^{(A_1^+)} = \frac{q_0 \cot \delta_0}{4\pi} - Z_{S+G}^{(A_1^+)} \quad \text{with } Z_{S+G}^{(A_1^+)}(q_0^2) = Z_{00}^{(A_1^+)}(q_0^2) + Z_{44}^{(A_1^+)}(q_0^2) \frac{q_0 \cot \delta_0}{q_0^9 \cot \delta_4} - \frac{4\pi}{q_0^9 \cot \delta_4} \left[Z_{00}^{(A_1^+)} Z_{44}^{(A_1^+)} - (Z_{04}^{(A_1^+)})^2 \right]$$



04 Numerical result

Modified Lüscher formula

➤ Quantization condition

- A_1^+ -irreps truncated at S -wave

Modified $D_S^{(A_1^+)} = \mathcal{K}_S - \Delta H_S^{(A_1^+)}$ Standard $D_S^{(A_1^+)} = \frac{q_0 \cot \delta_0}{4\pi} - Z_S^{(A_1^+)}$

- A_1^+ -irreps truncated at G -wave

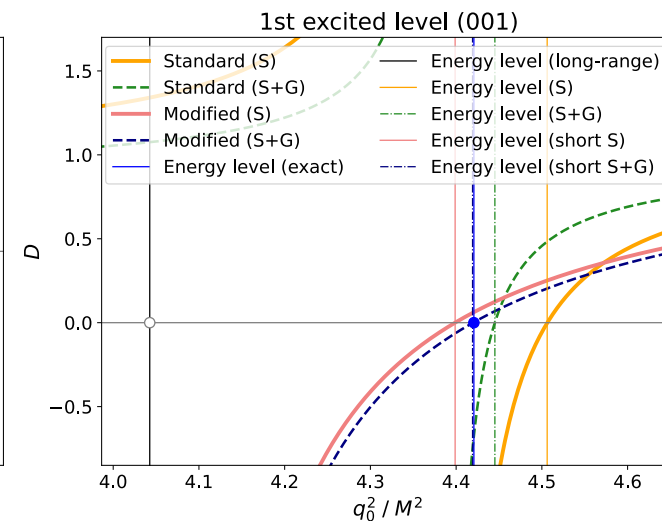
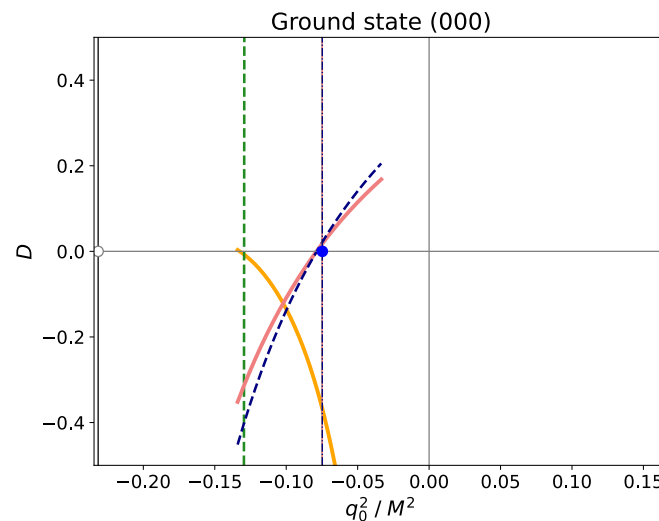
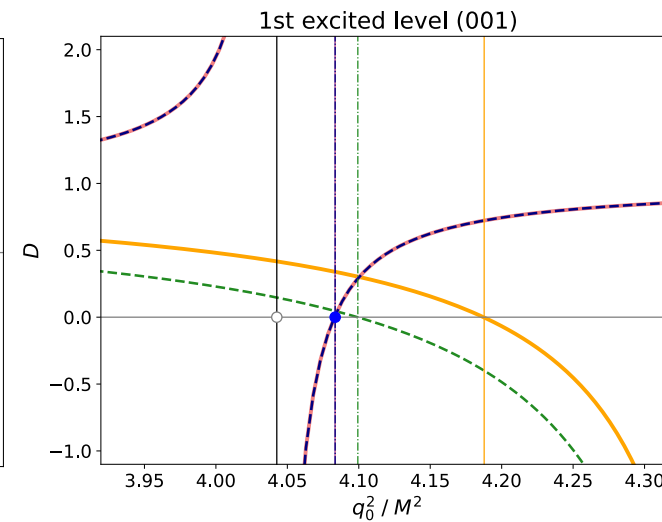
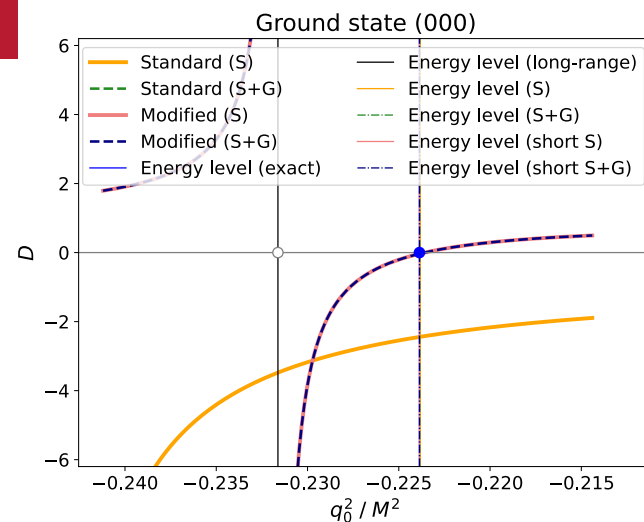
Modified $D_{S+G}^{(A_1^+)} = \mathcal{K}_{S+G} - \Delta H_{S+G}^{(A_1^+)}$ Standard $D_{S+G}^{(A_1^+)} = \frac{q_0 \cot \delta_0}{4\pi} - Z_{S+G}^{(A_1^+)}$

➤ Ground state

- Left-handed branch cut began with $q_0^2/M^2 = -0.25$
- Standard Lüscher formula does not work near left-handed cut
- Modified Lüscher formula predicts the ground state exactly and shows good convergence of partial-wave expansion

➤ 1st excited state

- For $M_S / M = 10$, modified Lüscher formula predicts the 1st excited state and shows good convergency of partial-wave expansion however standard one shows serious partial-wave mixing
- For $M_S / M = 2$, there is limited partial-wave mixing in modified Lüscher formula but it still works well up to G -wave



04 Numerical result

Modified phase shift term

➤ Quantization condition $\det \mathcal{A}^{(\Gamma)}(q_0^2) = 0$

- Modified quantization matrix

$$\mathcal{A}_{\ell t, \ell' t'}^{(\Gamma)}(q_0^2) = \frac{K_\ell^M(q_0^2) - \Re M_\ell(q_0)}{4\pi[(2\ell+1)!!]^2} \delta_{\ell t, \ell' t'} - \Delta H_{\ell t, \ell' t'}^{(\Gamma)}(q_0^2)$$

- Standard quantization matrix

$$\mathcal{A}_{\ell t, \ell' t'}^{(\Gamma)}(q_0^2) = \frac{q_0^{2\ell+1} \cot \delta_\ell}{4\pi} \delta_{\ell t, \ell' t'} - Z_{\ell t, \ell' t'}^{(\Gamma)}(q_0^2)$$

➤ Advantage of modified phase shift term

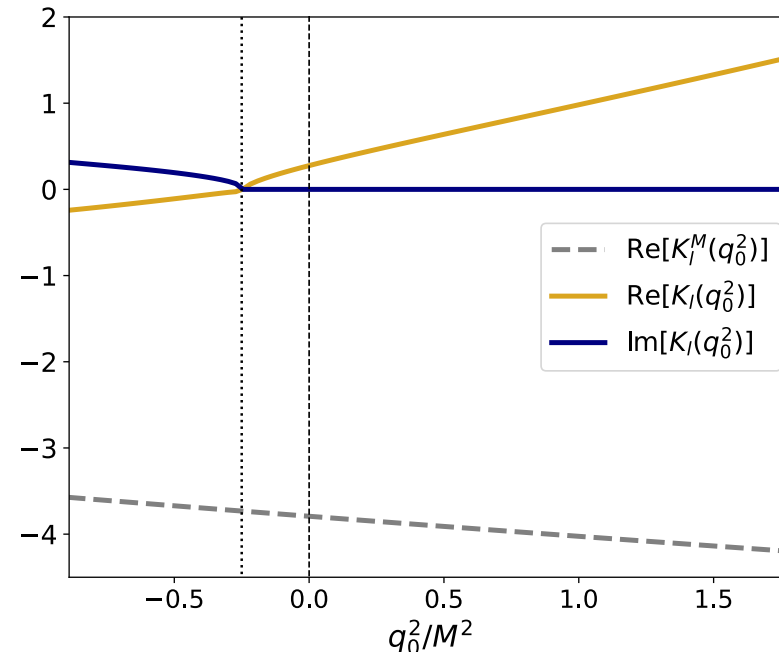
- Modified phase shift term

$$K_\ell^M(q_0^2) = M_\ell(q_0) + \frac{q_0^{2\ell+1} \cot(\delta_\ell - \sigma_\ell) - i q_0^{2\ell+1}}{|f_\ell(q_0)|^2} = -\frac{1}{\tilde{a}_\ell} + \frac{1}{2} \tilde{r}_\ell q_0^2 + O(q_0^4)$$

(much larger convergence radius, analytic continue below left-handed cut)

- Standard quantization matrix

$$q_0^{2\ell+1} \cot \delta_\ell = -\frac{1}{a_\ell} + \frac{1}{2} r_\ell q_0^2 + O(q_0^4) \quad (\text{poor convergence for long-range force and not work below left-handed cut})$$



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05 Conclusion and outlook

➤ Build up modified Lüscher formula

- Extract correct physical information near left-hand cut
- Fast convergency in partial-wave expansion
- Modified effective range expansion has much larger convergence radius

➤ Give fast convergent method to calculate modified zeta function

➤ Apply to OPE potential

➤ Outlook: apply modified Lüscher formula to NN and T_{cc}^+ (3875)

谢谢各位老师和同学！

