Progress on
Scattering Amplitudes & Beyond

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review based on works with N. Arkani-Hamed, Q. Cao, J. Dong, C. Figueiredo: 2312.16282, 2401.00041, 2401.05483, 2408.11891…

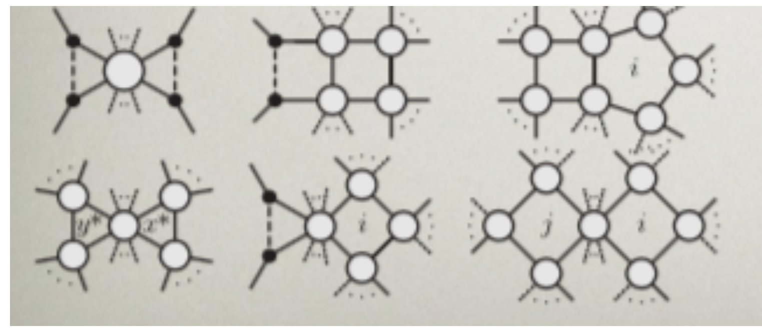
(see also w. N. Arkani-Hamed, Y. Bai, G.Yan: 2017; with N. Arkani-Hamed, T. Lam: + G. Salvatori, H. Thomas: 2019-22 …)

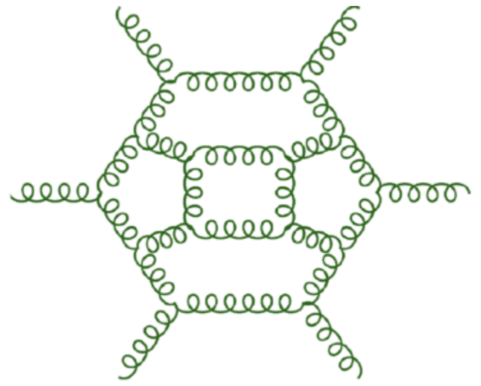
第十届海峡两岸粒子物理和宇宙学研讨会（广州）

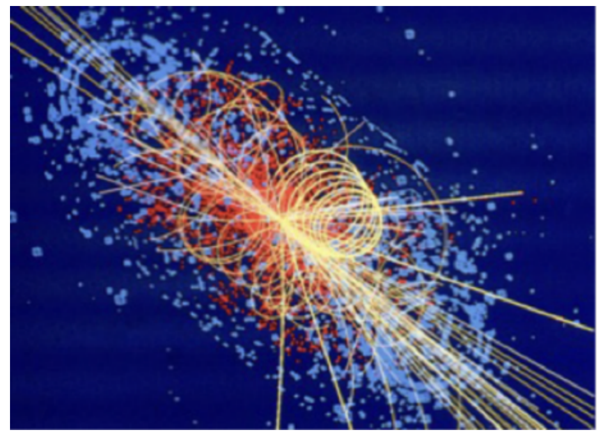
2026年1月

"Amplitudes"

On-shell, off-shell, weak/strong coupling…

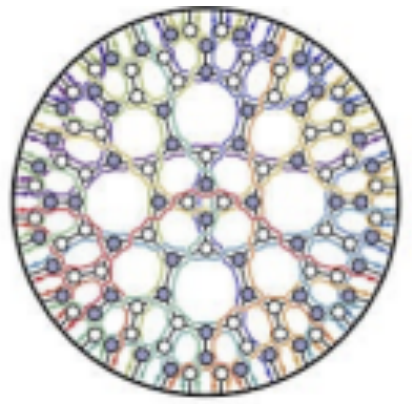






Formal QFT

Geometries, combinatorics, number theory, …

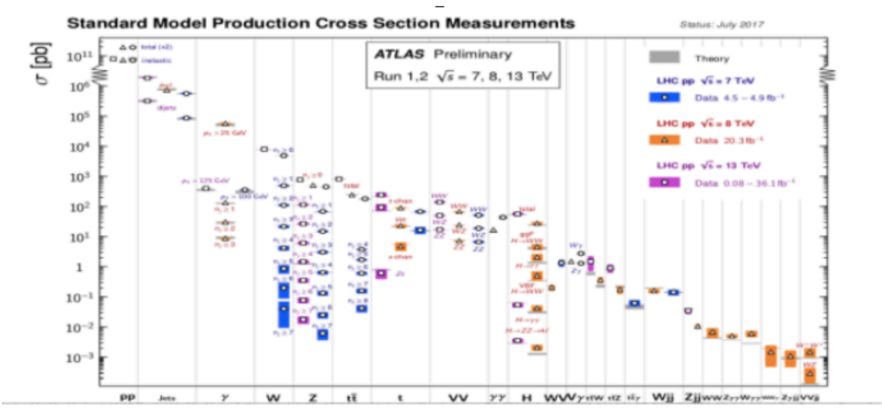


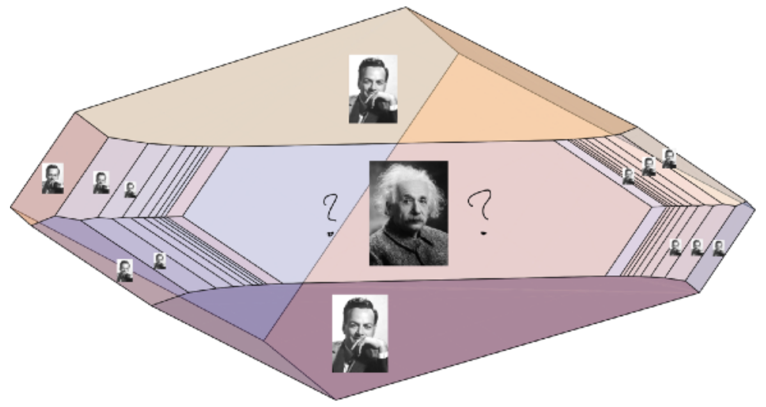
Mathematics

Amplitudes

precision frontier: loop integrands + integrals

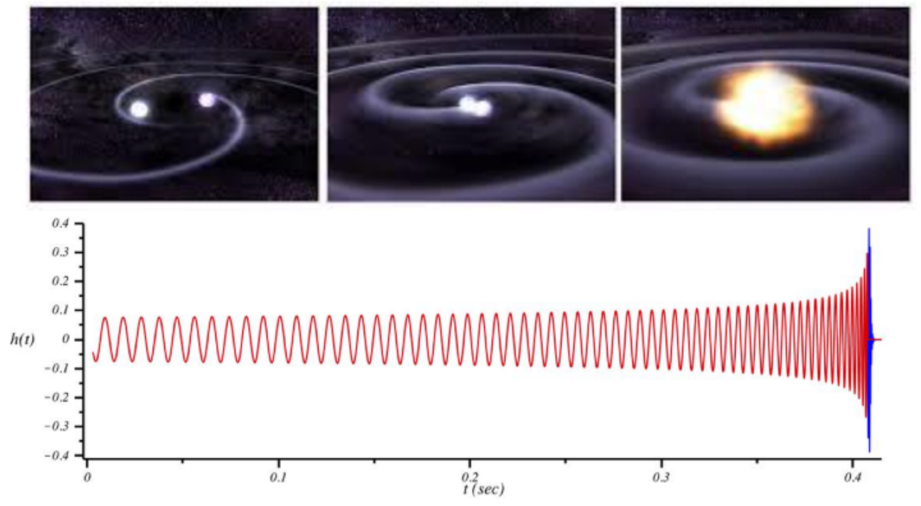
Collider Phenomenology



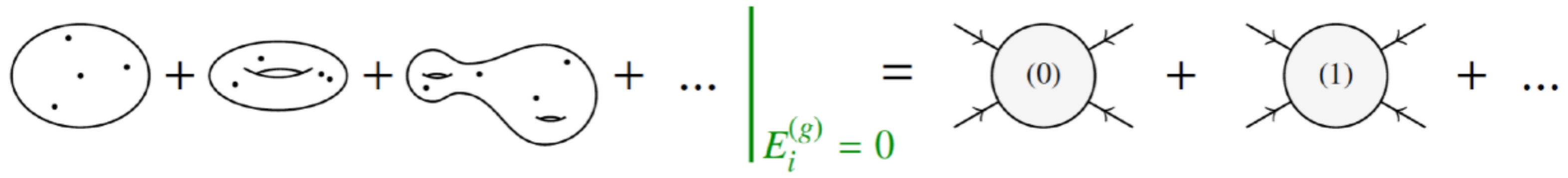


String Theory

Quantum/ Classical Gravity

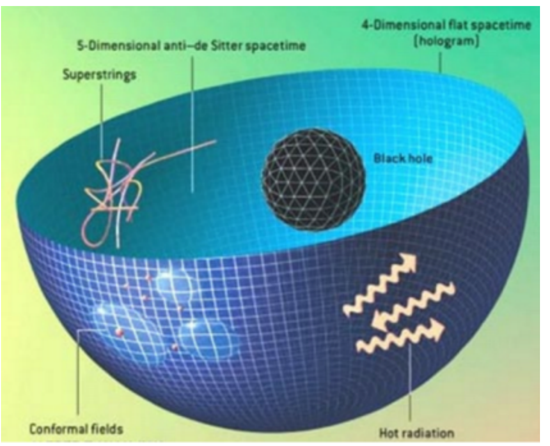


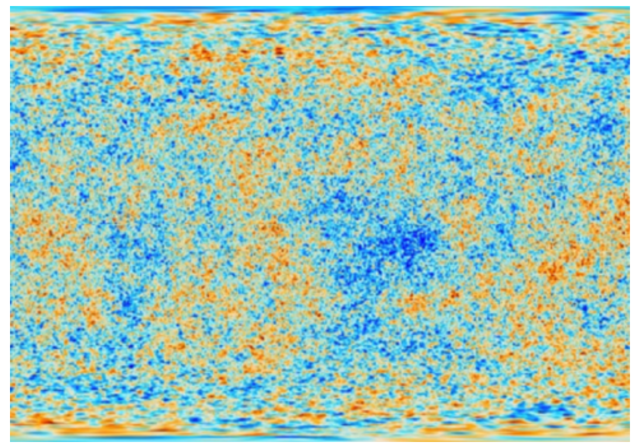
AdS/dS Cosmology



CFT + string perturbation, gauge/string dualities

Holography, curved background, inflation





gravity amps, black holes & GW

Combinatorial Geometries ("polytopes") underlying scattering amplitudes & beyond

- moduli space $\mathscr{M}_{g,n}$ for conventional & (ambi-)twistor strings [c.f. Witten,'04; Cachazo, SH, Yuan '13; Berkovits; Mason, Skinner ;…]
- positive Grassmannian $G_+(k,n)$, on-shell diagrams for planar N=4 SYM [Arkani-Hamed et al '12, …]

- Amplituhedron: all-loop integrands of N=4 SYM in momentum twistor space [Arkani-Hamed, Trnka '13 + Thomas '17;…]
- ABJM amplituhedron: all-loop integrands of ABJM (reduced from SYM) [SH, Kuo, Li, Zhang '22; SH, Huang, Kuo, '23,…]
- Correlahedron -> Wilson loops/cross-section in SYM/ABJM [SH, Huang, Kuo, '24-25] —> energy correctors [SH et al '24-25]

- kinematic associahedra (Tr ϕ^3 tree amps) & (Deligne-Mumford) worldsheet associahedra [Arkani-Hamed, Bai, SH, Yan, '17; …]
- surfacehedra + curve-integrals on surfaces => all-order Tr ϕ^3 amplitudes + (bosonic) string [Arkani-Hamed et al, 20-24,…]
- cosmological polytopes [Arkani-Hamed et al '17,…]—> wavefunction/correlator (FRW conformal scalars) [Arkani-Hamed et al '25, …]
- wider context: tropical geometries for Feynman integrals; positive geometries in dS/AdS (Mellin) amps etc [c.f. w. Cao, Li, Tang '24]

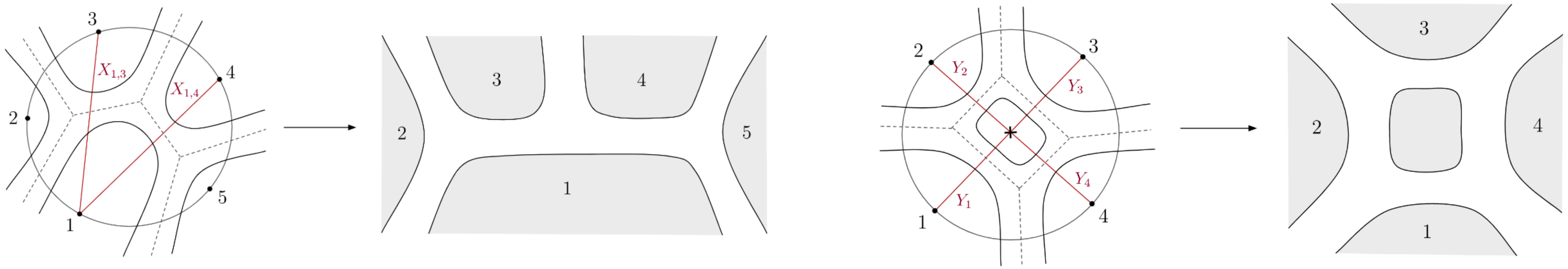
Tr ϕ^3 amplitudes [Arkani-Hamed, Bai, SH, Yan, ’17; Arkani-Hamed, Frost, Salvatori, Plamondon, Thomas, ’23,…]

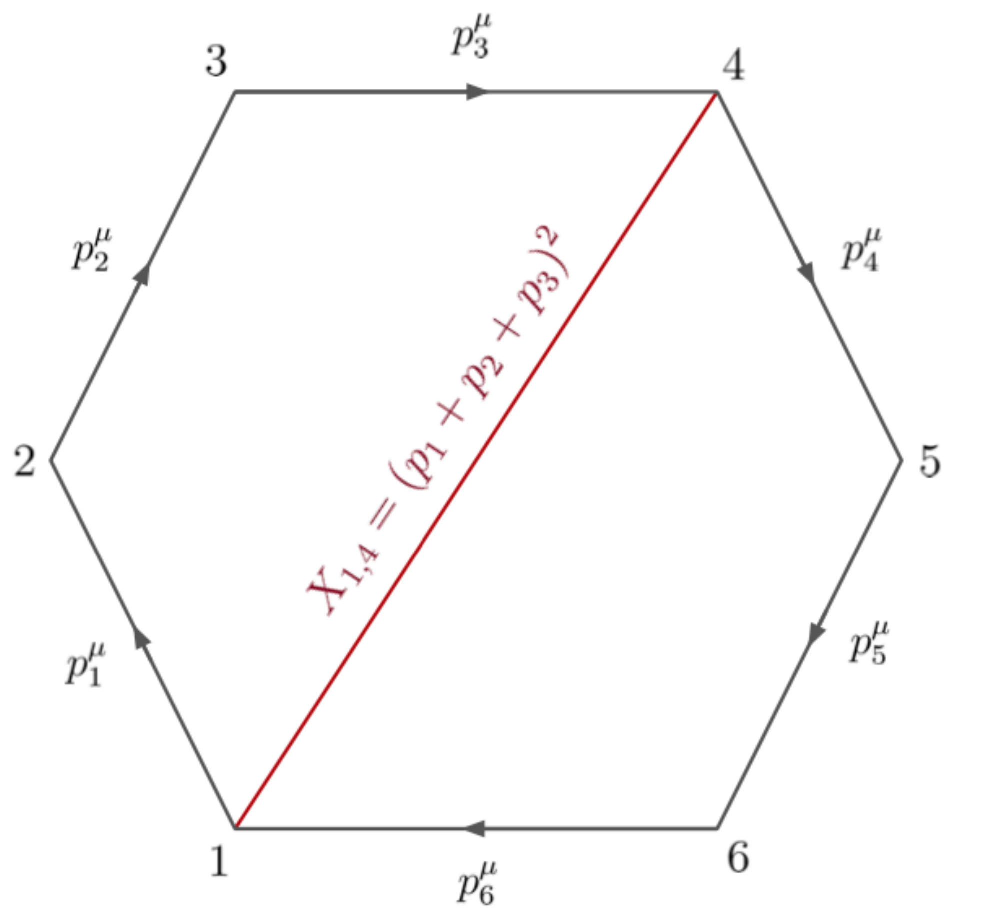
$$\mathcal{L}_{\mathrm{Tr}(\phi^3)} = \mathrm{Tr}(\partial\phi)^2 + g\,\mathrm{Tr}(\phi^3),$$

ϕ : N by N matrix -> fat graphs, genus expansion (only planar graphs for $N \to \infty$)

kinematics: e.g. tree amps (planar)

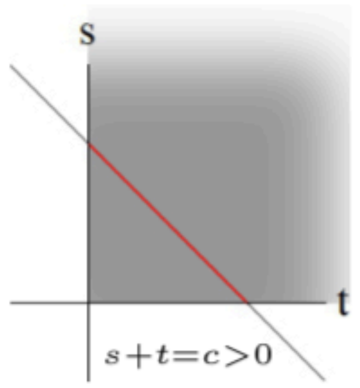
$$X_{i,j} = (p_i + \cdots + p_{j-1})^2.$$





tree amp = sum over n-gon triangulations = canonical form of associahedron

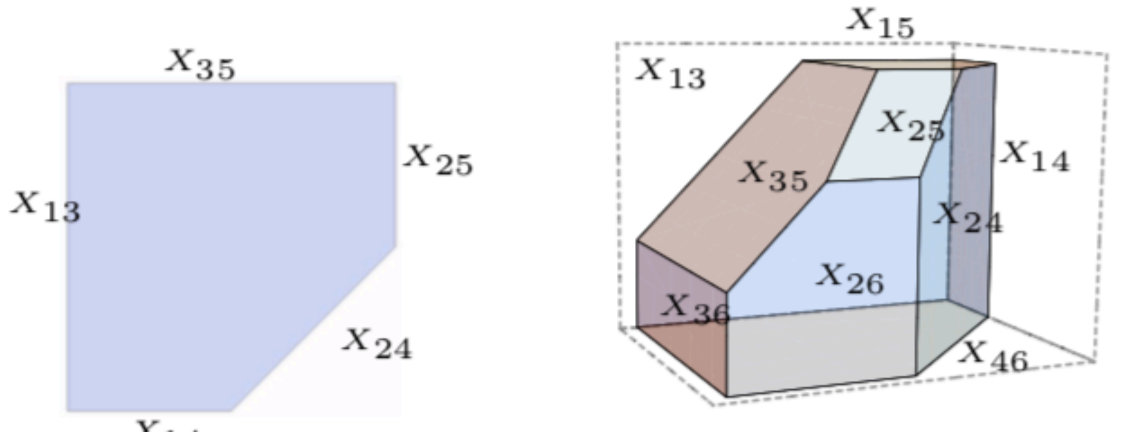
$$\mathcal{A}_4 = \frac{1}{X_{13}} + \frac{1}{X_{24}},$$



$e.g.\ \mathcal{A}_1 = \{s > 0, t > 0\} \cap \{-u = \mathrm{const} > 0\}$

$\mathcal{A}_2 = \{X_{13}, \cdots, X_{25} > 0\} \cap \{-s_{13} = c_{13}, -s_{14} = c_{14}, -s_{24} = c_{24}\}$

$$\mathcal{A}_5 = \frac{1}{X_{1,3}X_{1,4}} + \frac{1}{X_{2,4}X_{2,5}} + \frac{1}{X_{1,3}X_{3,5}} + \frac{1}{X_{1,4}X_{2,4}} + \frac{1}{X_{2,5}X_{3,5}}.$$



$$c_{i,j} := -2p_i \cdot p_j = X_{i,j} + X_{i+1,j+1} - X_{i+1,j} - X_{i,j+1},$$

Disk integral = stringy Tr ϕ^3 amp [c.f. Arkani-Hamed, SH, Lam, 19]

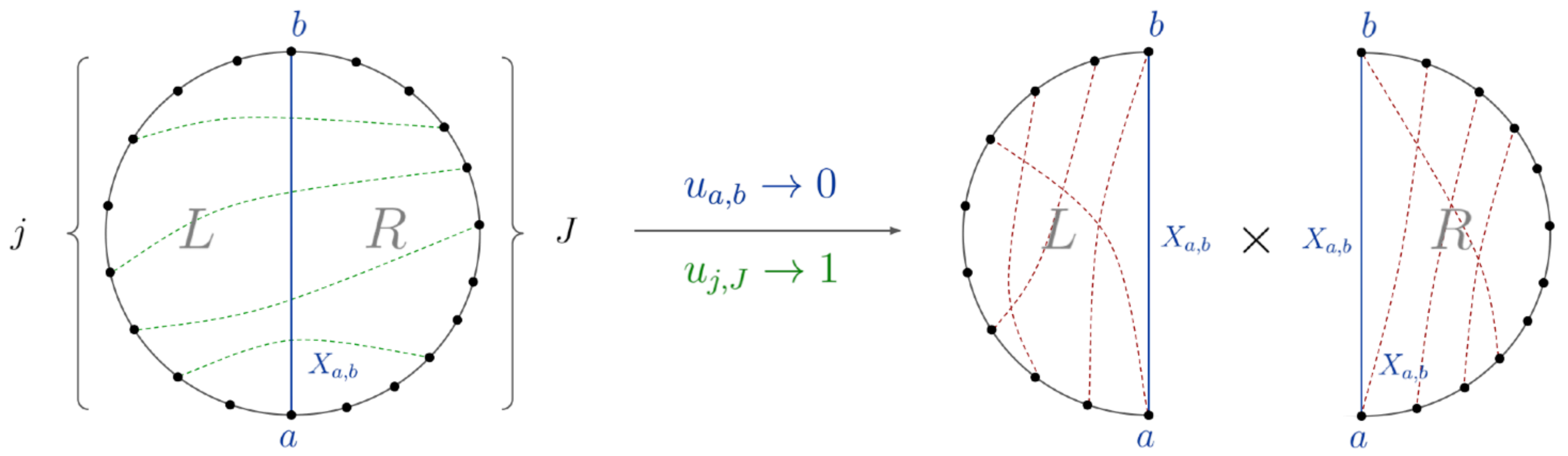
a la Veneziano-Koba-Nielsen; Z-theory [Carrasco, Mafra, Schlotterer 16',...]

$$\mathcal{I}_n^{\mathrm{Tr}\,\phi^3}(1,2,\ldots,n) = \int_{D(1\ldots n)} \frac{dz_1\ldots dz_n}{\mathrm{vol}\ \mathrm{SL}(2,\mathbb{R})} \underbrace{\frac{1}{z_{1,2}z_{2,3}\ldots z_{n,1}}}_{\mathrm{PT}(1,2,\ldots,n)} \times \underbrace{\prod_{i<j} z_{i,j}^{2\alpha' p_i\cdot p_j}}_{\text{Koba-Nielsen factor}}$$

u variables: $u_{a,b} := \dfrac{z_{a,b-1}z_{a-1,b}}{z_{a,b}z_{a-1,b-1}}$ [Koba, Nielsen, '69; ...]

=> $\displaystyle\int \frac{d^{n-3}z}{z_{12}\cdots z_{n1}} \prod_{a<b} u_{a,b}^{\alpha' X_{a,b}}$ w. positive parametrization $y_{I=1,\cdots,n-3}$ => $\displaystyle\mathcal{I}_n^{\mathrm{Tr}\phi^3} = \int_0^\infty \prod_I \frac{dy_I}{y_I} \prod_C u_C(y)^{\alpha' X_C}$ curve-integral on the disk!

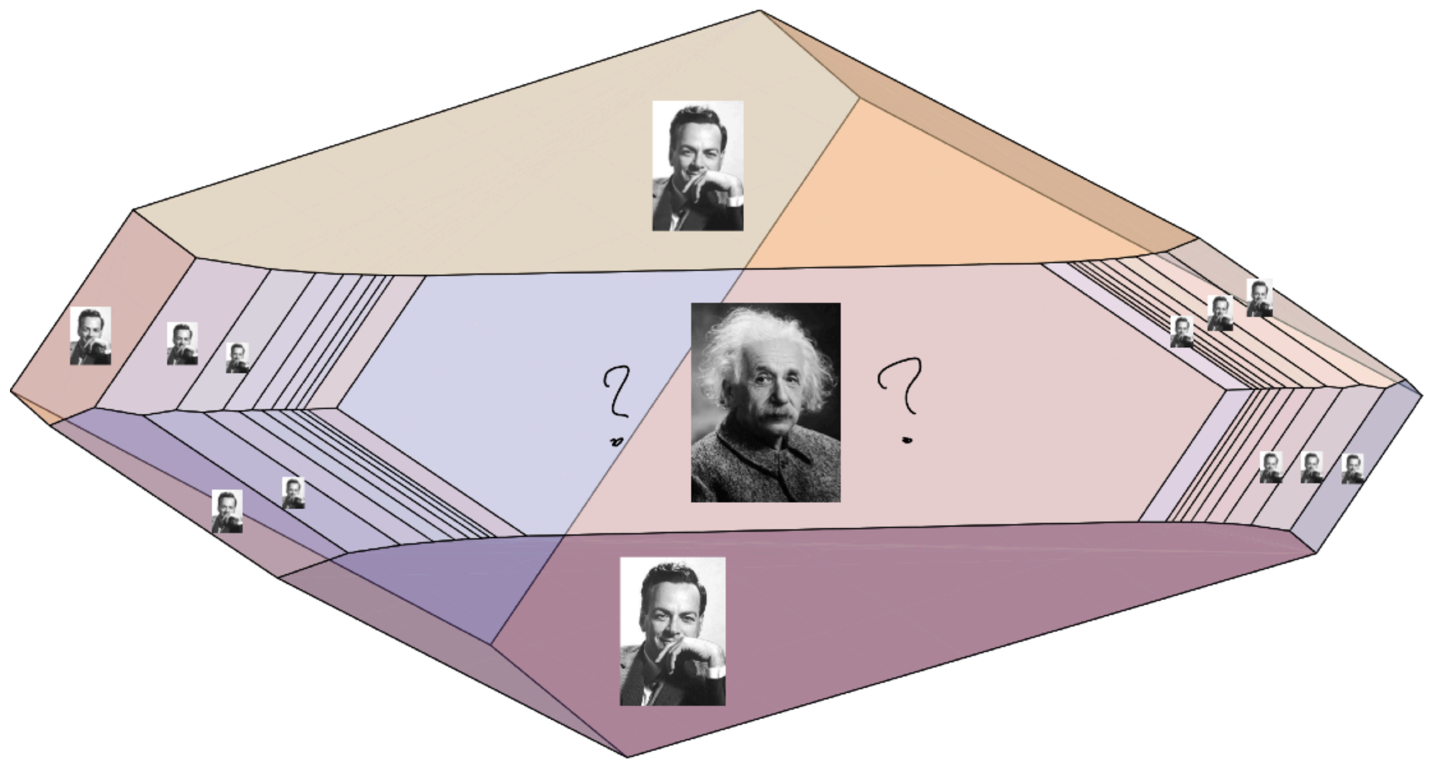
Binary geometry defined by u eqs: $\underbrace{u_{a,b}}_{\to 0} + \displaystyle\prod_{j\in L, J\in R} \underbrace{u_{j,J}}_{\to 1} = 1$ => all factorizations: $\mathcal{I}(1,\cdots,n) \to \mathcal{I}_L(a,\cdots,b) \otimes \mathcal{I}_R(b,\cdots,a)$



e.g. n=4: $\displaystyle\int_0^\infty \frac{dy}{y}\left(\frac{y}{1+y}\right)^{\alpha' X_{1,3}}\left(\frac{1}{1+y}\right)^{\alpha' X_{2,4}} = \frac{\Gamma(\alpha' X_{1,3})\Gamma(\alpha' X_{2,4})}{\Gamma(\alpha'(X_{1,3}+X_{2,4})}$

stringy extension of associahedra [Arkani-Hamed, SH, Lam, 19; ...] -> similar "u" variables (curves)+ positive para. (triangulation) for any surface!
-> combinatorial formulation of (bosonic) strings: $\alpha' \to 0$ (tropical) limit unifies all Feynman diagrams as a single object!!!

Curve integral: from toy model to Real World



A new way of thinking about particle/string amplitudes based on combinatorial & geometric objects without reference to Feynman diagrams/worldsheet physics

Curve integrals on surfaces: "counting problem" => "u" variables for bosonic string to all orders = stringy completion of (simplest colored scalars) Tr ϕ^3 amps (="surfacehera") [ABHY '17; Arkani-Hamed, Frost, Salvatori, Plamondon, Thomas: 2309.15913, 2311.09284 …]

- Manifest (tree) factorization (finite α', no blowup): massive residues, asymptotics etc. [c.f. Arkani-Hamed, Figueiredo, Remmen 2412.20639]
- All-loop cuts ($\alpha' \to 0$) from surfaces => all-loop recursion for Tr ϕ^3+ any colored-scalars [c.f. Arkani-Hamed, Frost, Salvatori 2412. 21027]
- Profound connections with math: positive geometries, quiver rep. theory, cluster algebra, Teichmueller theory, matrix model,…

Surprisingly, this toy model contains realistic theories: amplitudes of pions & gluons from stringy Tr ϕ^3 by kinematic shifts!

- The unity of ϕ^3, pions & gluons (tree) => all-loop non linear sigma model (+ mixed amps) $\subset \mathrm{Tr}\ \phi^3$
- Reveals novel features of all these theories, e.g. hidden patterns of zeros & factorization/splits near zeros
- "Combinatorial origin of Yang-Mills": scalar-scaffolded gluons => all-loop YM in stringy Tr ϕ^3

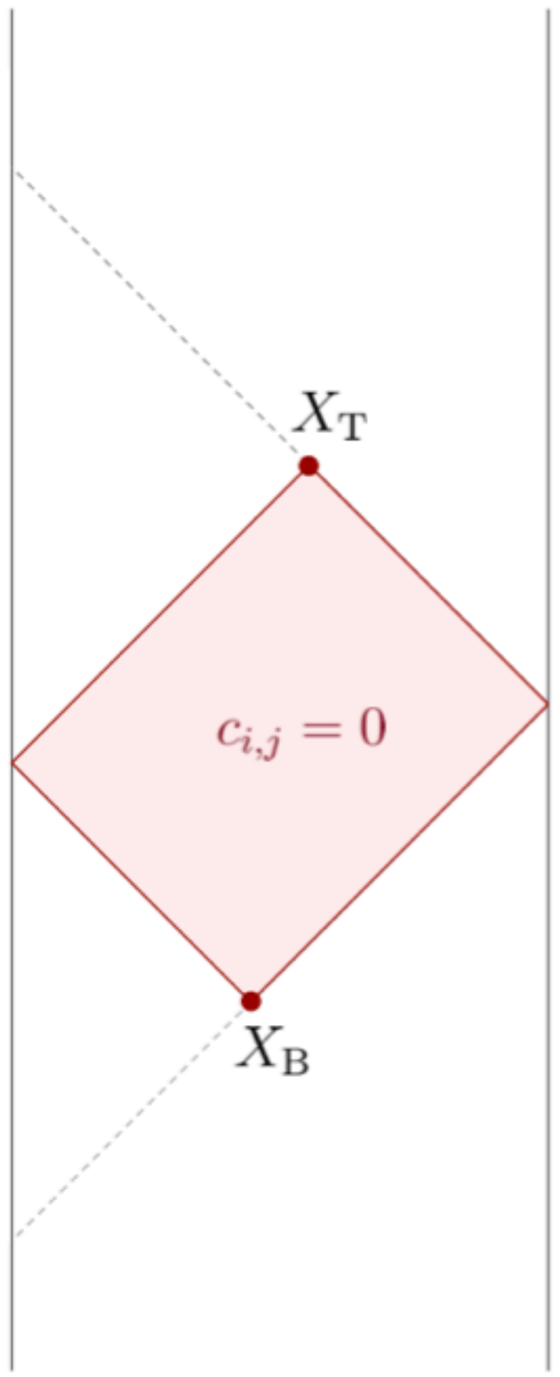
Zeros & splits of amplitudes [ACDFH 2023; see D'Adda, Sciuto, D'Auria, Gliozzi, 71]

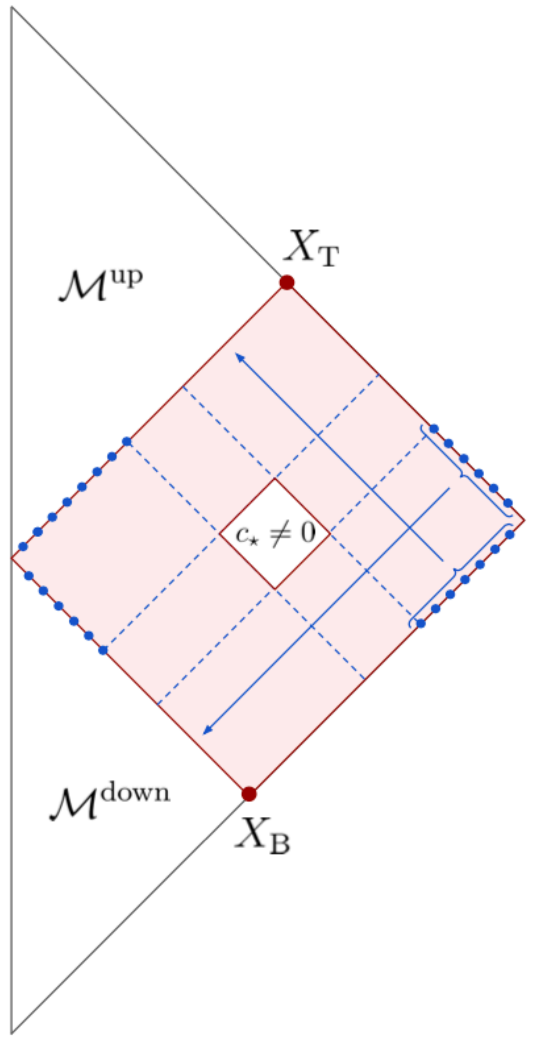
Zeros of Veneziano amp: by setting $\alpha' c_{1,3} = -n$, $\mathcal{I}_4^{\mathrm{Tr}(\phi^3)} \to \sum_{k=0}^{n} \underbrace{\int_{\mathbb{R}_{>0}} \frac{dy_{1,3}}{y_{1,3}} y_{1,3}^{\alpha' X_{1,3}+k}}_{=0} = 0.$

$$\mathcal{I}_n^{\mathrm{Tr}\,\phi^3} \to \sum_{k_{a_1,b_1},\dots,k_{a_N,b_N}=0}^{n_{a_1,b_1},\dots,n_{a_N,b_N}} (\text{remaining integrals}) \times \underbrace{\int_{\mathbb{R}_{>0}} \frac{dy_{1,i}}{y_{1,i}} y_{1,i}^{\alpha' X_{1,i}+k_{a_1,b_1}+\dots+k_{a_N,b_N}}}_{=0} = 0$$

$c_{i,j} = -n_{ij}, \quad 1 \leq i < a-1,\ a \leq j < n$

n(n-3)/2 infinite families of zeros





$$\mathcal{I}_n^{\mathrm{Tr}\,\phi^3} \to \mathcal{I}_i^{\mathrm{down},\mathrm{Tr}\,\phi^3} \times \mathcal{I}_{n-i+2}^{\mathrm{up},\mathrm{Tr}\,\phi^3} \times \mathcal{I}_4^{\mathrm{Tr}\,\phi^3}(\alpha' X_{1,i}, \alpha'(c_{km} - X_{1,i})).$$

$$X_{l,i} \to X_{l,i} + X_{1,i} = X_{l,n}, \text{for} \quad l = 2,\dots,k.$$

$$X_{i-1,j} \to X_{i-1,j} - X_{i-1,n} = X_{1,j}, \text{for} \quad j = m,\dots,n-1.$$

$$\mathcal{M}_n(c_\star \neq 0) = \left(\frac{1}{X_{\mathrm{B}}} + \frac{1}{X_{\mathrm{T}}}\right) \times \mathcal{M}^{\mathrm{down}} \times \mathcal{M}^{\mathrm{up}}.$$

shifted kinematics —> currents (with an off-shell leg)

$X = X_B + \tilde{X}. \qquad \tilde{X} = X_{\mathrm{T}} + X,$

Deformed to the real world [ACDFH 2023]

$$\mathcal{I}^{\delta}_{2n} = \int_{\mathbb{R}^{2n-3}_{>0}} \prod_{I=1}^{2n-3} \frac{dy_I}{y_I} \prod_{(a,b)} u_{a,b}^{\alpha' X_{a,b}} \left(\frac{\prod_{(e,e)} u_{e,e}}{\prod_{(o,o)} u_{o,o}} \right)^{\alpha'\delta}, \qquad \mathcal{I}^{\delta}_{2n} = \mathcal{I}^{\mathrm{Tr}\,\phi^3}_{2n}[\alpha' X_{e,e} \to \alpha'(X_{e,e}+\delta), \alpha' X_{o,o} \to \alpha'(X_{o,o}-\delta)].$$

key: all $c_{i,j} = X_{i,j} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j}$ are preserved => same zero + splits for deformed cases!

$\alpha'\delta = 0$

$$\mathcal{L}_{\mathrm{Tr}(\phi^3)} = \mathrm{Tr}(\partial\phi)^2 + g\,\mathrm{Tr}(\phi^3),$$

$0 < \alpha'\delta < 1 \quad (\text{or } \mathbb{R}/\mathbb{Z})$ $\qquad \alpha' \to 0$

$$\mathcal{L}_{\mathrm{NLSM}} = \frac{1}{8\lambda^2}\mathrm{Tr}\left(\partial_\mu \mathrm{U}^\dagger \partial^\mu \mathrm{U}\right), \quad \text{with} \quad \mathrm{U} = (\mathbb{I} + \lambda\Phi)(\mathbb{I} - \lambda\Phi)^{-1}$$

$\alpha'\delta = \pm 1, \ (\pm 2, \ldots)$

$$\mathcal{L}_{\mathrm{YMS}} = -\mathrm{Tr}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D^\mu\phi^I D_\mu\phi^I - \frac{g^2_{\mathrm{YM}}}{4}\sum_{I\neq J}[\phi^I, \phi^J]^2\right)$$

Tr ϕ^3 2n-pt string amps => 2n-scalar amp in NLSM or in YM + scalar: same function @ different pts!

All-loop NLSM contained in Tr ϕ^3 [ACDFH 2024]

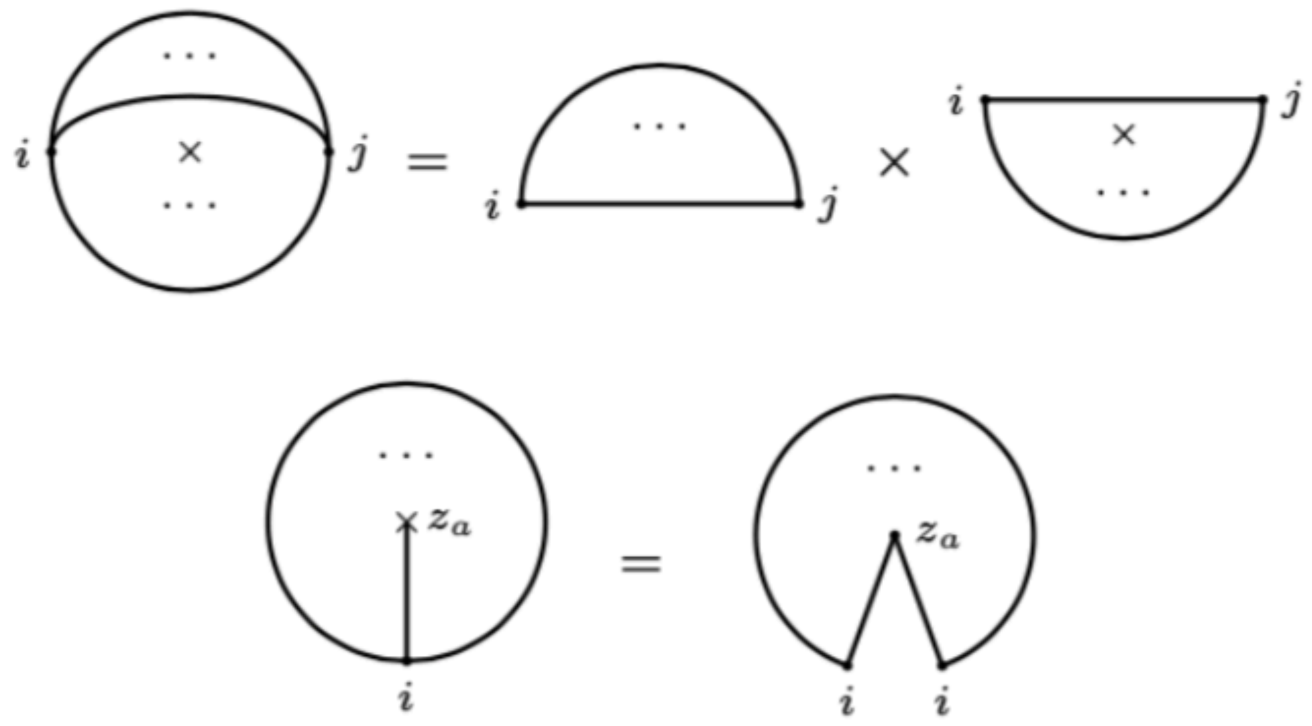
$$\mathcal{I}^{\delta}_{2n} = \int_{\mathbb{R}^{2n-3}_{>0}} \prod_{I=1}^{2n-3} \frac{dy_I}{y_I} \prod_{(e,e)} u_{e,e}^{\alpha'(X_{e,e}+\delta)} \times \prod_{(o,o)} u_{o,o}^{\alpha'(X_{o,o}-\delta)} \times \prod_{(o,e)} u_{o,e}^{\alpha' X_{o,e}}$$
$$\to \mathcal{A}^{\mathrm{Tr}\,\phi^3}_{2n}(X_{e,e} \to X_{e,e}+\delta, X_{o,o} \to X_{o,o}-\delta),$$

Field-theory directly take $\delta \to \infty$:

$$A^{\mathrm{NLSM}}_{2n} = \lim_{\delta\to\infty} \delta^{n-1} A^{\mathrm{Tr}\,\phi^3}_{2n}(X_{e,e} \to X_{e,e}+\delta, X_{o,o} \to X_{o,o}-\delta),$$

Same shift works for planar integrand of NLSM: $X_{e,e} \to X_{e,e} + \delta\,, \quad X_{o,o} \to X_{o,o} - \delta$ (inc. loop punctures)

$$\lim_{\delta\to\infty} \sum_{z_{a=1,\cdots,L}\ \text{even/odd}}^{2^L} (\delta)^{n+2L-2} A^{\delta}_{n,L} = A^{\mathrm{NLSM}}_{n,L}\,.$$

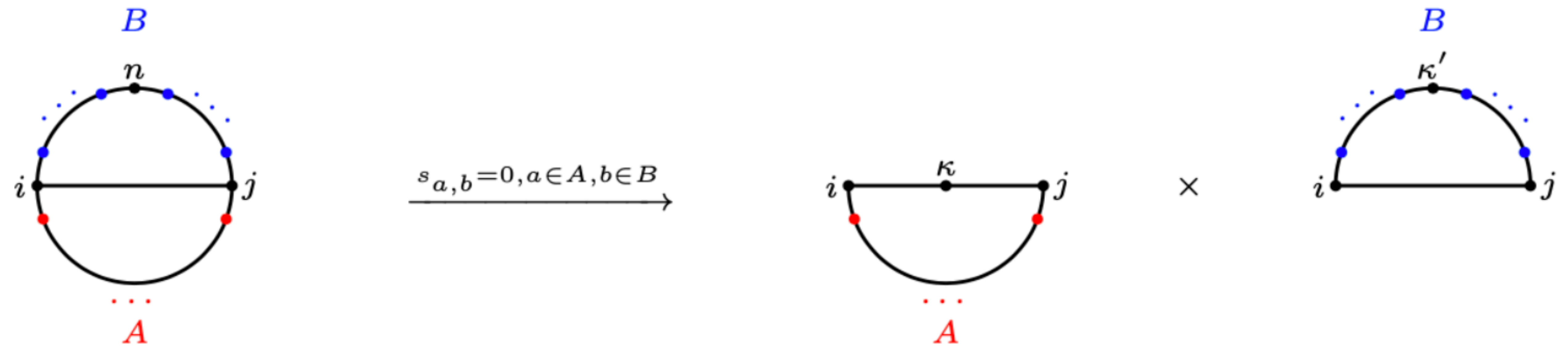


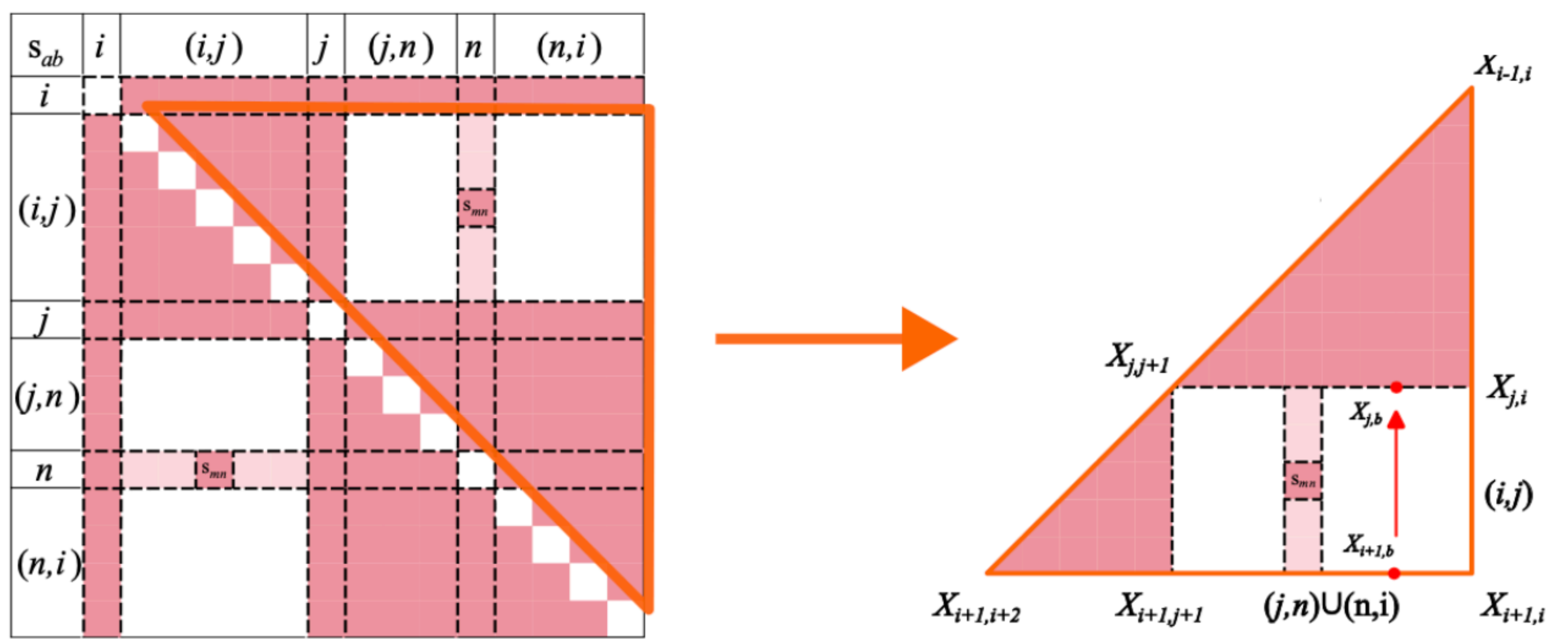
"Adler zero" manifest: soft limit -> scaleless integrals! very practical, e.g. 4-loop 4-pt NLSM integrand;
Lagrangian origin of NLSM $\subset \mathrm{Tr}\phi^3$ [Arkani-Hamed, Figueiredo, '24] => any quantities e.g. off-shell, cosmological, …

A universal splitting of string/particle amps [w. 曹趣, 董晋, 施灿欣, 朱凡, ’24]

A new behavior for a wide class of string/particle tree amp => factorizes into two off-shell currents (total #=n+3)

Universally hold for scalars (ϕ^3, NLSM, special Galileon, YMS/DBI) + gluons/gravitons in bosonic & superstring; implies & extends splittings near zeros etc. to all these theories [see also J. Trnka et al; L. Rodina; Y. Zhang;…]





$$\mathcal{S}_n = \sum_{a<b} s_{a,b}\log z_{a,b} = \sum_{a<b\neq k,(a,b)\neq(i,j)} s_{a,b}\log|ab| \qquad |ab| = z_{a,b} \quad \text{with } z_k\to\infty \text{ and } z_i=0, z_j=1,$$

$$s_{a,b}=0, \qquad \forall a\in A, b\in B\,,$$

$$\mathcal{S}_n \to \underbrace{(\mathcal{S}_A+\mathcal{S}_{i,A}+\mathcal{S}_{j,A})}_{\mathcal{S}_L(i,A,j;\kappa)} + \underbrace{(\mathcal{S}_B+\mathcal{S}_{i,B}+\mathcal{S}_{j,B})}_{\mathcal{S}_R(j,B,i;\kappa')}\,,$$

where $\mathcal{S}_A = \sum_{a<b,a,b\in A} s_{a,b}\log|ab|$, $\mathcal{S}_{i,A} = \sum_{a\in A} s_{i,a}\log|ia|$, $\mathcal{S}_{j,A} = \sum_{a\in A} s_{a,j}\log|aj|$ (in the gauge fixing above, $|ia| = z_a - z_i = z_a$, $|aj| = z_j - z_a = 1 - z_a$)

further set $s_{a,k}=0$ for all $a\in A$

except for $a=m$, and the left-potential further splits

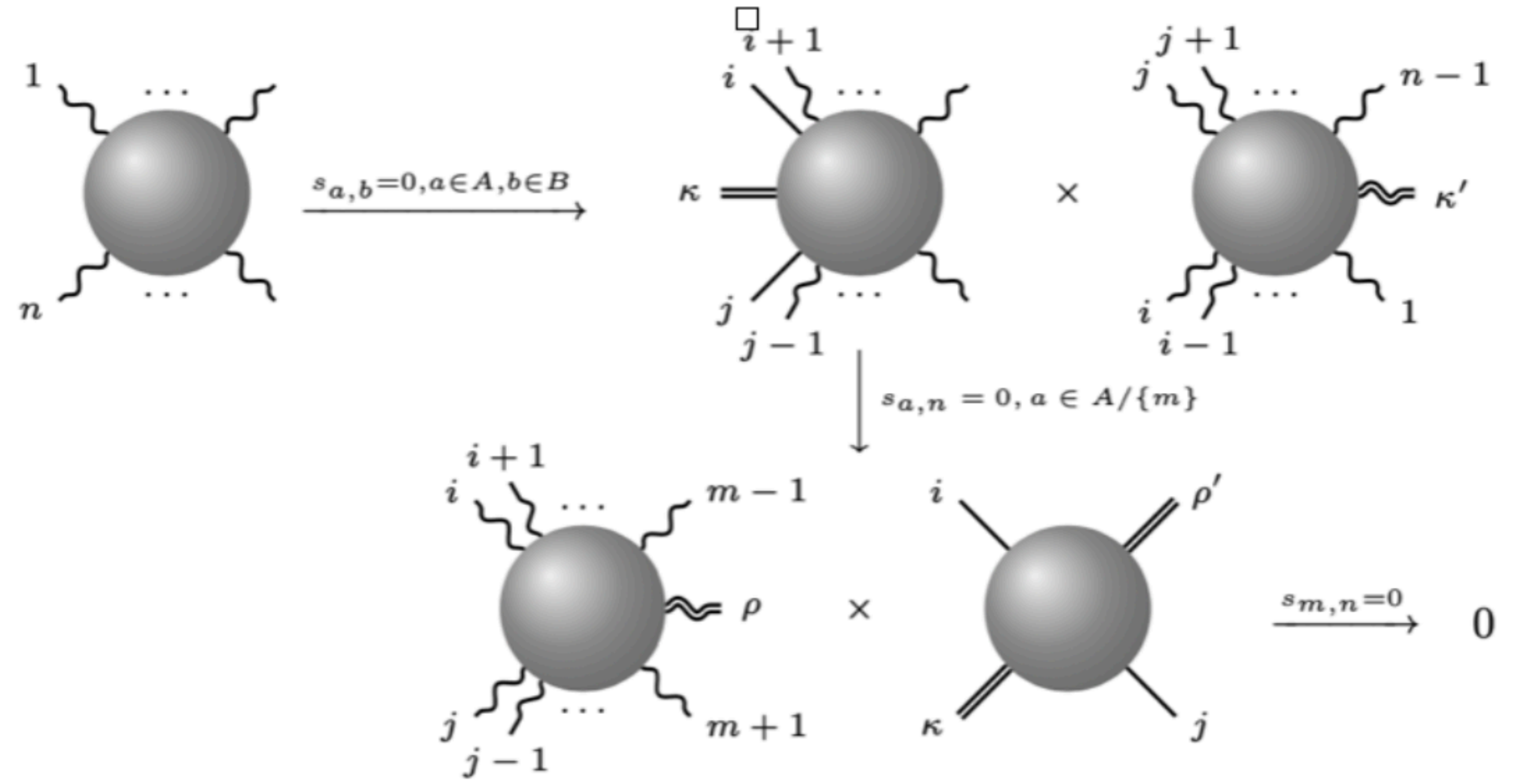
$$\mathcal{S}_L(i,A,j;\kappa) \to \mathcal{S}_L(i,A/\{m\},j;\rho) + \mathcal{S}(i,\rho',j,\kappa),$$

Very nicely, extends to all loops for stringy ϕ^3 & NLSM from “gluing surfaces” [Arkani-Hamed, Figueiredo, ’24]

Splitting & soft theorems [w. 曹趣, 董晋, 施灿欣, 朱凡 '24]

$$\epsilon_a \cdot \epsilon_b = \epsilon_a \cdot \epsilon_{i,j,k} = 0$$
$$p_a \cdot \epsilon_b = p_a \cdot \epsilon_{i,j,k} = 0$$
$$\epsilon_a \cdot p_b = p_a \cdot p_b = 0$$

$$\mathcal{M}_n^{\text{open}} \to \mathcal{J}^{\text{mixed}}(i^\phi, A, j^\phi; \kappa^\phi) \times \mathcal{J}(j, B, i; \kappa')_\mu \epsilon_n^\mu \,,$$



same for $\tilde{\epsilon}$: $$\mathcal{M}_n^{\text{closed}} \to \mathcal{J}(i^\phi, A, j^\phi; \kappa^\phi) \times \mathcal{J}(j, B, i; \kappa')_{\mu\nu} \epsilon_n^\mu \tilde{\epsilon}_n^\nu$$

swap a, b for $\tilde{\epsilon}$: $$\mathcal{M}_n^{\text{closed}} \to \mathcal{J}(i^g, A, j^g; \kappa^g)_\mu \epsilon_n^\mu \times \mathcal{J}(j^g, B, i^g; \kappa'^g)_\nu \tilde{\epsilon}_n^\nu \,.$$

simplest "skinny" splits => Weinberg's soft theorems for gluons/gravitons (string amp) + subleading [c.f. Cachazo, Strominger]

NLSM, DBI, sGal: enhanced Adler's zeros

general splits => multi-soft theorems

the "skinny" case: $A=\{a\}$: $$\mathcal{J}^{\text{mixed}}(i^\phi, a, j^\phi; \kappa^\phi) = \epsilon_a \cdot p_i B(s_{i,a}, s_{j,a}+1) - \epsilon_a \cdot p_j B(s_{i,a}+1, s_{j,a})$$

soft gluon limit: (n-1)-pt current-> amplitude

sum over choices of k (i,j fixed to be adjacent to a): $$\sum_{k\neq i,j,a} \mathcal{J}^{\text{mixed}} \times \mathcal{J}_{n-1} \to \left(\frac{\epsilon_a \cdot p_i}{p_a \cdot p_i} - \frac{\epsilon_a \cdot p_j}{p_a \cdot p_j}\right) \times \mathcal{M}_{n-1}^{\text{YM}},$$

similarly for gravity: sum over i, j ,k $$\sum_{k,i,j\neq a} \mathcal{J}^{\text{mixed}} \times \mathcal{J}_{n-1} \to \left(\sum_{b\neq a} \frac{\epsilon_a \cdot p_b \tilde{\epsilon}_a \cdot p_b}{p_a \cdot p_b}\right) \mathcal{M}_{n-1}^{\text{GR}}$$

Scalar-scaffolded gluons [ACDFH, 2023]

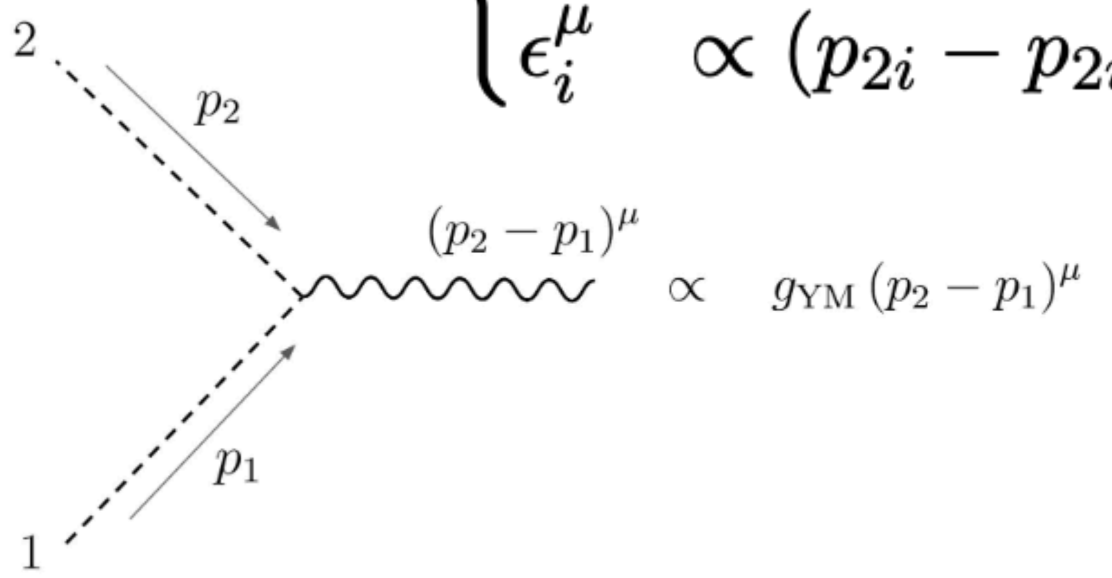
$\alpha'\delta = 1$ gives 2n-scalar stringy YMS amp = 2n-scalar in bosonic string!

$$\mathcal{A}_n^{\text{tree}}(1,2,\ldots,2n) = \int \frac{\mathrm{d}^{2n} z_i}{\mathrm{SL}(2,\mathbb{R})} \left(\prod_{i<j} z_{i,j}^{2\alpha' p_i \cdot p_j}\right) \exp\left(\sum_{i\neq j} 2\frac{\epsilon_i \cdot \epsilon_j}{z_{i,j}^2} - \frac{\sqrt{\alpha'}\epsilon_i \cdot p_j}{z_{i,j}}\right)\Bigg|_{\text{multi-linear in } \epsilon_i},$$

$$\begin{cases} q_i^\mu & = (p_{2i} + p_{2i-1})^\mu \\ \epsilon_i^\mu & \propto (p_{2i} - p_{2i-1})^\mu \end{cases},$$

$$p_i \cdot \epsilon_j = 0, \quad \forall\ (i,j) \in (1,\ldots,2n),$$

$$\epsilon_i \cdot \epsilon_j = \begin{cases} 1 & \text{if } (i,j) \in \{(1,2);(3,4);(5,6);\ldots;(2n-1,2n)\}, \\ 0 & \text{otherwise.} \end{cases}$$



$$\mathcal{A}_{2n}(1,2,\ldots,2n) \xrightarrow{\text{special kinematics}} \int \frac{\mathrm{d}^{2n} z_i}{\mathrm{SL}(2,\mathbb{R})} \prod_{i<j} z_{i,j}^{2\alpha' p_i \cdot p_j} \frac{1}{z_{1,2}^2 z_{3,4}^2 z_{5,6}^2 \cdots z_{2n-1,2n}^2}$$

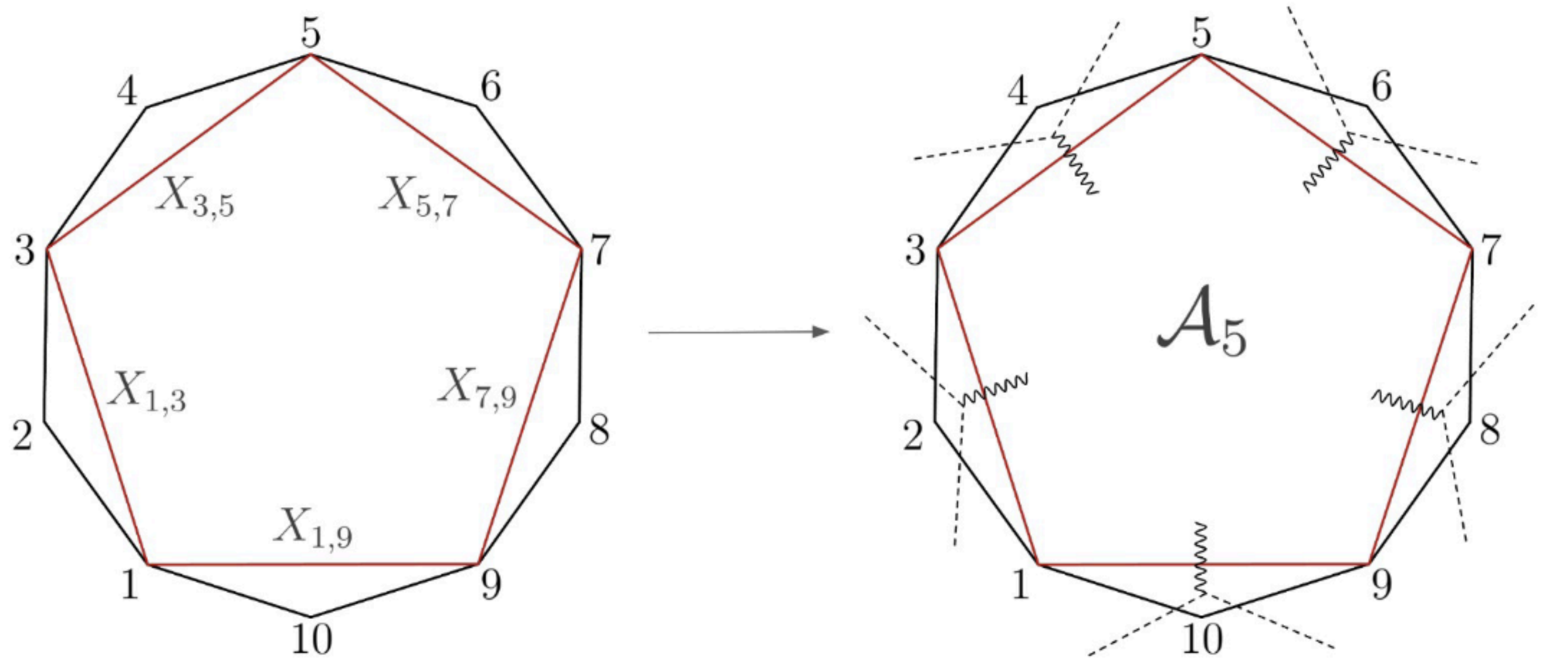
$$= \underbrace{\int \frac{\mathrm{d}^{2n} z_i}{\mathrm{SL}(2,\mathbb{R})} \frac{1}{z_{1,2} z_{2,3} z_{3,4} \cdots z_{2n,1}} \prod_{i<j} z_{i,j}^{2\alpha' p_i \cdot p_j}}_{\text{Stringy Tr}\,\phi^3} \frac{z_{2,3} z_{4,5} z_{6,7} \cdots z_{2n,1}}{z_{1,2} z_{3,4} z_{5,6} \cdots z_{2n-1,2n}} = \left(\prod u_{e,e} / \prod u_{o,o}\right) \quad (\alpha'\delta = 1)$$

taking n ``scaffolding residues'' $s_{1,2} = s_{3,4} = \cdots = 0$ => n-gluon bosonic string amps (in 2n-scalar language)

$$\mathcal{I}_{2n}^{\delta} = \int_{\mathbb{R}_{>0}^{2n-3}} \underbrace{\prod_{i=1}^{n} \frac{dy_{2i-1,2i+1}}{y_{2i-1,2i+1}^2} \prod_{I\in\mathcal{T}'} \frac{dy_I}{y_I^2} \prod_{(a,b)} u_{a,b}^{\alpha' X_{a,b}}}_{\Omega_{2n}},$$

$$X_{1,3} = X_{3,5} = \cdots = X_{1,2n-1} = 0$$

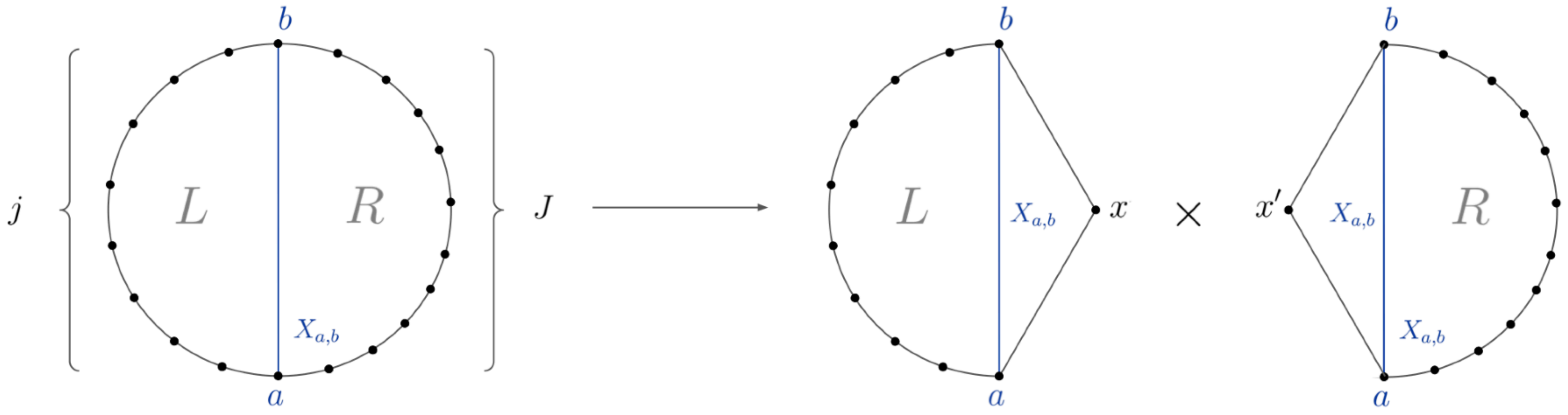
$$\mathcal{I}_n^{\text{gluon}} = \int_{\mathbb{R}_{>0}^{n-3}} \text{Res}_{y_{1,3}=0}\left(\text{Res}_{y_{3,5}=0}\left(\ldots\left(\text{Res}_{y_{1,2n-1}=0}\left(\Omega_{2n}\right)\right)\ldots\right)\right)\Big|_{X_{2i-1,2i+1}=0}$$



$$A_3^{\text{gluon}} = \alpha'^2\left(c_{1,3}c_{1,5} + c_{1,3}c_{2,5} + c_{1,3}c_{3,5} + c_{1,4}c_{3,5} + c_{1,5}c_{3,5} + c_{1,5}c_{3,6}\right) \quad + \\ - \alpha'^3\left(X_{1,4}X_{2,5}X_{3,6}\right)$$

surfaceology -> gauge invariance + gluon factorization (in X variables)

$$\mathcal{A}_n^{\text{gluon}} = \sum_{j\neq\{2i-1,2i,2i+1\}} \left(X_{2i,j} - X_{2i-1,j}\right) \times \mathcal{Q}_j \\ = \sum_{j\neq\{2i-1,2i,2i+1\}} \left(X_{2i,j} - X_{2i+1,j}\right) \times \tilde{\mathcal{Q}}_j ,$$



$$\text{Res}_{X_{a,b}=0}\,\mathcal{A}_n = -\sum_{j,J}\left(X_{j,J} - X_{j,b} - X_{a,J}\right) \underbrace{\prod_{X_L\in\mathcal{L}_j} \tilde{u}_{X_L}}_{Q_j^L} \cdot \underbrace{\prod_{X_R\in\mathcal{R}_J} \tilde{u}_{X_R}}_{Q_J^R}$$

$\mathcal{Q}_j = \tilde{\mathcal{Q}}_j := \partial_{X_{2i,j}} \mathcal{A}_n$ for gluon i (linear + gauge inv)

All-loop YM in stringy Tr ϕ^3 [ACDFH, 2024]

Surfaceology => generalize tree (disk) to loops (higher-genus surfaces):

$$A_n^{\text{gluon}} = \int_0^\infty \prod_i \frac{\mathrm{d}y_i}{y_i^2} \, \mathrm{Res}_{y_{s_1}=0}\left(\mathrm{Res}_{y_{s_2}=0}\left(\ldots\left(\mathrm{Res}_{y_{s_n}=0}\Omega_{2n}\right)\ldots\right)\right).$$

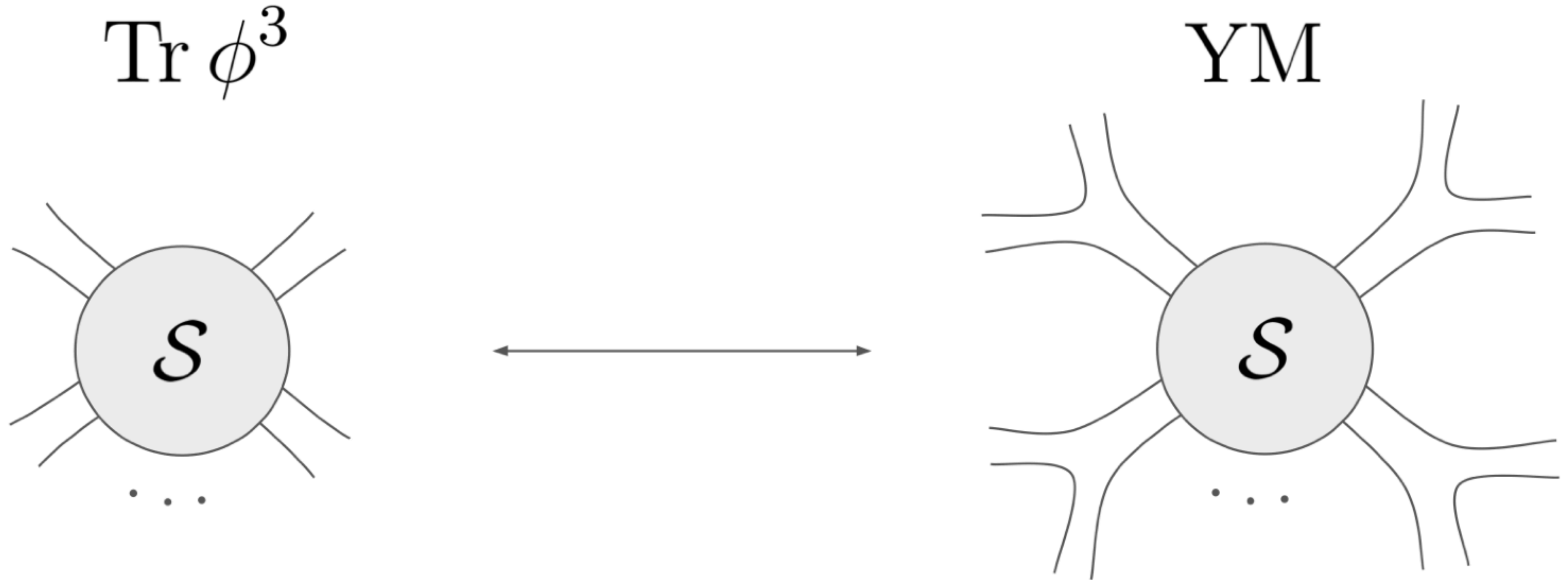
e.g. one-loop w. self-intersecting curves & closed curve Δ

$$\mathcal{I}_{2n}^{\text{1-loop}}(1,2,\ldots,2n) = \int_0^\infty \prod_i \frac{\mathrm{d}y_i}{y_i^2} \prod_C u_C^{\alpha' X_C} \times \prod_{C' \in \text{ s.i.}} u_{C'}^{\alpha' X_{C'}} \times u_\Delta^\Delta$$

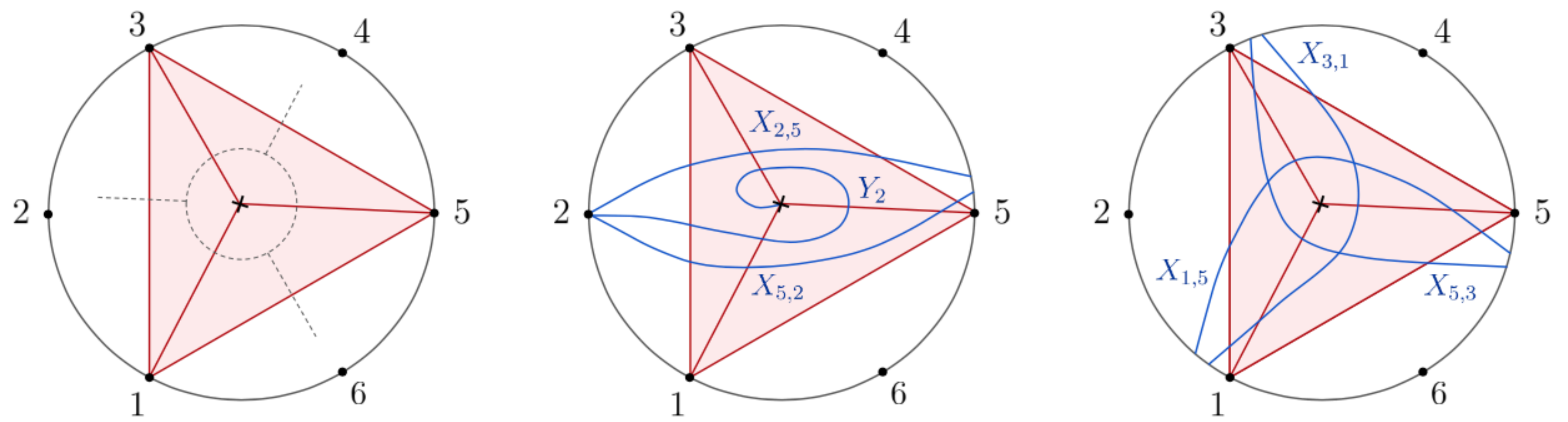
simpler than bosonic string loops, but gives all-loop YM integrands!

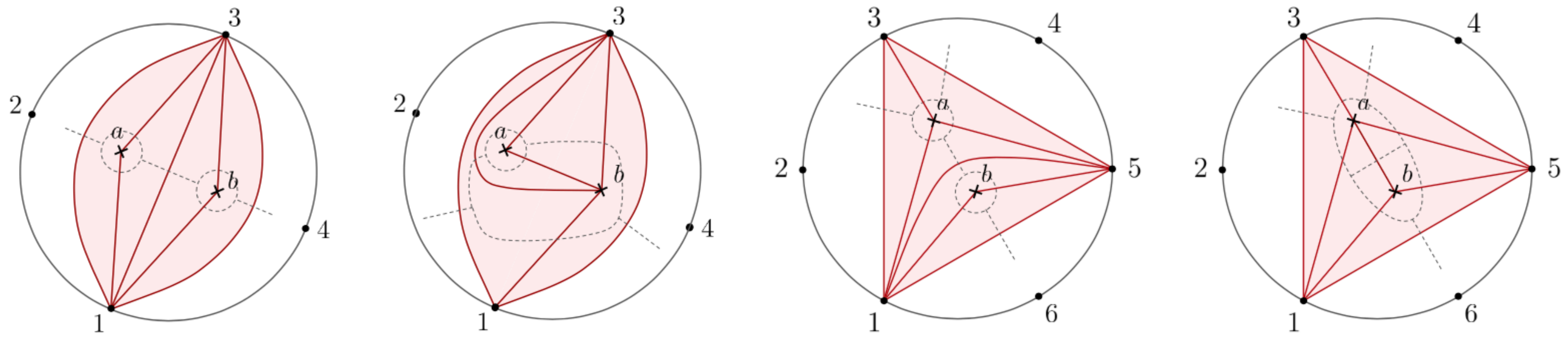
- extend the surface notion of gauge invariance + factorization/cuts
- proof for all-loop leading singularities (max. residue) with $\Delta = 1 - D$:

residue of $\int \prod \frac{dy}{y^2} \prod u^X$ = gluing of 3pt YM+ F^3 (in X space)



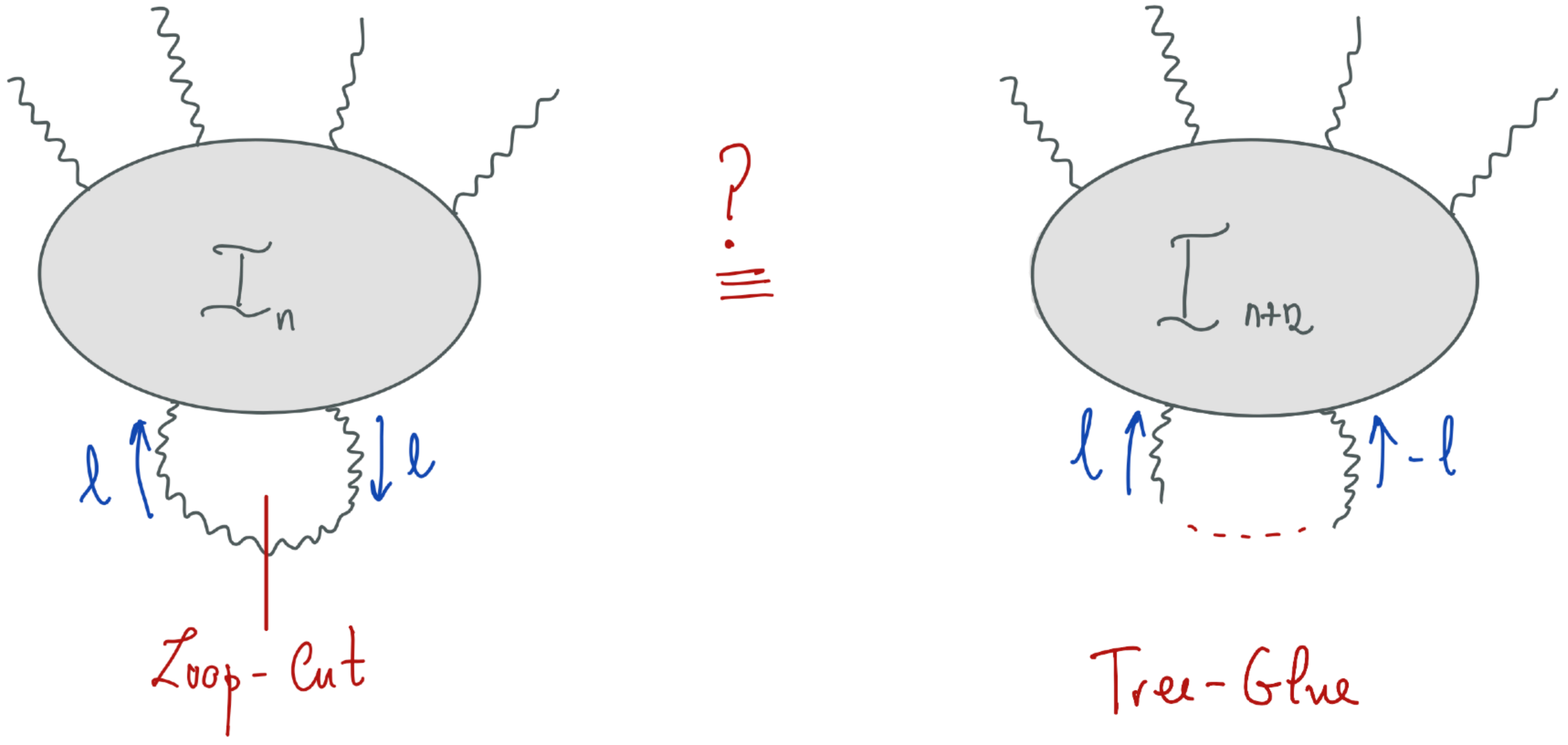
$$\int_0^\infty \prod \frac{\mathrm{d}y}{y} \prod u_X^{\alpha' X} \qquad\qquad \int_0^\infty \prod \frac{\mathrm{d}y}{y^2} \prod u_X^{\alpha' X}$$





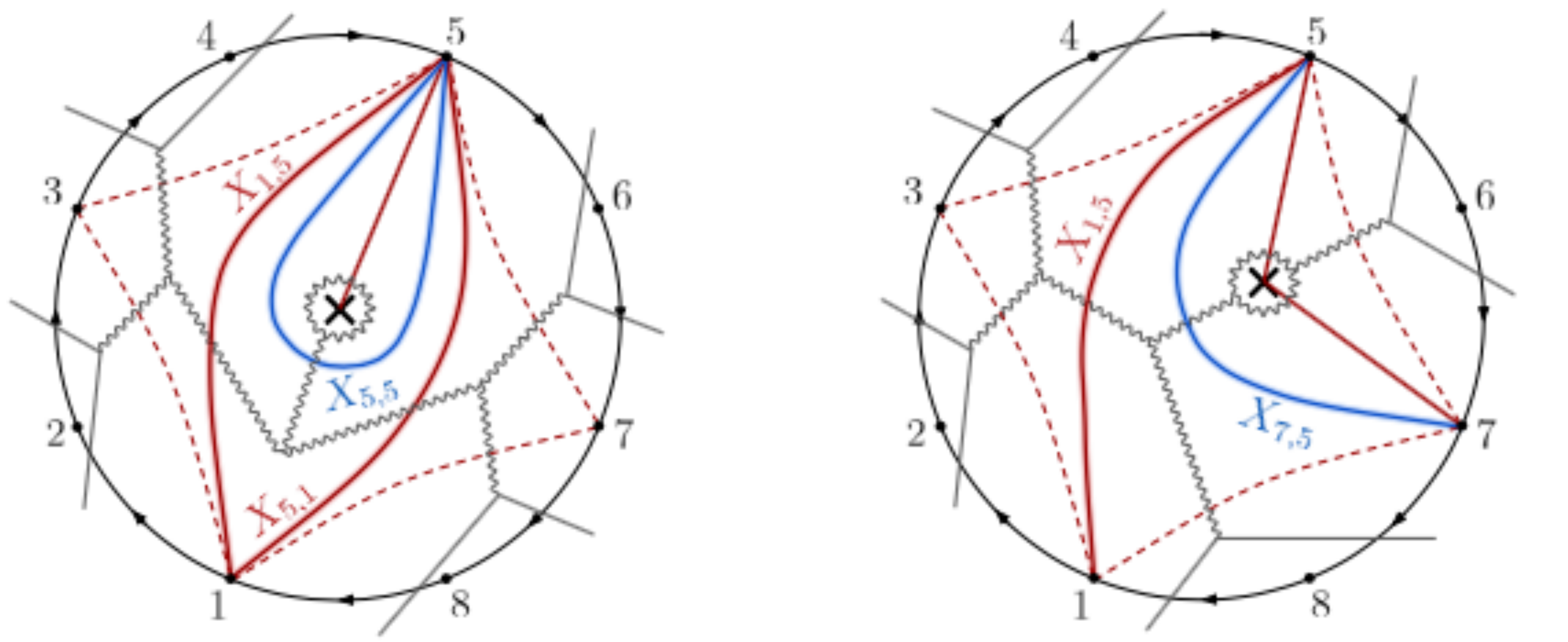
How to determine “the integrand” of YM?

Similar to tree factorizations, all we need are cuts: e.g. 1-loop single-cut = forward limit (gluing tree)



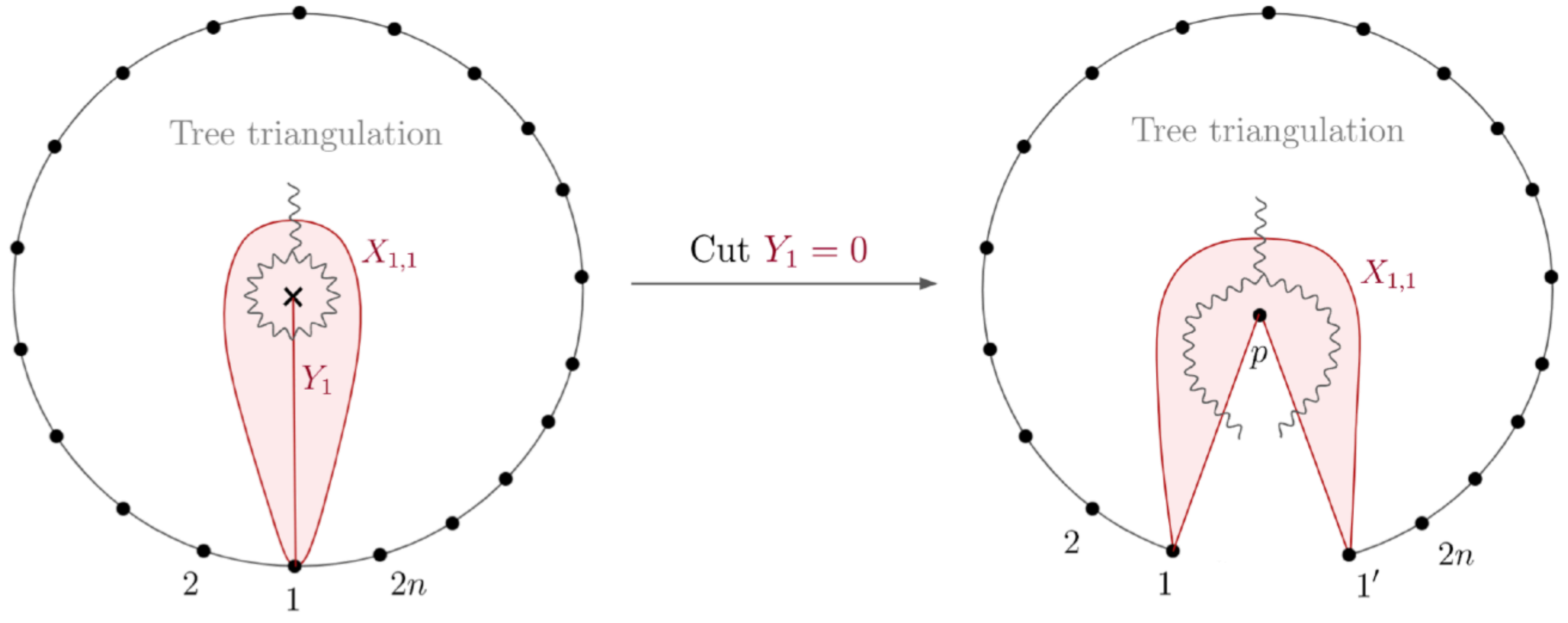
naively divergent => “the integrand”
(e.g. Adler zero, gauge inv.) ill defined!

no issues for scalars, but for gluons 1/0 !
(cancels in super-Yang-Mills)



surface provides a natural way out: curves without standard momentum (e.g. tadpoles) => “the integrand”

“doubling” variables: similar to Lorentzian -> complex in 4d tree kinematics



Loop recursions in YM [ACDFH, PRL 2024; w. 曹趣, 董晋, 朱凡 2025 + in progress]

Surface makes it clear "the integrand" to all loops exist (beyond planar limit)
=> the notion of surface gauge invariance+ cuts

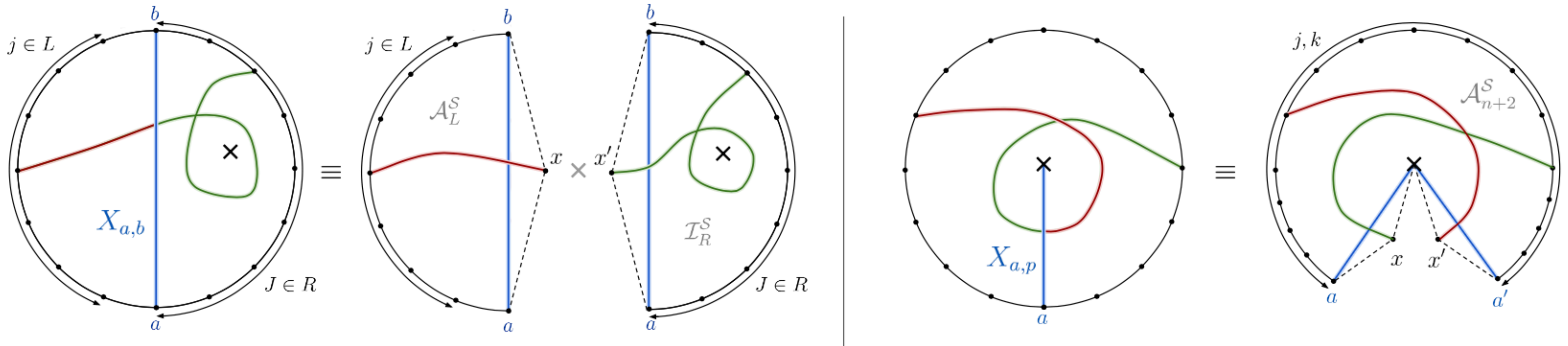
Loop integrands reconstructed from "residues" e.g. single-cut
=> forward-limit recursions for "the integrand" @ 1-loop & higher

$$\underset{X_{a,p}=0}{\text{Res}} \left(\mathcal{I}_n^{\mathcal{S}}\right) = \sum_{j,k} \left[(X_{j,p} + X_{k,p} - X_{k,j}) \frac{\partial^2 \mathcal{A}_{n+2}^{\mathcal{S}}}{\partial X_{j,x} \partial X_{k,x'}} \right] \Bigg|_{a' \to a} - d \frac{\partial \mathcal{A}_{n+2}^{\mathcal{S}}}{\partial X_{x',x}}; \quad j,k \in (a, a+1, \cdots, a-1, a'),$$

$$\mathcal{I}_n^{\mathcal{S}} = \sum_i \frac{\text{Res}_{X_{i,p}=0} \left(\mathcal{I}_n^{\mathcal{S}}(X_{j,p} \to X_{j,p} - X_{i,p})\right)}{X_{i,p}}.$$

Gauge-Invariance and Linearity (in gluon 1):

$$\mathcal{I}_n^{\mathcal{S}} = \sum_{j \neq 2} \left[(X_{2,j} - X_{1,j}) \frac{\partial \mathcal{I}_n^{\mathcal{S}}}{\partial X_{2,j}} + (X_{j,2} - X_{j,1}) \frac{\partial \mathcal{I}_n^{\mathcal{S}}}{\partial X_{j,2}} \right] + X_{2n,1} \times \left[\frac{\partial \mathcal{I}_n^{\mathcal{S}}}{\partial X_{2n,2}} + \frac{\partial \mathcal{I}_n^{\mathcal{S}}}{\partial X_{2n,1}} \Bigg|_{2 \to 1} \right]$$



I. Tree-Loop cut II. Loop cut

Discard scaleless integrals => physical integrands up to 2-loop 6-pt (-> 3-loop 4-pt in progress) [w. Cao, Dong, Zhu; 2503]

—> correct amplitudes after loop integration (in D dim), e.g. 1-loop helicity amps up to n=5; all-plus to all n?

enormous simplifications when reducing to 4d spinor-helicity: new results for higher loops?

Fermions, general gauge theories & SYM [w. 曹趣, 董晋, 朱凡, 2412 + in progress]

How to include matters ([c.f. De et al, 2406.04411] for Yukawa): fermions in the loop? nice structure obtained via worldsheet [w. Edison et al '20, '22]

"universal expansion" of gluon tree + 2 gluons/fermions/scalars
Forward Limit (surface) => 1-loop gluon amps in gauge theories

$$\mathcal{A}^{\rm YM}_{n+2}(-^g,+^g)=\sum_{m,\alpha\in S_m}\epsilon_-\cdot f_{\alpha_1}\cdot f_{\alpha_2}\cdots f_{\alpha_m}\cdot\epsilon_+\times\mathcal{A}^{\rm mixed}(+,\alpha,-)$$

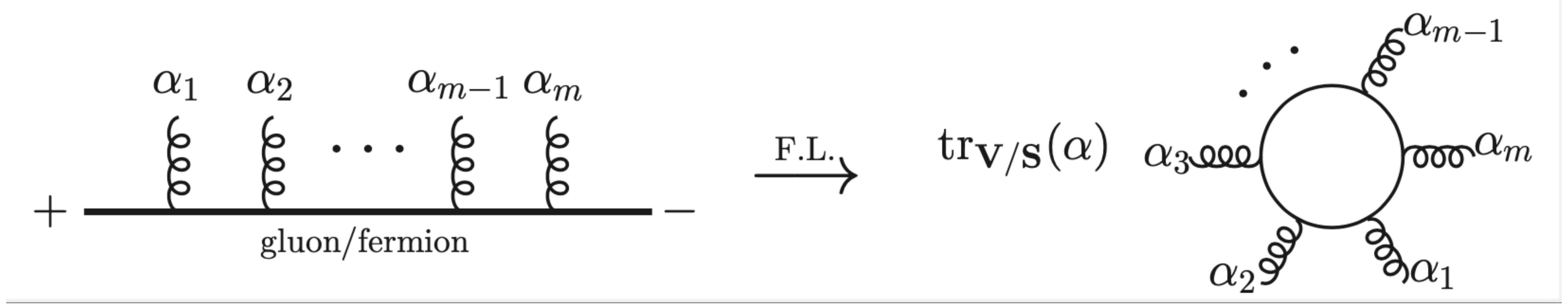
$$\mathcal{A}^{\rm gauge}_{n+2}(-^f,+^f)=\sum_{m,\alpha\in S_m}\bar{\chi}_-\not{f}_{\alpha_1}\not{f}_{\alpha_2}\cdots\not{f}_{\alpha_m}\xi_+\times\mathcal{A}^{\rm mixed}(+,\alpha,-)$$

$$\epsilon_-\cdot\epsilon_+\xrightarrow{\rm F.L.}D-2,$$

$$\epsilon_-\cdot f_{\alpha_1}\cdot f_{\alpha_2}\cdots f_{\alpha_m}\cdot\epsilon_+\xrightarrow{\rm F.L.}{\rm tr_V}(f_{\alpha_1}f_{\alpha_2}\cdots f_{\alpha_m}).$$

$$\bar{\chi}_-\xi_+\xrightarrow{\rm F.L.}2^{D/2-1},$$

$$\bar{\chi}_-\not{f}_{\alpha_1}\not{f}_{\alpha_2}\cdots\not{f}_{\alpha_m}\xi_+\xrightarrow{\rm F.L.}{\rm tr_S}(\not{f}_{\alpha_1}\not{f}_{\alpha_2}\cdots\not{f}_{\alpha_m}).$$



already new formula/relations for n-gluon amps: pure YM 1-loop = sum of mixed scalar-loop amps with Lorentz traces

$$\mathcal{A}^{\rm YM}_n=\sum_{m,\alpha}{\rm tr_V}(f_{\alpha_1}\cdots f_{\alpha_m})\mathcal{A}^{\rm scalar-loop}_\alpha$$

$$n=2:\quad (D-2)\mathcal{A}_\emptyset+{\rm tr_V}(f_1f_2)\mathcal{A}_{1,2},$$

$$n=3:\quad (D-2)\mathcal{A}_\emptyset+[{\rm tr_V}(f_1f_2)\mathcal{A}_{1,2}+2\ \ {\rm perms}]+{\rm tr_V}(f_1f_2f_3)\mathcal{A}_{1,2,3}.$$

Almost identical for gen. gauge theory! any multiplet in the loop = sum of mixed scalar-loop amps with "vector/spinor trace"

$$\mathcal{A}_n^{\text{gauge}} = \sum_{m,\alpha} \mathcal{T}_{\alpha_1,\cdots,\alpha_m} \mathcal{A}_\alpha^{\text{scalar-loop}}$$

$\mathscr{T}^{\mathbf{n}_{\mathrm{v,f,s}}}_{\alpha_1,\cdots\alpha_m} := \mathbf{n}_v \mathrm{tr}_V(\alpha_1\cdots\alpha_m) - \frac{\mathbf{n}_f}{2}\mathrm{tr}_S(\alpha_1\cdots\alpha_m)$ for $\mathbf{n}_v, \mathbf{n}_f, \mathbf{n}_s$ vectors, Weyl fermions, scalars

(m=0: $\mathscr{T}_\varnothing := (D-2)\mathbf{n}_v + \mathbf{n}_s - 2^{(D-2)/2}\frac{\mathbf{n}_f}{2}$, counts # of on-shell d.o.f)

universal 1-loop objects: scalar loop with tree blobs of m scalars (in α ordering) + (n-m) gluons attached

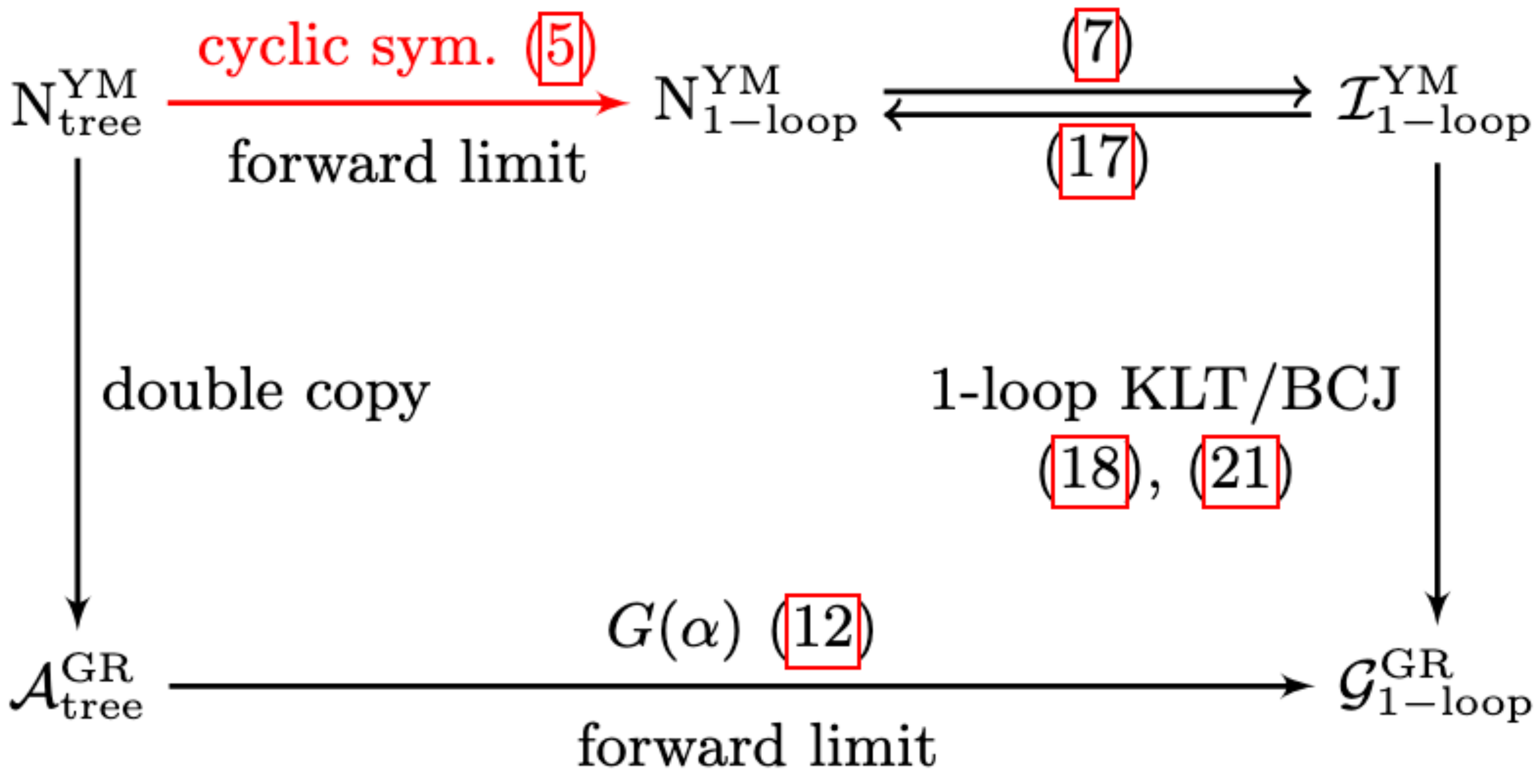
most important (m=0): $\mathscr{A}_\varnothing$ as the coefficient of "D-2" (large D limit), simply attaching n gluons to scalar loop
—>all $\mathscr{A}^{\text{mixed}}_{\alpha,m>0}$ obtained by differential operators w.r.t polarizations $\epsilon_{\alpha_1}, \cdots, \epsilon_{\alpha_m}$

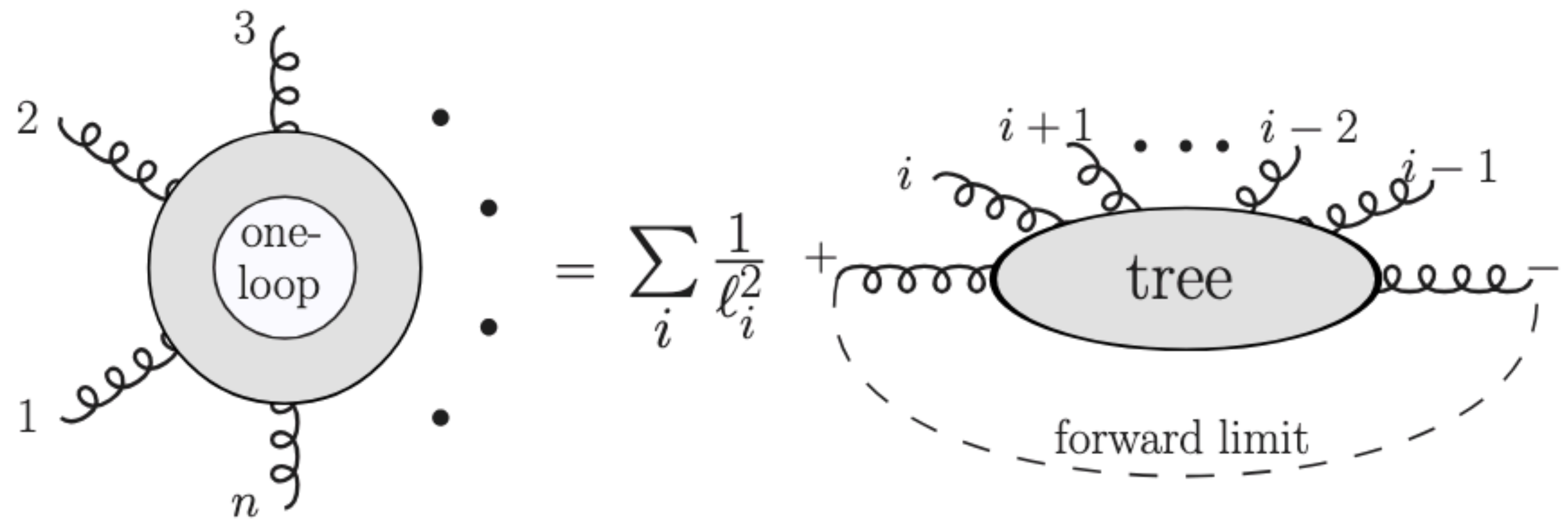
effective SUSY Ward identities! In particular, huge simplifications for SYM, e.g. w, maximal SUSY

$\mathscr{T}_{m<4} = 0$ since it is proportional to $\mathbf{n}_v - 2^{D/2-5}\mathbf{n}_f = 0$ (D=10, $\mathbf{n}_f = \mathbf{n}_v, \mathbf{n}_s = 0$, or D=4, $\mathbf{n}_f = 8\mathbf{n}_v, \mathbf{n}_s = 6\mathbf{n}_v$)

"no triangle/bubble/tadpole" + correct power-counting ℓ^{m-4} for the m-gon!
e.g. m=4: t_8 tensor for box numerator, $\mathrm{tr}_V - \frac{1}{2}\mathrm{tr}_S|_{D=10} = \frac{1}{2}[\mathrm{tr}_V(1,2,3,4) - \frac{1}{4}\mathrm{tr}_V(1,2)\mathrm{tr}_V(3,4) + \text{cyc.}]$

Loop-Level Double Copy Relations from Forward Limits [2509 w. 曹趣, 张勇, 朱凡]





$$\tilde{\mathcal{I}}_n(\mathbb{I}) = \sum_{i=1}^{n} \frac{1}{\ell_i^2} \hat{\mathcal{A}}_{n+2}(+, i, i{+}1, \ldots, i{-}1, -)\,.$$

$$\hat{\mathcal{A}}_{n+2}(+, i, \ldots, -) = \sum_{\rho \in S_n} \mathrm{N}_{\rho;\ell_i}\, m_{n+2}(+, i, \ldots, - \,|\, +, \rho, -)\,,$$

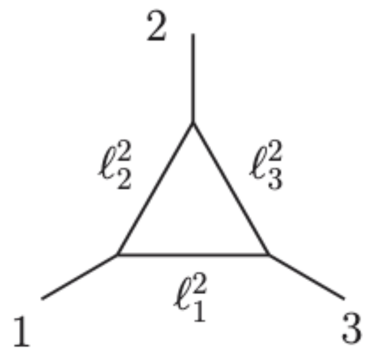
$$\tilde{\mathcal{G}} = \sum_{\alpha \in S_n} \mathrm{N}_{\alpha;\ell}\, \tilde{\mathcal{I}}_n(\alpha), \qquad \mathcal{G}_n = \sum_{\alpha \in S_{n-1}} \mathrm{N}_{1,\alpha;\ell}\, \mathcal{I}_n(1,\alpha) \cong \frac{1}{n} \tilde{\mathcal{G}}_n,$$

$$\mathrm{N}_{1,2,\ldots,n;\ell_1} = \mathrm{N}_{i,i+1,\ldots,i-1;\ell_i}\,,$$

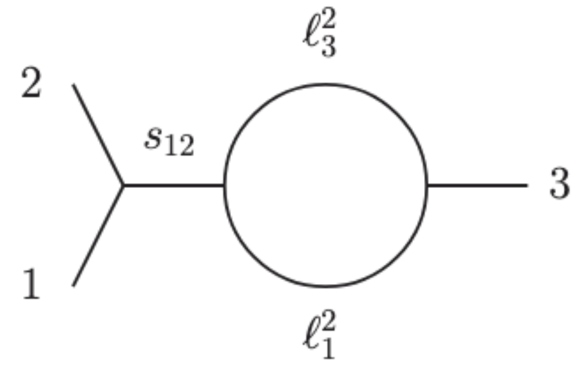
$$\mathcal{G}_n = \sum_{\alpha,\beta \in S_{n-1}} \mathcal{I}_n(1,\alpha)\mathcal{K}(1,\alpha|1,\beta)\mathcal{I}_n(1,\beta)\,. \qquad \mathcal{G}_n = \sum_{\alpha,\beta \in S_{n-1}} \mathrm{N}_{1,\alpha;\ell}\mathrm{N}_{1,\beta;\ell}\mathcal{M}_n(1,\alpha|1,\beta)\,.$$

$$\mathcal{G}_3 = (\mathcal{I}_{1,2,3}\ \ \mathcal{I}_{1,3,2}) \begin{pmatrix} \mathcal{M}_3(1,2,3|1,2,3) & \mathcal{M}_3(1,2,3|1,3,2) \\ \mathcal{M}_3(1,3,2|1,2,3) & \mathcal{M}_3(1,3,2|1,3,2) \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{I}_{1,2,3} \\ \mathcal{I}_{1,3,2} \end{pmatrix}$$

$P_{1,2,3}$:

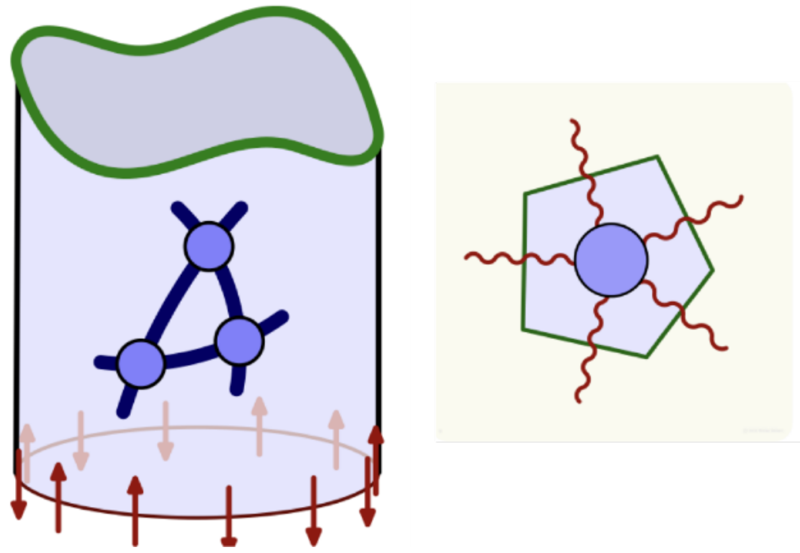


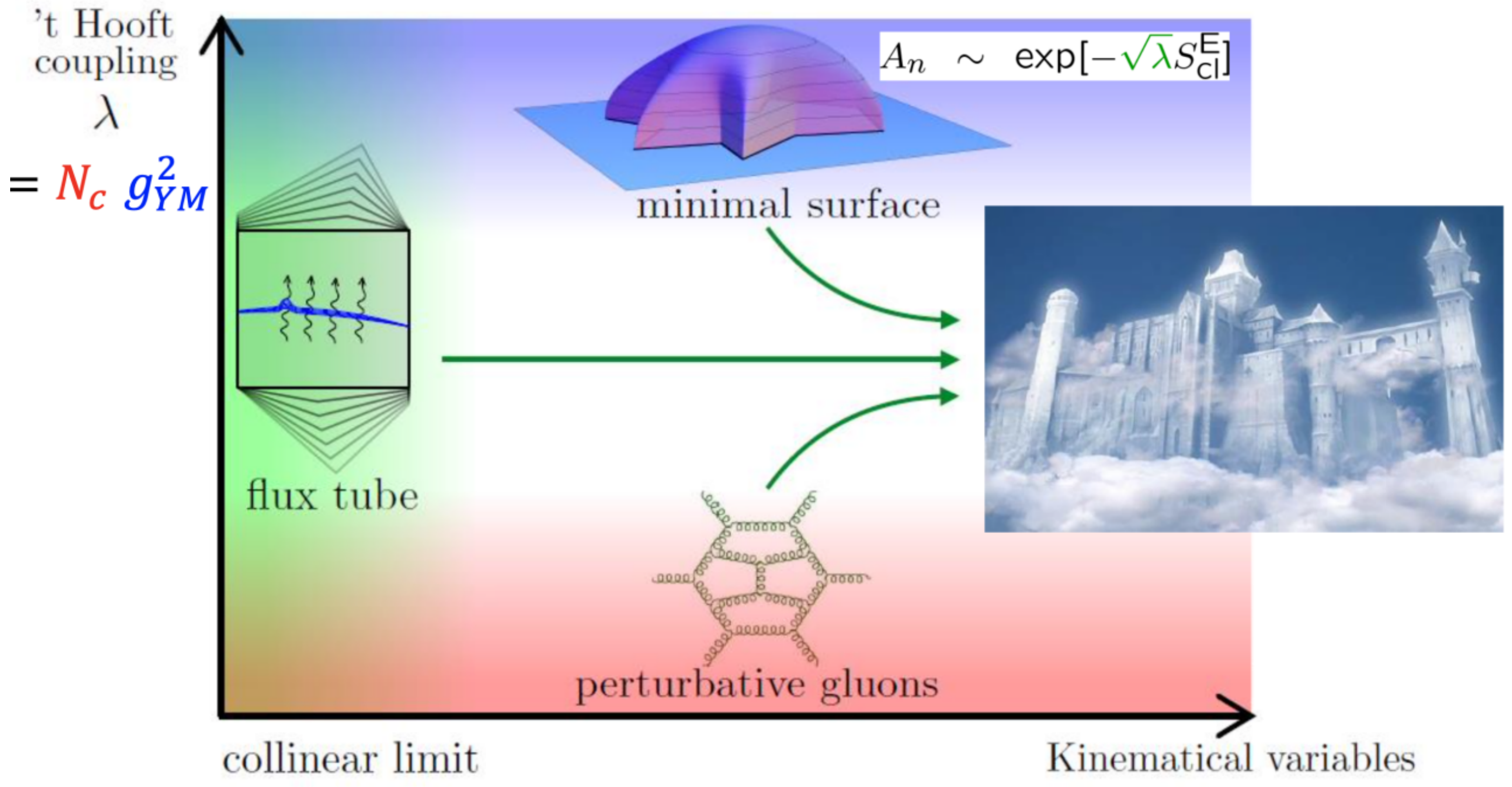
$P_{[1,2],3}$:

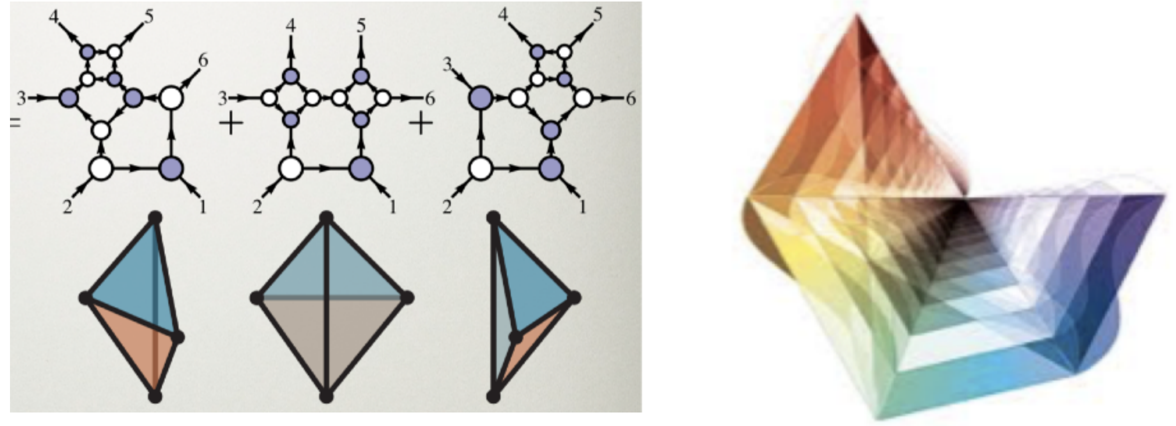


The simplest QFT

Numerous hidden structures & simplicity in (Amp, FF, correlators…) planar $\mathcal{N} = 4$ SYM



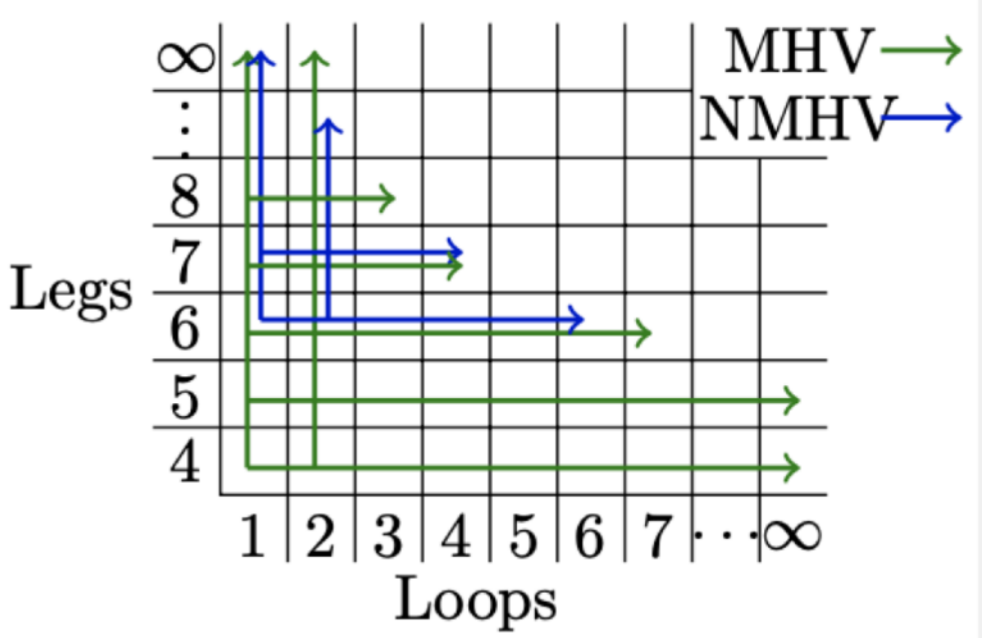




Integrability: correlators, Wilson loops & OPE, strong coupling via AdS/CFT, Yangian symmetry…

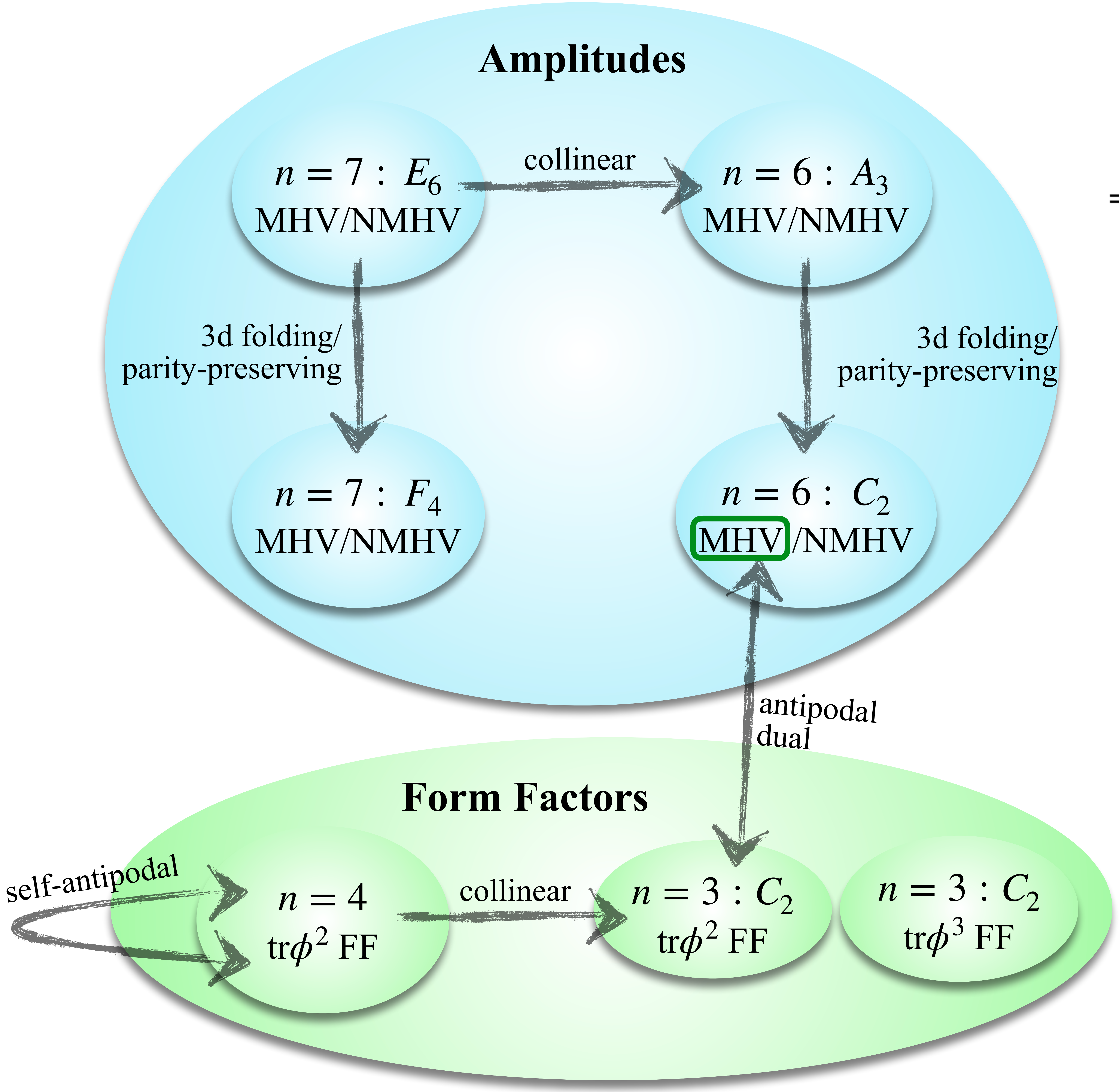
Integrands <-> amplituhedron+ positive Grassmannian, …
(rich math structures)

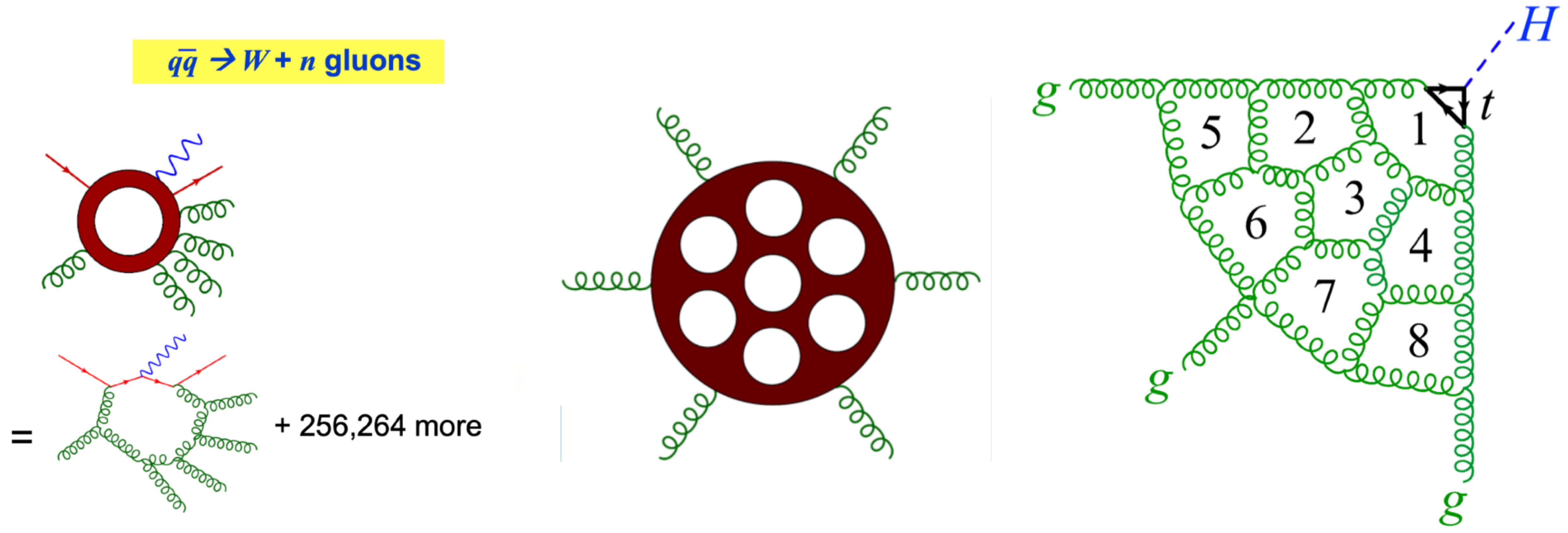
Integrated amplitudes: even richer structures + ideal laboratory for perturbative QFT!



e.g. powerful techniques for Feynman integrals, UT basis + diff eqs, polylogs & beyond (periods), symbology + bootstrap, cluster algebras, positive/tropical geometries, etc etc

A Web of Amps & FF





$n = 3 : C_2$ $\mathrm{tr}\phi^2$ FF [Dixon, Gurdogan, McLoed, Wilhelm, 22] $L = 8$

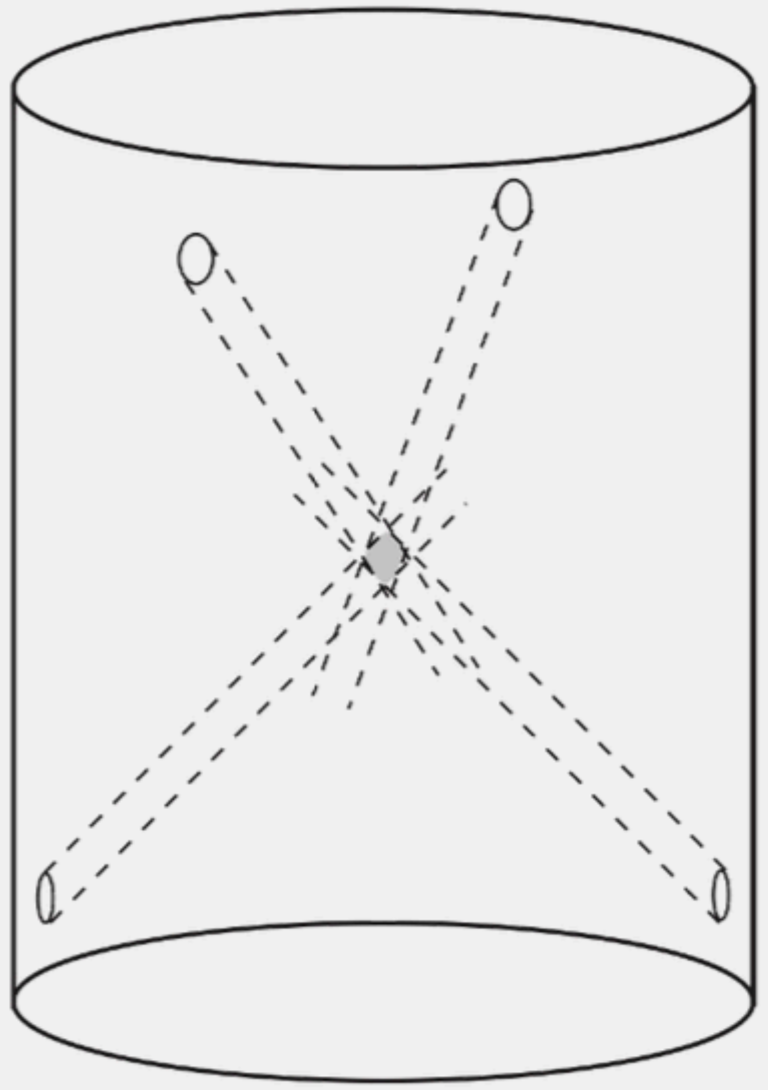
$n = 3 : C_2$ $\mathrm{tr}\phi^3$ FF [Basso, Dixon, Tumanov, 24] [Dixon, Li, private communication] $L = 6,(7,8)$

$n = 4$ $\mathrm{tr}\phi^2$ FF [Dixon, Gurdogan, Liu, McLoed, Wilhelm, 22] [Dixon, Li, private communication] $L = 2,3,(4)$

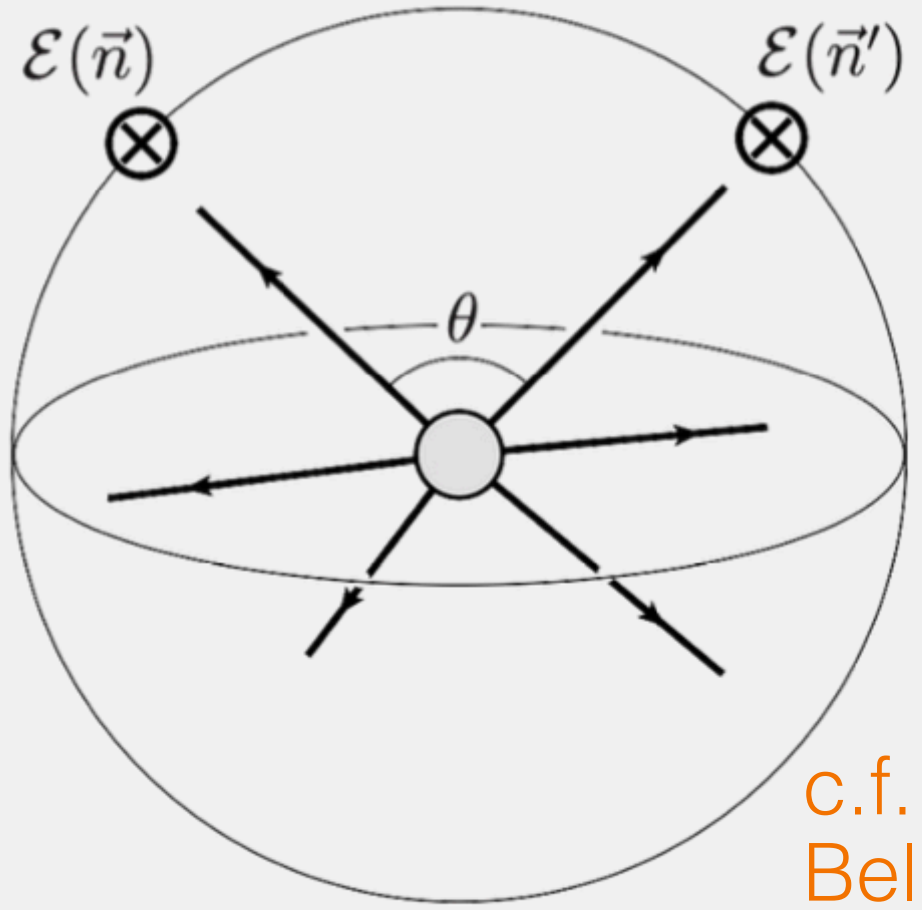
$n = 6 : A_3$
MHV: [Dixon et al…] [Dixon, Liu, 23] $L = 8$
NMHV: [Caron-Hout, Dixon, et al. 19] $L = 6,(7)$

$n = 7 : E_6$ MHV/NMHV [Drummond, Papathanasiou, Spradlin 14] [Dixon et al 16] [Drummond, Foster, Gurdogan, Papathanasiou, 18] [Dixon, Liu, 20] [SH, Jiang, Li, Liu, 2511.09669] $L = 4 \to L = 5$

AdS scattering
(strong)

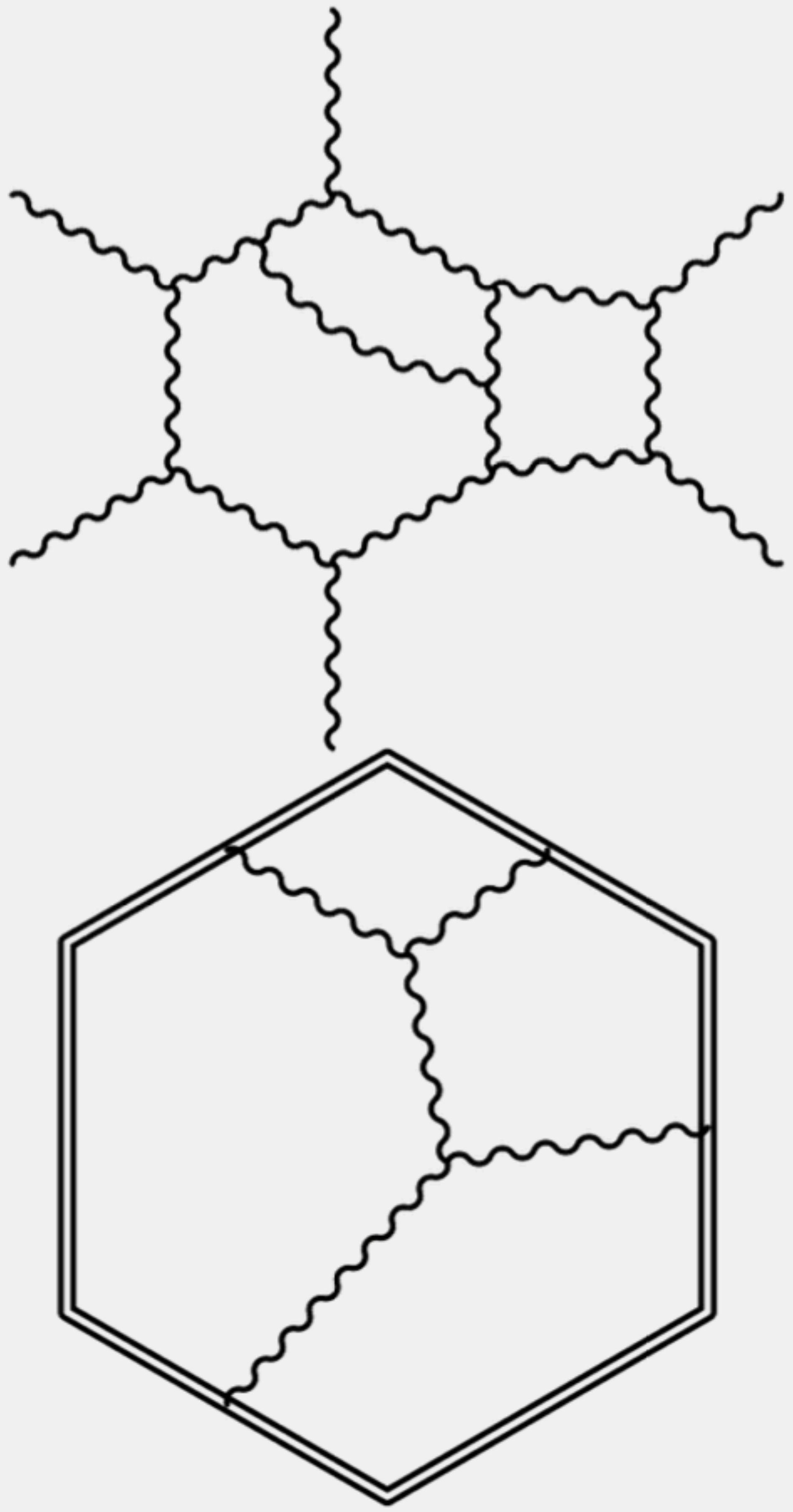


c.f. Alday, Maldacena 07,…



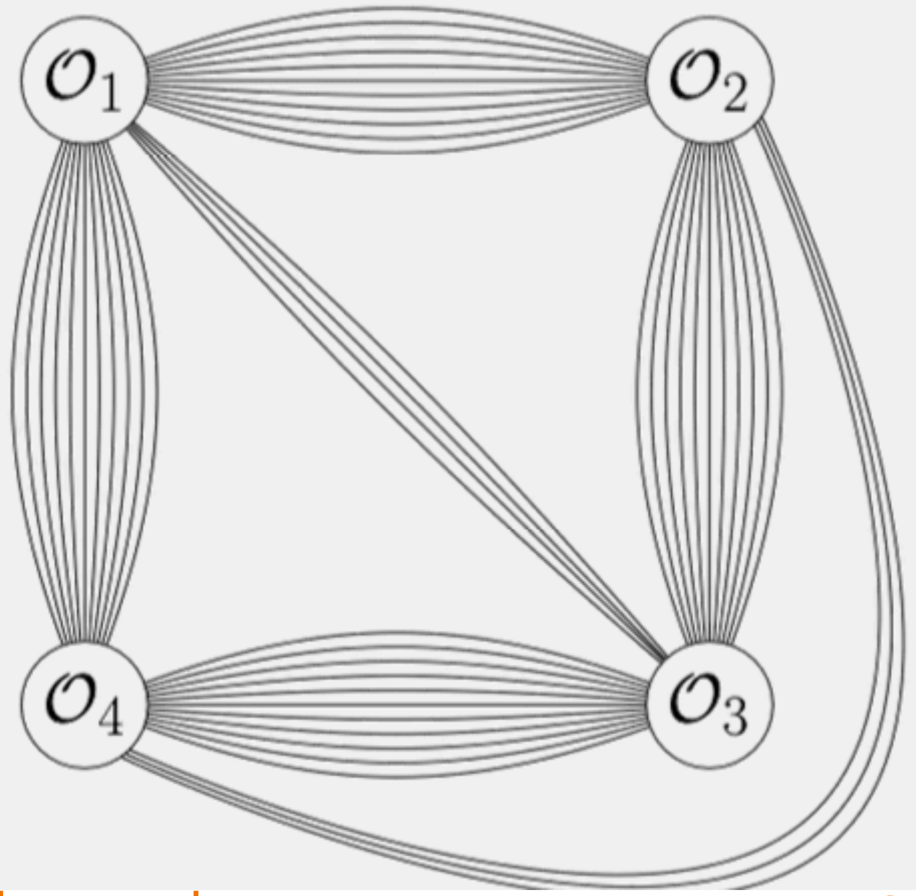
weighted
cross-sections
(finite)

c.f. Hoffmann, Maldacena ’08;
Belitsky et al ’13,…



$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4 \rangle$$

integrability
(finite)



CWA triality
(weak)

c.f. Coronado; Belitsky, Korchemsky;
Coronado, Caron-Huot ,…

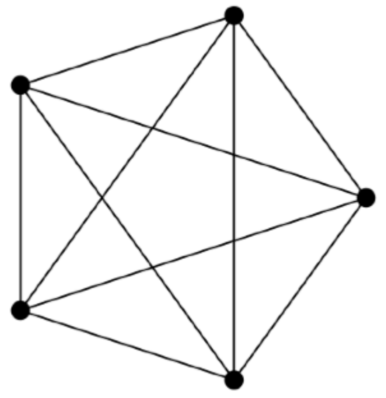
c.f. Drummond et al; Bern et al; Brandhuber et al, ’07…
Alday, et al; Eden, Korchemsky, Sokatchev, ’10, Heslop…
Caron-Huot, Mason, Skinner 10’ …

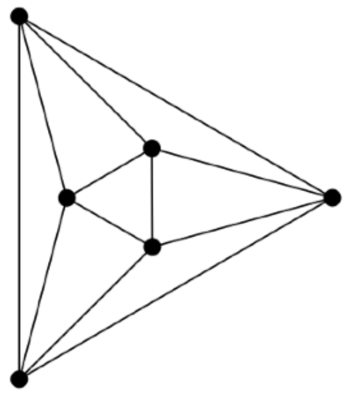
f-graphs: correlators& amplitudes to 12 loops

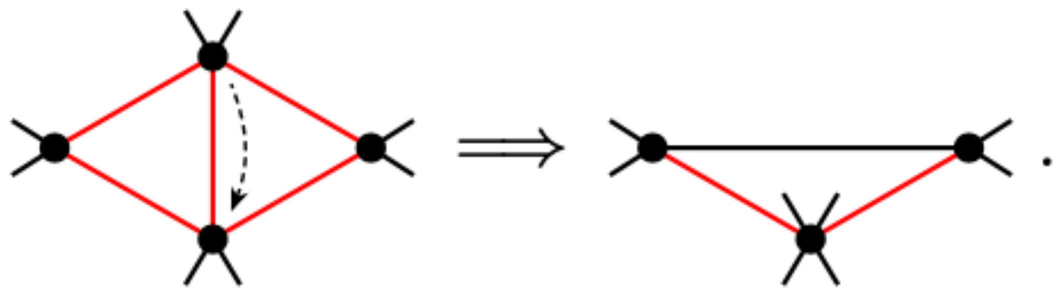
Bourjaily et al, 1512.07912+ 1609.00007;
w. 唐一朝、施灿欣、张耀奇, 2410.09859
+Bourjaily, 2503.15593

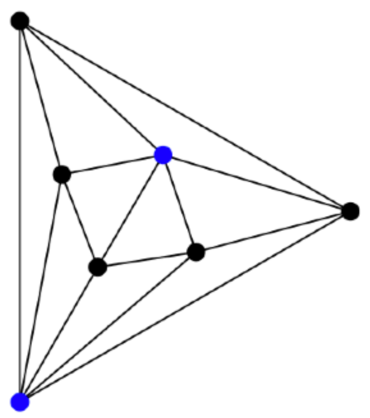
- ℓ-loop integrand computed by Born-level correlator with ℓ (chiral) Lagrangian insertions
- $F_{4+\ell}$ enjoys perm. symmetry of all $n \equiv 4+\ell$ points => represented graphically by f-graphs!
- bootstrap: "square rule" etc. (amps) + "triangle rule" (universal but weak) => 10 loops!
- A new universal behavior "cusp limit" => a single graphical rule => 12 loops!

n	all f-graphs $(\mathcal{N}_n)$	planar f-graphs (N_n)	
5	1	0	
6	1	1	
7	4	1	
8	32	3	
9	930	7	
10	189341	36	
11	$\cdots$	220	[1112.6432]
12		2707	[1512.07912]
13		42979	
14		898353	[1609.00007]
15		22024902	[2410.09859]
16		619981403	[2503.XXXXX]

$f^{(1)} =$ 

$f^{(2)} =$ 

$\mathcal{P}:$ 

$f^{(3)} =$  $= \dfrac{\frac{1}{20}\sum_{\sigma\in S_7} x^4_{\sigma_1\sigma_2}x^2_{\sigma_3\sigma_4}x^2_{\sigma_4\sigma_5}x^2_{\sigma_5\sigma_6}x^2_{\sigma_6\sigma_7}x^2_{\sigma_7\sigma_3}}{\prod_{1\le i<j\le 7} x^2_{ij}}$

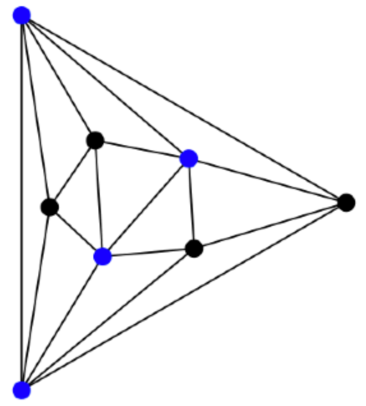
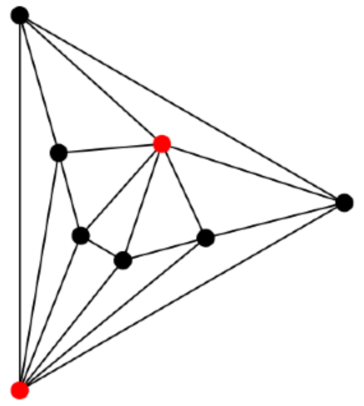
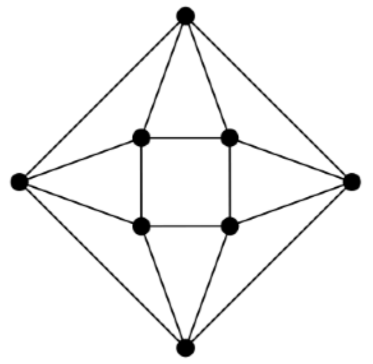
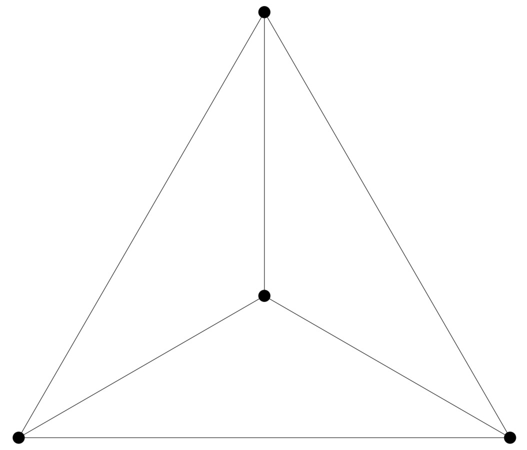
$f^{(4)} =$  $+$  $-$ 

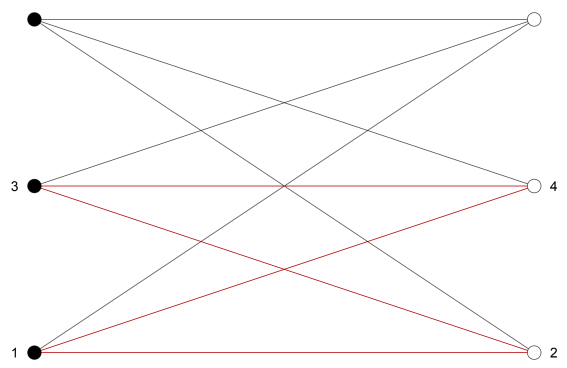
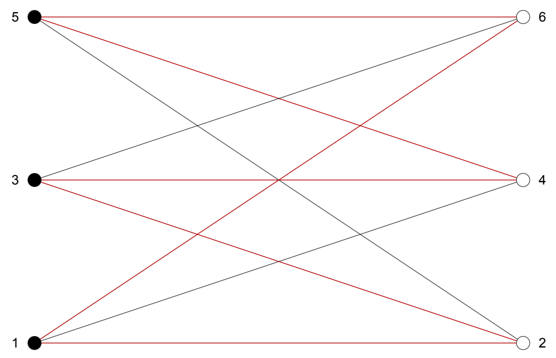
TABLE I. Numbers of terms required to represent the ℓ-loop amplitude via on-shell recursion, in terms of (dihedrally-symmetrized) dual-conformally invariant ('DCI') master integrals, or f-graphs—and how many have non-vanishing coefficients.

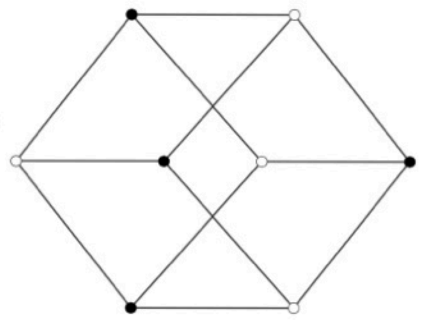
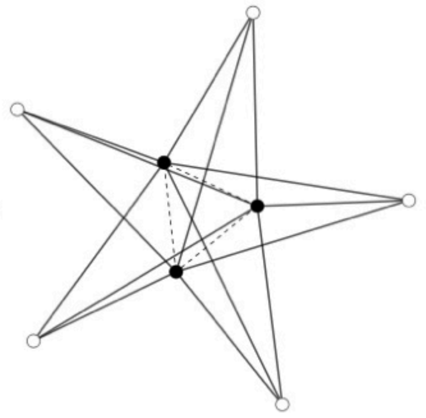
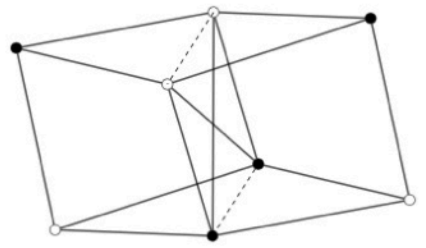
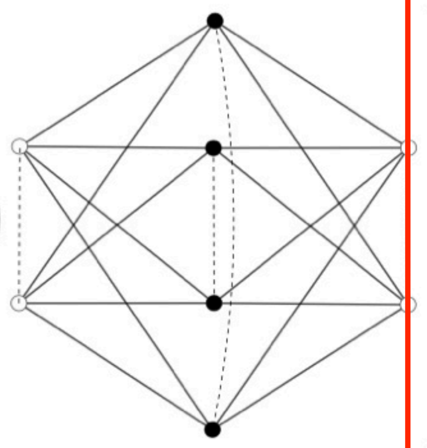
$\ell=$	1	2	3	4	5	6	7	8	9	10	11	12
recursed cells:	1	10	146	2,684	56,914	1,329,324	33,291,164	878,836,728	24,175,924,094	687,444,432,396	20,086,271,785,340	600,384,612,445,304
DCI integrals:	1	1	2	8	34	278	3,125	49,935	981,984	23,045,474	623,496,933	19,117,648,284
(contributing)	1	1	2	8	34	224	1,818	19,198	236,823	3,412,129	56,145,999	1,049,691,130
f-graphs:	1	1	1	3	7	36	220	2,707	42,979	898,353	22,024,902	619,981,403
(contributing)	1	1	1	3	7	26	127	1,060	10,525	136,433	2,048,262	35,503,735

|A|^2 in ABJM + bipartite f-graphs [w. C. Shi, Y. Tang, Y. Zhang, PRL 2025]



$$\Phi = \phi^4 + \eta^I \psi_I + \tfrac{1}{2}\epsilon_{IJK}\eta^I\eta^J\phi^K + \tfrac{1}{6}\epsilon_{IJK}\eta^I\eta^J\eta^K\psi_4\,,$$
$$\bar{\Phi} = \bar{\psi}^4 + \eta^I\bar{\phi}_I + \tfrac{1}{2}\epsilon_{IJK}\eta^I\eta^J\bar{\psi}^K + \tfrac{1}{6}\epsilon_{IJK}\eta^I\eta^J\eta^K\bar{\phi}_4\,.$$

"bipartite" unification: $M_4^{(2)} = \frac{1}{\xi_4'} \times$  $M_6^{(0)} = \frac{1}{\xi_6'} \times$ 

$\mathcal{F}_8 = c_2^{(8)}$  $+ c_3^{(8)} \Big(4$  $+$  $+ 0$  $\Big)$

Gram $=$ $\sum$ 48 planar f-graphs

@N=8: 61 planar vs. 4 bipartite (1 planar)

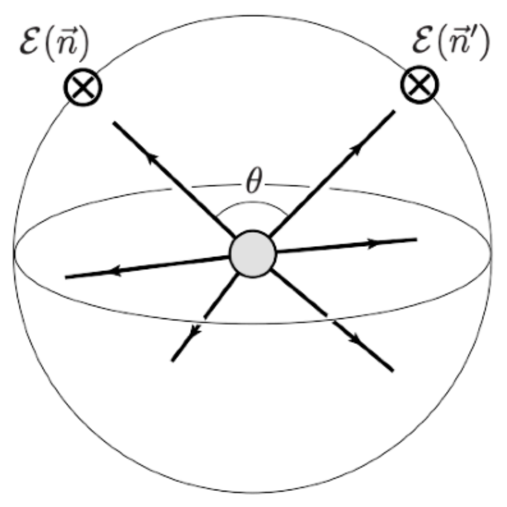
$$\mathcal{F}_{10} = \sum_{i=1}^{120} c_i^{(10)} f_i^{(10)} \xrightarrow{\cap\,\text{planar}} 3 \text{ free para.} + 31 \text{ Grams within bipartite } f\text{-graphs.}$$

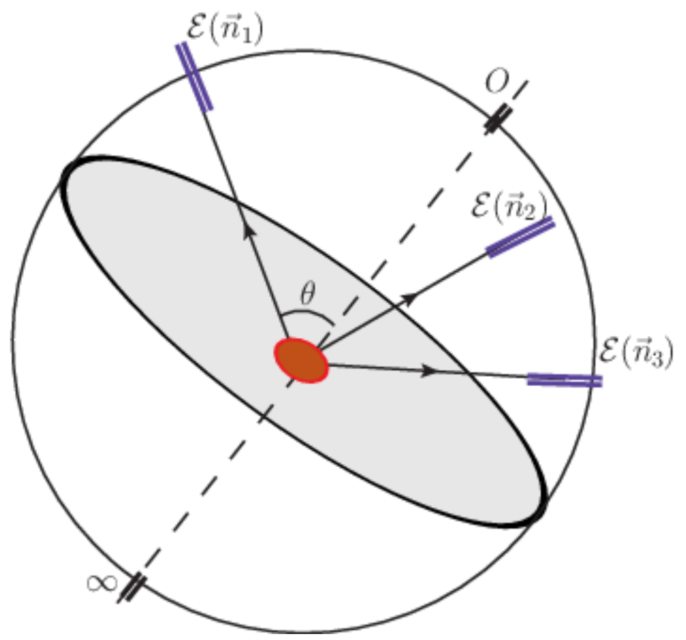
Bipartite ∩ Planarity: 3 free coeff / 31 Grams $\xrightarrow{\text{double quadrangle}}$ 1 free coeff / 27 Grams $\xrightarrow{\text{quadrangle}}$ 0 free coeff / 27 Grams

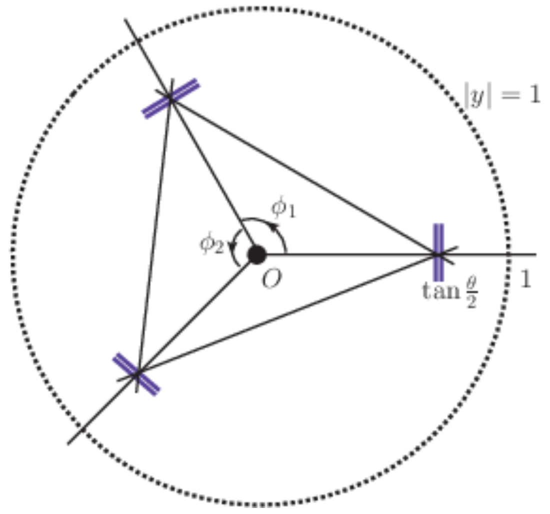
- $M_4^{(6)}$ New prediction!
- $M_6^{(4)}$ New prediction!
- $M_8^{(2)}$ [w. Huang, Kuo, Li, '22]
- $M_{10}^{(0)}$ [Gang et al '10; Brandhuber et al '12]

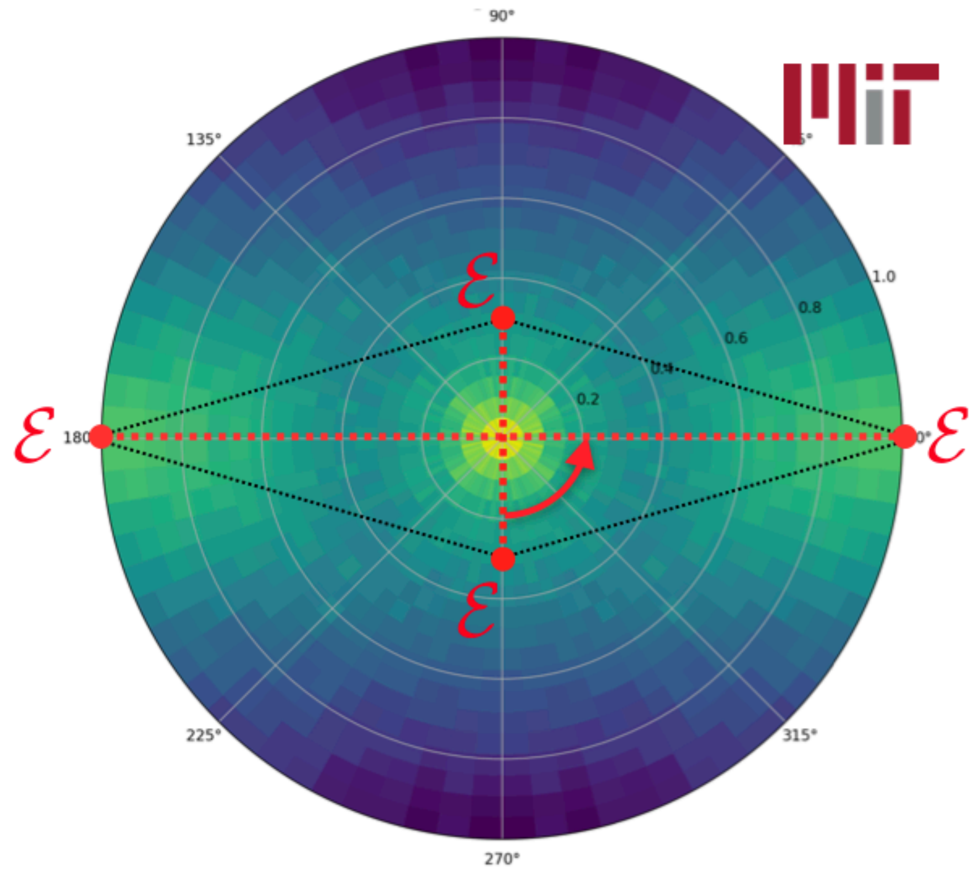
From amp^2 to energy correlators [Hofman, Maldacena; Belitsky, Hohenegger, Korchemsky, Sokatchev, Zhiboedov; c.f. H. Chen, L. Dixon, M. Luo, I. Moult, T. Yang, H. Zhu, K. Yan, H. Zhang,……] [w. 姜旭航、杨清霖、张耀奇, 2024]

- Correlation of energy flux: Infrared finite object measurable at experiments, lots of studies in QCD & N=4 SYM!
- Extensive studies & rich data for correlators w. $N \leq 3$, but very little is known beyond, c.f. [Checherin et al, ’24]
- N-pt EC=(N-1)-fold energy-integral of “splitting function” = tree amplitude^2 (or form factor^2)
- A beautiful conformal-invariant function, much simpler than super-amplitudes!









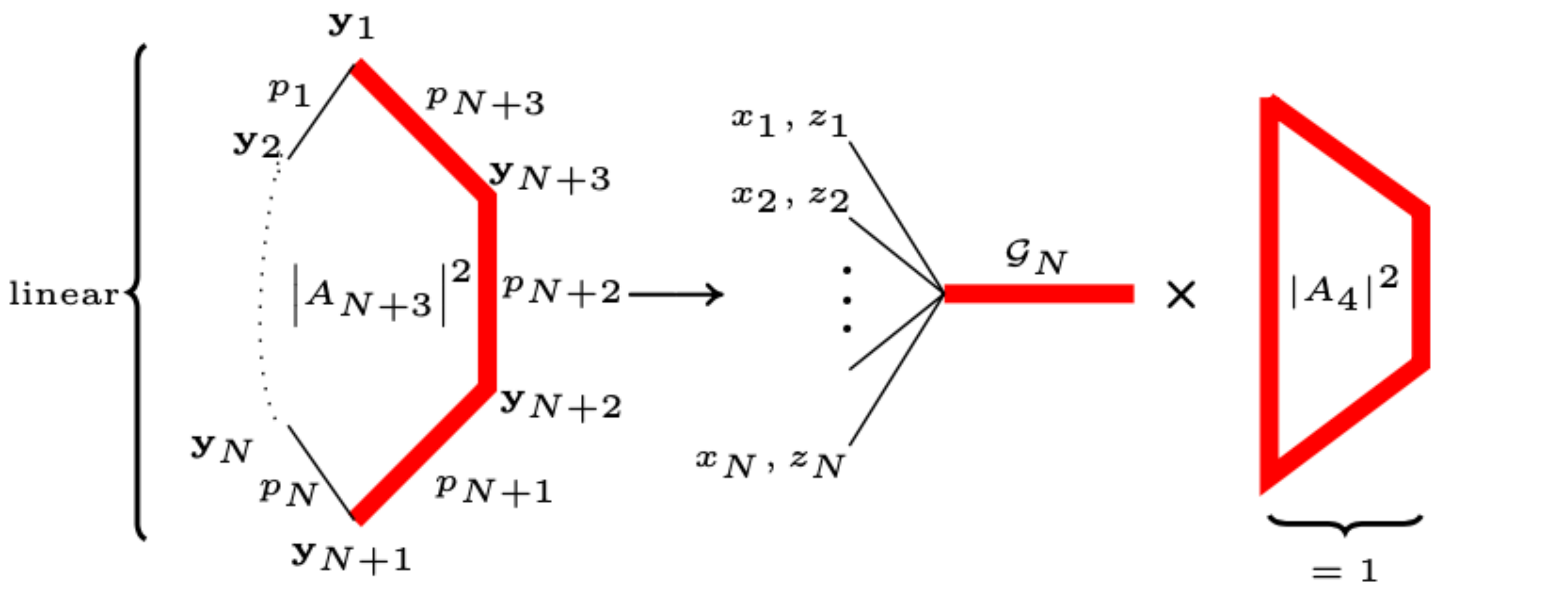


FIG. 1. The $1 \to N$ splitting function from collinear limit of squared amplitudes with $n = N{+}3$ legs.

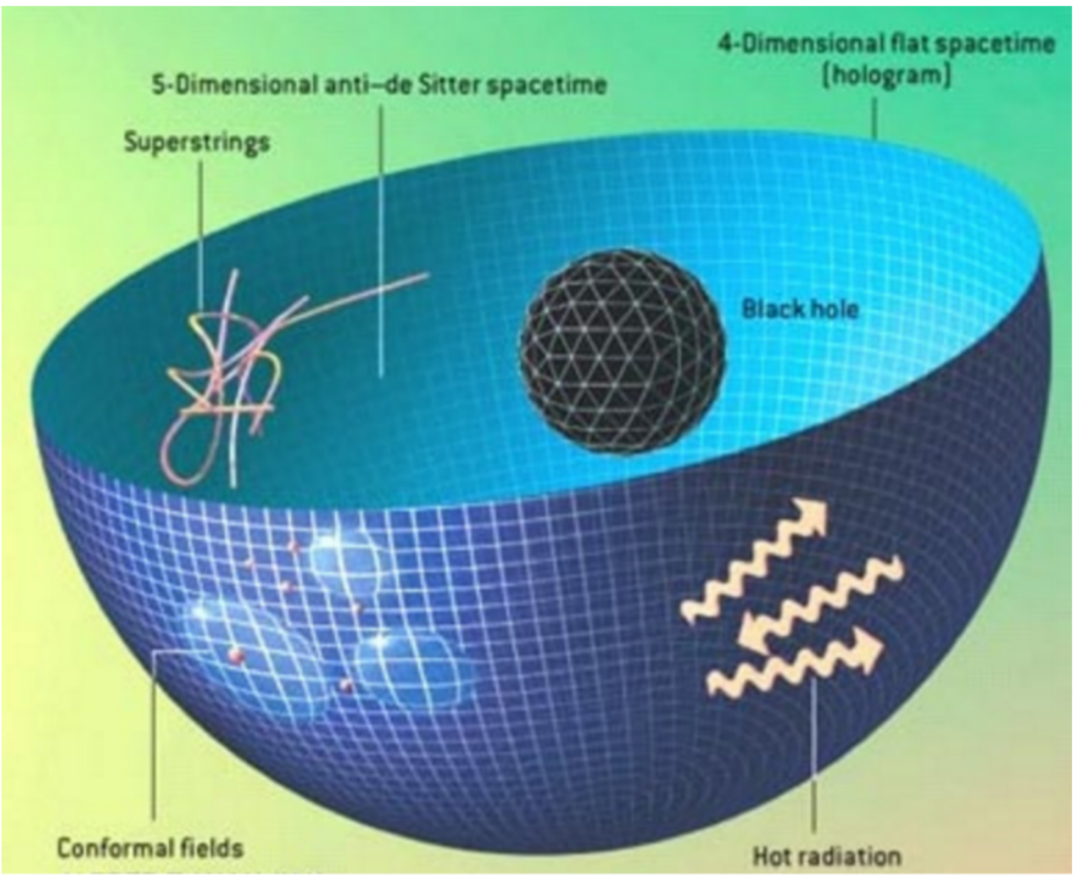
$$\mathbf{EC}^{(N)}(\{z_i\}) = \frac{I_N(z_1, \cdots, z_N)}{|z_{1,2} \cdots z_{N-1,N}|^2} + \text{perm}(1, 2, \cdots, N)\,,$$

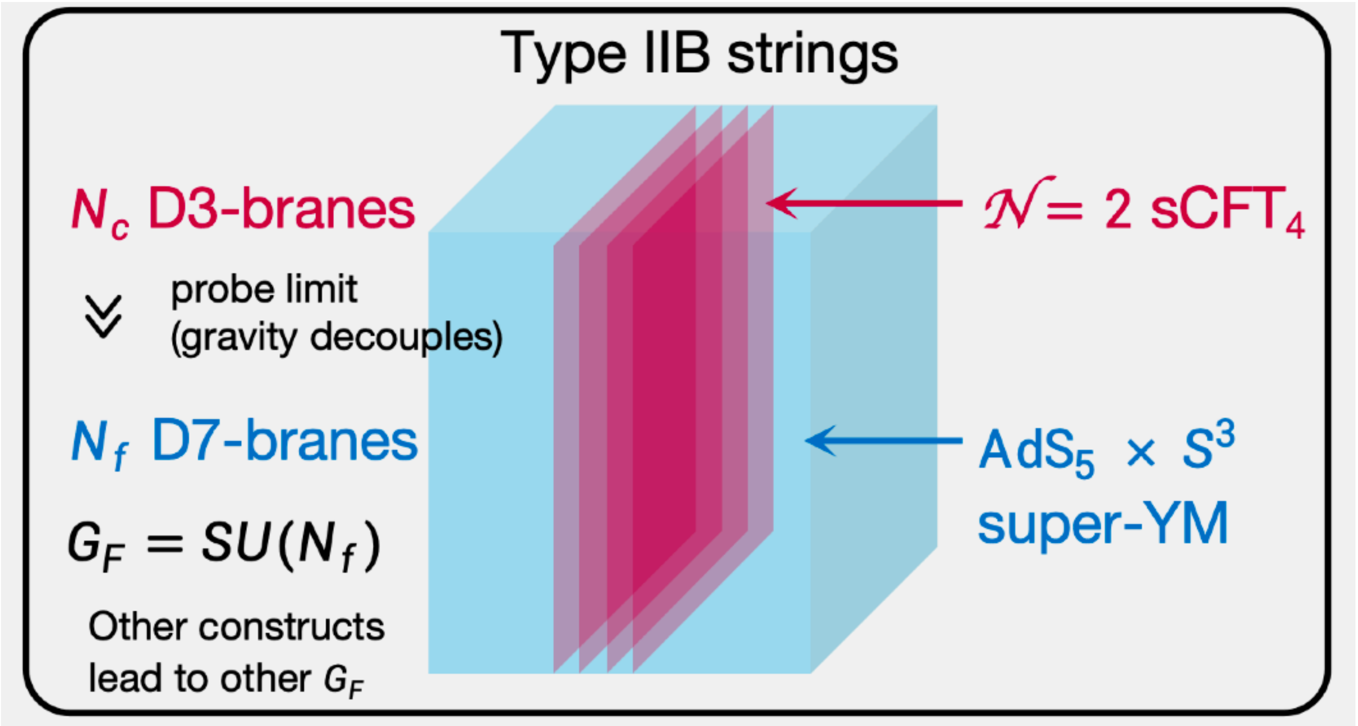
$$I_N := \int_0^\infty \frac{d^N x}{\text{GL}(1)} x_{12\cdots N}^{-N} \mathcal{G}_N(x_1, \cdots, x_N; z_1, \cdots, z_N)$$

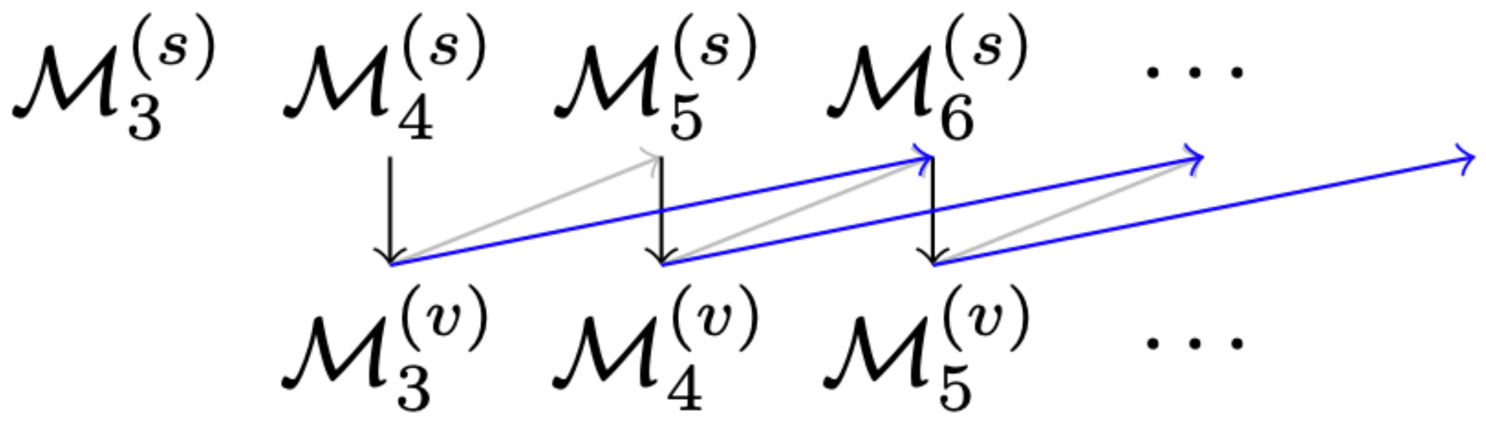
$$\mathcal{G}_N := \lim_{1||2\cdots||N} \frac{|A_n|^2}{|A_{n,\text{MHV}}|^2} = \lim_{1||2\cdots||N} \underbrace{\frac{1}{2} \sum_{k=0}^{n-4} \frac{A_{n,k} * A_{n,n-4-k}}{A_{n,0} * A_{n,n-4}}}_{r_n},$$

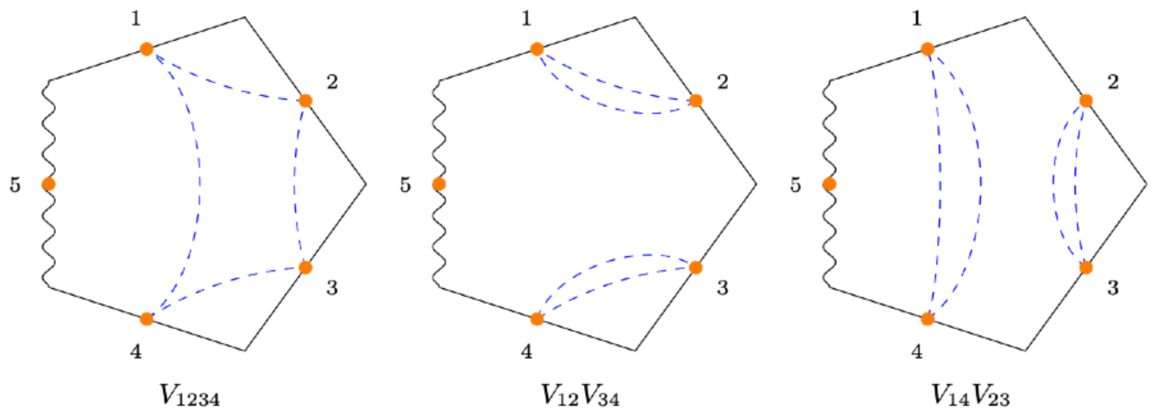
AdS amps=holographic correlators [w. 曹趣, 唐一朝, PRL 2023 + 李想, 2024]

- Supergluon amplitudes in AdS_5 x S^3 (tree-level): rich data for CFT_4 & "scattering in AdS"; known up to n=6 based on factorizations (OPE) + flat-space limit [Alday, Goncalves, Nocchi, Zhou 2023]
- We find a recursive algorithm for supergluon & spinning amps to all n ("AdS constructibility")









- Explicit, compact results up to n=8 (spinning for n=7), and the simplest R-symmetry case to all n
- New structures: general poles (truncation of descendents), nice Feynman rules, collinear/soft etc.
- They can be viewed as AdS generalizations of "scalar-scaffolded gluons" in flat-space!

Summary

Scattering amplitudes —> new structures in QFT + string theory: exciting frontier of hep-th
wide applications in particle physics, gravity + cosmology, strings, mathematics …

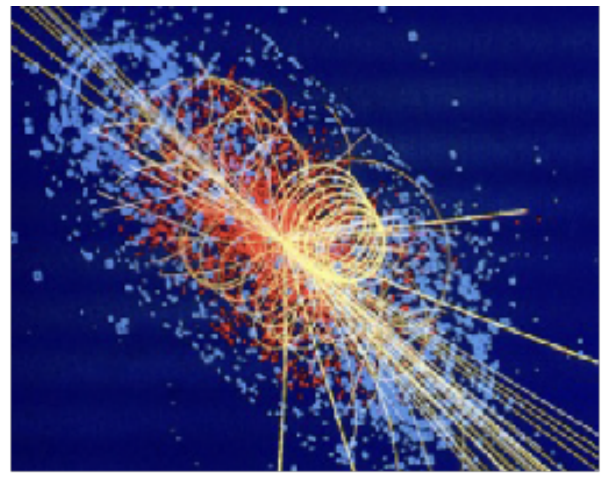
New relations: Goldstone particles, gluons, gravitons, strings… *double copy* for quantum & classical gravity

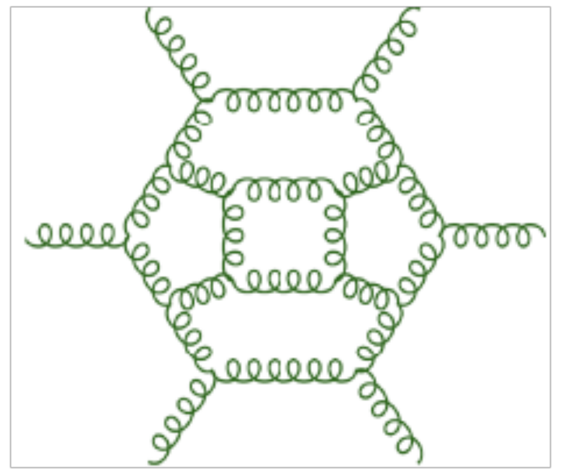
New formulations: twistor-strings, Grassmannian+ amplituhedron, CHY + duality for S-matrix on surfaces!

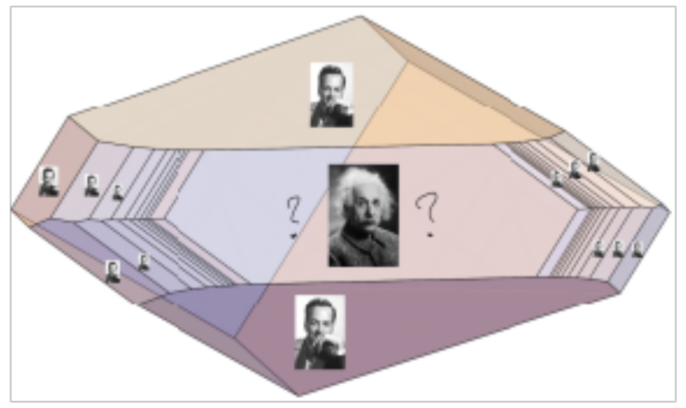
A New Theme: combinatorial geometries amplituhedron (in SYM & ABJM),
correlahedron -> half-BPS/energy correlators, AdS/dS + cosmological correlators…

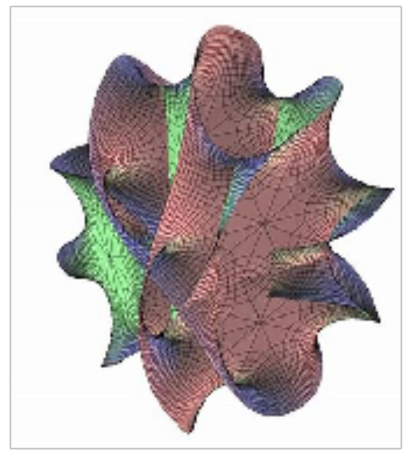
"surfaceology": stringy ϕ^3 from curve-integral
=> all-loop amps of pions, gluons etc. (real world)

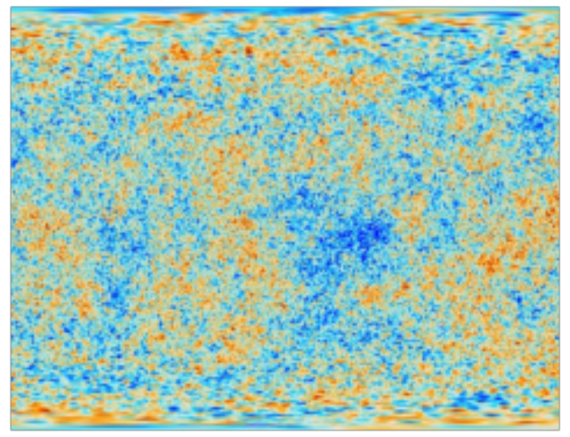
New principles? "theory at infinity" e.g. amplituhedra/surfacehedra:
geometry/combinatorics encoding QM + spacetime?

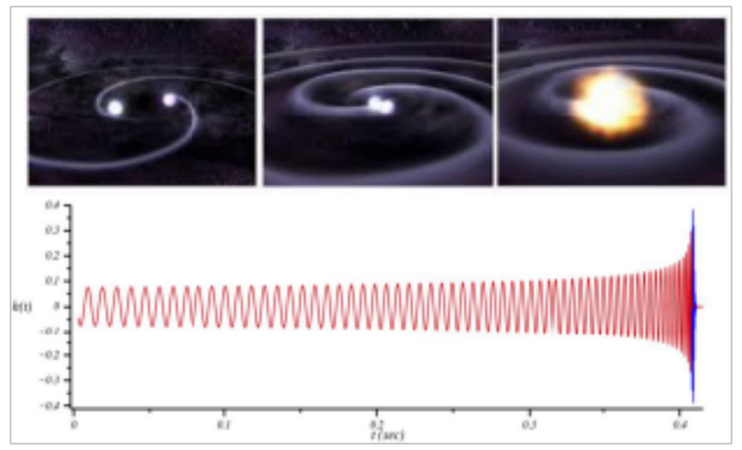












Thank you!