Quantum Relativity

and

Quantum Spacetime

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Reference Frame Transformations :-

- $\longleftrightarrow$ **Relativity Symmetry**

  — spacetime symmetry/reference frame

- physical/quantum frame Vs absolute/classical frame

  — relative 'uncertainty' / entanglement

- example of quantum spatial translation

$$\hat{x}_B^{(A)} \longrightarrow -\hat{x}_A^{(B)} \,, \qquad \hat{x}_C^{(A)} \longrightarrow \hat{x}_C^{(B)} - \hat{x}_A^{(B)} \,,$$
$$\hat{p}_B^{(A)} \longrightarrow -(\hat{p}_A^{(B)} + \hat{p}_C^{(B)}) \,, \quad \hat{p}_C^{(A)} \longrightarrow \hat{p}_C^{(B)} \,.$$

- quantum model of space(time) — the phase space

Unitary Q Spatial Translation :-

— generated by $\hat{p}_C$

Giacomini et.al. 19

- $e^{ix_B \hat{p}_C} \quad \rightarrow \quad e^{i\hat{x}_B^{(A)} \hat{p}_C^{(A)}} \quad \rightarrow \quad \hat{S}_x = \hat{\mathcal{P}}_{AB} e^{i\hat{x}_B^{(A)} \hat{p}_C^{(A)}}$

  — $\hat{\mathcal{P}}_{AB} : |x\rangle_B \otimes |y\rangle_C \rightarrow |-x\rangle_A \otimes |y\rangle_C$, $\quad \hat{x}_B^{(A)} \rightarrow -\hat{x}_A^{(B)}$

  $\mathcal{H}_B^{(A)} \otimes \mathcal{H}_C^{(A)} \rightarrow \mathcal{H}_A^{(B)} \otimes \mathcal{H}_C^{(B)}$

- *but* $\hat{x}_B^{(A)}$ generates $\hat{p}_B^{(A)}$ momentum translation

- Schrödinger picture : on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$

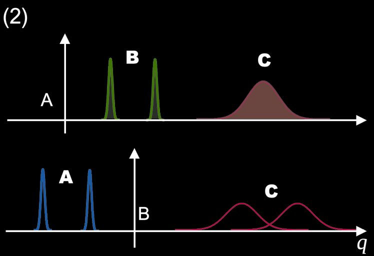
$$\hat{U}_x = \hat{\mathcal{S}}_{AB}^W \hat{I}_A \otimes \int dx' dy' \, |-x'\rangle\langle x'|_B \otimes |y' - x'\rangle\langle y'|_C$$

$$\hat{\mathcal{S}}_{AB}^W : |z\rangle_A \otimes |x\rangle_B \otimes |y\rangle_C \rightarrow |x\rangle_A \otimes |z\rangle_B \otimes |y\rangle_C$$

for $\quad |\psi\rangle = |\emptyset\rangle_A \otimes \int dx dy \, \psi(x,y) \, |x\rangle_B \otimes |y\rangle_C$

$$\hat{U}_x |\psi\rangle = \int dx dy \, \psi(x, y+x) \, |-x\rangle_A \otimes |\emptyset\rangle_B \otimes |y\rangle_C$$

The 4 scenarios: Case 2

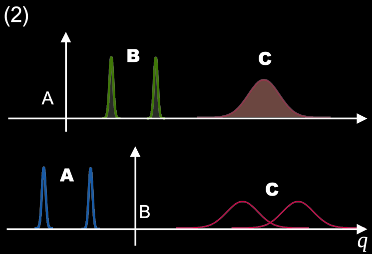


$$|\emptyset\rangle_A \otimes \frac{1}{\sqrt{2}}\left(|x_1\rangle + |x_2\rangle\right)_B \otimes \int \mathrm{d}y\, \psi(y)|y\rangle_C$$

$$\longrightarrow \frac{1}{\sqrt{2}}\Big(|-x_1\rangle_A \otimes |\emptyset\rangle_B \otimes \int \mathrm{d}y\, \psi(y)|y - x_1\rangle_C$$

$$+|-x_2\rangle_A \otimes |\emptyset\rangle_B \otimes \int \mathrm{d}y\, \psi(y)|y - x_2\rangle_C\Big)$$

The 4 scenarios: Case 2



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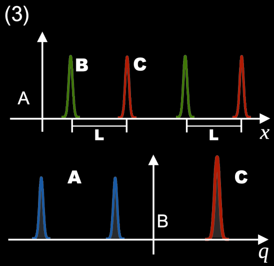
$$+|-x_2\rangle_A \otimes |\emptyset\rangle_B \otimes \int \mathrm{d}y\, \psi(y)|y - x_2\rangle_C\Big)$$

The 4 scenarios: Case 3



$$|\emptyset\rangle_A \otimes \left(c\,|x_1\rangle + s\,|x_2\rangle\right)_B \otimes \left(c\,|y_0 + x_1\rangle + s\,|y_0 + x_2\rangle\right)_C$$

$$\longrightarrow \left(c\,|-x_1\rangle + s\,|-x_2\rangle\right)_A \otimes |\emptyset\rangle_B \otimes |y_0\rangle_C$$

Distance Translated (A-frame to B-frame) ?

an Noncommutative Value :-

- classical $e^{ix_B \hat{p}_C}$: translation by x_B

  — say $x^i_B = 2$, $x^f_C = x^i_C - 2$, $x^f_A = -2$

- quantum : $[\hat{x}_B]^i_\phi$ as the value

  $$[\hat{x}_C]^f_{\phi'} = [\hat{x}_C]^i_\phi - [\hat{x}_B]^i_\phi \,, \quad [\hat{x}_A]^f_{\phi'} = -[\hat{x}_B]^i_\phi$$

  — $[\hat{x}_B]^i_\phi$ contains full quantum information of position

- evaluation $\hat{\mathcal{O}} \to [\hat{\mathcal{O}}]_\phi$: *algebraic homomorphism*

Quantum Relativity Principle :-

- **Penrose : Relativity Principle** $\rightarrow\!\otimes\!\leftarrow$ **Quantum**

- **Heisenberg picture** – $\hat{x}$ **and** $\hat{p}$ **as coordinates**

— **Noncommutative Geometry for Spacetime**

- **Rel. Sym.** $\leftarrow$ **Quantum Ref. Frame Transformations**

— ***e.g.*** **translation by the NC value of** $\hat{x}_A - \hat{x}_B$ **(ans. Penrose)**

- **Quantum Gravity as General Quantum Relativity**

Schrödinger Formulation :-

- 'equivalent' — only in Euclidean coordinates

- conceptually : old quantum theory thinking

- $\phi(x) \equiv \langle x|\phi\rangle$: $|\phi\rangle = \int dx\, |x\rangle\langle x|\phi\rangle = \sum_x \phi(x)|x\rangle$

— a set of coordinates on Hilbert space

— ??? configuration variables for a particle

- curvilinear $\hat{x}^i$ as x^i, no momentum vector

— metric loses its meaning in particle dynamics

Heisenberg & Dirac 1925/26 :-

— classical to quantum

only needs a new kinematic

- **H : physical quantities *not* real number variables**
- **Quine : real number as ‘convenient fiction’**
- **D : q-number as the new convenient fiction**

Concept of Numbers (in history) :

- $x + 2 = 0 \quad \longrightarrow \quad$ **negative numbers**
- $2x - 1 = 0 \quad \longrightarrow \quad$ **rational numbers**
- $x^2 - 2 = 0 \quad \longrightarrow \quad$ **real numbers**
- $x^2 + 1 = 0 \quad \longrightarrow \quad$ **complex numbers**
- $xy - x - i = 0 \quad \longrightarrow \quad$ (i,2), $(\frac{1}{i-1}$,-i), . . .
- $xy - yx - 1 = 0 \quad \longrightarrow \quad$ **noncommutative numbers**

★ $\hat{x}\hat{p} - \hat{p}\hat{x} - i\hbar = 0$

needs NC/q-number values for the variables

Evaluation as an Algebra Homomorphism :-

— real number is *only* an algebraic system

- classical $\qquad [\phi] : f(x_i, p_i) \rightarrow I\!R \qquad$ (observables have real values)

e.g. $\qquad E = p^2 + x^2 = pp + xx \qquad$ (1-D SHO $m = \frac{1}{2}$, $k = 2$)

$[\phi](x) = 2$, $[\phi](p) = 3 \qquad \Longrightarrow$

$$[\phi](E) = [\phi](p^2) + [\phi](x^2) = [\phi](p)[\phi](p) + [\phi](x)[\phi](x) = 13$$

$$[\phi](x_i p_i) = [\phi](x_i)[\phi](p_i) = [\phi](p_i)[\phi](x_i) = [\phi](p_i x_i)$$

- **quantum** $\qquad [\boldsymbol{\phi}] : \boldsymbol{\beta}(\hat{\boldsymbol{x}}_i, \hat{\boldsymbol{p}}_i) \rightarrow \boldsymbol{?}$

$$[\boldsymbol{\phi}](\hat{\boldsymbol{x}}_i)[\boldsymbol{\phi}](\hat{\boldsymbol{p}}_i) = [\boldsymbol{\phi}](\hat{\boldsymbol{p}}_i)[\boldsymbol{\phi}](\hat{\boldsymbol{x}}_i) + [\boldsymbol{\phi}](\boldsymbol{i\hbar\hat{I}})$$

$\Longrightarrow [\phi](\beta(\hat{x}_i, \hat{p}_i))$ **has to be a noncommutative algebra**

The Symplectic Geometry — NC Vs C :-

- **Heisenberg** — $\frac{d}{ds}\alpha(\hat{P}_\mu, \hat{X}_\mu) = \frac{1}{i\hbar}[\alpha(\hat{P}_\mu, \hat{X}_\mu), \hat{H}_s]$

- **Schrödinger** — $\frac{d}{ds} f_\alpha(z_n, \bar{z}_n) = \{f_\alpha(z_n, \bar{z}_n), f_{H_s(z_n, \bar{z}_n)}\}$

- $f_\alpha(z_n, \bar{z}_n) \equiv \frac{{}_g\langle\phi|\alpha(\hat{P}_\mu, \hat{X}_\mu)|\phi\rangle}{{}_g\langle\phi|\phi\rangle}$ $\quad \left(|\phi\rangle = \sum_n z_n |n\rangle\right)$

— as the pull-back of $\hat{\alpha}$ under $(z_n, \bar{z}_n) \longrightarrow (\hat{P}_\mu, \hat{X}_\mu)$

- $\rightarrow$ **bijective homomorphism between NC Poisson algebras**

— **NC Kähler product** $\quad f_\alpha \star_\kappa f_{\alpha'} = f_{\alpha\alpha'}$

Cirelli et.al 90

NC values of NC coordinates :-

— NC number as the new convenient fiction

- **state as evaluative homorphism**

— mapping observable algebra to algebra of their NC values

$[\hat{\alpha}]_\phi = \{ f_\alpha|_\phi, V_{\alpha_n}|_\phi \}$ $\quad (V_{\alpha_n} = \frac{\partial f_\alpha}{\partial z^n} = -f_\beta \bar{z}^n + \sum_m \bar{z}^m \langle m|\hat{\alpha}|n\rangle)$

- **Kähler product** — $\quad [\hat{\alpha}\hat{\alpha}']_\phi = [\hat{\alpha}]_\phi \star_\kappa [\hat{\alpha}']_\phi$

$f_{\alpha\alpha'} = f_\alpha f_{\alpha'} + \sum_n V_{\alpha_n} V_{\alpha'_{\bar{n}}}$, $\quad V_{\alpha\alpha'_n} = -f_{\alpha\alpha'} \bar{z}_n + \sum_{m,l} \bar{z}_m \langle m|\hat{\alpha}|l\rangle\langle l|\hat{\alpha}'|n\rangle$

Deutsch & Hayden 00 ; Kong 23

- **locality of quantum information** (**Heisenberg picture**)

Galvão & Hardy 03

- Substituting a **Qubit** for an Arbitrarily Large Number of Classical Bits’

Noncommutative Number Systems :-

- observables are dynamical *variables*

— a state is an evaluative homomorphism

- matrices as Dirac's q-numbers

- convention : representation of observables as

— DH-matrix value for a reference state, *e.g.*

$$[\hat{s}^i]_\phi = U_\phi^\dagger \sigma^i U_\phi \ , \quad |\phi\rangle = U_\phi \, |0\rangle \ , \quad U_\phi = \begin{pmatrix} c & -\bar{s} \\ s & \bar{c} \end{pmatrix}$$

- need to fix a 'reference frame' for the states

Quantum Physics is simply

a q-number version of classical physics

about the q-number reality

Quantum geometry is then

q-number geometry

— ***a true geometric picture of NCG***

Quantum Spacetime Geometry :-

— *NC (number) geometry*

- simple free particle motion $\hat{p}(t) = \hat{p}(0)$

$$\hat{x}(t) = \hat{p}(0)\frac{t}{m} + \hat{x}(0) \quad \Rightarrow \quad [\hat{x}(0), \hat{x}(t)] = \frac{i\hbar t}{m}$$

– NCNG : phase space → configuration space

- 'cotangent bundle' (metric independent)

— $\{\hat{x}^a, \hat{p}_b\} = \delta^a_b, \quad \{\hat{x}^a, \hat{x}^b\} = 0 = \{\hat{p}_a, \hat{p}_b\}$

$$\frac{\partial}{\partial \hat{x}^a} \equiv \{\cdot, \hat{p}_a\}, \qquad \frac{\partial}{\partial \hat{p}_a} \equiv -\{\cdot, \hat{x}^a\}$$

Prologue : Geometrodynamics

Spacetime Geometry

Noncommutative G. $\Leftrightarrow$ Quantum Gravity

Non-Euclidean G. $\Leftrightarrow$ Classical Gravity

Classical Geodesic from Hamiltonian: -

- $S_o = \int ds L_o(s) = \int ds \sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}}$

— manifold of any dimension and metric ($\pm$)

- Lagrangian $L(s) = -\frac{m}{2}L_o^2$ from free particle motion

- Hamiltonian : phase space as cotangent bundle

— symplectic structure independent of $g_{\mu\nu}(x)$

- Rindler frame: particle at $\rho(\tau) = 0$, acceleration a

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 = -\frac{a^2\rho^2}{c^2}d\tau^2 + d\rho^2 + \ldots$$

Goedesic from Free Particle Motion:-

- all positions coordinates Hermitian

- $\hat{H}_{\text{free}} = \frac{1}{2m}\, \hat{p}_{A}\, g^{Ab}(\hat{x})\, \hat{p}_{b}$, $\qquad \hat{p}^{b} = \hat{p}_{A}\, g^{Ab}(\hat{x})$

— four vectors : $\qquad \hat{V}'^{a} = \hat{V}^{i}\frac{\partial \hat{x}'^{a}}{\partial \hat{x}^{i}}$, $\quad \hat{W}'_{a} = \frac{\partial \hat{x}^{i}}{\partial \hat{x}'^{a}}\hat{W}_{i}$,

$$\hat{V}'^{A} \equiv \hat{V}'^{a\dagger} = \left(\frac{\partial \hat{x}'^{a}}{\partial \hat{x}^{i}}\right)^{\dagger}\hat{V}^{i\dagger} \equiv \left(\frac{\partial \hat{x}^{A}}{\partial \hat{x}'^{I}}\right)\hat{V}^{I} , \quad \hat{W}'_{A} = \hat{W}_{I}\left(\frac{\partial \hat{x}^{I}}{\partial \hat{x}'^{A}}\right).$$

— Schrödinger representation fails

- $\hat{x}^{a}$ and $\hat{p}^{a}$ as $\hat{g}$-Hermitian within the ref. frame

- Hamilton's Eqs. $\longrightarrow$ mass-indep. E.O.M.

. . .

- **Quantum Geodesic Equation :** $\frac{d^2\hat{x}^\mu}{ds^2} =$

$$\frac{d\hat{x}^\nu}{ds}\frac{\partial_\Lambda \hat{g}_{\nu\Omega}}{2}\frac{d\hat{x}^\Omega}{ds}\hat{g}^{\Lambda\mu} - \frac{d\hat{x}^\nu}{ds}\frac{\partial_\Omega \hat{g}_{\nu\Lambda}}{2}\hat{g}^{\Lambda\mu}\frac{d\hat{x}^\Omega}{ds} - \frac{d\hat{x}^\xi}{ds}\hat{g}_{\xi\Omega}\frac{d\hat{x}^\nu}{ds}\hat{g}^{\Omega\zeta}\frac{\partial_\zeta \hat{g}_{\nu\Lambda}}{2}\hat{g}^{\Lambda\mu}$$

— s **a real number Hamiltonian evolution parameter**

— **proper time a quantum observable**

— $\hat{p}^\mu = m\frac{d\hat{x}^\mu}{ds}$, $g(\hat{x}^\mu)$**-Hermitian (frame-dependent)**

- $\hat{g}_{\mu\nu} = \frac{\partial\hat{x}^a}{\partial\hat{x}^\mu}\eta_{ab}\frac{\partial\hat{x}^b}{\partial\hat{x}^\nu}$ — **all** $\hat{x}$ **commute and Hermitian**

$$\frac{\partial}{\partial\hat{x}^\mu} \equiv \{\cdot, \hat{p}_\mu\}, \qquad \frac{\partial}{\partial\hat{p}_\mu} \equiv -\{\cdot, \hat{x}^\mu\}$$

— $\{\hat{x}^\mu, \hat{p}^\nu\} = \{\hat{x}^\mu, \hat{g}^{\nu\lambda}\hat{p}_\lambda\} = \hat{g}^{\mu\nu}$, $\{\hat{x}^\mu, \hat{p}_\nu\} = \delta^\mu_\nu$

$\{\hat{x}^\mu, \hat{x}^\nu\} = 0 = \{\hat{p}_\mu, \hat{p}_\nu\}$ **invariant (with** $\hat{p}_\mu = \frac{\partial\hat{x}^a}{\partial\hat{x}^\mu}\hat{p}_a$**)**

Metric Operator $\hat{g}$ on Krein Space :-

• proper **inner product with Minkowski Signature**

— effectively, bra as ${}_g\langle\cdot| = \langle\cdot|\,\hat{g}$

• **observables (pseudo-)Hermitian** $\quad {}_\eta\Big\langle\cdot\Big|\hat{A}^{\dagger^\eta}\cdot\Big\rangle = {}_\eta\Big\langle\hat{A}\cdot\Big|\cdot\Big\rangle$

— $\hat{X}_a = \hat{\eta}\hat{X}^a\hat{\eta}^{-1}$ and $\hat{P}_a = \hat{\eta}\hat{P}^b\hat{\eta}^{-1}$

metric operator $\leftrightarrow$ metric tensor $\quad \hat{\eta} \leftrightarrow \hat{\eta}_{ab}$

• **noncommutative geometric picture** : $\quad g$-Hermitian

— $\hat{X}^\mu$ and $\hat{P}^\mu$ as coordinates, $\quad \hat{A}^{\dagger^g} = \hat{A}$

nontrivial $\quad \hat{g}_{\mu\nu} = g_{\mu\nu}(\hat{x}^\zeta) \quad \longrightarrow \quad \hat{g} \quad (?)$

Lorentz Covariant Quantum Physics :-

- Schrödinger wavefunction $\phi(x^a)$

— basic operators x^a and $-i\hbar\partial_a$

- abstract operators as Minkowski four-vectors

— $\hat{X}_i \longrightarrow \hat{X}_a$ and $\hat{P}_i \longrightarrow \hat{P}_a$

— $[\hat{X}_a, \hat{P}_b] = i\hbar\eta_{ab}$

- Heisenberg-Weyl symmetry — $[Y_a, E_b] = i\hbar\, c\, \eta_{ab}\, M$

— M **is an effective Casimir element** $\rightarrow$ **Newtonian mass** m

— $m\hat{X}_a \longleftarrow Y_a$, different m for different irr. representations

— $\hat{P}_a \longleftarrow \frac{1}{c}E_a$, constant c (... $c \rightarrow \infty$ limit)

Quantum Rindler Frame :-

$$\hat{x} = \hat{\rho}\cosh\frac{\hat{a}_N\hat{\tau}}{c}, \qquad c\hat{t} = \hat{\rho}\sinh\frac{\hat{a}_N\hat{\tau}}{c}$$

- eigenstates : $\hat{x}$ & $\hat{t}$ $\quad\longrightarrow\quad$ $\hat{\rho}$ & $\hat{a}_N\hat{\tau}$

  entanglement between $\hat{\tau}$ and $\hat{a}_N$

- metric : $\hat{g}_{c\hat{\tau},c\hat{\tau}} = \frac{\hat{a}_N^2\hat{\rho}^2}{c^4}$

- quantum geodesic equations :

$$\frac{d^2c\hat{\tau}}{ds^2} + \frac{dc\hat{\tau}}{ds}\frac{1}{\hat{\rho}}\frac{d\hat{\rho}}{ds} + \frac{d\hat{\rho}}{ds}\frac{1}{\hat{\rho}}\frac{dc\hat{\tau}}{ds} = 0\ ,$$

$$\frac{d^2\hat{\rho}}{ds^2} + \frac{dc\hat{\tau}}{ds}\frac{\hat{a}_N^2\hat{\rho}}{c^4}\frac{dc\hat{\tau}}{ds} = 0\ .$$

$\longrightarrow$ maintaining (Weak) Equivalence Principle

Future (H & D $\longrightarrow$ . . .) :-

- **q-number physics – Q gravity as GQR**

  more NC physics : $[\hat{x}^i, \hat{x}^j] \neq 0$?

- **q-number geometry – NC Geo. as symplectic**
  coordinate picture : 'Euclidean NC Geo.'

- **q-number theory – algebra beyond algebra**

- **q-number technology – q-information**

  Quine: To be is to be the (q-number) value of a (physical) variable.

THANK YOU !