



Model-independent determination of gravitational form factors from dispersion relations

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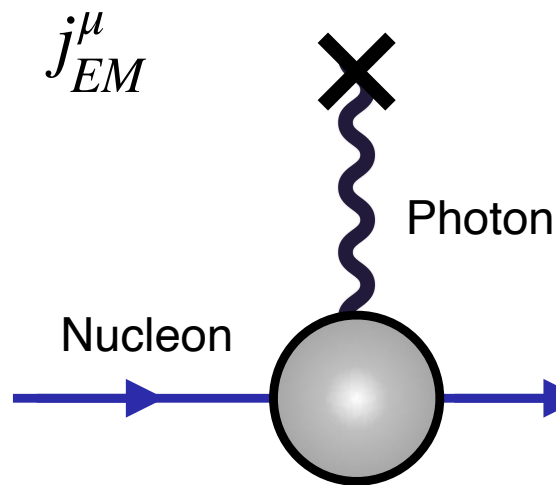
Institute of Theoretical Physics, Chinese Academy of Sciences

XHC, Feng-Kun Guo, Qu-Zhi Li & De-Liang Yao, **Dispersive determination of nucleon gravitational form factors**, *Nat. Commun.* **16**, 6979 (2025)

XHC, Feng-Kun Guo, Qu-Zhi Li, Bo-Wen Wu & De-Liang Yao, **Gravitational form factors of pions, kaons and nucleons from dispersion relations**, arXiv: 2507.05375

2025/8/22

EM interactions: Hadrons in external field



● Definition: $t \equiv q^2 = (p' - p)^2$

$$\langle N(p') | j_{EM}^{\mu} | N(p) \rangle = \bar{u}(p') \left[F_1(t) \gamma^{\mu} + i \frac{F_2(t)}{2m_N} \sigma^{\mu\nu} q_{\nu} \right] u(p)$$

● Sachs form factors: $G_E = F_1 + \frac{t}{4m^2} F_2$, $G_M = F_1 + F_2$

● Charge radius (proton):

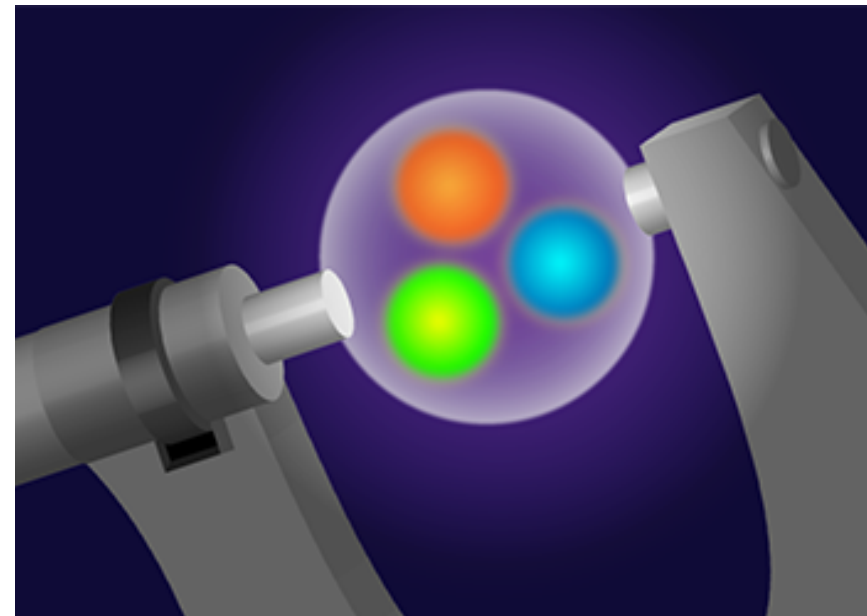
$$G_E(t) = 1 + t \langle r_C^2 \rangle / 6 + \dots$$

● Extracted from the lepton-nucleon elastic scattering or the hydrogen(-like) atom spectroscopy

“proton radius puzzle”

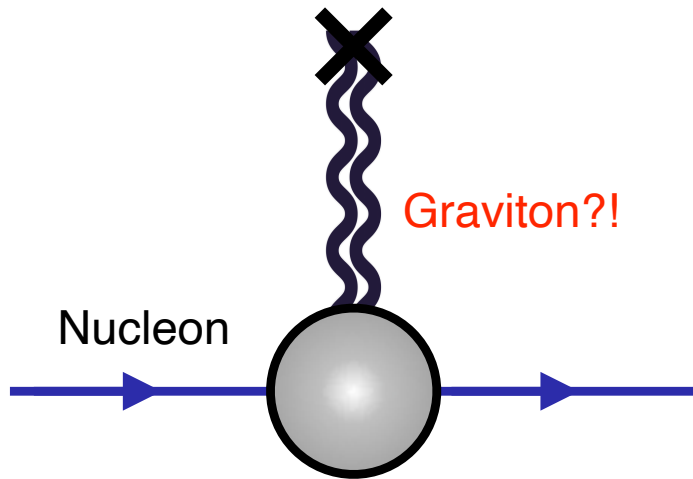
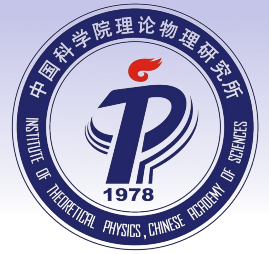
0.84 fm v.s. 0.88 fm

Gao and Vanderhaeghen, RMP (2022)



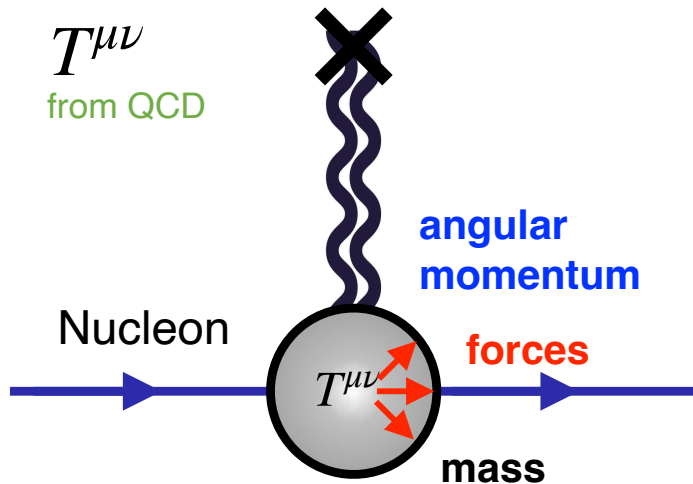
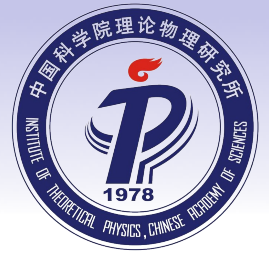
<https://physics.aps.org/articles/v12/s28>

Gravitational structure of nucleons



- No direct experiment for detection of the nucleon-graviton interaction (10^{-39} times weaker than electromagnetic interaction)

Gravitational structure of nucleons



- Gravity couples to matter through energy-momentum tensor (EMT) $T^{\mu\nu}$

$$T_{\text{QCD}}^{\mu\nu} = T_q^{\mu\nu} + T_g^{\mu\nu}$$

study gravitational forces within hadrons



study mechanical properties of hadrons



Definition:

$$\langle N(p') | T^{\mu\nu} | N(p) \rangle = \frac{1}{4m_N} \bar{u}(p') \left[A(t) P^\mu P^\nu + J(t) \left(i P^{\{\mu} \sigma^{\nu\} \rho} \Delta_\rho \right) + D(t) \left(\Delta^\mu \Delta^\nu - g^{\mu\nu} \Delta^2 \right) \right] u(p)$$

Kobzarev, Okun (1962)
Pagels (1966)

Mass normalization:

$$m_N = \int d^3r T_{00}(r)$$

$$A(0) = 1$$

Spin normalization:

$$J^i = \epsilon_{ijk} \int d^3r r_j T_{0k}(r)$$

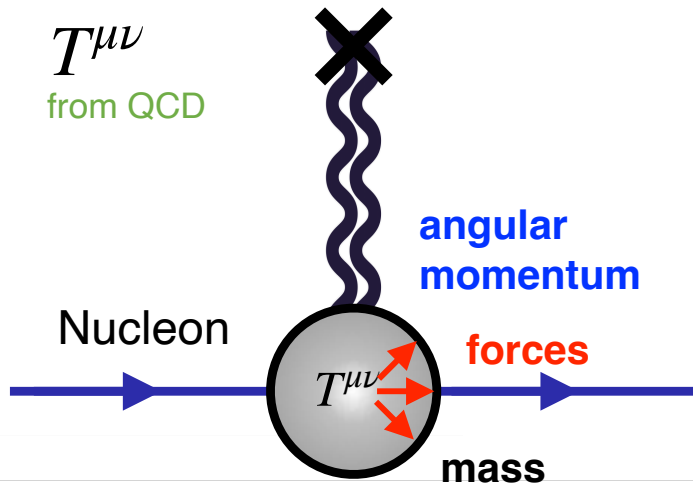
$$J(0) = 1/2$$

D-term: $D \equiv D(0)$

$$D = -\frac{m_N}{2} \int d^3r \left(r_i r_j - \frac{1}{3} \delta_{ij} \right) T_{ij}(r)$$

$$D = ?$$

Gravitational structure of nucleons



**D-term as the
“last unknown global property”**

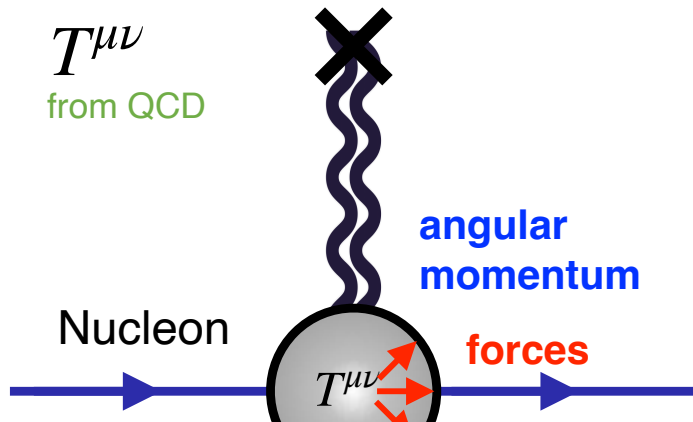
Polyakov, and Schweitzer, (2018)

em:	$\partial_\mu J_{\text{em}}^\mu = 0$	$\langle N' J_{\text{em}}^\mu N \rangle$	$\longrightarrow$	$Q = 1.602176487(40) \times 10^{-19} \text{C}$
				$\mu = 2.792847356(23) \mu_N$

weak: PCAC	$\langle N' J_{\text{weak}}^\mu N \rangle$	$\longrightarrow$	$g_A = 1.2694(28)$
			$g_p = 8.06(55)$

gravity:	$\partial_\mu T_{\text{grav}}^{\mu\nu} = 0$	$\langle N' T_{\text{grav}}^{\mu\nu} N \rangle$	$\longrightarrow$	$m = 938.272013(23) \text{ MeV}/c^2$
				$J = \frac{1}{2}$
				$D = ?$

Gravitational structure of nucleons



“..... there is very little hope of learning anything about the detailed **mechanical structure** of a particle, because of the extreme weakness of the gravitational interaction”

H. Pagels 1966

$$\text{em: } \partial_\mu J_{\text{em}}^\mu = 0 \quad \langle N' | J_{\text{em}}^\mu | N \rangle \longrightarrow Q = 1.602176487(40) \times 10^{-19} \text{C}$$

$$\mu = 2.792847356(23) \mu_N$$

$$\text{weak: PCAC} \quad \langle N' | J_{\text{weak}}^\mu | N \rangle \longrightarrow g_A = 1.2694(28)$$

$$g_p = 8.06(55)$$

$$\text{gravity: } \partial_\mu T_{\text{grav}}^{\mu\nu} = 0 \quad \langle N' | T_{\text{grav}}^{\mu\nu} | N \rangle \longrightarrow m = 938.272013(23) \text{ MeV}/c^2$$

$$J = \frac{1}{2}$$

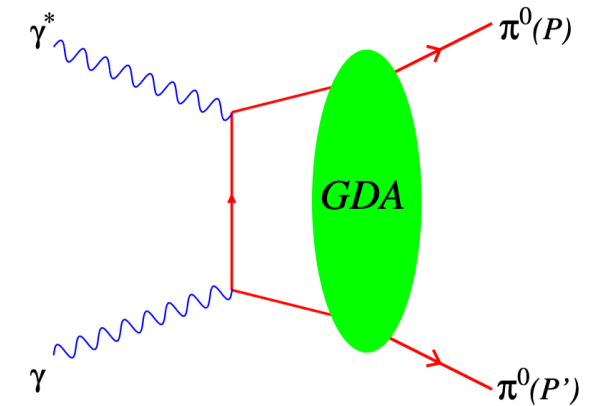
$$D = ?$$

First insights from experiment

- π^0 : $\gamma^*\gamma \rightarrow \pi^0\pi^0$ in e^+e^- Belle data: PRD 93, 032003 (2016)
 $D_{\pi^0}^Q \simeq -0.7$ at $\langle Q^2 \rangle = 16.6 \text{ GeV}^2$

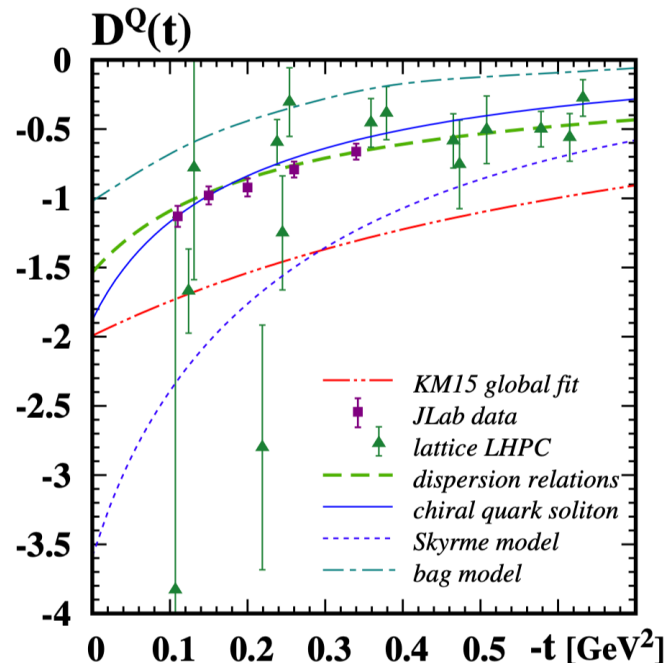
Kumano, Song, Teryaev, PRD (2018)

Chiral symmetry: total $D_{\pi^0} \simeq -1$ (gluons contribute the rest)



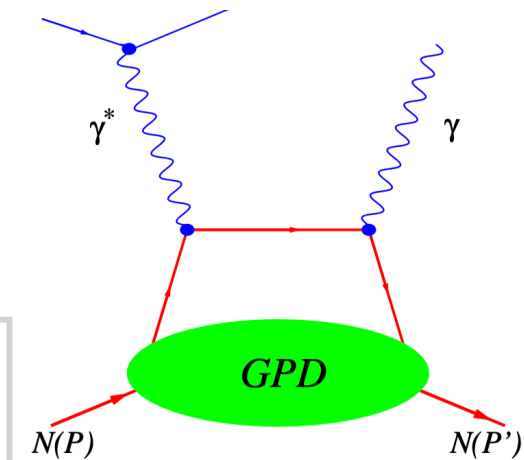
- Proton: Burkert, Elouadrhiri, Girod, Nature 557, 396 (2018)

JLab data: PRL 100, 162002 (2008) & PRL 115, 212003 (2015)



Polyakov, and Schweitzer, (2018)

subtraction function in
dispersion relations for
DVCS amplitudes



● $\Delta(t, \mu) \rightarrow D^Q(t, \mu)$ scale-dependent

model-dependent

● Explore Q^2 range at EIC & ElcC

Scalar radius

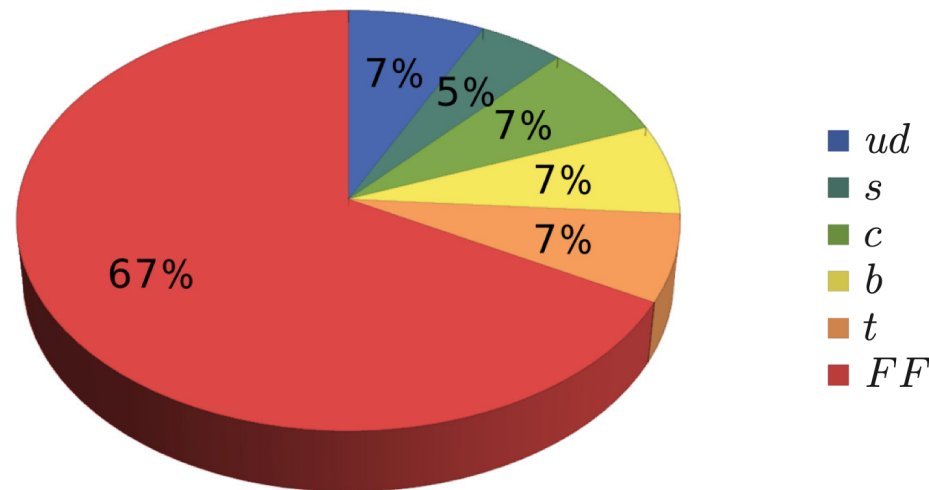


- Trace FF:

$$\left\langle N(p') \left| T_{\mu}^{\mu} \right| N(p) \right\rangle = m_N \bar{u}(p') \left[A(t) - \frac{t}{4m_N^2} [A(t) - 2J(t) + 3D(t)] \right] u(p) \equiv \bar{u}(p') \Theta(t) u(p)$$

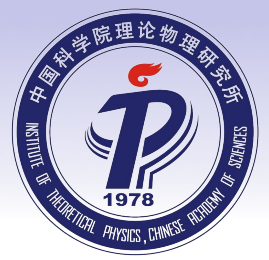
- Trace anomaly in QCD: $T_{\mu}^{\mu} = \frac{\beta(g)}{2g} F^{a,\mu\nu} F_{\mu\nu}^a + (1 + \gamma_m) \sum_q m_q \bar{\psi}_q \psi_q$

- Scalar radius: $\Theta(t) = m_N (1 + t \langle r_{\Theta}^2 \rangle / 6 + \dots)$



Hoferichter & Ruiz de Elvira, 2506.3902

Scalar radius puzzle



- Trace FF:

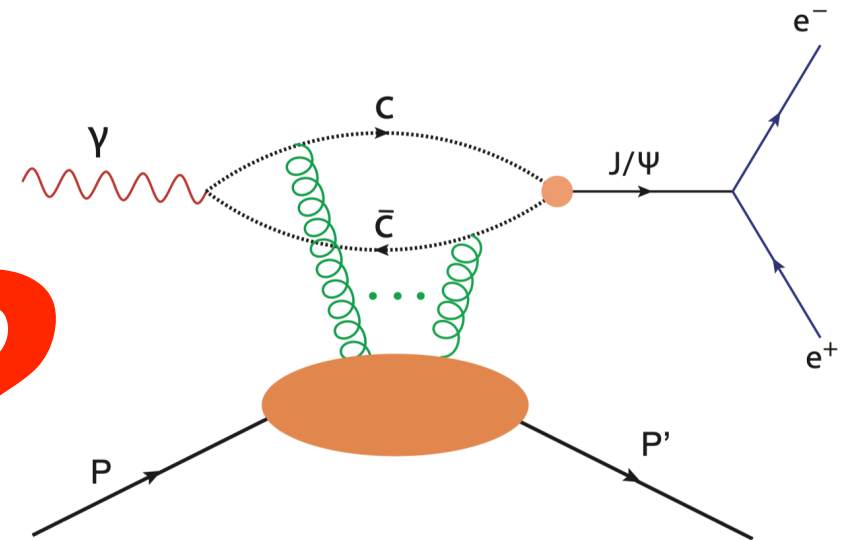
$$\left\langle N(p') \left| T_{\mu}^{\mu} \right| N(p) \right\rangle = m_N \bar{u}(p') \left[A(t) - \frac{t}{4m_N^2} [A(t) - 2J(t) + 3D(t)] \right] u(p) \equiv \bar{u}(p') \Theta(t) u(p)$$

- Trace anomaly in QCD: $T_{\mu}^{\mu} = \frac{\beta(g)}{2g} F^{a,\mu\nu} F_{\mu\nu}^a + (1 + \gamma_m) \sum_q m_q \bar{\psi}_q \psi_q$

- Scalar radius: $\Theta(t) = m_N (1 + t \langle r_{\Theta}^2 \rangle / 6 + \dots)$

🧑 Kharzeev proposed it can be extracted from the threshold photoproduction of the vector-meson, e.g., J/ψ , and the fit result is ~ 0.55 fm
Kharzeev, PRD (2021).....

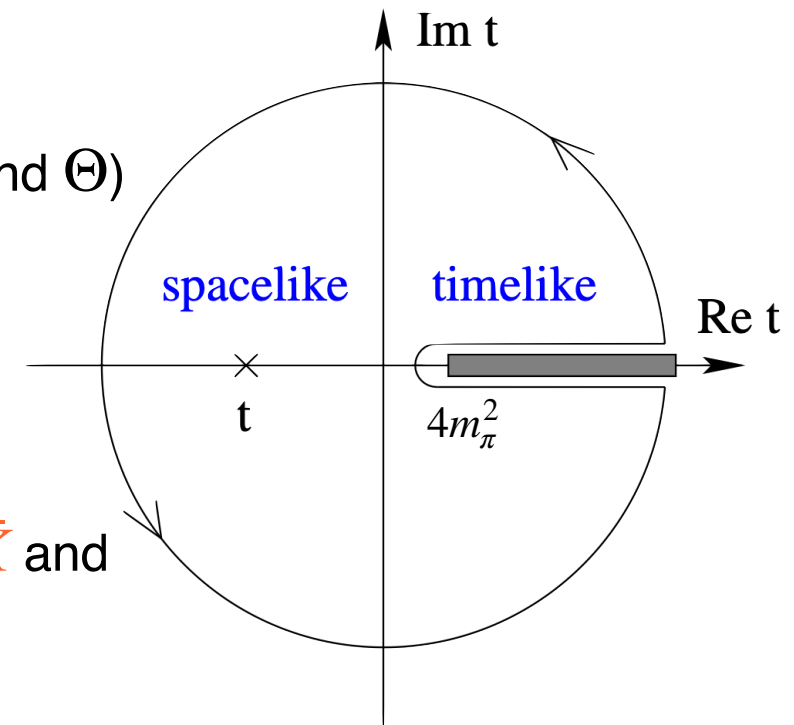
🧑 Recently, two LQCD calculations at near physical quark mass result in a large scalar radius, ~ 1 fm
Hackett et al., PRL (2024); Wang et al. [χQCD], PRD (2024)



fits to low- t data $\Rightarrow$ dispersive analyses of "all" data

Why data-driven dispersion theory?

- Based on fundamental principles: **unitarity**, **analyticity** and **crossing symmetry**
- Simultaneous analysis of all four FFs (A , J , D and Θ)
- Connects FFs over full range of momentum transfers: **timelike** and **spacelike** data
- Connects to data from other processes ($\pi\pi$, $K\bar{K}$ and πN , KN scatterings...)
- Constraints from **ChPT**, pQCD



Unitarity relation for pion GFFs

- Gravitational form factors (GFFs) for spin-0 particles, e.g., for pion:

$$\langle \pi^a(p') | \hat{T}^{\mu\nu}(0) | \pi^b(p) \rangle = \frac{\delta^{ab}}{2} \left[A^\pi(t) P^\mu P^\nu + D^\pi(t) (\Delta^\mu \Delta^\nu - t g^{\mu\nu}) \right]$$

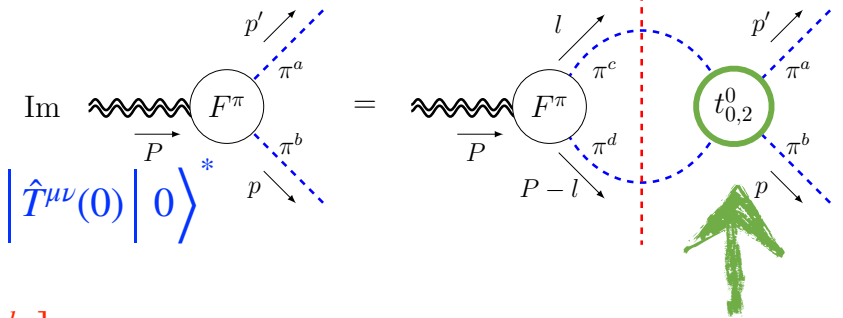
$$P^\mu = p'^\mu + p^\mu, \quad \Delta^\mu = p'^\mu - p^\mu$$

Crossing symmetry
for constructing dispersion relations

$$\langle \pi^a(p') \pi^b(p) | \hat{T}^{\mu\nu}(0) | 0 \rangle = \frac{\delta^{ab}}{2} \left[A^\pi(t) \Delta^\mu \Delta^\nu + D^\pi(t) (P^\mu P^\nu - t g^{\mu\nu}) \right]$$

- Unitarity for full amplitude $\Rightarrow$ discontinuity (imaginary part) of the pion GFFs

$$\begin{aligned} & \text{Disc} \langle \pi^a(p') \pi^b(p) | \hat{T}^{\mu\nu}(0) | 0 \rangle \\ &= \frac{1}{2} \frac{i}{(4\pi)^2} \frac{p_\pi}{\sqrt{t}} \int d\Omega_l \langle \pi^a(p') \pi^b(p) | \pi^c(l) \pi^d(P-l) \rangle \langle \pi^c(l) \pi^d(P-l) | \hat{T}^{\mu\nu}(0) | 0 \rangle^* \\ &= \frac{1}{2} \frac{i}{(4\pi)^2} \frac{p_\pi}{\sqrt{t}} \int d\Omega_l \left[A(t, s, u) \delta^{ab} \delta^{cd} + A(s, t, u) \delta^{ac} \delta^{bd} + A(u, s, t) \delta^{ad} \delta^{bc} \right] \\ & \quad \times \frac{\delta^{cd}}{2} \left[(A^\pi(t))^* (2l - P)^\mu (2l - P)^\nu + (D^\pi(t))^* (P^\mu P^\nu - t g^{\mu\nu}) \right] \\ &= \frac{1}{2} \frac{i}{(4\pi)^2} \frac{p_\pi}{\sqrt{t}} \int d\Omega_l \frac{\delta^{ab}}{2} A^{I=0}(t, s, u) \left[(A^\pi(t))^* \underbrace{(2l - P)^\mu (2l - P)^\nu}_{S-, D\text{-waves}} + (D^\pi(t))^* \underbrace{(P^\mu P^\nu - t g^{\mu\nu})}_{S\text{-wave}} \right] \end{aligned}$$



$\pi\pi$ scattering

Unitarity relation for pion GFFs

- Discontinuity of A^π and D^π : associated only with t_0^0, t_2^0 partial waves.

$$\begin{aligned} \text{Disc} \left\langle \pi^a(p') \pi^b(p) \right| \hat{T}^{\mu\nu}(0) | 0 \rangle &= \frac{\delta^{ab}}{2} \left[\text{Disc} A^\pi(t) \Delta^\mu \Delta^\nu + \text{Disc} D^\pi(t) (P^\mu P^\nu - t g^{\mu\nu}) \right] \\ &= 2i \frac{2p_\pi}{\sqrt{t}} \frac{\delta^{ab}}{2} \left[(A^\pi(t))^* \left(\frac{4}{3t} p_\pi^2 \boxed{t_0^0(t) - t_2^0(t)} (P^\mu P^\nu - t g^{\mu\nu}) + \boxed{t_2^0(t)} \Delta^\mu \Delta^\nu \right) + (D^\pi(t))^* \boxed{t_0^0(t)} (P^\mu P^\nu - t g^{\mu\nu}) \right] \end{aligned}$$

Partial-wave amplitudes of $\pi\pi$ scattering

- Partial-wave unitarity

$$\begin{aligned} \text{Im} A^\pi(t) &= \frac{2p_\pi}{\sqrt{t}} (t_2^0(t))^* A^\pi(t) \\ \text{Im} D^\pi(t) &= \frac{2p_\pi}{\sqrt{t}} \left[\frac{4}{3} \frac{p_\pi^2}{t} (t_0^0(t) - t_2^0(t))^* A^\pi(t) + (t_0^0(t))^* D^\pi(t) \right] \end{aligned}$$

- Decomposition into $J^{PC} = 0^{++}, 2^{++}$, matrix elements (conserved separately):

$$\left\langle \pi^a(p') \pi^b(p) \right| \hat{T}^{\mu\nu}(0) | 0 \rangle = \delta^{ab} \left\{ \frac{1}{3} \left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2} \right) \Theta^\pi(t) + \left[\Delta^\mu \Delta^\nu + \frac{\Delta^2}{3t} (P^\mu P^\nu - t g^{\mu\nu}) \right] A^\pi(t) \right\} \quad \text{Raman, PRD (1971)}$$

$$\text{Trace part: } \left\langle \pi^a(p') \pi^b(p) \right| \hat{T}_\mu^\mu(0) | 0 \rangle = \delta^{ab} \Theta^\pi(t), \quad \Theta^\pi(t) = -\frac{1}{2} (4p_\pi^2 A^\pi(t) + 3t D^\pi(t))$$

Unitarity relation for pion GFFs

- Discontinuity of A^π and D^π : associated only with t_0^0, t_2^0 partial waves.

$$\begin{aligned} \text{Disc} \left\langle \pi^a(p') \pi^b(p) \right| \hat{T}^{\mu\nu}(0) | 0 \rangle &= \frac{\delta^{ab}}{2} \left[\text{Disc} A^\pi(t) \Delta^\mu \Delta^\nu + \text{Disc} D^\pi(t) (P^\mu P^\nu - t g^{\mu\nu}) \right] \\ &= 2i \frac{2p_\pi}{\sqrt{t}} \frac{\delta^{ab}}{2} \left[(A^\pi(t))^* \left(\frac{4}{3t} p_\pi^2 \boxed{t_0^0(t) - t_2^0(t)} (P^\mu P^\nu - t g^{\mu\nu}) + \boxed{t_2^0(t)} \Delta^\mu \Delta^\nu \right) + (D^\pi(t))^* \boxed{t_0^0(t)} (P^\mu P^\nu - t g^{\mu\nu}) \right] \end{aligned}$$

Partial-wave amplitudes of $\pi\pi$ scattering

- Partial-wave unitarity

$$\begin{aligned} \text{Im} A^\pi(t) &= \frac{2p_\pi}{\sqrt{t}} (t_2^0(t))^* A^\pi(t) \\ \text{Im} D^\pi(t) &= \frac{2p_\pi}{\sqrt{t}} \left[\frac{4}{3} \frac{p_\pi^2}{t} (t_0^0(t) - t_2^0(t))^* A^\pi(t) + (t_0^0(t))^* D^\pi(t) \right] \end{aligned} \quad \longrightarrow \quad \text{Im} \Theta^\pi(t) = \frac{2p_\pi}{\sqrt{t}} (t_0^0(t))^* \Theta^\pi(t)$$

- Decomposition into $J^{PC} = 0^{++}, 2^{++}$, matrix elements (conserved separately):

$$\left\langle \pi^a(p') \pi^b(p) \right| \hat{T}^{\mu\nu}(0) | 0 \rangle = \delta^{ab} \left\{ \frac{1}{3} \left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2} \right) \Theta^\pi(t) + \left[\Delta^\mu \Delta^\nu + \frac{\Delta^2}{3t} (P^\mu P^\nu - t g^{\mu\nu}) \right] A^\pi(t) \right\}$$

Trace part: $\left\langle \pi^a(p') \pi^b(p) \right| \hat{T}_\mu^\mu(0) | 0 \rangle = \delta^{ab} \Theta^\pi(t), \quad \Theta^\pi(t) = -\frac{1}{2} (4p_\pi^2 A^\pi(t) + 3t D^\pi(t))$

A^π : D -wave single channel

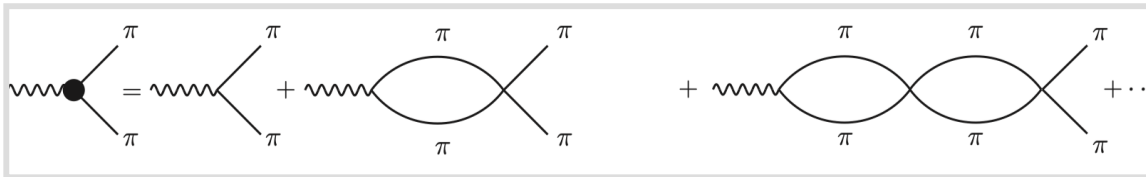
- **Single-channel**: Watson's theorem $\Rightarrow$ phase of FF = scattering phase shift

$$\text{disc } A^\pi(t) = 2iA^\pi(t)\sin\delta_2^0(t)e^{-i\delta_2^0(t)}\theta(t-t_\pi)$$

$$\left|t_2^0\right| e^{i\phi_2^0} = \frac{\eta_2^0 e^{2i\delta_2^0} - 1}{2i\sigma_\pi}$$

⊗ Omnes solution

$$A^\pi(t) = P_2^\pi(t)\Omega_2^0(t), \quad \Omega_2^0(t) \equiv \exp \left\{ \frac{t}{\pi} \int_{t_\pi}^{\infty} \frac{dt'}{t'} \frac{\phi_2^0(t')}{t' - t} \right\}$$



δ_2^0 replaced by the phase of $\pi\pi$ partial wave ϕ_2^0 to account for inelasticity

Hoferichter et al., Phys. Rept. (2016)

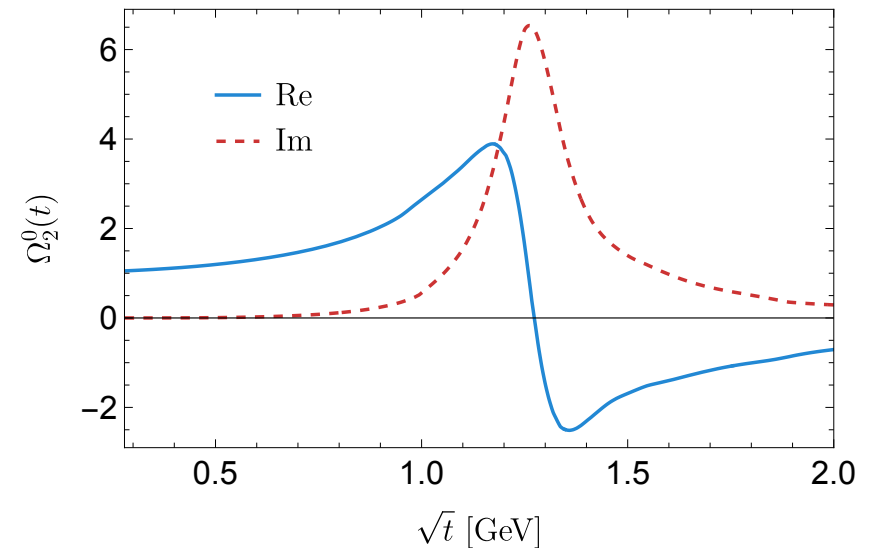
• δ_2^0, η_2^0 up to $E_0 \simeq 2$ GeV from dispersive analysis Bydžovský et al., PRD (2016)

• Polynomial: $P_2^\pi(t) = 1 + \alpha t$ matched to NLO ChPT

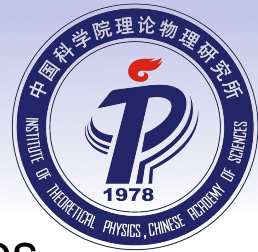
$$\Rightarrow A^\pi(t) = 1 - \frac{2L_{12}^r}{F_\pi^2} t$$

Tensor-meson dominance estimate: $L_{12}^r = -\frac{F_\pi^2}{2m_{f_2}^2}$

$$P_2^\pi(t) = 1 + \left[\frac{1}{m_{f_2}^2} - \dot{\Omega}_2^0(0) \right] t \simeq 1 - (0.01 \text{ GeV}^{-2}) t$$



Θ^π : S -wave $\pi\pi - K\bar{K}$ coupled channel



- $\pi\pi$ S -wave phase shifts known precisely from Roy(-like) equation analyses

Bern group; Madrid-Krakov group

- Generalisation to coupled channels: isoscalar-scalar $f_0(500), f_0(980)$ mesons

Unitarity relation for $\Theta^\pi \Rightarrow$ Matrix relation for coupled channels (both pion and kaon trace GFFs):

$$\text{Im } \Theta^\pi(t) = \frac{2p_\pi}{\sqrt{t}} (t_0^0(t))^* \Theta^\pi(t) \quad \Rightarrow \quad \text{Im } \Theta(t) = [\mathbf{T}_0^0(t)]^* \Sigma_0^0(t) \Theta(t) \quad \Theta(t) = \begin{pmatrix} \Theta^\pi(t) \\ \frac{2}{\sqrt{3}} \Theta^K(t) \end{pmatrix}$$

Phase-space factor

$$\Sigma_0^0(t) \equiv \text{diag}(\sigma_\pi \theta(t - t_\pi), \sigma_K \theta(t - t_K))$$

$$\text{With } \sigma_i(t) = \sqrt{1 - 4m_i^2/t} \quad (i = \pi, K)$$

$\pi\pi$ - $K\bar{K}$ T-matrix

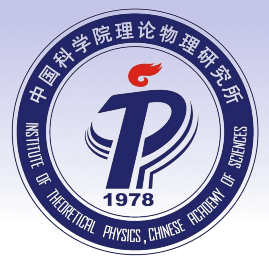
$$\mathbf{T}_0^0(t) = \begin{pmatrix} \frac{\eta_0^0(t) e^{2i\delta_0^0(t)} - 1}{2i\sigma_\pi} & |g_0^0(t)| e^{i\Psi_0^0(t)} \\ |g_0^0(t)| e^{i\Psi_0^0(t)} & \frac{\eta_0^0(t) e^{2i(\Psi_0^0(t) - \delta_0^0(t))} - 1}{2i\sigma_K} \end{pmatrix}.$$

$$\eta_0^0(t) = \sqrt{1 - 4\sigma_\pi \sigma_K |g_0^0(t)|^2 \theta(t - t_K)}$$

Muskhelishvili-Omnès problem

Donoghue, Gasser, Leutwyler, NPB (1991)

Muskhelishvili-Omnes representation



- **Coupled-channel:** solution known as the Muskhelishvili-Omnes (MO) representation

🔧 Take isoscalar-scalar $\pi\pi - K\bar{K}$ as example (matching point ~ 1.3 GeV)

$$\Omega_0^0(t) = \frac{1}{\pi} \int_{t_\pi}^{\infty} \frac{dt'}{t' - t} [\mathbf{T}_0^0(t)]^* \Sigma_0^0(t) \Omega_0^0(t')$$

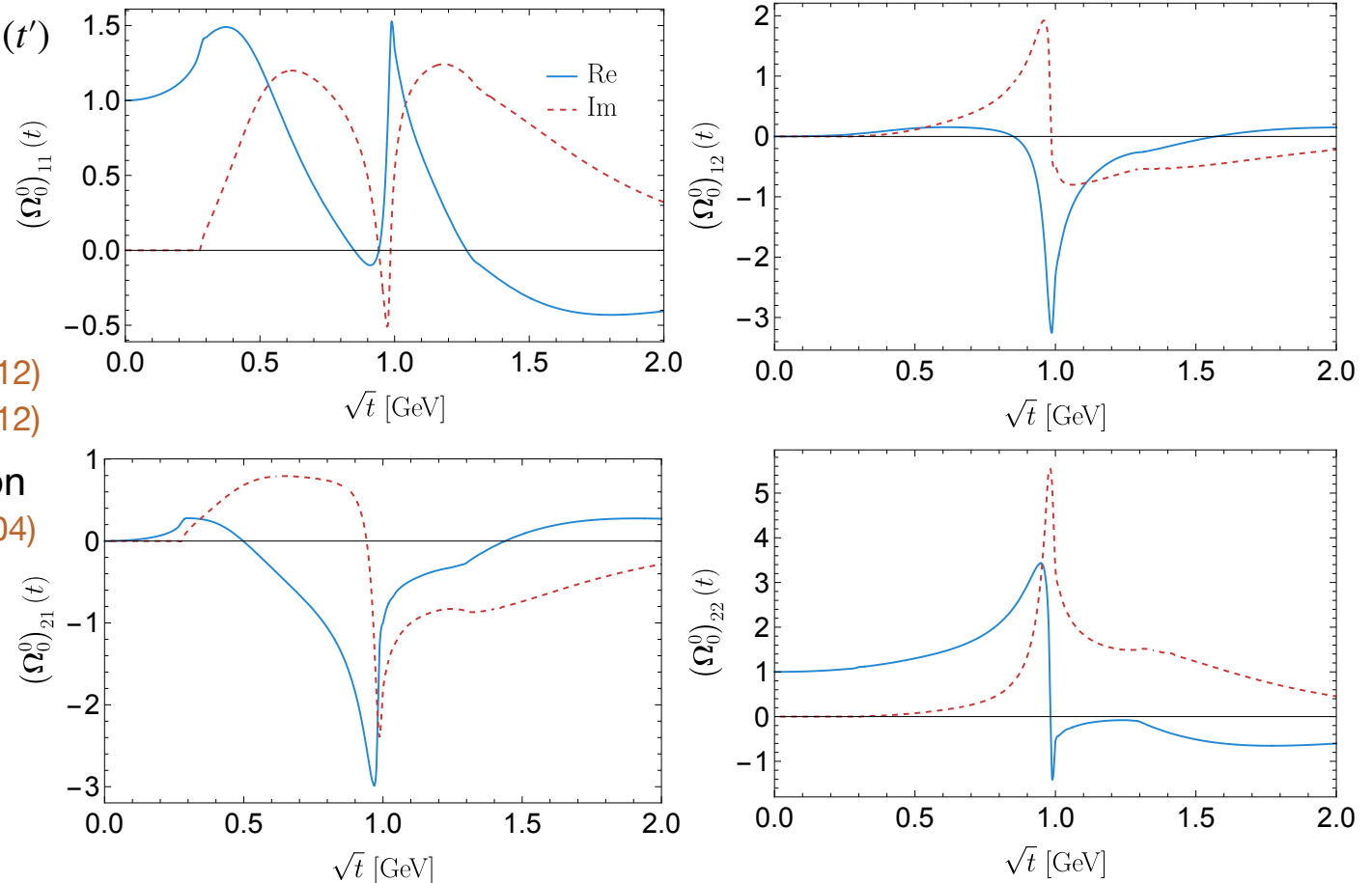
$\pi\pi$ phase shift: Roy equation

Ananthanarayan et al., Phys. Rept. (2012)

Caprini et al., EPJC (2012)

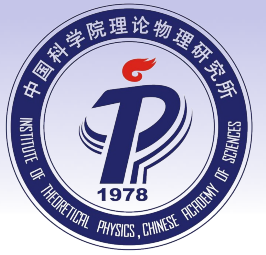
$\pi\pi \rightarrow K\bar{K}$: Roy-Steiner equation

Büttiker et al., EPJC (2004)



See also Hoferichter et al., JHEP (2012)

ChPT matching



● Pion and kaon trace GFFs

$$[\Theta(t)]^T = [P_0(t)]^T \Omega_0^0(t), \quad P_0(t) = \begin{pmatrix} 2m_\pi^2 + \beta_\pi t \\ \frac{2}{\sqrt{3}} (2m_K^2 + \beta_K t) \end{pmatrix}$$

$$\beta_\pi = \dot{\Theta}^\pi(0) - 2m_\pi^2 \left(\dot{\Omega}_0^0 \right)_{11}(0) - \frac{4m_K^2}{\sqrt{3}} \left(\dot{\Omega}_0^0 \right)_{12}(0),$$

$$\beta_K = \dot{\Theta}^K(0) - \sqrt{3}m_\pi^2 \left(\dot{\Omega}_0^0 \right)_{21}(0) - 2m_K^2 \left(\dot{\Omega}_0^0 \right)_{22}(0),$$

● Matching to **NLO ChPT** Donoghue, Leutwyler, ZPC 52 (1991) 343

$$\dot{\Theta}^\pi(0) = 1 - 4L_{12}^r \frac{m_\pi^2}{F_\pi^2} - 24(L_{11}^r - L_{13}^r) \frac{m_\pi^2}{F_\pi^2} - \frac{3}{2} \frac{m_\pi^2}{F_\pi^2} I_\pi + \frac{m_\pi^2}{2F_\pi^2} I_\eta = 0.98(2)$$

$$\dot{\Theta}^K(0) = 1 - 4L_{12}^r \frac{m_K^2}{F_\pi^2} - 24(L_{11}^r - L_{13}^r) \frac{m_K^2}{F_\pi^2} - \frac{m_K^2}{F_\pi^2} I_\eta = 0.94(14) \quad \Rightarrow \text{chiral logs: } I_i = \frac{1}{48\pi^2} \left(\ln \frac{\mu^2}{m_i^2} - 1 \right)$$

● Similar coupled-channel analysis for D -wave

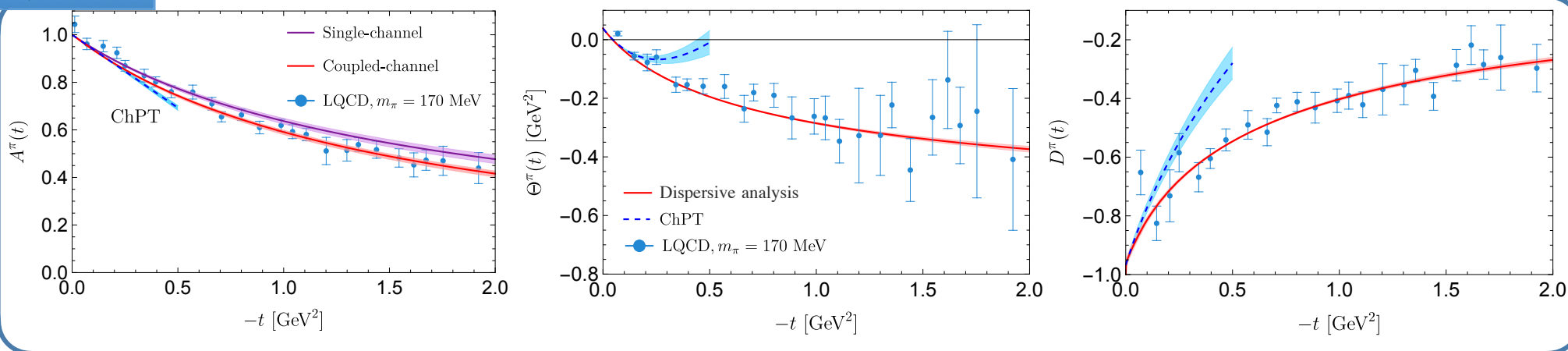
$\Rightarrow$ coupled channel results for A^π, A^K and D^π, D^K

Pion and Kaon GFFs



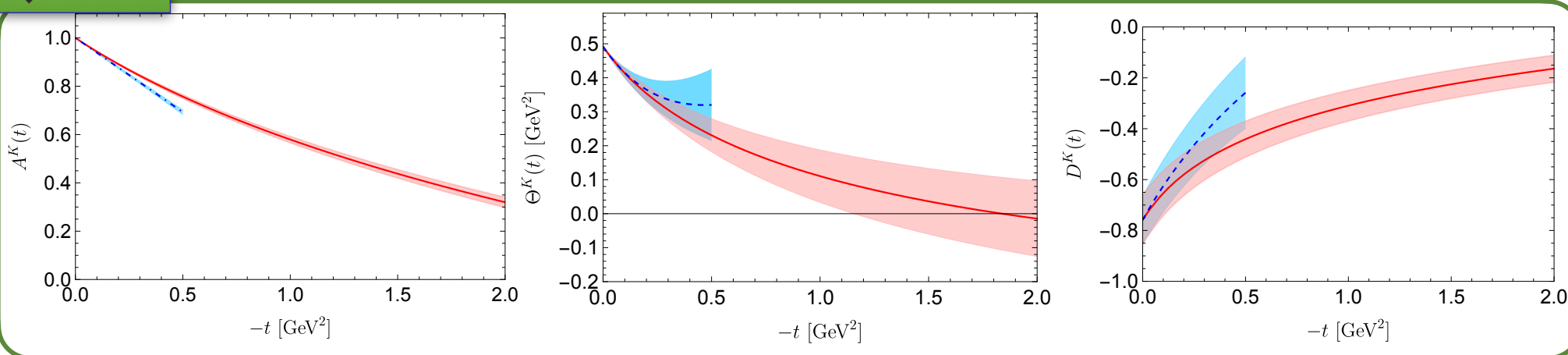
Prediction, Not fit

Pion

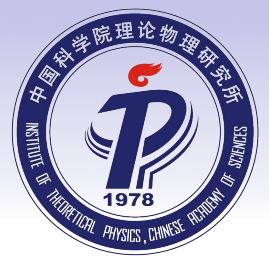


LQCD ($m_\pi \simeq 170$ MeV): Hackett et al., PRD (2023)

Kaon

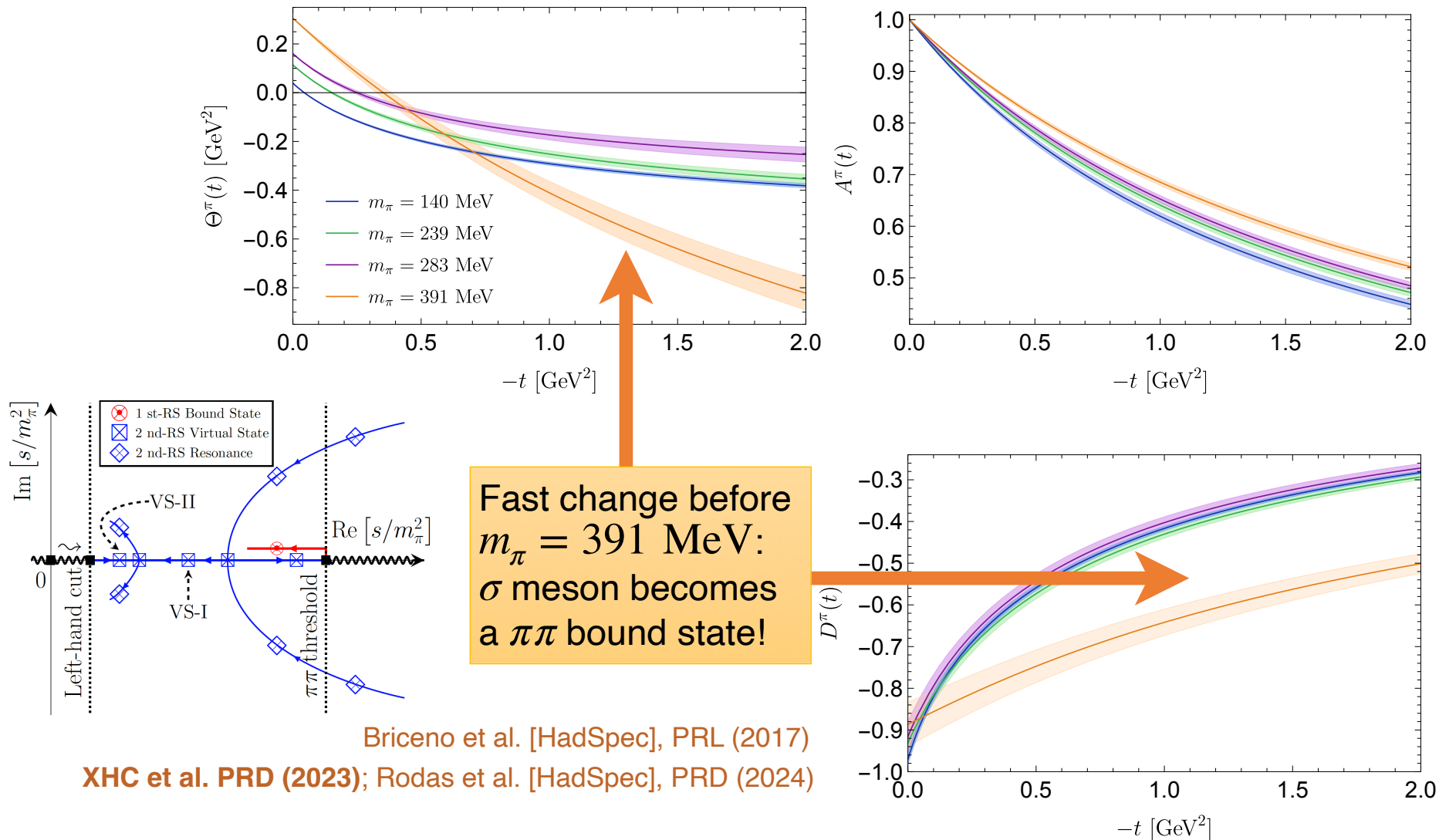


Pion GFFs: pion mass dependence

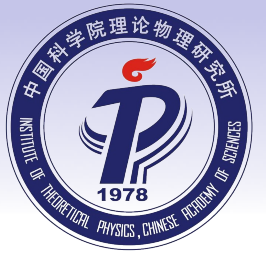


- Using the scattering phase shifts at unphysical pion masses (239 MeV, 283 MeV, 391 MeV) obtained from Roy equation analyses

XHC et al. PRD (2023); Rodas et al. [HadSpec], PRD (2024)



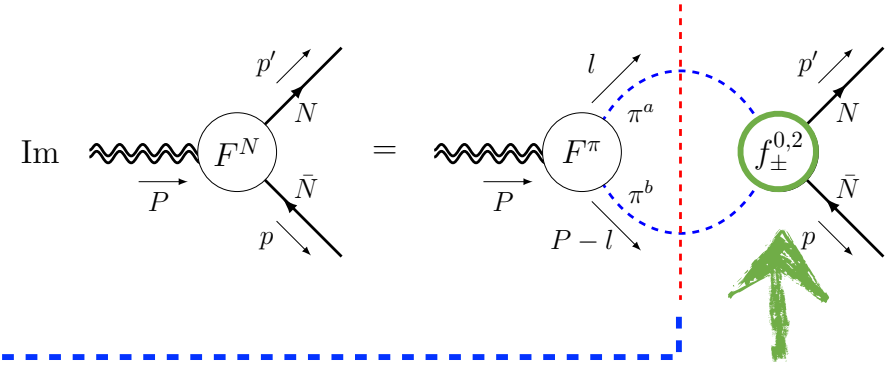
Unitarity relation for nucleon GFFs



● Discontinuity

$$\text{Disc} \langle N(p') \bar{N}(p) | \hat{T}^{\mu\nu}(0) | 0 \rangle$$

$$\propto \sum_n \langle N(p') \bar{N}(p) | n \rangle \langle n | \hat{T}^{\mu\nu}(0) | 0 \rangle^* \delta^4(p + p' - p_n)$$



● In the elastic region, only $\pi\pi$ intermediate state

$\pi\pi(K\bar{K}) \rightarrow N\bar{N}$ scattering

$$\text{Disc} \langle N(p') \bar{N}(p) | \hat{T}^{\mu\nu}(0) | 0 \rangle$$

$$= \frac{1}{4m_N} \bar{u}(p') \left[\text{Disc} \hat{A}(t) \Delta^\mu \Delta^\nu + \text{Disc} \hat{J}(t) \left(i \Delta^{\{\mu} \sigma^{\nu\} \rho} P_\rho \right) + \text{Disc} \hat{D}(t) (P^\mu P^\nu - t g^{\mu\nu}) \right] v(p)$$

$$= \frac{1}{2} \frac{i}{(4\pi)^2} \frac{p_\pi}{\sqrt{t}} \int d\Omega_l \langle N(p') \bar{N}(p) | \pi^a(l) \pi^b(P-l) \rangle \langle \pi^a(l) \pi^b(P-l) | \hat{T}^{\mu\nu}(0) | 0 \rangle^*$$

$$= \frac{1}{2} \frac{i}{(4\pi)^2} \frac{p_\pi}{\sqrt{t}} \int d\Omega_l \bar{u}(p') \left[\underbrace{\delta^{ab} \mathbb{1} \left(A^+ + \frac{(\not{P} - 2\not{l})}{2} B^+ \right)}_{\text{Isospin-even}} + i \epsilon_{bac} \tau^c \left(A^- + \frac{(\not{P} - 2\not{l})}{2} B^- \right) \right] v(p)$$

$$\times \frac{\delta^{ab}}{2} \left[(A^\pi(t))^* (2l - P)^\mu (2l - P)^\nu + (D^\pi(t))^* (P^\mu P^\nu - t g^{\mu\nu}) \right]$$

$$= \frac{1}{2} \frac{i}{(4\pi)^2} \frac{p_\pi}{\sqrt{t}} \int d\Omega_l \bar{u}(p') \frac{3}{2} \left(A^+ + \frac{(\not{P} - 2\not{l})}{2} B^+ \right) v(p) \left[(A^\pi(t))^* (2l - P)^\mu (2l - P)^\nu + (D^\pi(t))^* (P^\mu P^\nu - t g^{\mu\nu}) \right]$$

$A^\pm, B^\pm$: Lorentz invariant πN scattering amplitudes

Unitarity relation for nucleon GFFs

● Discontinuity of the nucleon GFFs

$$\Gamma^2 = m_N^2 \sqrt{\frac{2}{3}} f_-^2 - f_+^2$$

$$\text{Im } A(t) = \frac{3p_\pi^5}{\sqrt{6}t} \left[f_-^2(t) + \sqrt{\frac{3}{2}} \frac{m_N}{p_N^2} \Gamma^2(t) \right]^* A^\pi(t)$$

$$\text{Im } J(t) = \frac{3p_\pi^5}{2\sqrt{6}t} [f_-^2(t)]^* A^\pi(t)$$

$$\text{Im } D(t) = -\frac{3m_N p_\pi}{2p_N^2 \sqrt{t}} \left[\frac{4p_\pi^2}{3t} \left((f_+^0(t))^* - (p_\pi p_N)^2 (f_+^2(t))^* \right) A^\pi(t) + (f_+^0(t))^* D^\pi(t) \right]$$

$f_\pm^J: \pi\pi \rightarrow K\bar{K}$ partial wave amp. With $+/-$ for parallel/anti-parallel $N\bar{N}$ helicities

Frazer & Fulco (1960); Höhler (1983)

● Decomposition into $J^{PC} = 0^{++}, 2^{++}$, matrix elements

$$\langle N(p') \bar{N}(p) | \hat{T}^{\mu\nu}(0) | 0 \rangle = \bar{u}(p') \left(T_S^{\mu\nu} + T_T^{\mu\nu} \right) v(p)$$

$$T_S^{\mu\nu} = \frac{1}{3} \left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2} \right) \Theta(t)$$

$$T_T^{\mu\nu} = \frac{1}{4m_N} \left[\Delta^\mu \Delta^\nu + \frac{\Delta^2}{3t} (P^\mu P^\nu - t g^{\mu\nu}) \right] A(t) + \left[i \Delta^{\{\mu} \sigma^{\nu\} \rho} P_\rho + \frac{2i \sigma^{\rho\kappa} \Delta_\rho P_\kappa}{3t} (P^\mu P^\nu - t g^{\mu\nu}) \right] J(t)$$

Unitarity relation for nucleon GFFs

● Discontinuity of the nucleon GFFs

$$\Gamma^2 = m_N^2 \sqrt{\frac{2}{3}} f_-^2 - f_+^2$$

$$\text{Im } A(t) = \frac{3p_\pi^5}{\sqrt{6t}} \left[f_-^2(t) + \sqrt{\frac{3}{2}} \frac{m_N}{p_N^2} \Gamma^2(t) \right]^* A^\pi(t)$$

$$\Theta(t) = \frac{1}{4m_N} [-4p_N^2 A(t) + 2tJ(t) - 3tD(t)]$$

$$\text{Im } J(t) = \frac{3p_\pi^5}{2\sqrt{6t}} [f_-^2(t)]^* A^\pi(t)$$

$$\text{Im } \Theta(t) = -\frac{3p_\pi}{4p_N^2 \sqrt{t}} [f_+^0(t)]^* \Theta^\pi(t)$$

$$\text{Im } D(t) = -\frac{3m_N p_\pi}{2p_N^2 \sqrt{t}} \left[\frac{4p_\pi^2}{3t} \left((f_+^0(t))^* - (p_\pi p_N)^2 (f_+^2(t))^* \right) A^\pi(t) + (f_+^0(t))^* D^\pi(t) \right]$$

$f_\pm^J: \pi\pi \rightarrow K\bar{K}$ partial wave amp. With $+/-$ for parallel/anti-parallel $N\bar{N}$ helicities

Frazer & Fulco (1960); Höhler (1983)

● Decomposition into $J^{PC} = 0^{++}, 2^{++}$, matrix elements

Coupled-channel

$$\langle N(p') \bar{N}(p) | \hat{T}^{\mu\nu}(0) | 0 \rangle = \bar{u}(p') \left(T_S^{\mu\nu} + T_T^{\mu\nu} \right) v(p)$$

$$T_S^{\mu\nu} = \frac{1}{3} \left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2} \right) \Theta(t)$$

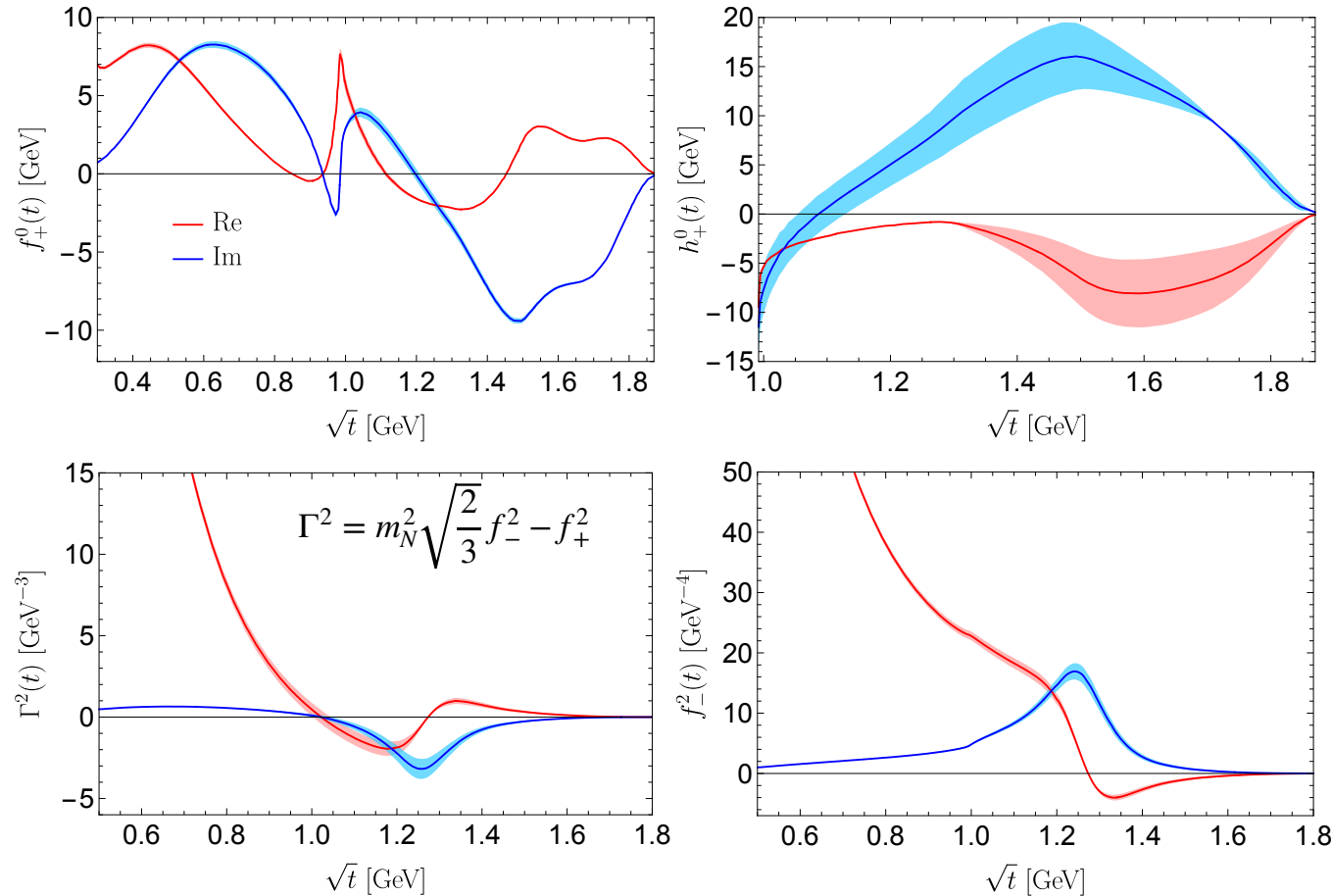
$$\text{Im } \Theta(t) = -\frac{3}{4p_N^2 \sqrt{t}} \left\{ p_\pi [f_+^0(t)]^* \Theta^\pi(t) \theta(t - t_\pi) + \frac{4}{3} p_K [h_+^0(t)]^* \Theta^K(t) \theta(t - t_K) \right\}$$

$$T_T^{\mu\nu} = \frac{1}{4m_N} \left[\Delta^\mu \Delta^\nu + \frac{\Delta^2}{3t} (P^\mu P^\nu - t g^{\mu\nu}) \right] A(t) + \left[i \Delta^{\{\mu} \sigma^{\nu\} \rho} P_\rho + \frac{2i \sigma^{\rho\kappa} \Delta_\rho P_\kappa}{3t} (P^\mu P^\nu - t g^{\mu\nu}) \right] J(t)$$

$\pi\pi/K\bar{K} \rightarrow N\bar{N}$ partial wave amplitudes



- Inputs: $\pi\pi/K\bar{K} \rightarrow N\bar{N}$ partial wave amp. $f_{\pm}^{0,2}, h_{+}^0$ from Roy-Steiner analyses
Hite & Steiner, Nuove Cimento A (1973)



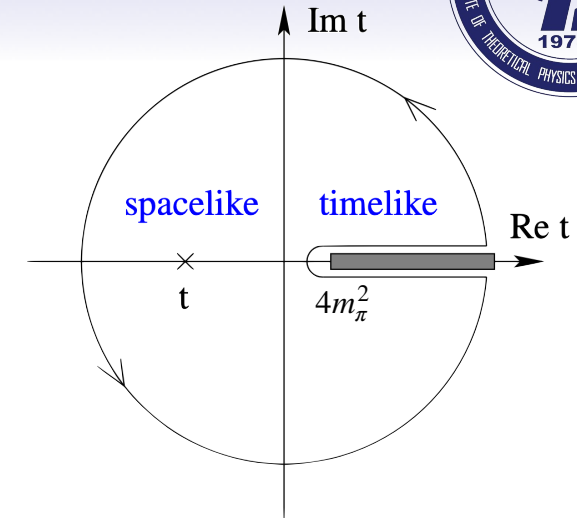
πN amplitudes from modern Roy-Steiner equation analyses

Ditsche, et al., JHEP (2012); Hoferichter, et al., PRL 115, 092301(2015); PRL 115, 192301 (2015); Phys. Rept. (2016);
XHC, Q.-Z. Li & H.-Q. Zheng, JHEP (2022); Hoferichter, et al., PLB (2024)

Gravitational form factors of the nucleon

- Dispersive relations (DRs) for the nucleon GFFs :

$$(A, J, \Theta)(t) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} dt' \frac{\text{Im}(A, J, \Theta)(t')}{t' - t}$$



- Constraints:

Normalizations: mass m_N , spin $1/2$

$\Rightarrow$ sum rules saturated by $\pi\pi$, $K\bar{K}$ continuum and some higher mass states

Belushkin et al., PRC (2007); Hoferichter et al., EPJA (2016) ...

$$\frac{1}{\pi} \int_{4m_\pi^2}^{\infty} dt' \frac{\text{Im}(A, J, \Theta)(t')}{t'} = \left(1, \frac{1}{2}, m_N\right)$$

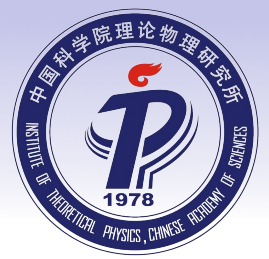
Introduce S -wave 0^{++} and D -wave (2^{++}) poles to the spectral functions:

$\pi c_{S,D} m_{S,D}^2 \delta(t - m_{S,D}^2)$ to satisfy the sum rules

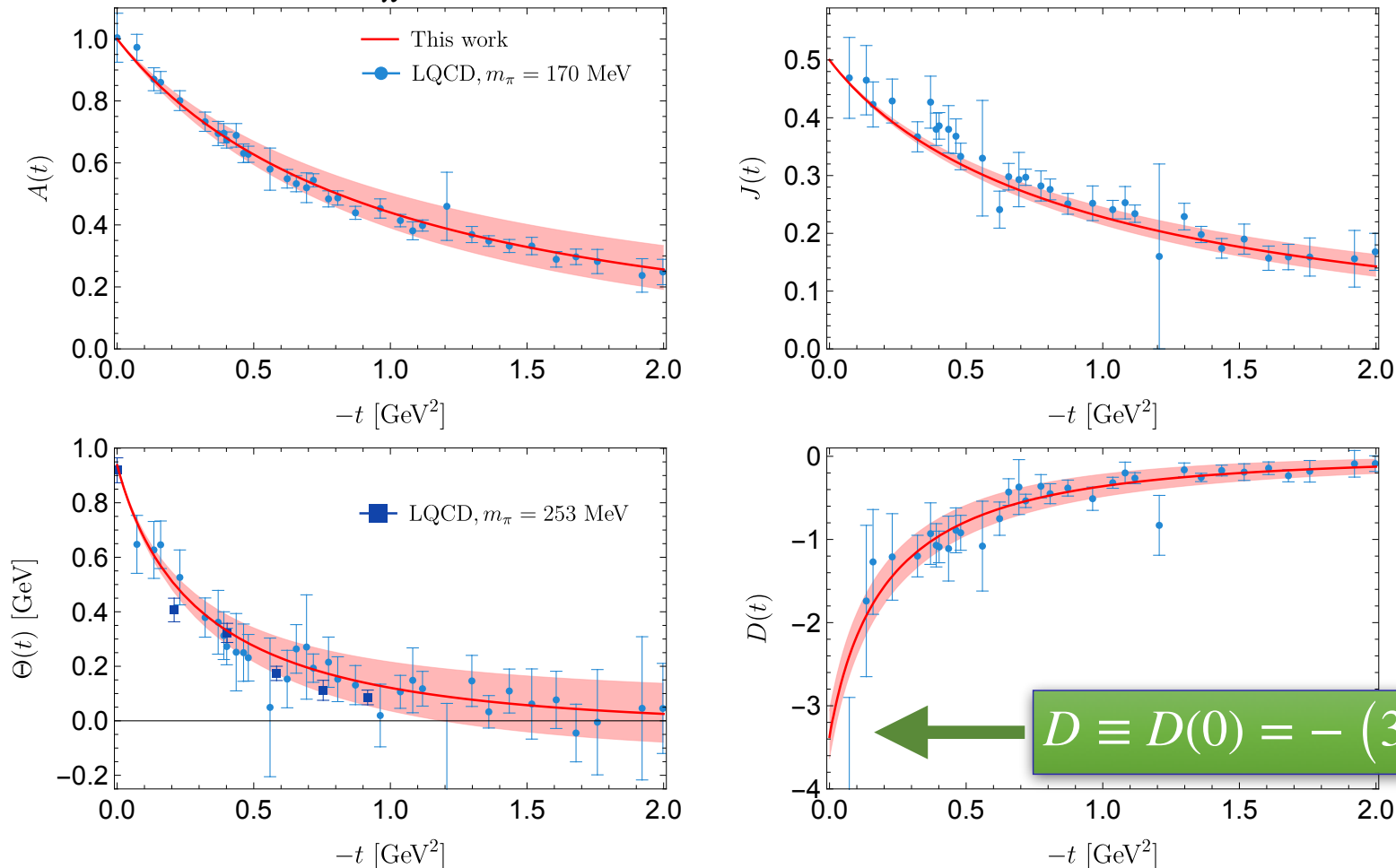
✓ 0^{++} : $m_S \in (1.5, 1.8)$ GeV to cover $f_0(1500)$ and $f_0(1710)$

✓ 2^{++} : $m_D \in (1.5, 2.2)$ GeV to cover $f_2(1500)$, $f_2(1950)$ and $f_2(2010)$

Nucleon GFFs



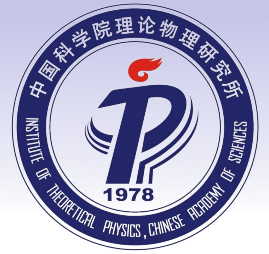
● Predictions: LQCD ($m_\pi \simeq 170$ MeV): Hackett et al., PRL (2024)



LQCD ($m_\pi \simeq 253$ MeV): Wang et al. [χ QCD], PRD (2024)

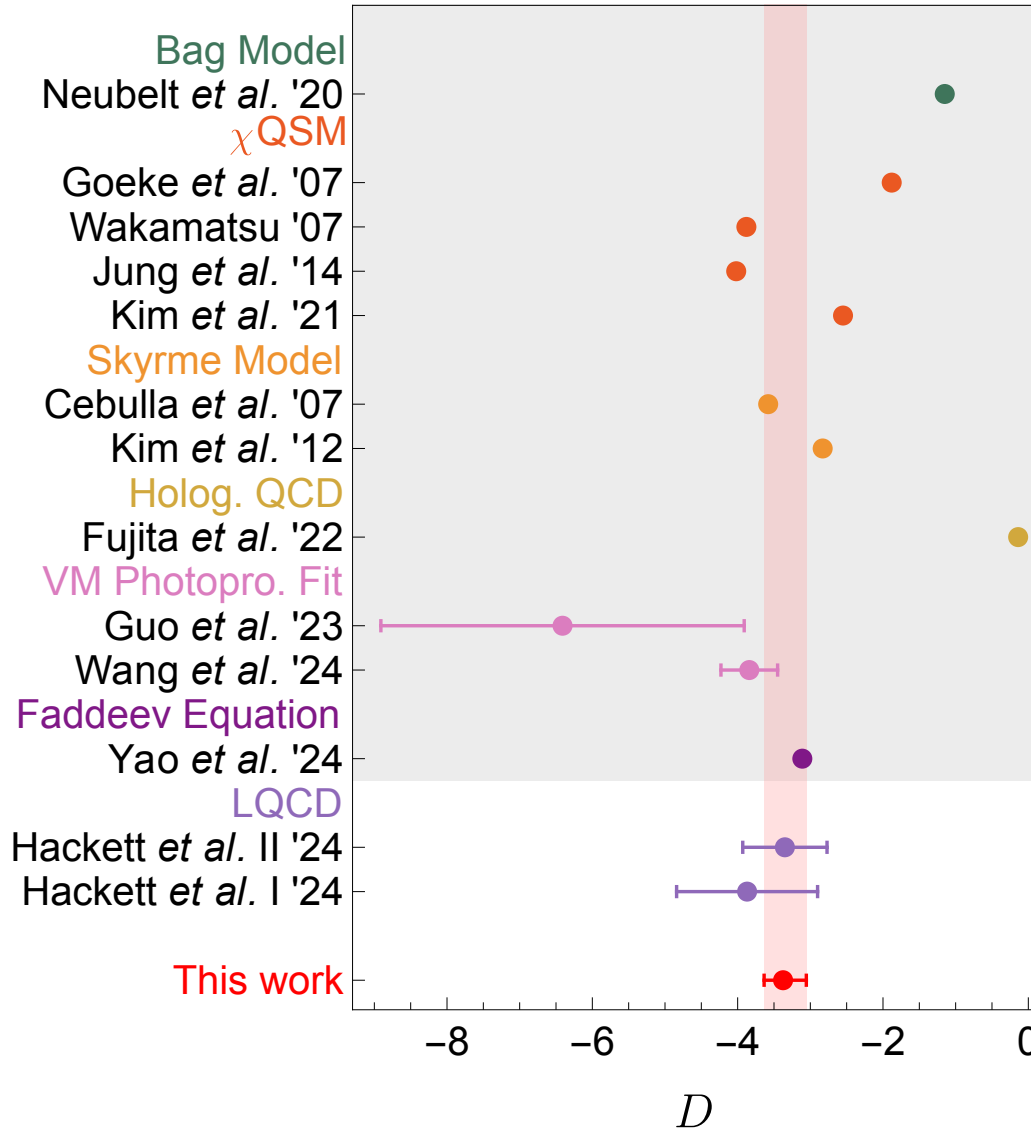
- ☑ GFFs be **model-independently** predicted using data-driven dispersion relations + ChPT
- ☑ Not a fit!

Nucleon D-term



- Nucleon D-term $D \equiv D(0)$:

$$D = - \left(3.38^{+0.34}_{-0.35} \right)$$



- Positivity bound:

$$D < -0.2$$

Gegelia & Polyakov, PLB (2021)

“The stability condition”

$$\text{normal force } \frac{2}{3}s(r) + p(r) \geq 0$$

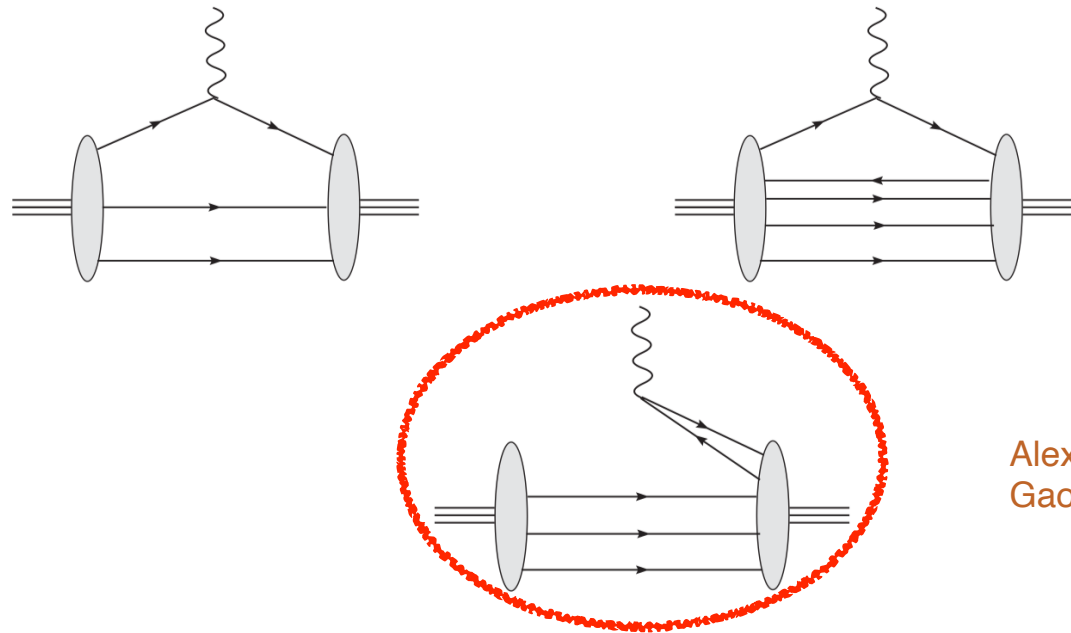
$$\widetilde{D}(r) = \int \frac{d^3\Delta}{(2\pi)^3} e^{-i\Delta r} D(-\Delta^2) < 0$$

3D spatial mass (energy) distribution

- Nucleon 3D spatial mass distribution (T^{00}) in Breit frame (BF):

$$\rho_{\text{Mass}}(r) = m_N \int \frac{d^3\Delta}{(2\pi)^3} e^{-ir \cdot \Delta} \left[A(t) - \frac{t}{4m_N^2} [A(t) - 2J(t) + D(t)] \right] = \int \frac{d^3\Delta}{(2\pi)^3} e^{-ir \cdot \Delta} \left[\Theta(t) + \frac{t}{2m_N} D(t) \right]$$

- Interpretation around **probabilistic distributions** of FFs in the BF are known to be plagued by relativistic recoil corrections Burkardt, 2000; Miller, 2018; Jaffe, 2020.....



QFT perspective

Alexandrou et al., RMP (2012)
Gao and Vanderhaeghen, RMP (2022)

- 3D probabilistic interpretation (BF) is not exact, valid only for heavy system (Compton wave length $\ll$ size)
- 2D light-front approach is exact, valid for all systems Miller, 2007
- 3D phase-space approach is exact, valid for all systems, but has no probabilistic interpretation Lorcé, 2020
- 3D zero average momentum frame (ZAMF) approach of sharp localisation Epelbaum et al., 2022

Spatial density profiles: mass & angular momentum



Consider various densities

- Breit frame (BF) Polyakov, PLB 555 (2003) 57; Polyakov, Schweitzer, IJMPA 33 (2018) 183005; Lorcé et al., PLB 776 (2018) 38; Lorcé et al., EPJC 79 (2019) 89; Ji, Front. Phys. (Beijing) 16 (2021) 64601; ...

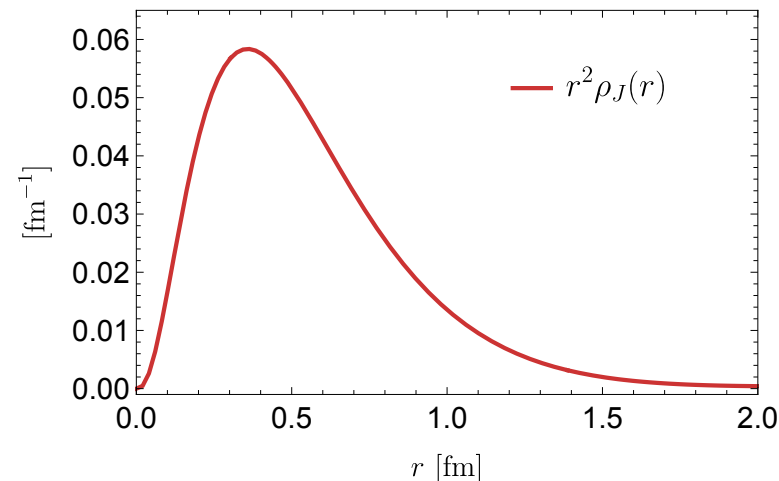
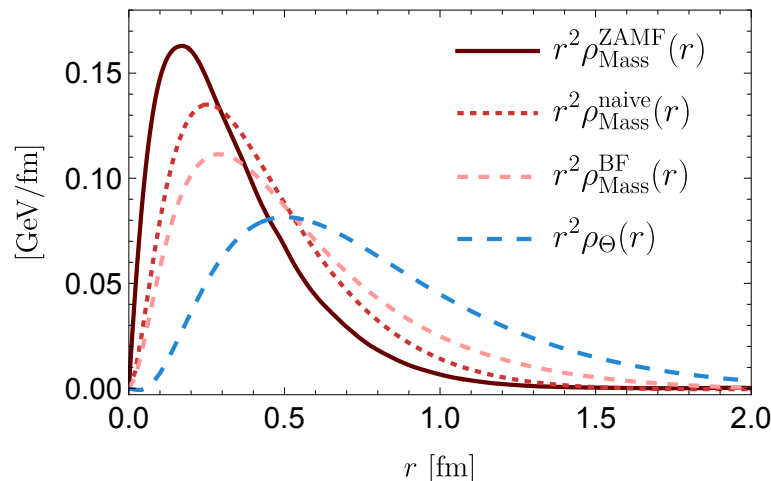
$$\rho_{\text{Mass}}(r) = m_N \int \frac{d^3\Delta}{(2\pi)^3} e^{-ir\cdot\Delta} \left[A(t) - \frac{t}{4m_N^2} [A(t) - 2J(t) + D(t)] \right] = \int \frac{d^3\Delta}{(2\pi)^3} e^{-ir\cdot\Delta} \left[\Theta(t) + \frac{t}{2m_N} D(t) \right]$$

$$\rho_{\Theta}(r) = m_N \int \frac{d^3\Delta}{(2\pi)^3} e^{-ir\cdot\Delta} \left[A(t) - \frac{t}{4m_N^2} [A(t) - 2J(t) + 3D(t)] \right] = \int \frac{d^3\Delta}{(2\pi)^3} e^{-ir\cdot\Delta} \Theta(t)$$

$$\rho_J(r) = \int \frac{d^3\Delta}{(2\pi)^3} e^{-ir\cdot\Delta} \left[J(t) + \frac{2}{3} t \frac{d}{dt} J(t) \right]$$

- Zero-average-momentum frame (ZAMF) Epelbaum et al., PRL 129 (2022) 012001; Panteleeva et al., EPJC 83 (2023) 617; ...

$$\rho_{\text{Mass}}^{\text{ZAMF}}(r) = \frac{m_N}{4\pi r} \int_0^\infty d\Delta \Delta \sin(\Delta r) \int_{-1}^1 d\alpha A \left[(\alpha^2 - 1) \Delta^2 \right], \quad \rho_{\text{Mass}}^{\text{naive}}(r) = m_N \int \frac{d^3\Delta}{(2\pi)^3} e^{-ir\cdot\Delta} A(t)$$



Spatial density profiles: “mechanical force”

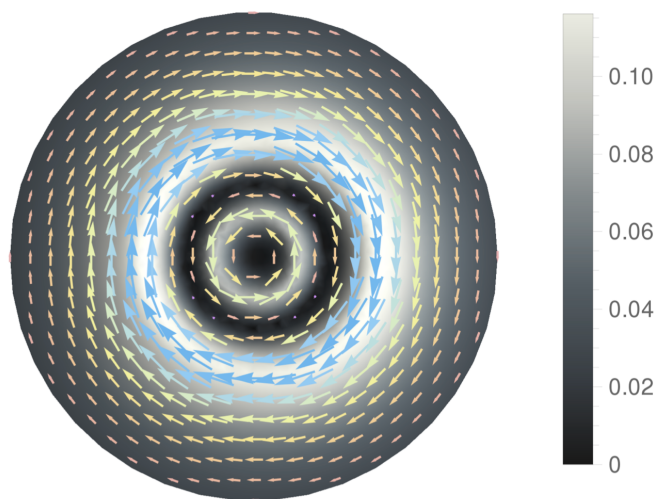
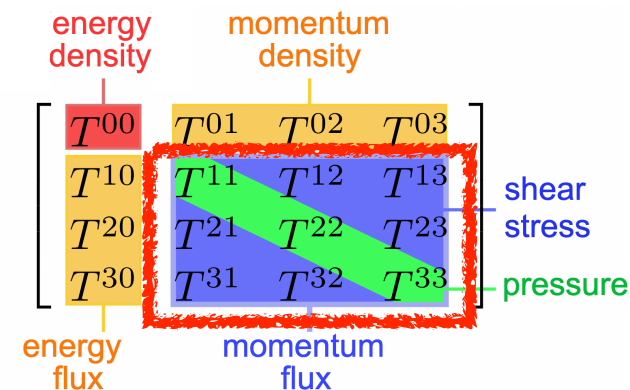
• $T_{ij}(\vec{r}) = \underbrace{s(r)}_{\text{Shear force}} \left(\frac{r_i r_j}{r^2} - \frac{1}{3} \delta_{ij} \right) + \underbrace{p(r)}_{\text{Pressure}} \delta_{ij} = \text{symmetric } 3 \times 3 \text{ matrix} \rightarrow \text{diagonalize}$

Shear force

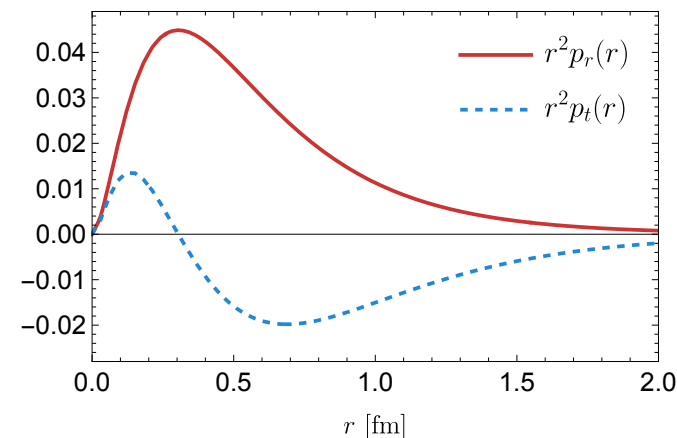
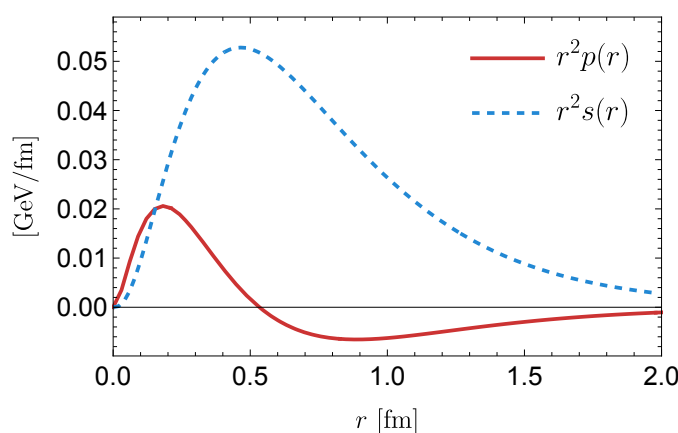
Pressure

$\frac{2}{3}s(r) + p(r) = p_r(r)$ normal force (eigenvector $\vec{e}_r$)

$-\frac{1}{3}s(r) + p(r) = p_t(r)$ tangential force ($\vec{e}_\theta, \vec{e}_\phi$, degenerate for spin 0 and $\frac{1}{2}$)



Tangential force from
Polyakov and Schweitzer, (2018)



Other 3D interpretation, see

Ji & Yang, 2503.01991; Ji & Liu PRD 106, 034028 (2022)

Freese, 2505.06135; PRD 111, 034047 (2025)

Li, Wang & P. Vary, 2405.06892.....

Recent review: Lorcé & Schweitzer, Acta Phys.Polon.B 56, A17 (2025)

Nucleon GFFs: D-term & radii

Various radii in the Breit frame

Radius of the scalar trace density

$$\langle r_{\Theta}^2 \rangle = \frac{6\dot{\Theta}(0)}{m_N} = 6\dot{A}(0) - \frac{9D}{2m_N^2}$$

Radius of the mass (energy) density

$$\langle r_{\text{Mass}}^2 \rangle = 6\dot{A}(0) - \frac{3D}{2m_N^2}$$

Mechanical radius Polyakov, PLB (2003)
Polyakov, & Schweitzer, JMPA (2018)

$$\langle r_{\text{Mech}}^2 \rangle = \frac{6D}{\int_{-\infty}^0 dt D(t)}$$

Radius of the angular momentum (spin) density $J(t) + \frac{2}{3}t \frac{dJ(t)}{dt}$

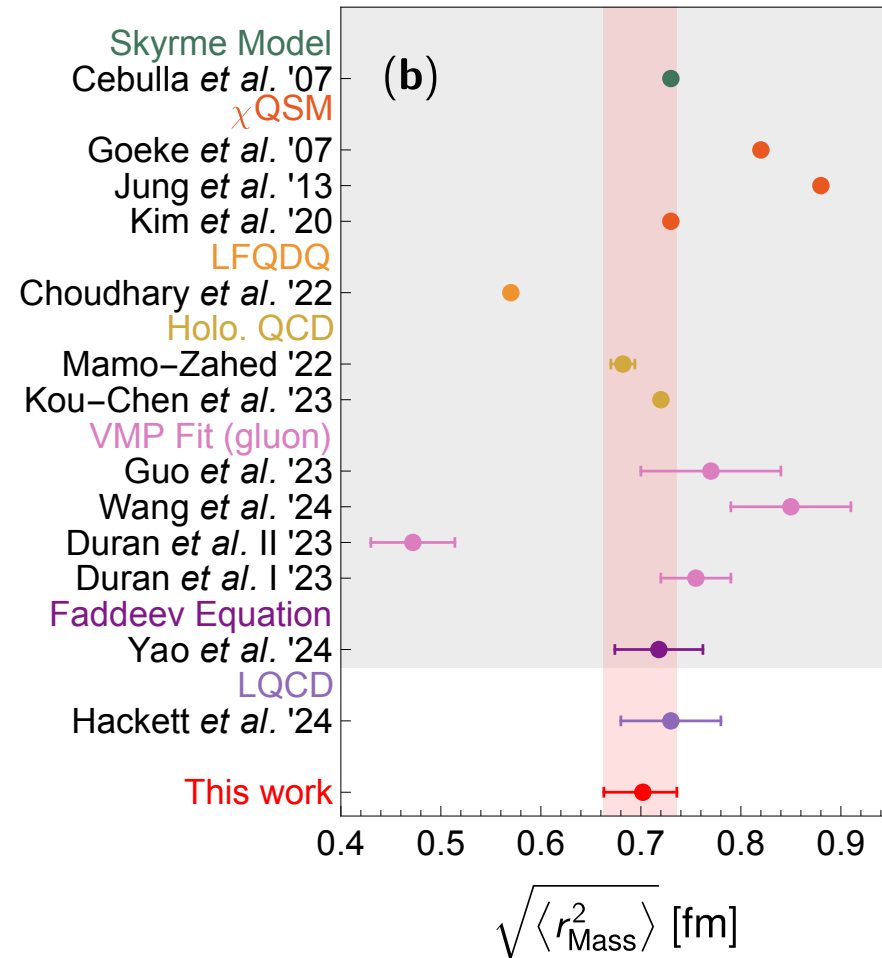
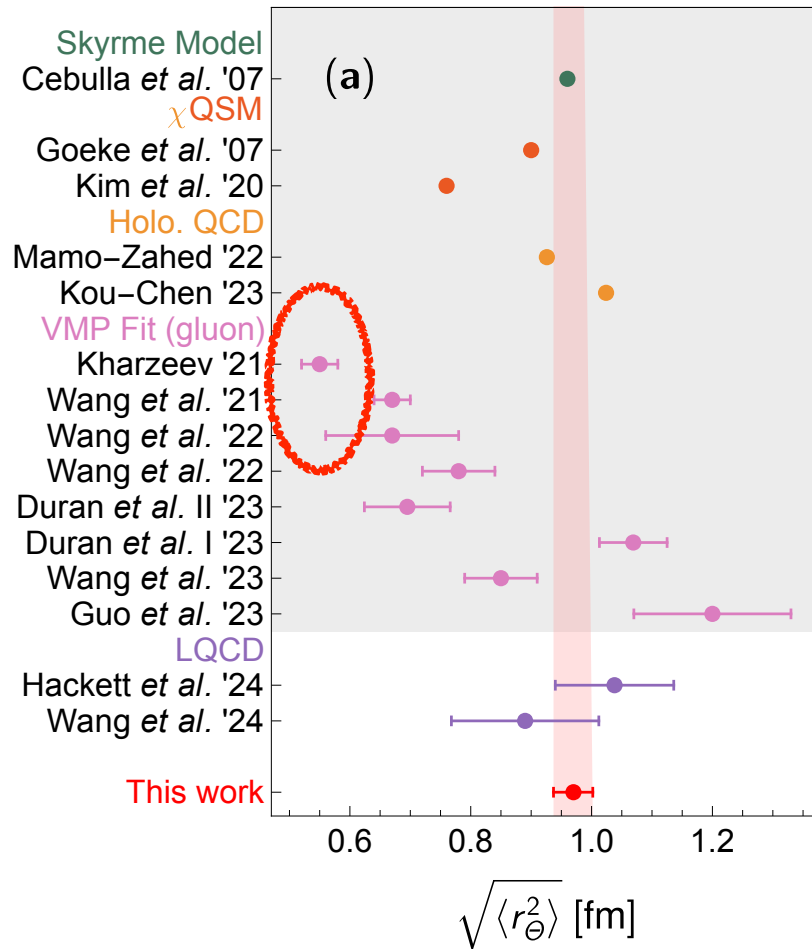
$$\langle r_J^2 \rangle = 20J'(0)$$

Polyakov, PLB (2003)
Lorcé et al., PLB (2018)

Quantity	Result	Error budget
D-term	$-3.38^{+0.34}_{-0.35}$	$+(0.18)_{\text{ChPT}}(0.12)_{\text{PWA}}(0.26)_{\text{eff}}$ $-(0.16)_{\text{ChPT}}(0.12)_{\text{PWA}}(0.29)_{\text{eff}}$
$\sqrt{\langle r_{\Theta}^2 \rangle} [\text{fm}]$	$0.97^{+0.03}_{-0.03}$	$+(0.01)_{\text{ChPT}}(0.01)_{\text{PWA}}(0.03)_{\text{eff}}$ $-(0.02)_{\text{ChPT}}(0.01)_{\text{PWA}}(0.26)_{\text{eff}}$
$\sqrt{\langle r_{\text{Mass}}^2 \rangle}$	$0.70^{+0.03}_{-0.04}$	$+(0.02)_{\text{ChPT}}(0.01)_{\text{PWA}}(0.02)_{\text{eff}}$ $+(0.02)_{\text{ChPT}}(0.01)_{\text{PWA}}(0.03)_{\text{eff}}$
$\sqrt{\langle r_{\text{Mech}}^2 \rangle}$	$0.72^{+0.09}_{-0.08}$	$+(0.02)_{\text{ChPT}}(0.00)_{\text{PWA}}(0.09)_{\text{eff}}$ $-(0.03)_{\text{ChPT}}(0.01)_{\text{PWA}}(0.07)_{\text{eff}}$
$\sqrt{\langle r_J^2 \rangle}$	$0.70^{+0.02}_{-0.02}$	$+(0.01)_{\text{ChPT}}(0.01)_{\text{PWA}}(0.01)_{\text{eff}}$ $-(0.01)_{\text{ChPT}}(0.00)_{\text{PWA}}(0.02)_{\text{eff}}$

ChPT: NLO ChPT inputs
pwa: $\pi\pi/K\bar{K} \rightarrow N\bar{N}$
eff: effective poles m_S, m_D

Comparison with other results



Physics of the scalar (trace) radius

- The scalar (trace) radius is larger than the mass radius as long as $D < 0$

$$T_{\mu}^{\mu}$$

$$\sqrt{\langle r_{\Theta}^2 \rangle} = 0.97^{+0.03}_{-0.03} \text{ fm}$$



“the largest radius in strong interaction”

$$> \sqrt{\langle r_{C,p}^2 \rangle} \simeq 0.84 \text{ fm}$$

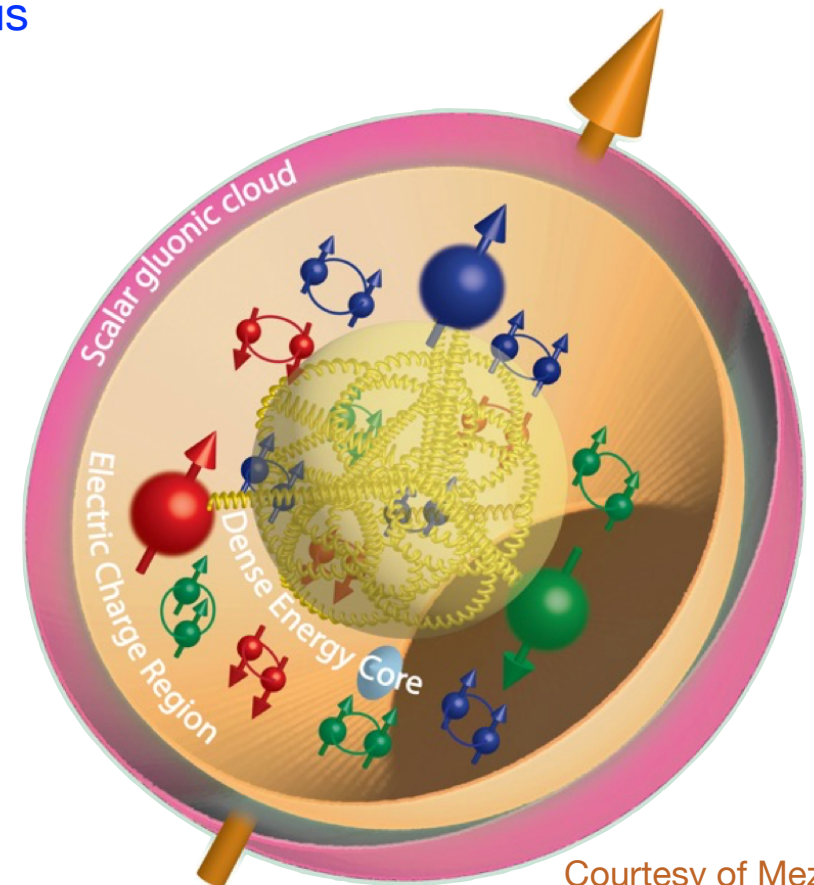
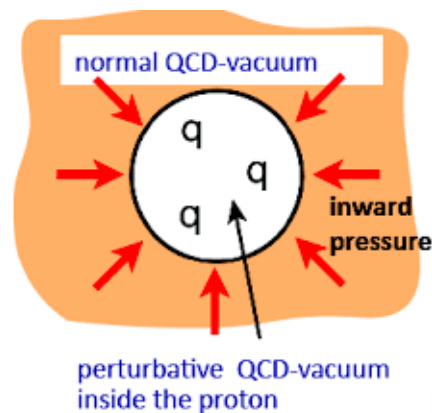
proton charge radius

$$T^{00}$$

$$> \sqrt{\langle r_{\text{Mass}}^2 \rangle} = 0.70^{+0.03}_{-0.04} \text{ fm}$$

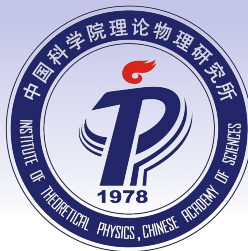
- In the MIT bag model, scalar radius may be considered as the bag (confinement) radius $\sim 1 \text{ fm}$ (physical boundary of confinement)

Ji, Front.Phys.(Beijing) (2021)



Courtesy of Mezziani

Effective chiral action in external gravitational field



Contains known LECs

UG Meissner et al. 2000

$$S_{\pi N} = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} \bar{\Psi} i e_a^\mu \gamma^a \nabla_\mu \Psi - \frac{1}{2} \nabla_\mu \bar{\Psi} i e_a^\mu \gamma^a \Psi - m \bar{\Psi} \Psi + \frac{g_A}{2} \bar{\Psi} e_a^\mu \gamma^a \gamma_5 u_\mu \Psi \right. \\ + c_1 \langle \chi_+ \rangle \bar{\Psi} \Psi - \frac{c_2}{8m^2} g^{\mu\alpha} g^{\nu\beta} \langle u_\mu u_\nu \rangle (\bar{\Psi} \{ \nabla_\alpha, \nabla_\beta \} \Psi + \{ \nabla_\alpha, \nabla_\beta \} \bar{\Psi} \Psi) + \frac{c_3}{2} g^{\mu\nu} \langle u_\mu u_\nu \rangle \bar{\Psi} \Psi \\ + \frac{ic_4}{4} \bar{\Psi} e_a^\mu e_b^\nu \sigma^{ab} [u_\mu, u_\nu] \Psi + c_5 \bar{\Psi} \hat{\chi}_+ \Psi + \frac{c_6}{8m} \bar{\Psi} e_a^\mu e_b^\nu \sigma^{ab} F_{\mu\nu}^+ \Psi + \frac{c_7}{8m} \bar{\Psi} e_a^\mu e_b^\nu \sigma^{ab} \langle F_{\mu\nu}^+ \rangle \Psi$$

$$+ \left\{ \frac{c_8}{8} R \bar{\Psi} \Psi + \frac{ic_9}{m} R^{\mu\nu} (\bar{\Psi} e_\mu^a \gamma_a \nabla_\nu \Psi - \nabla_\nu \bar{\Psi} e_\mu^a \gamma_a \Psi) \right\}$$

D-term is related to interaction with space-time curvature

New LECs: interaction with curvature

Alharazin et al., 2020; Alharazin, 2024

$$\frac{D(0)}{m_N} = c_8 + \frac{g_A^2}{16\pi F^2} M_\pi + \frac{-3g_A^2/m_N + 2(-4c_1 + c_2 + 2c_3)}{8\pi^2 F^2} M_\pi^2 \ln \left(\frac{M_\pi}{m_N} \right) + \mathcal{O}(M_\pi^3)$$

Belitsky and Ji, 2002

Diehl, Manashov and Schafer, 2006

Chiral expansion of D-term

Alharazin et al., 2020; Alharazin, 2024

Matching dispersion relation to ChPT

- NLO ($\mathcal{O}(p^2)$) matching:

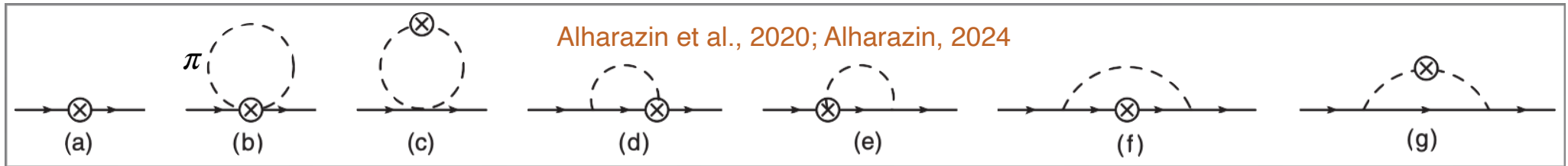
$$A^{(2)}(t) = 1 - \frac{2c_9}{m_N}t, \quad J^{(2)}(t) = \frac{1}{2} - \frac{c_9}{m_N}t, \quad D^{(2)}(t) = c_8 m_N$$

- LECs: $c_8 = -3.60^{+0.37}_{-0.38} \text{ GeV}^{-1}$, $c_9 = -0.58^{+0.03}_{-0.03} \text{ GeV}^{-1}$

Matching dispersion relation to ChPT

- NLO ($\mathcal{O}(p^2)$) matching: $c_8 = -3.60_{-0.38}^{+0.37} \text{ GeV}^{-1}$, $c_9 = -0.58_{-0.03}^{+0.03} \text{ GeV}^{-1}$
- NNLO ($\mathcal{O}(p^3)$) matching: a new chiral $\mathcal{O}(p^3)$ operator

$$\mathcal{S}_{\pi N}^{(3)} = \int d^4x \sqrt{|g|} \left\{ i \tilde{d}_{g4} \nabla_\beta R^{\mu\nu\alpha\beta} \left(\bar{\psi} \sigma_{\mu\nu} e_\mu^a \gamma_a \nabla_\nu \psi - \nabla_\nu \bar{\psi} \sigma_{\mu\nu} e_\mu^a \gamma_a \psi \right) \right\}$$

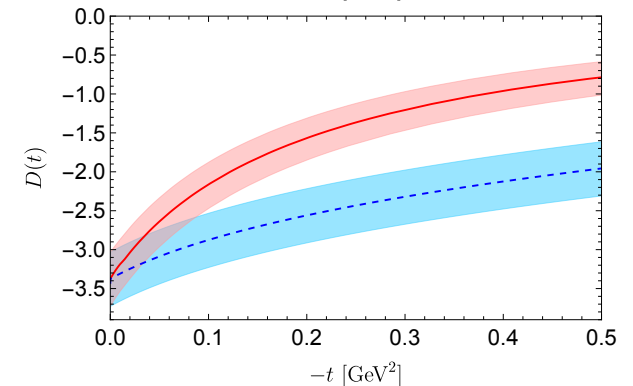
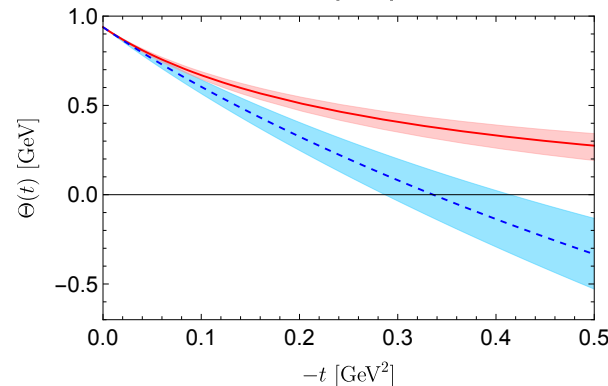
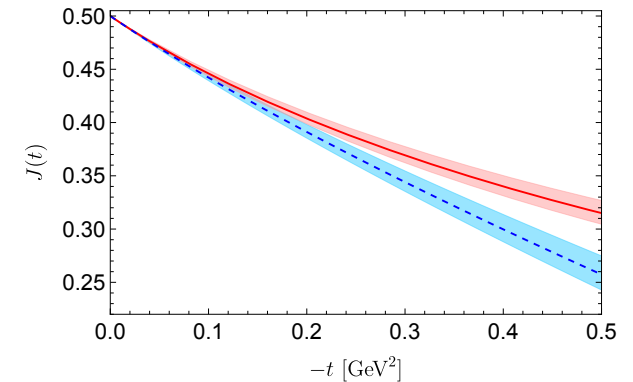
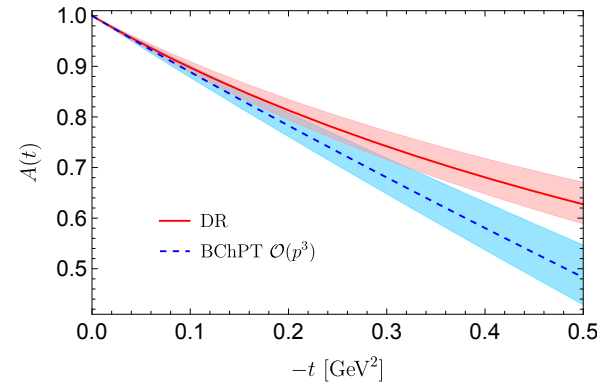


● LECs:

$$c_8 = -4.28_{-0.38}^{+0.37} \text{ GeV}^{-1}$$

$$c_9 = -0.68_{-0.05}^{+0.06} \text{ GeV}^{-1}$$

$$\tilde{d}_{g4} = -0.04_{-0.02}^{+0.01} \text{ GeV}^{-3}$$



Many other applications:

1. **Soft hadron reactions** in grav. field, e.g. pion gravitoproduction
2. Low energy gravitoproduction can be probed in a lab, e.g. **non-diagonal DVCS** (ongoing analysis at CLAS 12, plans for EIC&ElcC).....

Summary and outlook

✓ The pion, kaon and nucleon GFFs are precisely determined using model-independent dispersive method

✓ Quantification of **systematic and theoretical uncertainties**

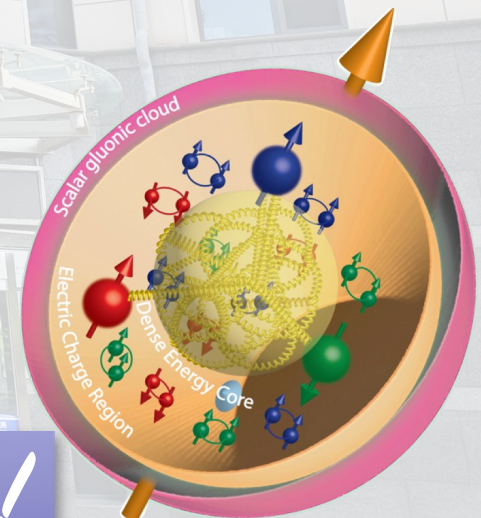
✓ Nucleon static properties:

$$D = -3.38^{+0.34}_{-0.35}, \quad \sqrt{\langle r_{\Theta}^2 \rangle} = 0.97^{+0.03}_{-0.03} \text{ fm} > \sqrt{\langle r_{C,p}^2 \rangle} \simeq 0.84 \text{ fm} > \sqrt{\langle r_{\text{Mass}}^2 \rangle} = 0.70^{+0.03}_{-0.04} \text{ fm}$$

proton charge radius

□ Pion-mass dependence of nucleon GFFs

□ **Hyperon** gravitational structure.....



Thank you for your attention!