



Scalar-induced vector perturbations as a new way to constrain non-singular cosmology

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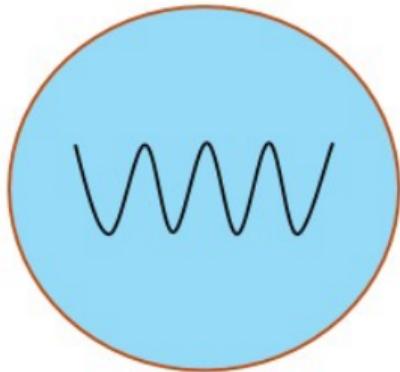
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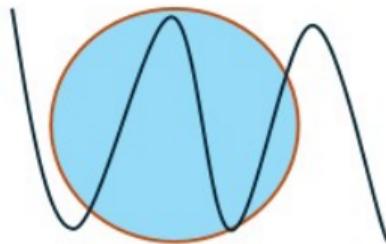
1. Introduction to cosmological perturbation theory
2. Non-linear fluctuations
3. Scalar induced vector perturbations (SIVP) in non-singular cosmology
4. Matter bounce as an example
5. Conclusion and outlook

Cosmological perturbation theory



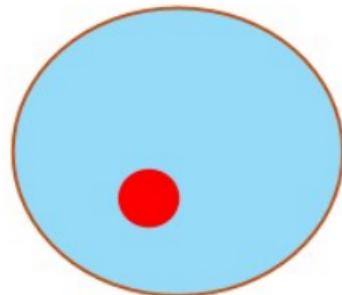
Quantum fluctuation
inside the horizon:

$$\frac{a}{k} < H^{-1}$$



Quantum fluctuation to
Classical perturbation at
horizon crossing:

$$\frac{a}{k} = H^{-1}$$



Local overdense
region

$$\delta \equiv \frac{\delta\rho}{\rho_{bg}}$$

Figure: Classical perturbations from quantum fluctuations in early universe.

SVT decomposition in cosmology

Cosmological principle sets the background to be FLRW

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j . \quad (1)$$

The perturbed metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} , \quad (2)$$

The most general decomposition of $h_{\mu\nu}$ into its irreducible representations under 3d rotations with a subsequent Helmholtz decomposition

$$h_{00} = 2A , \quad h_{0i} = B_i^T + \partial_i B , \quad (3)$$

$$h_{ij} = \frac{D}{3} \delta_{ij} + E_{ij}^{TT} + \partial_{(i} E_{j)}^T + \left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \partial^2 \right) E . \quad (4)$$

Observational evidence: CMB

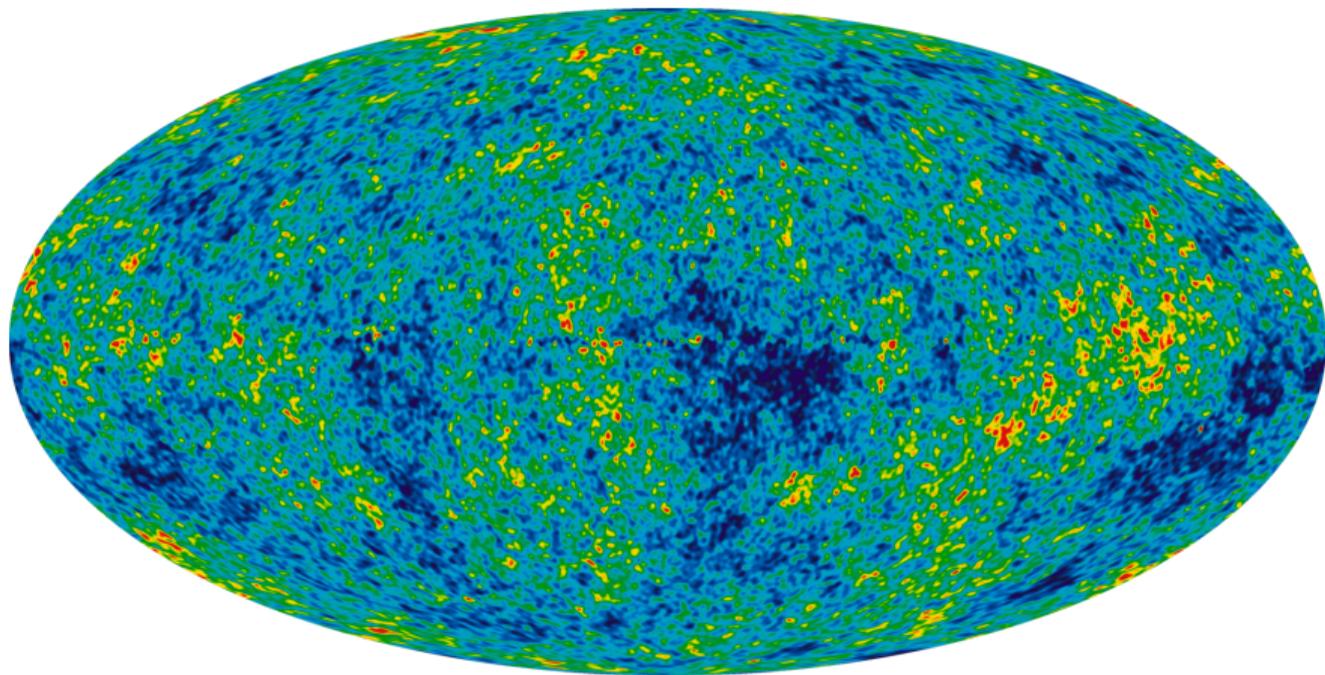


Figure: Cosmic Microwave background radiations, almost homogeneously distributed, with small temperature anisotropies at the scale $\Delta T/T \simeq 10^{-5}$.

Descriptions of CMB anisotropy

We describe the CMB anisotropy using

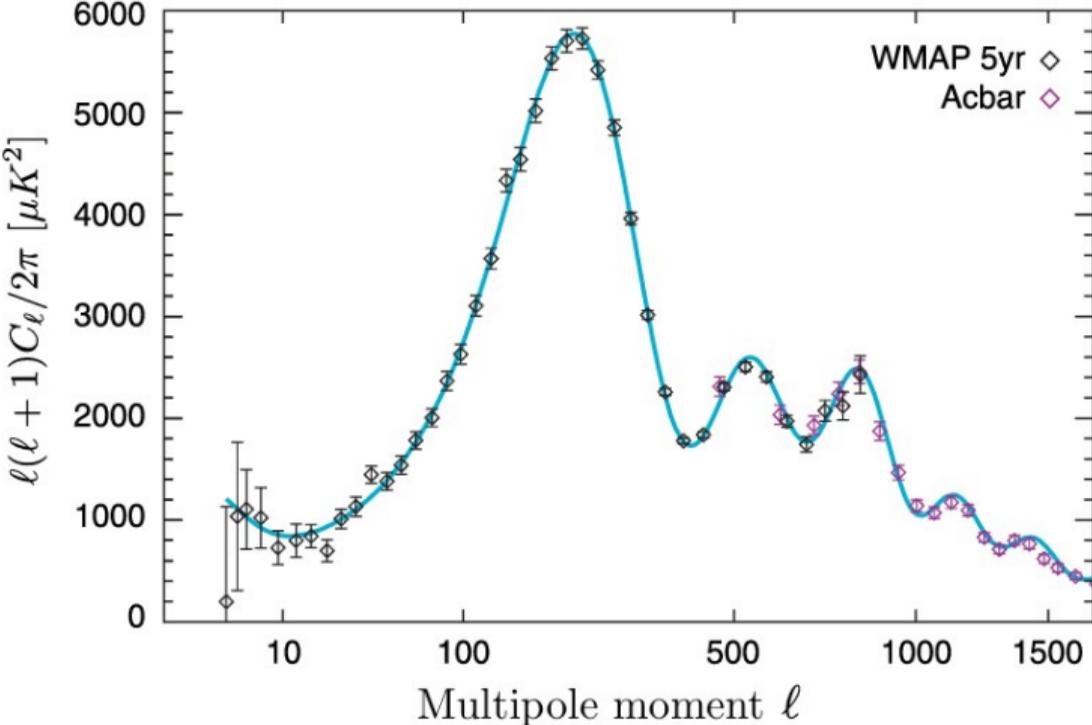
$$\Theta(\hat{n}) \equiv \frac{\Delta T(\hat{n})}{T_0} = \frac{T(\hat{n}) - T_0}{T_0} = \sum_{lm} a_{lm} Y_{lm}(\hat{n}) , \quad (5)$$

$Y_{lm}(\hat{n})$: Spherical harmonics functions.

We characterise the statistic properties of a_{lm} by its correlation functions. Specifically the two-point correlation functions (2ptcf):

$$\langle a_{lm}^* a_{pq} \rangle \equiv C_l^{TT} \delta_{lp} \delta_{mq} . \quad (6)$$

Precise detection of CMB anisotropy



1. Introduction to cosmological perturbation theory **Figure: Measurement of CMB anisotropy.**

Curvature fluctuation

A metric with only spatial fluctuations:

$$ds^2 = -dt^2 + a^2(t)[1 - 2\mathcal{R}(x_i)]\delta_{ij}dx^i dx^j . \quad (7)$$

After some computation [1], we found the relation between 2ptcf of \mathcal{R} and C_l^{TT} :

$$C_l^{TT} = \frac{2}{\pi} \int k^2 dk P_{\mathcal{R}}(k) \Delta_{T_l}(k) \Delta_{T_l}(k) . \quad (8)$$

$\Delta_{T_l}(k) \equiv \frac{1}{3}j_l(k[\tau_0 - \tau_{\text{rec}}])$: transfer function

$P_{\mathcal{R}}(k)$: power spectrum of curvature fluctuation:

$$\langle \mathcal{R}_{\vec{k}} \mathcal{R}_{\vec{q}} \rangle \equiv (2\pi)^3 \delta^{(3)}(\vec{k} + \vec{q}) \frac{2\pi^2}{k^3} P_{\mathcal{R}}(k) . \quad (9)$$

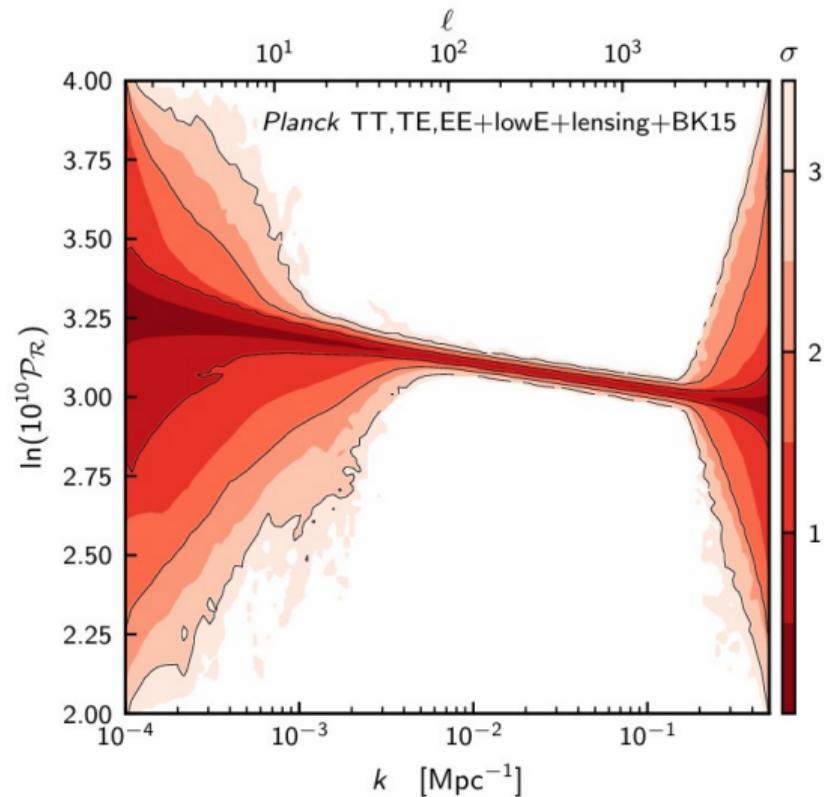


Figure: Constraints on primordial curvature power spectrum from CMB observation from PLANCK results [2].

Dynamics of scalar fluctuations

Working in k-essence theory

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2} + K(\phi, X) \right], \quad X \equiv -\frac{\partial_\mu \phi \partial^\mu \phi}{2}, \quad (10)$$

The quadratic action

$$S_\zeta^{(2)} = \int d\tau d^3x \frac{3a^2}{2c_s^2} \left[\zeta'^2 - c_s^2 (\partial_i \zeta)^2 \right]. \quad (11)$$

Define Mukhanov-Sasaki (MS) variable:

$$v_k = z_s \zeta_k, \quad z_s \equiv \sqrt{3}a/c_s, \quad (12)$$

The MS equation

$$v_k'' + \left(c_s^2 k^2 - \frac{2}{\tau^2} \right) v_k = 0. \quad (13)$$

Solution of curvature fluctuations

Impose vacuum initial condition

$$v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}, \quad -k\tau \gg 1. \quad (14)$$

Assuming the parametrization of scale factor

$$a(\tau) \propto (\tau/\tau_0)^n, \quad (15)$$

The equation becomes

$$v_k'' + \left(c_s^2 k^2 - \frac{n(n-1)}{\tau^2} \right) v_k = 0. \quad (16)$$

Solution: Hankel function

$$v_k = CH_\nu^{(1)}(c_s k \tau), \quad \nu \equiv n - \frac{1}{2}. \quad (17)$$

Scale-invariant curvature power spectrum

Expanding Hankel function around $k\tau \sim 0$:

$$H_{\nu}^{(1)}(x) \propto x^{-|\nu|}, \quad (18)$$

so for super-horizon fluctuations

$$v_k \propto k^{-|\nu|}, \quad \zeta_k \propto k^{-|\nu|}; \quad \langle \zeta_k \zeta_k \rangle \propto k^{-|\nu|}. \quad (19)$$

Scale-invariant requires

$$\mathcal{P}_{\zeta}(k) \propto k^0, \quad \langle \zeta_k \zeta_k \rangle \propto k^{-3}; \quad |\nu| = \frac{3}{2}. \quad (20)$$

Which scenarios match?

Case $n = -1$: $a \propto \tau^{-1} \propto e^{Ht}$, an exponentially expanding universe: inflation

Case $n = 2$: $a = (\tau/\tau_0)^2$, EoS $w \equiv p/\rho = 0$, a contracting universe?

The non-linearity of GR

General Relativity (GR) is non-linear

$$M_p^2 G_{\mu\nu} = T_{\mu\nu} , \quad (21)$$

Einstein tensor contains non-linear terms e.g., $g\partial^2 g$

The non-linearity of GR lead to distinctive features like the non-renormalizability in perturbative QFT.

In cosmology, the non-linearity implies the existence of secondary fluctuations in cosmological perturbation theory, resulting in new phenomenology.

Scalar-induced gravitational waves

$$ds^2 = -e^{2\psi} dt^2 + a(t)^2 e^{2\Phi} (e^h)_{ij} dx^i dx^j, \quad (22)$$

Action of gravitational waves up to cubic order

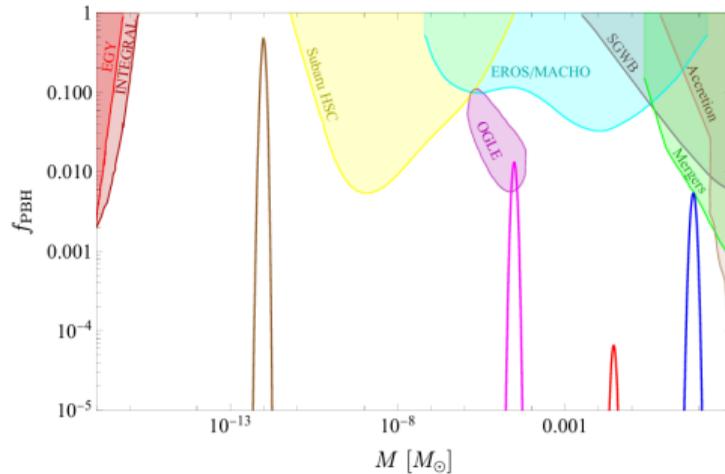
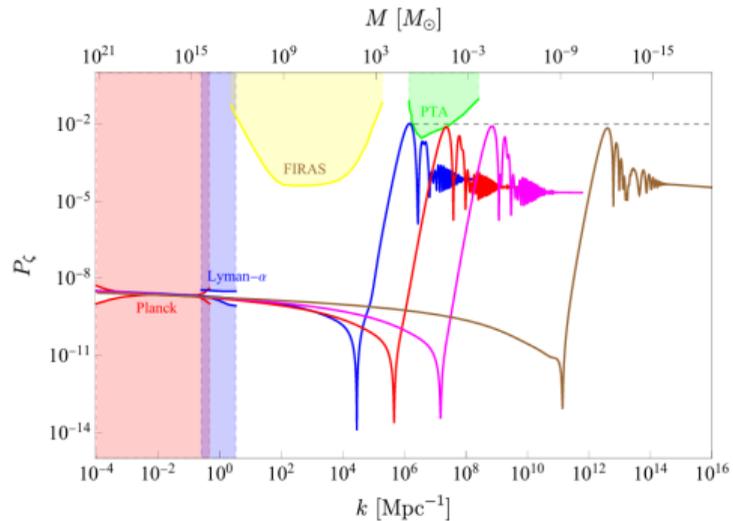
$$S_h = \int d^3x dt a \left\{ a^2 \dot{h}_{ij} \dot{h}^{ij} - \frac{1}{8} \partial_i h^{kl} \partial^i h_{kl} \right. \\ \left. - 2h^{ij} \partial_i (\Phi + \Psi) \partial_j \Phi + h^{ij} \partial_i \Phi \partial_j \Phi + \frac{h^{ij}}{2} \partial_i \delta\varphi \partial_j \delta\varphi \right\}. \quad (23)$$

First line: linear perturbations; Second line: non-linear perturbations

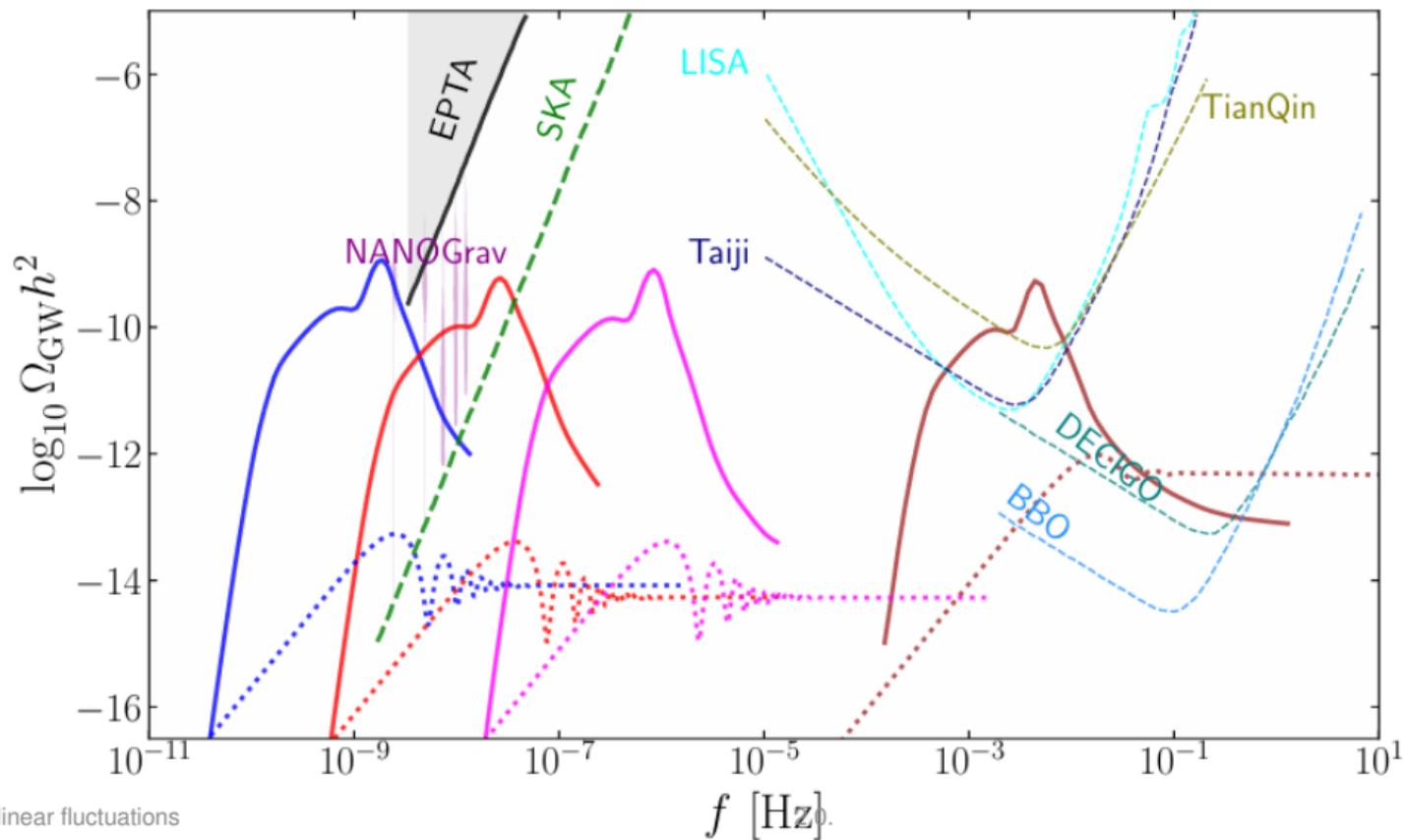
The non-linear interactions introduce new source term in EoM

$$h''_k + 2\mathcal{H}h'_k + k^2 h_k = S_k. \quad (24)$$

SIGW associated with PBHs



SIGW associated with PBHs



Vector modes?

So far, we have talked about scalar and tensor fluctuations, as well as their vital importance in cosmology.

Question: where is vectors?

Answer: vectors scale as a^{-2} , thus dilute as universe expands

Vectors are important but...

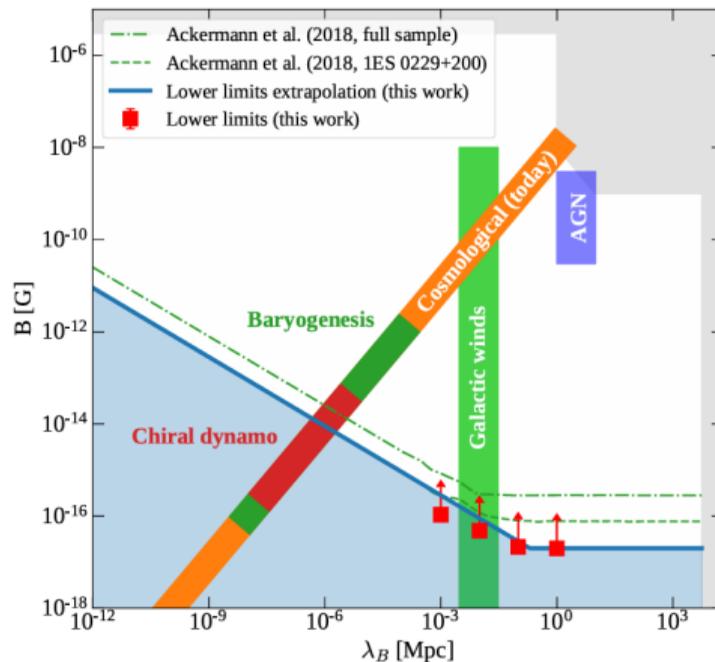


Figure: Detection of intergalactic magnetic fields from MAGIC collaborations
Astron.Astrophys. 670 (2023) A145

Inflation

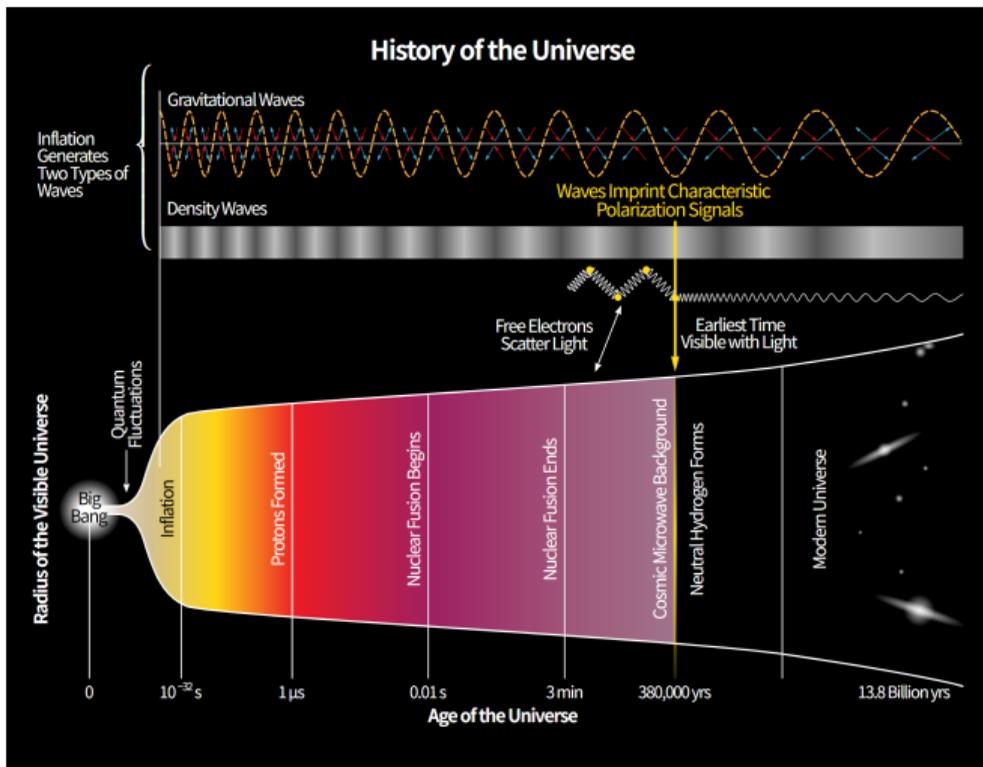


Figure: History of our Universe. Credit to Drbogdan & Yinweichen, <https://commons.wikimedia.org/w/index.php?curid=31825049>

Why non-singular cosmology?

Penrose–Hawking singularity theorems: cosmological singularity exists in GR when matter satisfies strong energy condition (SEC). For perfect fluid it reduces to

$$\rho + p \geq 0, \quad \rho + 3p \geq 0. \quad (25)$$

In inflation, matter field violates the SEC, can inflation avoid cosmological singularity?

Answer: No![3]

Inflationary Spacetimes Are Incomplete in Past Directions

[Arvind Borde](#)^{1,2}, [Alan H. Guth](#)^{1,3}, and [Alexander Vilenkin](#)¹

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Resolution: non-singular (classical) cosmology?

The proof is purely geometric: the past completeness must exist provided that the "averaged expansion rate of universe" is positive

$$\int_{-\infty}^t H(\lambda) d\lambda > 0 . \quad (26)$$

Resolution: introduce a non-expansion stage in far past.

Challenges: over-production of vectors?

Starting point

How to resolve vector instabilities in bouncing cosmology? [4]

- Assuming no presence of vector modes in contraction phase.
- Fine-tuning of initial conditions of vector modes.

Our observation: both approaches are not applicable for SIVP [5]

- SIVP originates from non-linear interactions of vectors and scalars due to the non-linear nature of GR. Cannot be eliminated as long as scalar fluctuations exist.
- The amplitude of SIVP is determined by the amplitude of scalar fluctuations, which is precisely measured by CMB on large scales and there's no room for fine-tuning!

Einstein equations for SIVP

The perturbed metric

$$ds^2 = a^2(\tau) \left[-e^{2\alpha} d\tau^2 + e^{2\zeta} (dx^i + e^{-2\zeta} (\partial_k \beta - G_k) d\tau) \right. \\ \left. \times (dx^j + e^{-2\zeta} (\partial_l \beta - G_l) d\tau) \right] , \quad (27)$$

SIVP is determined by the Einstein Eq. $G_{0i}^{(2)} = T_{0i}^{(2)}$ or $G_{ij}^{(2)} = T_{ij}^{(2)}$ with $i \neq j$. After a lengthy computation we get

$$G^\lambda{}_\nu(\tau, \vec{k}) + 2\mathcal{H}G^\lambda(\tau, \vec{k}) = S^\lambda(\tau, \vec{k}) , \quad (28)$$

$$S^\lambda(\tau, \vec{k}) = \frac{ik^i}{k^2} e^{\lambda,j}(\hat{k}) S_{ji}(\tau, \vec{k}) . \quad (29)$$

$$G_i(\tau, \vec{x}) = \sum_\lambda \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} G^\lambda(\tau, \vec{k}) e_i^\lambda(\hat{k}) , \quad (30)$$

Power spectrum of SIVP

The power spectrum of SIVP is

$$\langle G^\lambda(\vec{k})G^s(\vec{p}) \rangle \equiv (2\pi)^3 \delta(\vec{k} + \vec{p}) \delta_{\lambda s} \frac{2\pi^2}{k^3} \mathcal{P}_G(\tau, \vec{k}), \quad (31)$$

The general solution of G_i is given by

$$G^\lambda(\tau, \vec{k}) = \frac{1}{a(\tau)^2} \int^\tau a(\tilde{\tau})^2 S^\lambda(\tilde{\tau}, \vec{k}) d\tilde{\tau}. \quad (32)$$

After another lengthy calculation

$$\begin{aligned} \langle G^\lambda(\tau, \vec{k})G^s(\tau, \vec{k}') \rangle &= 2 \frac{(2\pi^2)^2}{k^4} \int_{\tau_{\text{ini}}}^\tau d\tilde{\tau}_1 a(\tilde{\tau}_1)^2 \int_{\tau'_{\text{ini}}}^\tau d\tilde{\tau}_2 a(\tilde{\tau}_2)^2 \int d^3 p \\ &\times e^\lambda(\vec{k}, \vec{p}) e^s(\vec{k}, \vec{p}) f^*(\vec{p}, \vec{k}, \tilde{\tau}_1) f(\vec{p}, \vec{k}, \tilde{\tau}_2) \frac{\mathcal{P}_\zeta(\vec{p}, \tau_0) \mathcal{P}_\zeta(\vec{k} - \vec{p}, \tau_0)}{p^3 |\vec{k} - \vec{p}|^3}. \end{aligned} \quad (33)$$

Power spectrum of SIVP

Summing the polarization tensors:

$$\begin{aligned} \langle G^\lambda(\tau, \vec{k}) G^s(\tau, \vec{k}') \rangle &= \frac{8\pi^5}{k^3} \delta^{\lambda s} \delta(\vec{k} + \vec{k}') \int_0^\infty dy \int_{|1-y|}^{1+y} dx \left[1 - \left(\frac{1+y^2-x^2}{2y} \right)^2 \right] \\ &\times \frac{y^2}{x^2} \left(\frac{1+y^2-x^2}{2y} \right)^2 \mathcal{P}_\zeta(ky, \tau_0) \mathcal{P}_\zeta(kx, \tau_0) \left| \int_{z_{\text{ini}}}^z d\tilde{z} \frac{a(\tilde{z})^2}{a(z)^2} f(x, y, \tilde{z}) \right|^2. \end{aligned} \quad (34)$$

The final power spectrum

$$\begin{aligned} \mathcal{P}_G(\tau, \vec{k}) &= \int_{\frac{1}{\sqrt{2}}}^\infty dt \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} ds \frac{(1-2s^2)(2t^2-1)(2st+1)^2}{4(t^2-s^2)^2} \\ &\times \mathcal{P}_\zeta\left(\frac{k}{\sqrt{2}}(t-s), \tau_0\right) \mathcal{P}_\zeta\left(\frac{k}{\sqrt{2}}(t+s), \tau_0\right) |\mathcal{I}(s, t, z)|^2, \end{aligned} \quad (35)$$

$$\mathcal{I}(s, t, z) = \int_{z_{\text{ini}}}^z d\tilde{z} \frac{a(\tilde{z})^2}{a(z)^2} f(s, t, \tilde{z}), \quad (36)$$

Where we used...

We defined

$$f(k, p, \tau) \equiv \frac{\alpha_{\vec{p}}(\tau)\alpha_{\vec{k}-\vec{p}}(\tau) + \zeta_{\vec{p}}(\tau)\zeta_{\vec{k}-\vec{p}}(\tau) - 2\zeta'_{\vec{p}}(\tau)\beta_{\vec{k}-\vec{p}}(\tau)}{\zeta_{\vec{p}}(\tau_0)\zeta_{\vec{k}-\vec{p}}(\tau_0)}. \quad (37)$$

And eliminate α and β by using Hamiltonian/Momentum constraints:

$$\alpha_{\vec{k}} = \frac{\zeta'_{\vec{k}}}{\mathcal{H}}, \quad \beta_{\vec{k}} = -\frac{\zeta_{\vec{k}}}{\mathcal{H}} - \frac{3\zeta'_{\vec{k}}}{2c_s^2 k^2}. \quad (38)$$

We also assume a Gaussian distributions of ζ :

$$\begin{aligned} \langle \zeta_{\vec{p}} \zeta_{\vec{k}-\vec{p}} \zeta_{\vec{q}} \zeta_{\vec{k}'-\vec{q}} \rangle(\tau_0) &= \langle \zeta_{\vec{p}} \zeta_{\vec{q}} \rangle(\tau_0) \langle \zeta_{\vec{k}-\vec{p}} \zeta_{\vec{k}'-\vec{q}} \rangle(\tau_0) + \langle \zeta_{\vec{p}} \zeta_{\vec{k}'-\vec{q}} \rangle(\tau_0) \langle \zeta_{\vec{k}-\vec{p}} \zeta_{\vec{q}} \rangle(\tau_0) \\ &= (2\pi)^6 \frac{2\pi^2}{p^3} \frac{2\pi^2}{|\vec{k}-\vec{p}|^3} \delta(\vec{k} + \vec{k}') \left[\delta(\vec{p} + \vec{q}) + \delta(\vec{q} + \vec{k} - \vec{p}) \right] \mathcal{P}_\zeta(\vec{p}, \tau_0) \mathcal{P}_\zeta(\vec{k} - \vec{p}, \tau_0). \end{aligned}$$

Physical meaning of vector power spectrum

For simplicity we concern on the gradient energy of vectors

$$\rho_V(\vec{x}, \tau) = \frac{1}{4a^2} \partial_i G_j(\vec{x}, \tau) \partial^i G^j(\vec{x}, \tau), \quad (39)$$

Working in Fourier space:

$$\rho_V(\tau) = 2 \times \frac{1}{4a^2} \int k \mathcal{P}_G(\tau, \vec{k}) dk = \frac{1}{2a^2} \int k \mathcal{P}_G(\tau, \vec{k}) dk. \quad (40)$$

The ratio of vector energy versus background one:

$$\delta_V \equiv \frac{\rho_V(\tau_0)}{\rho_{\text{bg}}(\tau_0)} = \frac{1}{24} \int (k_{\tau_0}) \mathcal{P}_G(k, \tau_0) d(k_{\tau_0}), \quad (41)$$

When δ_V approaches 1, perturbation theory breaks down!

Curvature fluctuation in matter bounce scenario

In the matter bounce case, $a \propto (\tau/\tau_0)^2$, we can compute

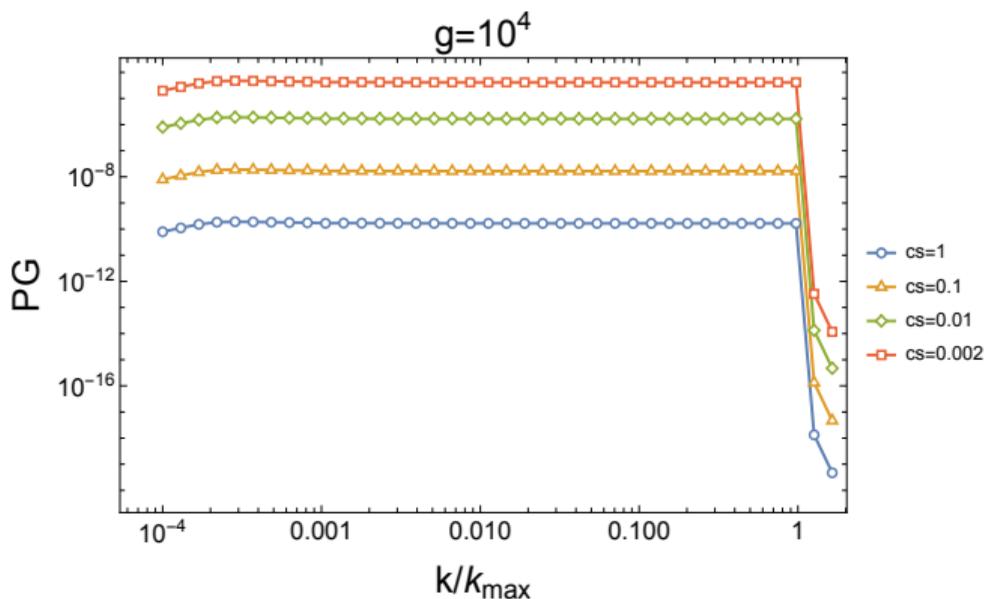
$$\zeta_k(\tau) \equiv \frac{v_k(\tau)}{z_s} = \frac{e^{-ikc_s\tau} c_s}{\sqrt{6c_s k}} \left(1 - \frac{i}{c_s k \tau}\right) \left(\frac{\tau_0}{\tau}\right)^2. \quad (42)$$

The amplitude of curvature perturbations

$$\frac{1}{12\pi^2 c_s \tau_0^2} = A_s = 2.1 \times 10^{-9}. \quad (43)$$

Vector power spectrum

$$\mathcal{P}_G(\tau_0, k) = \mathcal{A}_G \left(\frac{k}{k_{\max}} \right)^{n_G}, \quad k < k_{\max}. \quad (44)$$



Ratio of vector energy density

$$\delta_V(\tau_0) = C \mathcal{A}_s^2 \frac{g^2}{c_s^4}, \quad C \simeq 3.1 \times 10^{-2}, \quad (45)$$

Model parameters		Outcomes		
c_s	g	\mathcal{A}_G	n_G	$\delta_V(\tau_0)$
2×10^{-3}	10^4	4.1×10^{-5}	0.01	8.5×10^{-1}
10^{-2}	10^4	1.6×10^{-6}	0.01	1.3×10^{-3}
10^{-1}	10^4	1.6×10^{-8}	0.01	1.3×10^{-7}
1	10^4	1.6×10^{-10}	0.02	1.3×10^{-11}
1	10^5	1.6×10^{-8}	0.02	1.3×10^{-9}

Table: The values of δ_V for various values of c_s and g in the uniform field gauge.

Observational constraints

Consistency relation

$$r = 24c_s, \quad r < 0.044 \rightarrow c_s < 0.002. \quad (46)$$

CMB and Large scale structure observations from 10^{-4}Mpc^{-1} to $\mathcal{O}(1)\text{Mpc}^{-1}$, so $g > 10^4$.

Therefore, $\delta_V > 0.85$ in matter bounce in k-essence framework. Those models suffer from strong back-reaction from SIVP and thus theoretically inconsistent.

Conclusion and outlook

- SIVP is a powerful tool examining the robustness of non-singular bounce models
- Applications to other models?
- Gauge dependence problem?

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