The Gauged Soft Recursion Relation

Ming-Lei Xiao 肖明磊

Sun Yat-Sen University 中山大学 · 深圳

Joint work with: Ian Low and Yu-Hui Zheng [2511.09403, accepted by JHEP]

第十届海峡两岸粒子物理和宇宙学研讨会 · 广州

On-Shell Bootstrap of S-Matrix

Focus on the physical observables – **scattering amplitudes**.

- 1960s: S-matrix Program, fails to compete with Quantum Field Theory
- 2000s: BCFW recursion, simplicity in calculation (YM, SYM, SUGRA)
- 2010s: Systematic generalization (EFT, massive...), hidden structures (double copy, positivity..)

The First Principles:

- Unitarity
- Lorentz Symmetry
- Locality & Causality

Outline

Outline

Recursion Relations

1. Complex momentum shift: $p_i \to \hat{p}_i(z)$

$$\hat{p}_i(0) = p_i\,, \quad \hat{p}_i(z)^2 = p_i^2\,, \quad \sum_i \hat{p}_i(z) = 0\,.$$

Recursion Relations

1. Complex momentum shift: $p_i \to \hat{p}_i(z)$

$$\hat{p}_i(0) = p_i\,, \quad \hat{p}_i(z)^2 = p_i^2\,, \quad \sum_i \hat{p}_i(z) = 0\,.$$

2. Locality & Causality

$$\begin{aligned}\mathcal{A}_{\text{phy}} &= \frac{1}{2\pi \mathrm{i}} \oint_{z=0} \frac{\hat{\mathcal{A}}(z)}{z} \\ &= -\sum_{\substack{\text{physical} \\ z\text{-poles}}} \operatorname{Res} \frac{\hat{\mathcal{A}}(z)}{z} + C_\infty\end{aligned}$$

Recursion Relations

1. Complex momentum shift: $p_i \to \hat{p}_i(z)$

$$\hat{p}_i(0) = p_i\,, \quad \hat{p}_i(z)^2 = p_i^2\,, \quad \sum_i \hat{p}_i(z) = 0\,.$$

2. Locality & Causality

3. Unitarity: Factorization

$$\begin{aligned} \mathcal{A}_{\text{phy}} &= \frac{1}{2\pi\text{i}} \oint_{z=0} \frac{\hat{\mathcal{A}}(z)}{z} \\ &= -\sum_{\substack{\text{physical} \\ z\text{-poles}}} \text{Res} \frac{\hat{\mathcal{A}}_{\text{L}}(z) \times \hat{\mathcal{A}}_{\text{R}}(z)}{z \hat{P}(z)^2} + C_\infty \end{aligned}$$

Recursion Relations

1. Complex momentum shift: $p_i \to \hat{p}_i(z)$

$$\hat{p}_i(0) = p_i\,, \quad \hat{p}_i(z)^2 = p_i^2\,, \quad \sum_i \hat{p}_i(z) = 0\,.$$

2. Locality & Causality

3. Unitarity: Factorization

$$\mathcal{A}_{\text{phy}} = \frac{1}{2\pi \mathrm{i}} \oint_{z=0} \frac{\hat{\mathcal{A}}(z)}{z}$$

$$= -\sum_{\substack{\text{physical} \\ z\text{-poles}}} \text{Res}\, \frac{\hat{\mathcal{A}}_{\text{L}}(z) \times \hat{\mathcal{A}}_{\text{R}}(z)}{z \hat{P}(z)^2} + C_\infty$$

Key Properties and Benefits

- Apply first principles without off-shell Feynman diagrams
- Reduction of combinatorial complexity
- Simplicity of on-shell object

$$\mathcal{A}_{\text{MHV}} = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

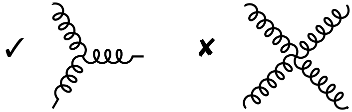
- Rely on the vanishing of C_∞

On-Shell Constructibility

A theory is **On-Shell Constructible** $\Leftrightarrow$ $C_\infty = 0$

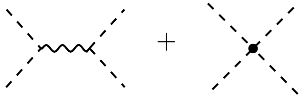
- The recursive construction depends on residues on the physical poles.
- Contributions without poles – contact terms – should be included as seed amplitudes.

Each seed amplitude must respect the symmetries:



- If factorizable contributions and contact terms coexist — a sign of inconstructibility.

Scalar QED: $A_4 =$



Effective Field Theories (EFT) are usually not on-shell constructible.

Non-Linear Symmetry and Goldstone Bosons

Spontaneous Symmetry Breaking is an important phenomenon in nature.

- ✘ The symmetry is **broken**
- ✓ The symmetry is **non-linearly realized** (shift symmetry)

The impact of **non-linear symmetry**: $\partial^\mu U \partial_\mu U \Rightarrow \{\pi^4\partial^2, \pi^6\partial^2, \dots\}$

Higher point contact terms are NOT independent contributions, just like A^4 in Yang-Mills!

Adler's Zero Condition

$$\lim_{p\to 0} \mathcal{M}(\dots, \pi(p), \dots) = 0$$

Soft Recursion Relation [C. Cheung, *et.al.*, 2016]

How to impose Adler's Zero condition in the recursion relation?

Soft Recursion Relation [C. Cheung, *et.al.*, 2016]

How to impose Adler's Zero condition in the recursion relation?

Key Steps:

1. All-line shift: $p_i \to \hat{p}_i = (1 - a_i z) p_i$
2. Divide by the scaling factors: $F(z) = \prod_i (1 - a_i z)$

$$\mathcal{A}_{\text{phy}} = \hat{\mathcal{A}}(z=0) = \frac{1}{2\pi \mathrm{i}} \oint_{z=0} \frac{\hat{\mathcal{A}}(z)}{z F(z)}$$

Soft Recursion Relation [C. Cheung, *et.al.*, 2016]

How to impose Adler's Zero condition in the recursion relation?

Key Steps:

1. All-line shift: $p_i \to \hat{p}_i = (1 - a_i z) p_i$
2. Divide by the scaling factors: $F(z) = \prod_i (1 - a_i z)$

$$\mathcal{A}_{\text{phy}} = \hat{\mathcal{A}}(z=0) = \frac{1}{2\pi \mathrm{i}} \oint_{z=0} \frac{\hat{\mathcal{A}}(z)}{z F(z)}$$

- $F(z)$ in the denominator suppresses the large-z behavior.
- Amplitudes not satisfying Adler's Zero condition have extra poles from $F(z)$.
- Condition for $C_\infty = 0$: $\lim_{z\to\infty} \hat{\mathcal{A}}(z) < F(z) \sim z^n$

Soft Recursion Relation [C. Cheung, *et.al.*, 2016]

How to impose Adler's Zero condition in the recursion relation?

Key Steps:

1. All-line shift: $p_i \to \hat{p}_i = (1 - a_i z) p_i$
2. Divide by the scaling factors: $F(z) = \prod_i (1 - a_i z)$

$$\mathcal{A}_{\rm phy} = \hat{\mathcal{A}}(z=0) = \frac{1}{2\pi \mathrm{i}} \oint_{z=0} \frac{\hat{\mathcal{A}}(z)}{z F(z)}$$

- $F(z)$ in the denominator suppresses the large-z behavior.
- Amplitudes not satisfying Adler's Zero condition have extra poles from $F(z)$.
- Condition for $C_\infty = 0$: $\lim_{z\to\infty} \hat{\mathcal{A}}(z) < F(z) \sim z^n$

NLSM is on-shell constructible from $\mathcal{A}_4 \sim 1/f^2$

Gauge Interaction of Goldstone Bosons

Goldstone bosons $\pi^a \in G/H$ and form representation of the unbroken group H

- If G/H is gauged $\Rightarrow$ Higgs mechanism
- If H is (partially) gauged $\Rightarrow$ **charged** Pseudo-Goldstone $(\pi^{\pm})$

$$\lim_{p\to 0} \mathcal{M}(\ldots, \pi^{\pm}(p), \ldots; \gamma) \neq 0$$

Adler's Zero condition is lost, is the theory still on-shell constructible?

Outline

Weinberg's and Low's Soft Photon Theorem

- Weinberg's Soft Photon Theorem (multiplicative, universal):

$$\lim_{k\to 0}\mathcal{M}_{n+1}(\ldots;\gamma^{\pm}(k)) \simeq S^{(0)}\mathcal{M}_n \equiv \sum_i q_i \frac{\varepsilon_{\pm}\cdot p_i}{k\cdot p_i}\mathcal{M}_n \sim O(k^{-1})$$

Weinberg's and Low's Soft Photon Theorem

- Weinberg's Soft Photon Theorem (multiplicative, universal):

$$\lim_{k\to 0} \mathcal{M}_{n+1}(\ldots;\gamma^{\pm}(k)) \simeq S^{(0)}\mathcal{M}_n \equiv \sum_i q_i \frac{\varepsilon_{\pm}\cdot p_i}{k\cdot p_i}\mathcal{M}_n \sim O(k^{-1})$$

- Next to Leading Order Soft Theorem (derivative, spin-sensitive):

$$S^{(1)}\mathcal{M}_n \equiv \sum_i q_i \frac{k_\mu \epsilon_\nu}{k\cdot p_i}\hat{J}_i^{\mu\nu}\mathcal{M}_n \sim O(k^0)$$

Weinberg's and Low's Soft Photon Theorem

- Weinberg's Soft Photon Theorem (multiplicative, universal):

$$\lim_{k\to 0}\mathcal{M}_{n+1}(\ldots;\gamma^{\pm}(k))\simeq S^{(0)}\mathcal{M}_n\equiv\sum_i q_i\frac{\varepsilon_{\pm}\cdot p_i}{k\cdot p_i}\mathcal{M}_n\sim O(k^{-1})$$

- Next to Leading Order Soft Theorem (derivative, spin-sensitive):

$$S^{(1)}\mathcal{M}_n\equiv\sum_i q_i\frac{k_\mu\epsilon_\nu}{k\cdot p_i}\hat{J}_i^{\mu\nu}\mathcal{M}_n\sim O(k^0)$$

Infrared property of scattering amplitude

Soft particles	**Symmetry**	**Infrared property**
Goldstone bosons	Non-linear Sym.	Adler's Zero
gauge bosons	Asym.Gauge Sym.	Soft Theorem

Probing the Soft Limit

The soft photon limit can also be probed by the shift $k \to \hat{k} = (1 - a_k z) k$

$$\operatorname*{Res}_{z=1/a_k} \frac{\hat{\mathcal{M}}_{n+1}(z)}{z} = \operatorname*{Res}_{z=1/a_k} \left[\frac{S^{(0)} \hat{\mathcal{M}}_n(z)}{z(1 - a_k z)} \right]$$

Probing the Soft Limit

The soft photon limit can also be probed by the shift $k \to \hat{k} = (1 - a_k z)k$

$$\underset{z=1/a_k}{\text{Res}} \frac{\hat{\mathcal{M}}_{n+1}(z)}{z(1-a_k z)} = \underset{z=1/a_k}{\text{Res}} \left[\frac{S^{(0)} \hat{\mathcal{M}}_n(z)}{z(1-a_k z)^2} + \frac{S^{(1)} \hat{\mathcal{M}}_n(z)}{z(1-a_k z)} \right]$$

Probing the Soft Limit

The soft photon limit can also be probed by the shift $k \to \hat{k} = (1 - a_k z)k$

$$\operatorname*{Res}_{z=1/a_k} \frac{\hat{\mathcal{M}}_{n+1}(z)}{z(1-a_k z)} = \operatorname*{Res}_{z=1/a_k} \left[\frac{S^{(0)} \hat{\mathcal{M}}_n(z)}{z(1-a_k z)^2} + \frac{S^{(1)} \hat{\mathcal{M}}_n(z)}{z(1-a_k z)} \right]$$

1. Global Symmetries $\Rightarrow$ amplitude seeds (Soft Blocks [Low&Yin 2019])
2. Locality + Causality $\Rightarrow$ Cauchy's theorem
3. Unitarity $\Rightarrow$ **residue on hard poles**
4. Infrared $\Rightarrow$ **residue on soft poles** $\begin{cases} \text{Goldstone bosons: Adler's Zero} \\ \text{Gauge bosons: Soft Theorem} \end{cases}$

Probing the Soft Limit

The soft photon limit can also be probed by the shift $k \to \hat{k} = (1 - a_k z) k$

$$\underset{z=1/a_k}{\text{Res}} \frac{\hat{\mathcal{M}}_{n+1}(z)}{z(1-a_k z)} = \underset{z=1/a_k}{\text{Res}} \left[\frac{S^{(0)} \hat{\mathcal{M}}_n(z)}{z(1-a_k z)^2} + \frac{S^{(1)} \hat{\mathcal{M}}_n(z)}{z(1-a_k z)} \right]$$

1. Global Symmetries $\Rightarrow$ amplitude seeds (Soft Blocks [Low&Yin 2019])
2. Locality + Causality $\Rightarrow$ Cauchy's theorem
3. Unitarity $\Rightarrow$ **residue on hard poles**
4. Infrared $\Rightarrow$ **residue on soft poles** $\begin{cases} \text{Soft Recursion Relation} \\ \text{Gauged Soft Recursion Relation} \end{cases}$

Example: $\mathcal{M}_{4+1}$

$\mathcal{M}_{4+1}(\pi_1,\pi_2,\pi_3,\pi_4;\gamma(k))$ only has a soft pole at $k\to 0$:

$$\begin{aligned}\mathcal{M}_{4+1} &= -\operatorname*{Res}_{z=1/a_k}\left[\frac{S^{(0)}\hat{\mathcal{M}}_4(z)}{z(1-a_k z)^2}+\frac{S^{(1)}\hat{\mathcal{M}}_4(z)}{z(1-a_k z)}\right]\\ &= \operatorname*{Res}_{z=0}\left[\frac{S^{(0)}\hat{\mathcal{M}}_4(z)}{z(1-a_k z)^2}+\frac{S^{(1)}\hat{\mathcal{M}}_4(z)}{z(1-a_k z)}\right]\\ &= \left(S^{(0)}+S^{(1)}\right)\mathcal{M}_4\end{aligned}$$

Example: $\mathcal{M}_{4+1}$

$\mathcal{M}_{4+1}(\pi_1,\pi_2,\pi_3,\pi_4;\gamma(k))$ only has a soft pole at $k\to 0$:

$$\begin{aligned}\mathcal{M}_{4+1} &= -\operatorname*{Res}_{z=1/a_k}\left[\frac{S^{(0)}\hat{\mathcal{M}}_4(z)}{z(1-a_kz)^2}+\frac{S^{(1)}\hat{\mathcal{M}}_4(z)}{z(1-a_kz)}\right]\\ &= \operatorname*{Res}_{z=0}\left[\frac{S^{(0)}\hat{\mathcal{M}}_4(z)}{z(1-a_kz)^2}+\frac{S^{(1)}\hat{\mathcal{M}}_4(z)}{z(1-a_kz)}\right]\\ &= \left(S^{(0)}+S^{(1)}\right)\mathcal{M}_4\end{aligned}$$

For example: $\mathcal{M}_{4+1}(\pi^+,\pi^-,\pi^0,\pi^0;\gamma^+) = \dfrac{e\langle 12\rangle}{\langle 15\rangle\langle 25\rangle}s_{34}$

- $\lim_{p_3,p_4\to 0}\mathcal{M}_{4+1}=0$ **neutral scalars still satisfy Adler's Zero**
- $\lim_{p_1,p_2\to 0}\mathcal{M}_{4+1}\neq 0$ **charged scalars do not satisfy Adler's Zero**

Generalization: $\mathcal{M}_{n+l}$

For $n \geq 6$, $\mathcal{M}_{n+l}$ has both soft poles $k_s \to 0$ and hard poles:

$$\mathcal{M}_{n+l} = \frac{1}{2\pi \mathrm{i}} \oint_{z=0} \frac{\hat{\mathcal{M}}_{n+l}(z)}{zF(z)} \qquad F(z) = \prod_i (1 - a_i z)$$

$$= -\sum_{s=1}^{l} \operatorname*{Res}_{z=1/a_s} \left[\frac{S_s^{(0)} \hat{\mathcal{M}}_{n+l-1}(z)}{zF(z)(1-a_s z)} + \frac{S_s^{(1)} \hat{\mathcal{M}}_{n+l-1}(z)}{zF(z)} \right] - \sum_I \operatorname*{Res}_{z=z_I^\pm} \frac{\hat{\mathcal{M}}^{I\mathrm{L}}(z) \times \hat{\mathcal{M}}^{I\mathrm{R}}(z)}{zF(z)\hat{P}_I(z)^2}$$

Generalization: $\mathcal{M}_{n+l}$

For $n \geq 6$, $\mathcal{M}_{n+l}$ has both soft poles $k_s \to 0$ and hard poles:

$$\mathcal{M}_{n+l} = \frac{1}{2\pi \mathrm{i}} \oint_{z=0} \frac{\hat{\mathcal{M}}_{n+l}(z)}{zF(z)} \qquad F(z)=\prod_i (1-a_i z)$$

$$= -\sum_{s=1}^{l} \operatorname*{Res}_{z=1/a_s} \left[\frac{S_s^{(0)} \hat{\mathcal{M}}_{n+l-1}(z)}{zF(z)(1-a_s z)} + \frac{S_s^{(1)} \hat{\mathcal{M}}_{n+l-1}(z)}{zF(z)} \right] - \sum_I \operatorname*{Res}_{z=z_I^{\pm}} \frac{\hat{\mathcal{M}}^{I\mathrm{L}}(z) \times \hat{\mathcal{M}}^{I\mathrm{R}}(z)}{zF(z)\hat{P}_I(z)^2}$$

How to choose $F(z)$?

- Large-z behavior should be able to suppress C_∞.
- May contain $(1-a_s z)$ from the gauge bosons.
- May NOT contain $(1-a_i z)$ from the charged scalars.

Outline

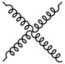
Simplicity of Gauge Theory

Why is on-shell method particularly powerful for gauge theories (and gravity)?

- The contribution from 4-point vertex  is automatically included.

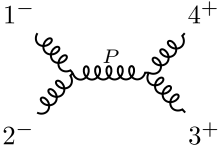
Simplicity of Gauge Theory

Why is on-shell method particularly powerful for gauge theories (and gravity)?

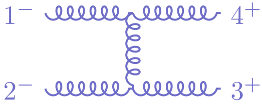
- The contribution from 4-point vertex  is automatically included.

- **Some channels are recovered from others (Why?)**

$\mathcal{A}_{\rm MHV}(1^-,2^-,3^+,4^+) =$



$+$

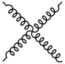


invisible

(shift $[1,4\rangle$) $$= \frac{\langle 12\rangle^3}{\langle 1P\rangle\langle 2P\rangle} \times \frac{1}{P^2} \times \frac{[34]^3}{[3P][4P]} = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

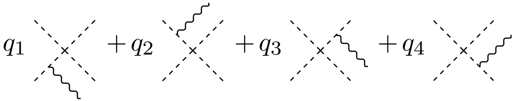
Simplicity of Gauge Theory

Why is on-shell method particularly powerful for gauge theories (and gravity)?

- The contribution from 4-point vertex  is automatically included.
- Some channels are recovered from others **(Why?)**
- The factorization channels are **not independent**:
  - One cannot have one factorization channel without the others.
  - Only the combination is **gauge invariant**.
  - We may construct **minimal** gauge invariant amplitude components.

Independent Amplitude Components

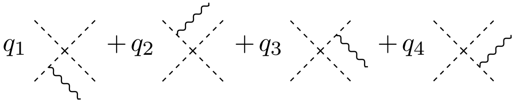
Diagrams with **minimally coupled** gauge boson attaching to different charges are not independent.



Why? charge conservation $q_1 + q_2 + q_3 + q_4 = 0$

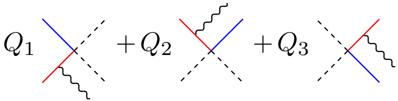
Independent Amplitude Components

Diagrams with **minimally coupled** gauge boson attaching to different charges are not independent.



Why? charge conservation $q_1 + q_2 + q_3 + q_4 = 0$

Some combinations of diagrams are gauge invariant.



- **Independent charges:** $Q_1 = q_1$, $Q_2 = q_1 + q_2$, $Q_3 = q_1 + q_2 + q_3$.
- Each component is **gauge invariant** $\mathcal{A}_i \equiv \partial \mathcal{M} / \partial Q_i$

Charge Decomposition

The amplitude with generic charges $\{q_i\}$ can be decomposed as

$$\mathcal{M}_{n+1}(1^{q_1}, 2^{q_2}, \ldots, n^{q_n}; \gamma) = \sum_{i=1}^{n-1} Q_i \mathcal{A}_i(\ldots, i^{+1}, (i+1)^{-1}, \ldots; \gamma)$$

- We can individually compute $\mathcal{A}_i$ via recursion relation.
- Each of them has $n-2$ **neutral scalars** with Adler's Zero condition!
- Counting power of z under the shift $\hat{p}_j = (1 - a_j z) p_j$:

$$\lim_{z \to \infty} \hat{\mathcal{M}}_{n+1}(z) \sim z^{m-1} , \quad \lim_{z \to \infty} F(z) \sim z^{(n-2)+1} ,$$

On-Shell Constructible $\iff$ $m < n$ $(m = 2$ in NLSM$)$

Example: $\mathcal{M}_{6+1}$

1. Charge Decomposition:

$$\mathcal{M}_{6+1}(1^{q_1},2^{q_2},\ldots,6^{q_n};\gamma)=\sum_{i=1}^{5}Q_i\mathcal{A}_i(\ldots,i^{+1},(i+1)^{-1},\ldots;\gamma)$$

2. Each component is determined by its soft pole and hard poles:

$$\mathcal{A}_1=\frac{1}{2\pi\mathrm{i}}\oint_{z=0}\frac{\hat{\mathcal{A}}_1(z)}{zF(z)} \qquad F(z)=\prod_{i=3,4,5,6}(1-a_i z)$$

$$=-\operatorname*{Res}_{z=1/a_7}\frac{S^{(0)}\hat{\mathcal{M}}_6(z)}{zF(z)(1-a_7z)}-\sum_I\operatorname*{Res}_{z=z_I^{\pm}}\frac{\hat{\mathcal{M}}_4^I(z)\times\hat{\mathcal{M}}_{4+1}^I(z)}{zF(z)\hat{P}_I(z)^2}$$

can be constructed recursively!

Outline

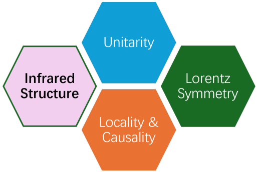
Summary: On-Shell Construction of EFT

- The on-shell methods for EFTs have practical and theoretical significance.
  1. Simplify the calculation in EFT.
  2. Expand the scope of application of the on-shell methods.
  3. Improve the understanding of the scattering amplitudes.

Summary: On-Shell Construction of EFT

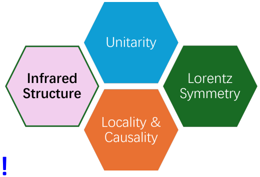
- The on-shell methods for EFTs have practical and theoretical significance.
  1. Simplify the calculation in EFT.
  2. Expand the scope of application of the on-shell methods.
  3. Improve the understanding of the scattering amplitudes.

- A thorough mastery of the infrared structure of scattering amplitudes is essential for the S-matrix program.
  1. Adler's Zero condition $\longrightarrow$ Soft Recursion Relation
  2. Soft photon theorems $\longrightarrow$ **Gauged** Soft Recursion Relation



Summary: On-Shell Construction of EFT

- The on-shell methods for EFTs have practical and theoretical significance.
  1. Simplify the calculation in EFT.
  2. Expand the scope of application of the on-shell methods.
  3. Improve the understanding of the scattering amplitudes.
- A thorough mastery of the infrared structure of scattering amplitudes is essential for the S-matrix program.
  1. Adler's Zero condition $\longrightarrow$ Soft Recursion Relation
  2. Soft photon theorems $\longrightarrow$ **Gauged** Soft Recursion Relation



Thank you for your attention!