# The long-range force in a finite volume

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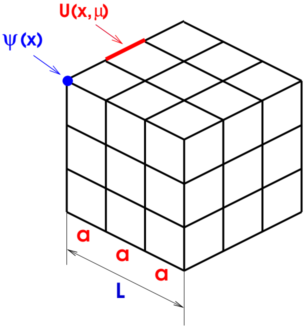
# Plan

- Introduction
- Scale separation and the choice of the EFT
- Quantization condition: Lüscher equation, three-body sector
- Two- and three-body decays
- Implications of the relativistic invariance
- Long-range forces in a finite volume (JHEP 05 (2024) 168, arXiv:2507.18399)
- Conclusions, outlook

# QCD on the lattice

In QCD, the structure of hadrons and their interactions at low energies cannot be studied in perturbation theory $\rightarrow$ *QCD on the lattice.*

- Fermion fields $\bar{\psi}(x)$, $\psi(x)$, gluon field $U(x,\mu)$ are defined on an Euclidean lattice.
- The QCD action is discretized.
- The path integral for a *finite number* of the degrees of freedom reduces to the ordinary integral, is calculated by using the Monte-Carlo technique.
- The continuum limit $a \to 0$ and the infinite-volume limit $L \to \infty$ have to be performed.



# Calculation of the spectrum of stable particles

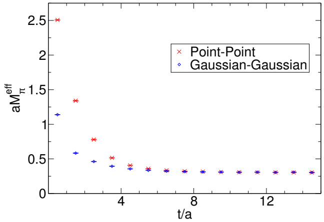
- The Euclidean path integral

$$D(t) = \sum_{\boldsymbol{x}} \langle 0 | T\Phi(t, \boldsymbol{x})\Phi^\dagger(0, \mathbf{0}) | 0 \rangle = \frac{\int \mathscr{D}U \mathscr{D}\psi \mathscr{D}\bar{\psi} e^{-S} \sum_{\boldsymbol{x}} \Phi(t, \boldsymbol{x}) \Phi^\dagger(0)}{\int \mathscr{D}U \mathscr{D}\psi \mathscr{D}\bar{\psi} e^{-S}}$$

- If $t \to \infty$, then

$$D(t) \to |\langle 0|\Phi(0)|n\rangle|^2 e^{-E_n t} + \cdots$$

$$M_{\text{eff}}(t) = \ln \frac{D(t)}{D(t+a)} \to aE_n + \cdots$$

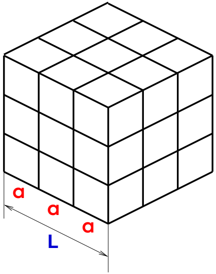



S. Dürr *et al.*, Science 322 (2008) 1224

# "Scattering" in a finite volume

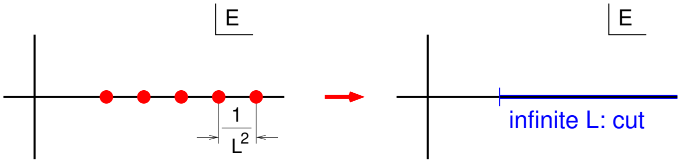
- No-go theorem: The scattering $S$-matrix elements cannot be directly extracted from the amplitudes in the Euclidean theory (Maiani and Testa, 1990)
- Impose (periodic) boundary conditions
- The spatial size of the box, $L$, is finite
- Assume the temporal size $L_t \gg L$, $L_t \to \infty$
- Three-momenta are quantized $\boldsymbol{p} = \dfrac{2\pi}{L}\boldsymbol{n}$, $\quad \boldsymbol{n} \in \mathbb{Z}^3$
- Discrete energy levels: $E_{n+1} - E_n = O(L^{-2})$
- In a finite volume, the three-momentum is quantized

  $\hookrightarrow$ states lying above threshold can be reached



# There is no free lunch...

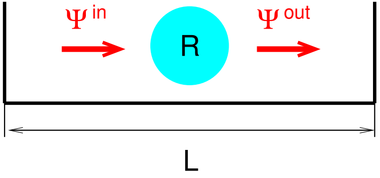
- The structure of spectrum is different in a finite and infinite volume:




- No asymptotic scattering states in a finite volume
- No regular infinite-volume limit at fixed energy for the calculated matrix elements

How does one extract the scattering observables:
phase shifts, cross sections, resonance poles,...
from the measured quantities on the lattice?

# The place where lattice meets NREFT




Scale separation: use EFT to describe the large-distance behavior of hadrons:

- When $R \ll L$, well-separated hadrons can be formed, $\Psi_{in/out}$ are close to asymptotic states
- Justifying the use of the non-relativistic EFT: since $p \sim 1/L$ and $R \sim 1/m$, then $p \ll m$

  Polarization effects, caused by creation/annihilation of the particles, are exponentially small and can be neglected

# Non-relativistic EFT: essentials

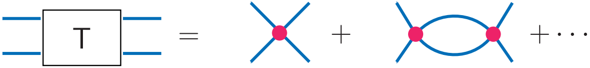
- Propagator:

$$\frac{1}{m^2-p^2} = \underbrace{\frac{1}{2w(\boldsymbol{p})(w(\boldsymbol{p})-p^0-i\varepsilon)}}_{\text{particle}} + \underbrace{\frac{1}{2w(\boldsymbol{p})(w(\boldsymbol{p})+p^0-i\varepsilon)}}_{\text{anti-particle, integrated out}}$$

- The vertices in the Lagrangian conserve particle number:

$$\mathscr{L} = \phi^\dagger(i\partial_t - w)(2w)\phi + \underbrace{\frac{C_0}{4}\phi^\dagger\phi^\dagger\phi\phi + \frac{D_0}{36}\phi^\dagger\phi^\dagger\phi^\dagger\phi\phi\phi + \cdots}_{\text{relativistic-invariant}}$$

- Only bubble diagrams:




$K$-matrix

# Matching of the EFT couplings

- Couplings $C_0, D_0, \ldots$ describe the short-range physics in the two- and three-particle sectors.
- Matching: using dimensional regularization and a particular renormalization prescription (threshold expansion), EFT couplings in the two-body sector can be matched to the scattering phase or effective range expansion parameters:

$$K^{-1}(p) = p \cot \delta(p) = -\frac{1}{a} + \frac{1}{2} r p^2 + O(p^4)\,, \qquad C_0 \leftrightarrow a\,, \ldots$$

- Matching can be performed in a relativistic-invariant manner, despite integrating out antiparticles
- Crucial point: $R \ll L$, the energy spectrum can be calculated by using the same EFT in a finite volume (decoupling theorem)

# A loop in a finite volume

- The energy spectrum is given by the poles of the $T$-matrix in a finite volume

$$\int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \to \frac{1}{L^3}\sum_{\boldsymbol{k}}, \qquad \boldsymbol{k}_n = \frac{2\pi}{L}\boldsymbol{n}, \ \ \boldsymbol{n} \in \mathbb{Z}^3$$

- Loop diagram in a finite volume

$$ip_0 \to \quad \frac{2}{\sqrt{\pi}L\gamma} Z_{00}^{\boldsymbol{P}}(1; q_0^2)\,, \qquad q_0 = \frac{p_0 L}{2\pi}$$

($Z_{00}$ is an irregular function, poles at free two-particle energies)

# The Lüscher equation (Lüscher, 1991)

- The Lüscher equation (in the absence of partial-wave mixing):

$$T \propto \frac{1}{p\cot\delta(p) - ip} \to \frac{1}{p\cot\delta(p) - \frac{2}{\sqrt{\pi}L\gamma} Z_{00}^{\boldsymbol{P}}(1;q_0^2)}$$

$$\hookrightarrow \quad \underbrace{p\cot\delta(p)}_{\text{short-range}} = \frac{2}{\sqrt{\pi}L\gamma} \quad \underbrace{Z_{00}^{\boldsymbol{P}}(1;q_0^2)}_{\text{geometry of a box}}$$

$\hookrightarrow$ measuring energy levels, one extracts phase shift at the same energy

- Relativistic-invariant: can be used in moving frames $\boldsymbol{P} \neq 0$
- Resonances: analytic continuation into the complex plane

NREFT serves as a bridge between finite and infinite volume

# EFT in a finite folume: Sino-German collaboration

1. R. Bubna, H.-W. Hammer, B.-L. Hoid, J.-Y. Pang, A. Rusetsky and J.-J. Wu, “Modified Lüscher zeta-function and the modified effective range expansion in the presence of a long-range force,” [arXiv:2507.18399 [hep-lat]].
2. R. Bubna, H.-W. Hammer, F. Müller, J.-Y. Pang, A. Rusetsky and J.-J. Wu, “Lüscher equation with long-range forces,” JHEP **05** (2024) 168.
3. J.-Y. Pang, R. Bubna, F. Müller, A. Rusetsky and J.-J. Wu, “Lellouch-Lüscher factor for the $K \to 3\pi$ decays,” JHEP **05** (2024) 269.
4. F. Müller, J. Y. Pang, A. Rusetsky and J. J. Wu, “Three-particle Lellouch-Lüscher formalism in moving frames,” JHEP **02** (2023) 214.
5. J. Y. Pang, M. Ebert, H.-W. Hammer, F. Müller, A. Rusetsky and J. J. Wu, “Spurious poles in a finite volume,” JHEP **07** (2022) 019.
6. F. Müller, J.-Y. Pang, A. Rusetsky and J.-J. Wu, “Relativistic-invariant formulation of the NREFT three-particle quantization condition,” JHEP **02** (2022) 158.
7. J.-Y. Pang, J.-J. Wu, H.-W. Hammer, U.-G. Meißner and A. Rusetsky, “Energy shift of the three-particle system in a finite volume,” Phys. Rev. D **99** (2019) 074513.

# EFT in a finite folume: Sino-German collaboration

1. M. Döring, H.-W. Hammer, M. Mai, J.-Y. Pang, A. Rusetsky and J.-J. Wu, "Three-body spectrum in a finite volume: the role of cubic symmetry," Phys. Rev. D **97** (2018) 114508.
2. Y. Meng, C. Liu, U.-G. Meißner and A. Rusetsky, "Three-particle bound states in a finite volume: unequal masses and higher partial waves," Phys. Rev. D **98** (2018) 014508.
3. H.-W. Hammer, J.-Y. Pang and A. Rusetsky, "Three particle quantization condition in a finite volume: 2. general formalism and the analysis of data," JHEP **10** (2017) 115.
4. H.-W. Hammer, J.-Y. Pang and A. Rusetsky, "Three-particle quantization condition in a finite volume: 1. The role of the three-particle force," JHEP **09** (2017) 109.
5. Z.-H. Guo, L. Liu, U.-G. Meißner, J.-A. Oller and A. Rusetsky, "Chiral study of the $a_0(980)$ resonance and $\pi\eta$ scattering phase shifts in light of a recent lattice simulation," Phys. Rev. D **95** (2017) 054004.
6. D. Agadjanov, F.-K. Guo, G. Ríos and A. Rusetsky, "Bound states on the lattice with partially twisted boundary conditions," JHEP **01** (2015) 118.

## From two to three particles

Why three particles on the lattice?

- Three-pion decays of $K, \eta, \omega$
- $a_1(1260) \to \rho\pi \to 3\pi$ and $a_1(1420) \to f_0(980)\pi \to 3\pi$
- Properties of exotica: $T_{cc}^+(3875)$ ($DD^*$ scattering), $X(3872)$ ($D\bar{D}^*$ scattering), ...
- Roper resonance: $\pi N$ and $\pi\pi N$ final states
- Few-body physics: reactions with the light nuclei

# Lattice vs. continuum: observables

Infinite volume:

- Three-particle bound states
- Elastic scattering
- Rearrangement reactions, breakup
- The mass and width of the three-particle resonances

Finite volume:

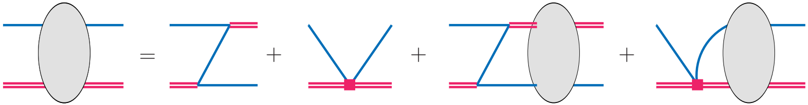
- Two- and three-particle energy levels

How does one connect these two sets? EFT serves as a bridge!

# Three-particle quantization condition

- Three different but equivalent formulations of the three-particle quantization condition are available
  - RFT (Relativistic Field Theory): Hansen & Sharpe, 2014
  - NREFT (Non-Relativistic Effective Field Theory): Hammer, Pang & AR, 2017, see also K. Polejaeva and AR, 2012
  - FVU (Finite-Volume Unitarity): Mai & Döring, 2017
- Enables one to extract scattering observables in the three-body sector from the measured finite-volume spectrum

# The scattering equation in the infinite volume (CM frame)



Bethe-Salpeter equation

$$\mathcal{M}(\boldsymbol{p},\boldsymbol{q};E) = Z(\boldsymbol{p},\boldsymbol{q};E) + 8\pi\int^{\Lambda}\frac{d^3\boldsymbol{k}}{(2\pi)^3 2w(\boldsymbol{k})}\,Z(\boldsymbol{p},\boldsymbol{k};E)\tau(\boldsymbol{k};E)\mathcal{M}(\boldsymbol{k},\boldsymbol{q};E)$$

$$Z(\boldsymbol{p},\boldsymbol{q};E) = \frac{1}{2w(\boldsymbol{p}+\boldsymbol{q})(w(\boldsymbol{p})+w(\boldsymbol{q})+w(\boldsymbol{p}+\boldsymbol{q})-E)} + \tilde{H}_0 + \cdots$$

2-body amplitude (dimer): $4w(k^*)\tau^{-1}(\boldsymbol{k};E) = k^*\cot\delta(k^*) + \underbrace{\sqrt{\frac{s_2}{4}-m^2}}_{=k^*}$

# Finite volume

$$\mathcal{M}_L(\boldsymbol{p},\boldsymbol{q};E) = Z(\boldsymbol{p},\boldsymbol{q};E) + \frac{8\pi}{L^3}\sum_{\boldsymbol{k}}^{\Lambda} Z(\boldsymbol{p},\boldsymbol{q};E)\tau_L(\boldsymbol{k};E)\mathcal{M}_L(\boldsymbol{k},\boldsymbol{q};E)$$

$$\tau_L^{-1}(\boldsymbol{k};E) = k^*\cot\delta(k^*) - \frac{4\pi}{L^3}\sum_{\boldsymbol{l}} \frac{1}{\boldsymbol{k}^2+\boldsymbol{l}^2+\boldsymbol{k}\boldsymbol{l}-mE}$$

- Poles in the amplitude $\rightarrow$ finite-volume energy spectrum
- Quantization condition: $\det(\tau_L^{-1} - Z) = 0$

Workflow:

- Two-body interactions as an input: $k^*\cot\delta(k^*)$ fitted in the two-particle sector
- Extracting short-range quantities encoded in the three-body couplings $\tilde{H}_0,\ldots$ – should be fitted to the three-particle energies
- Finally, solve the equations in the infinite volume to arrive at the $S$-matrix elements!

# Relativistic invariance in the three-particle sector

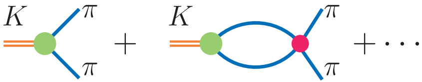
- $\hookrightarrow$ Three-dimensional formalism, manifest Lorentz invariance is lost, even in the infinite volume!
- $\hookrightarrow$ Only Lorentz-invariant operators in the Lagrangian?
- $\hookrightarrow$ Proliferation of the independent couplings that should be extracted from lattice data in different moving frames?
  - In two-particle sector, the problem is solved by dim.reg.+threshold expansion. Sectors with more particles?

A manifestly Lorentz-invariant formulation of the three-particle scattering equations can be found even in the absence of anti-particles:

F. Müller, J.-Y. Pang, AR and J.-J. Wu JHEP 02 (2022) 158

# Two-particle decays: the Lellouch-Lüscher formula (Lellouch & Lüscher, 2001)

- Final-state interactions lead to an irregular $L$-dependence of the matrix element




- The non-relativistic Lagrangian

$$\mathscr{L} = \phi^\dagger(i\partial_t - w)(2w)\phi + \frac{C_0}{4}\phi^\dagger\phi^\dagger\phi\phi + \cdots + K^\dagger(i\partial_t - w_K)(2w_K)K$$
$$+ \; g(K^\dagger\phi\phi + \text{h.c.})$$

- Calculate the decay matrix element in a finite and in the infinite volume, extract $g$
- Matrix elements are related through

$$\langle n|H_W|K\rangle_L = \underbrace{\Phi_2(L)}_{\textit{depends on phase shift}} \langle\pi\pi; out|H_W|K\rangle_\infty$$

# Three-particle decays

(F. Müller and AR, JHEP 03 (2021) 152, F. Müller, J.-Y. Pang, AR and J.-J.Wu, JHEP 02 (2023) 214)

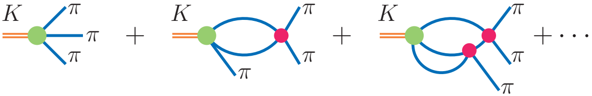
a) Decays through the weak or electromagnetic interactions; isospin-breaking decays: pole on the real axis

   Example: $K \to 3\pi$

b) Decays through strong interactions, the pole moves into the complex plane

   Example: $N(1440) \to \pi\pi N$

- Final-state interactions lead to the irregular volume-dependence in the matrix element




An analog of the LL formula in the three-particle sector?

# The 3-particle LL factor

$$\langle \pi(k_1)\pi(k_2)\pi(k_3); out|H_W|K\rangle_\infty = \Phi_3(\{k\})L^{3/2}\langle n|H_W|K\rangle_L$$

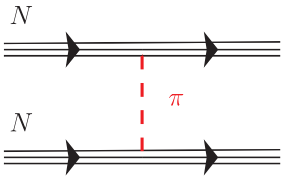
- The factor $\Phi_3(\{k\})$ depends on the $\pi\pi$, $\pi\pi\pi$ interactions and on $L$, but not on the coupling $g_0$ that describes the short-range part of the $K \to 3\pi$ amplitude!
- The derivative couplings emerge at higher orders. The three-particle LL factor becomes a matrix
- Higher partial waves, derivative couplings, isospin, ... can be systematically taken into account

## $K \to 3\pi$ decays (J.-Y. Pang, R. Bubna, F. Müller, AR and J.-J.Wu, JHEP 05 (2024) 269)

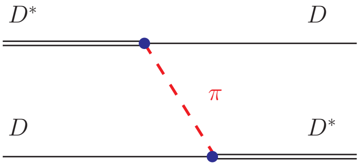
- $K \to 3\pi$ decays: important source of information about $CP$ violation
- Three pion final state: challenging but realistic
- The crucial question: how big is the contribution of the three-body force?

Contribution of the (unknown) three-body force to the LL factor is very small. A crude estimate suffices to reliable extract the $K \to 3\pi$ amplitude from lattice data!

# Inclusion of the long-range forces: change of the paradigm?



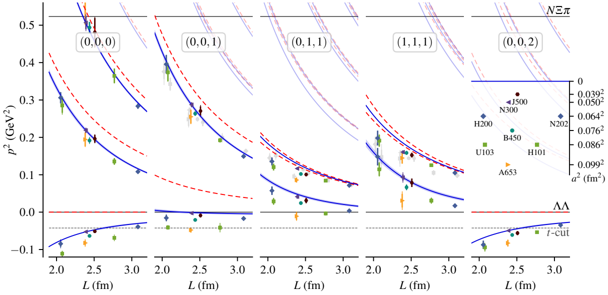

$NN$ scattering




$T_{cc}^{+}(3875)$

- Scale separation upset!
- Left hand cut close to threshold: the energy levels below the left-hand branch point cannot be used
- Slowly converging partial-wave expansion: expecting strong admixture of higher partial waves in the quantization condition
- Exponentially suppressed corrections still sizable

# Left-hand cut: case of $NN$ scattering




J.R. Green *et al.*, PRL 127(24) (2021) 242003

$$V = \frac{1}{2}\int_{-1}^{1} d\cos\theta \frac{g^2}{M_\pi^2 + (\boldsymbol{p}-\boldsymbol{q})^2}$$

- Left-hand cut: $-\infty < s \leq \underbrace{(2m_N)^2 - M_\pi^2}_{=(1875\,\mathrm{MeV})^2}$; right-hand cut: $\underbrace{(2m_N)^2}_{=(1880\,\mathrm{MeV})^2} \leq s < +\infty$
- Phase shift *real* below the left-hand branch point?

# Plane-wave basis (Meng & Epelbaum, 2021)

- Describe the system in terms of the parameters of the effective Lagrangian which, by definition, encode only faraway singularities
- Work in the plane wave basis; do not resort to the partial-wave expansion
  - For the *NN* scattering, it was shown that, at the physical quark masses, the partial-wave mixing is sizable (Meng & Epelbaum, 2021)
  - A consistent fit of the $DD^*$ scattering phases to lattice data in the left-hand cut region has been performed (Meng *et al.*, 2023)

# Alternative approaches

- Splitting long- and short-range interactions (Hansen & Raposo, 2023)
  - Fit short-range part to the scattering data, get full amplitude through solving integral equations
  - Quantization condition is written down both in the plane-wave basis and the partial-wave basis
- Applying three-particle formalism to the $DD\pi$ system (Hansen, Romero-Lopez and Sharpe, 2024)
  - Two-particle quantization condition for a stable $D^*$
  - Plane wave basis is used
- Using Lüscher equation plus EFT with long-range force in the infinite volume above the left-hand cut (Collins *et al.*, 2024)
- Using $N/D$ method (Du, Guo & Wu, 2024; S. Dawid *et al.*, 2025)
- HAL QCD approach (Lyu *et al.*, 2023)

# Modified Lüscher equation in the presence of a long-range force

R. Bubna, H.-W. Hammer, F. Müller, J.-Y. Pang, AR & J.-J. Wu, JHEP 05 (2024) 168

- Splitting of the potential

$$V(r) = \underbrace{V_L(r)}_{\text{known, local}} + \underbrace{V_S(r)}_{\text{unknown}}$$

$$\langle \boldsymbol{p}|V_S|\boldsymbol{q}\rangle = C_0^{00} + 3C_1^{00}\boldsymbol{pq} + C_0^{10}(\boldsymbol{p}^2 + \boldsymbol{q}^2) + \cdots$$

- Modified Lüscher equation: finite-volume version of the modified effective range expansion (van Haeringen & Kok, 1982)

# Modified effective range expansion (van Haeringen & Kok, 1982)

Standard effective-range expansion: very small radius of convergence...

- The long-range Jost function: $f_\ell(q) = \dfrac{q^\ell e^{-i\ell\pi/2}(2\ell+1)}{(2\ell+1)!!} \lim_{r\to 0} r^\ell f_\ell(q,r)$
- The loop function: $M_\ell(q) = \dfrac{1}{\ell!}\left(-\dfrac{iq}{2}\right)^\ell \lim_{r\to 0} \dfrac{d^{2\ell+1}}{dr^{2\ell+1}} \dfrac{f_\ell(q,r)}{f_\ell(q)}$
- Larger radius of convergence for the modified effective-range function:

$$K_\ell^M(q^2) = M_\ell(q) + \frac{q^{2\ell+1}}{|f_\ell(q)|^2}\left(\cot(\delta_\ell(q) - \sigma_\ell(q)) - i\right) = -\frac{1}{\tilde{a}_\ell} + \frac{1}{2}\tilde{r}_\ell q^2 + O(q^4)$$

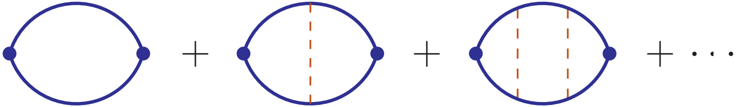
- Relation between $K_\ell^M(q^2)$ and the full phase $\delta_\ell(q)$ is *algebraic!*

# Scattering on two potentials: the EFT framework

$$
\begin{aligned}
T &= T_L + (1 + T_L G_0) T_S (1 + G_0 T_L) \\
T_S &= V_S + V_S G_L T_S
\end{aligned}
$$

- The Green function with the long-range potential only: $G_L = G_0 + G_0 V_L G_L$

# Modified Lüscher zeta-function



- Sums up all insertions of the long-range potential in a finite volume
- Modified Lüscher equation:

$$\det \mathscr{A}_{\ell m,\ell' m'} = 0\,, \qquad \mathscr{A}_{\ell m,\ell' m'} = \delta_{\ell\ell'}\delta_{mm'} K_\ell^M(q_0^2) - H_{\ell m,\ell' m'}(q_0)$$

# Implementation of the modified Lüscher formalism

R. Bubna, H.-W. Hammer, B.-L. Hoid, J.-Y. Pang, AR & J.-J. Wu, arXiv:2507.18399

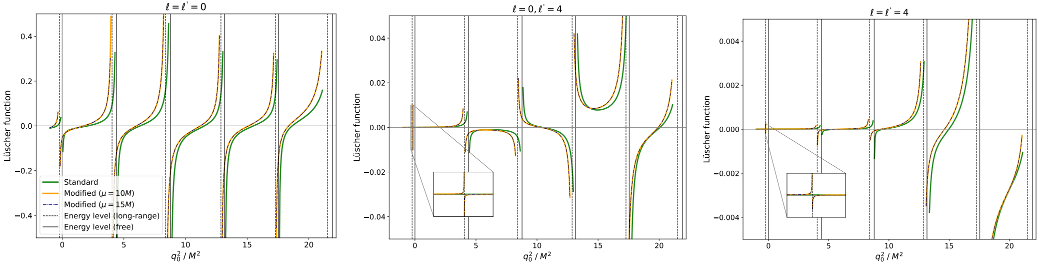
- The model: the potential is a sum of long- and short-range parts:

$$V = \frac{4\pi g}{M^2 + (\boldsymbol{p} - \boldsymbol{q})^2} + \frac{4\pi g_S}{M_S^2 + (\boldsymbol{p} - \boldsymbol{q})^2}\,, \qquad M_S = 2M \text{ or } M_S = 10M$$

- Dimensional regularization vs. cutoff regularization: renormalization constants of natural size, numerically stable results for higher partial waves.
- Perturbative apprroach for divergent loop integrals, resumming all convergent integrals via the Lippmann-Schwinger equation.

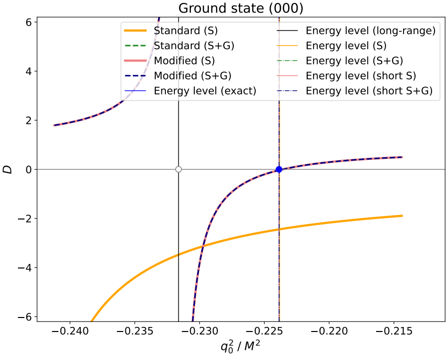
# Implementation of the modified Lüscher formalism

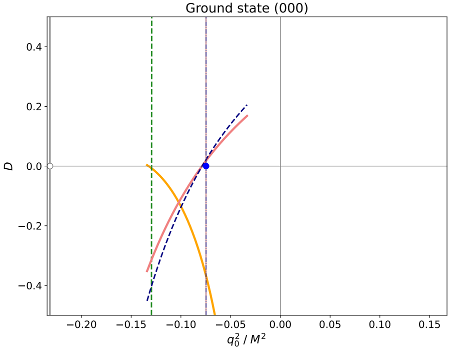
The modified Lüscher zeta-function:



# The modified vs. standard Lüscher approach

The ground-state energy:

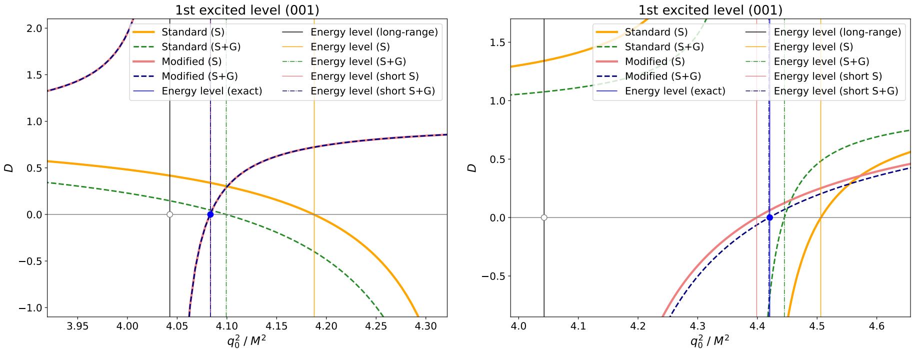



$\hookrightarrow$ Practically no partial-wave mixing in the modified approach!

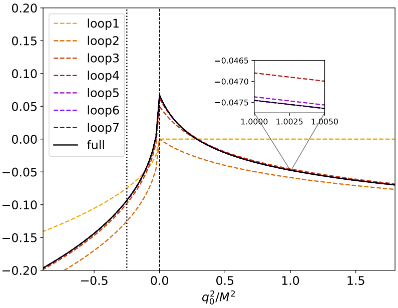
# The modified vs. standard Lüscher approach

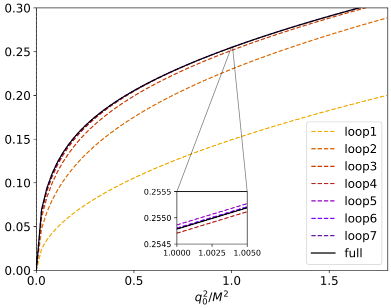
The first excited level:




$\hookrightarrow$ Practically no partial-wave mixing in the modified approach!

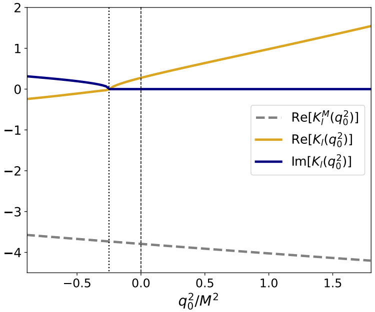
# Infinite-volume loop with any number of long-range insertions




- Perturbative apprroach for divergent loop integrals, resumming all convergent integrals via the Lippmann-Schwinger equation.
- Using VEGAS routine to perform integrals over Feynman parameters.

# The modified vs. standard Lüscher approach




- The effective-range function in the standard approach is singular at the beginning of the left-hand cut, develops the imaginary part.
- On the contrary, the effective-range function in the modified Lüscher approach is real and regular across the left-hand cut.

# Conclusions & outlook

- In the analysis of lattice data, EFT can be used to systematically relate the finite- and infinite-volume observables. This facilitates the extraction of scattering observables from lattice data in the two- and three-particle sectors
- The crucial point: decoupling of short- and long-range physics
- Explicitly Lorentz-invariant formalism in the two- and three-body sector: spectrum and decays ✓
- Including long-range forces in a finite volume, two-body sector ✓
- Outlook:
  - Long-range forces in the two-particle sector: Coulomb force, $C^*$ boundary conditions
  - Long-range forces in the three-particle sector
  - The Roper resonance
  - Boxed exotica: analysis of data for the $T_{cc}^+(3875)$...