Simulating scalar-induced gravitational waves

Xiang-Xi Zeng

- Scalar-induced gravitational waves with non-Gaussianity up to all orders, arXiv: 2508.10812, **X-X Zeng**, Z Ning, R-G Cai, and S-J Wang.
- Probing the Primordial Universe: Scalar-Induced Gravitational Waves Including Isocurvature Perturbations with Lattice Simulations, arXiv: 2510.xxxxx, **X-X Zeng**.

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Contents

Contents

What is SIGW?

Starting from the conformal Newtonian gauge

Up to second order, this is all we need

$$\mathrm{d}s^2 = a(\tau)^2 \left\{ -(1+2\Phi)\mathrm{d}\tau^2 + \left[(1-2\Phi)\delta_{ij} + \frac{1}{2}h_{ij} \right] \mathrm{d}x^i \mathrm{d}x^j \right\}$$

$$\Downarrow$$

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = -4S^{\mathrm{TT}}_{ij}$$

$$S_{ij} = 4\Phi\partial_i\partial_j\Phi + 2\partial_i\Phi\partial_j\Phi - \frac{4}{3(1+\omega)\mathcal{H}^2}\partial_i(\Phi' + \mathcal{H}\Phi)\partial_j(\Phi' + \mathcal{H}\Phi)$$

$$\Phi'' + 3\mathcal{H}(1+c_s^2)\Phi' + (2\mathcal{H}' + (1+3c_s^2)\mathcal{H}^2 - c_s^2\nabla^2)\Phi = 0$$

for adiabatic perturbation

After doing the Fourier transformation, you will find

$$\langle h_{ij}h_{ij}\rangle \sim \int\int d^3q d^3k \langle\zeta\zeta\zeta\zeta\rangle$$

For adiabatic initial condition

$$\Phi_i = \frac{3+3\omega}{5+3\omega}\zeta_i$$

How to deal with the four-point correlation function?

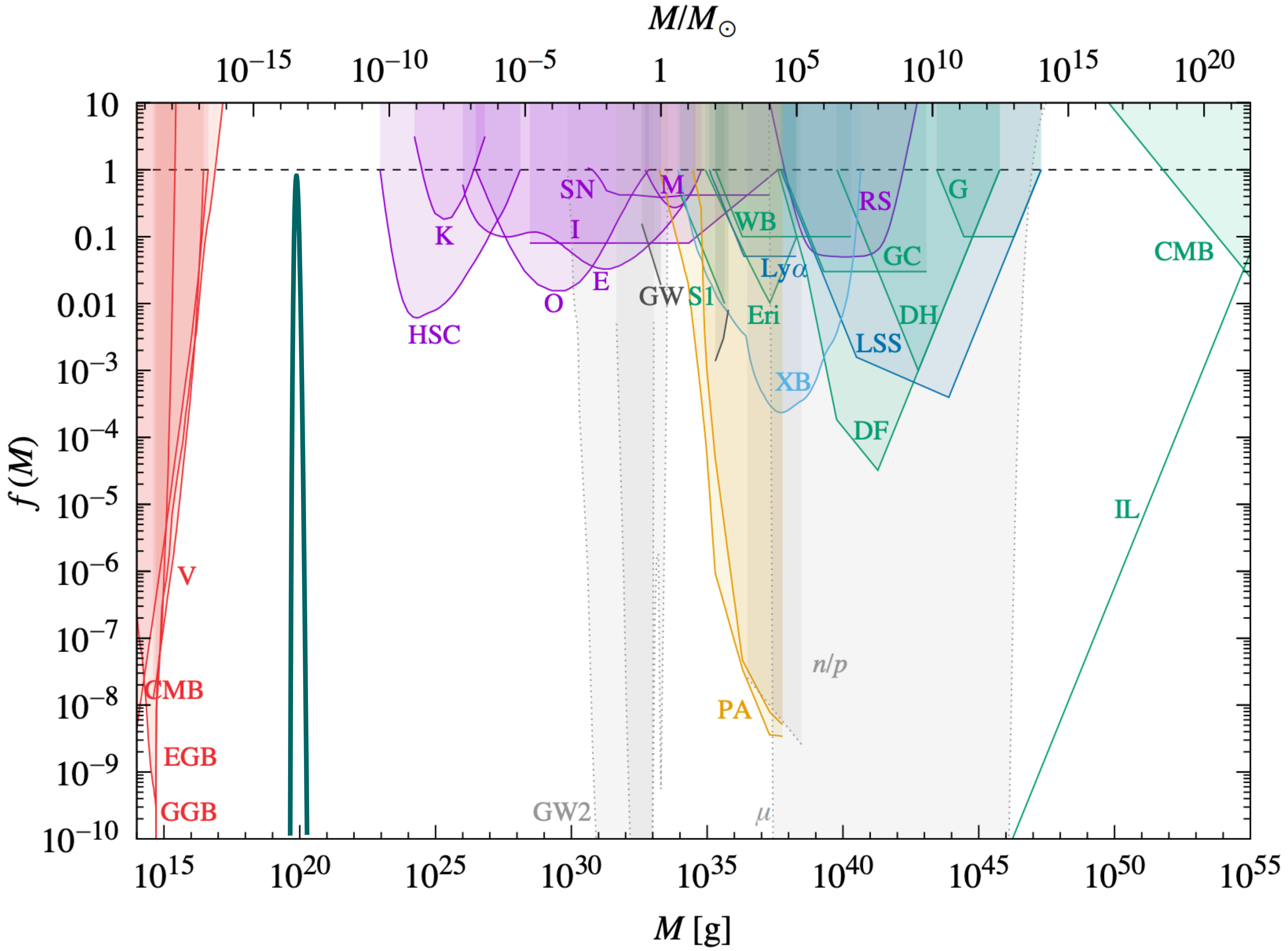
Typically $$\langle\zeta\zeta\zeta\zeta\rangle = \int D\zeta \mathscr{P}[\zeta]\zeta\zeta\zeta\zeta$$

For local type field $$\zeta = \zeta_g + F_{\rm NL}\zeta_g^2 + G_{\rm NL}\zeta_g^3 + H_{\rm NL}\zeta_g^4 + \dots$$

⇓

$$\zeta = f(\zeta_g) \qquad \langle\zeta\zeta\zeta\zeta\rangle = \langle(\zeta_g + F_{\rm NL}\zeta_g^2 + \dots)^4\rangle$$

Here is a problem



B. Carr, K. Kohri, Y. Sendouda and J. Yokoyama, Rept. Prog. Phys. 2002.12778

For a peak-like (monochromatic PBH) power spectrum to explain the all dark matter

$$\mathcal{P}_{\zeta_g}(k) = A\frac{(k/k_*)^3}{\sqrt{2\pi}e}\exp\left[-\frac{(k/k_*-1)^2}{2e^2}\right]$$

⇓

$$A \sim O(0.01)$$

⇓

$$\zeta_g \sim O(0.1)$$

⇓ Seems not good

$$\zeta = 0.1 + 5\times 0.01 + 3\times 0.001 + \dots$$

Contents

2 Simulation setup

$$\zeta(x) = f(\zeta_g(x))$$

On superhorizon

Generating the initial $\zeta_g(x)$ → Generating the initial $\zeta(x)$ → Eloving the equations → Deriving the GW energy spectrum

$$\zeta_g(\vec{x}) = \int \frac{\mathrm{d}^3 x}{(2\pi)^3} \zeta_g(\vec{k}) e^{i\vec{k}\cdot\vec{x}}$$

$$\mathscr{P}[\zeta_g(\vec{k})] = \frac{1}{\pi\sigma_k^2} \exp\left[-\frac{|\zeta_g(\vec{k})|^2}{\sigma_k^2}\right] \longleftarrow \langle \zeta_g(\vec{k})\zeta_g(\vec{q})\rangle = \sigma_k^2 \delta^3(\vec{k}+\vec{q}) \longleftarrow \sigma_k^2 = (2\pi)^3 P_{\zeta_g}(k)$$

2 Simulation setup

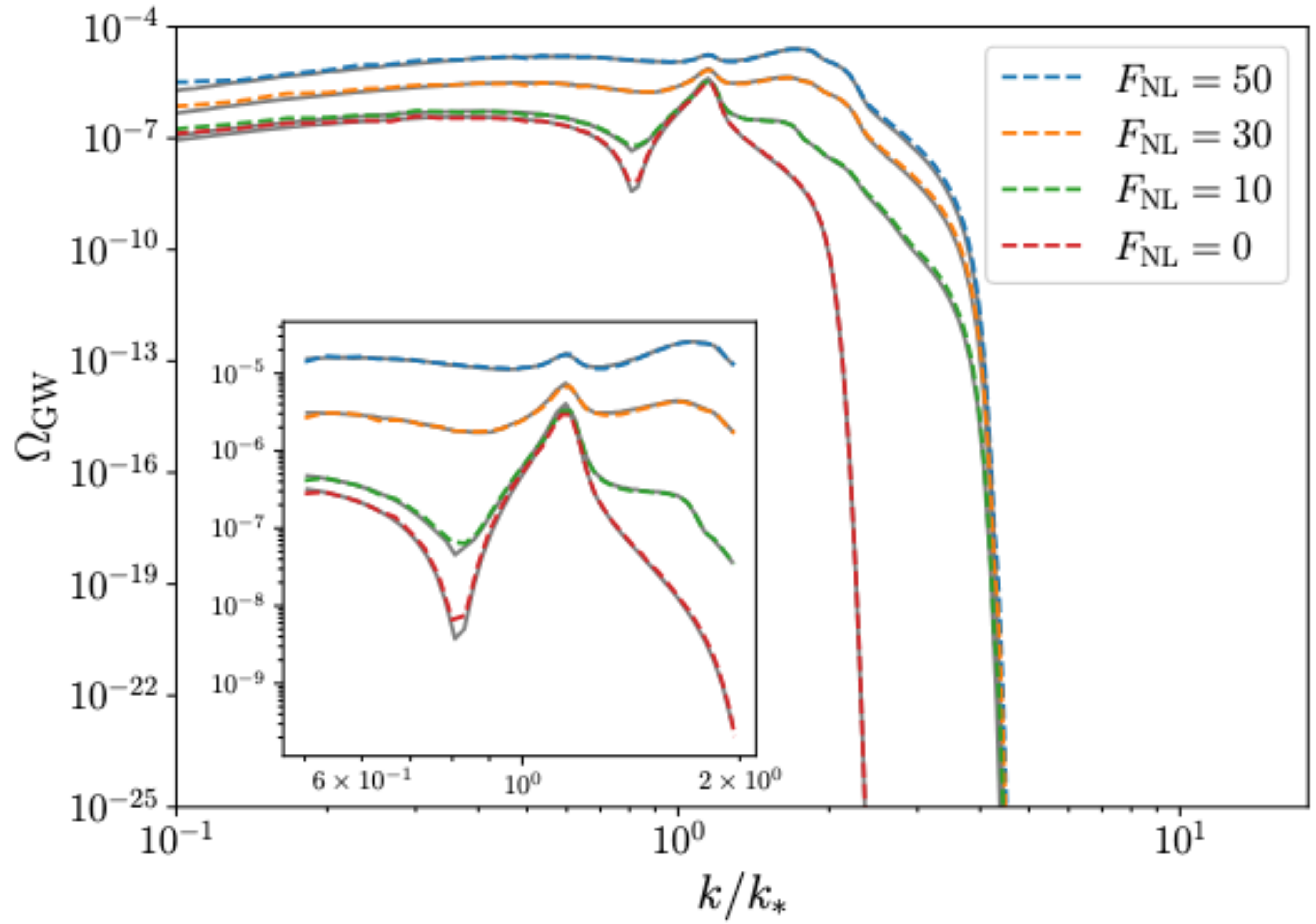
Benchmark with the semi-analytical results

$$\mathcal{P}_{\zeta_g}(k) = A\frac{(k/k_*)^3}{\sqrt{2\pi}e}\exp\left[-\frac{(k/k_*-1)^2}{2e^2}\right]$$

$A = 0.01,\ e = 1/30$

Semi-analytical results

Simulation results



Two results are consistent with each other!

The efficiency of the lattice simulation

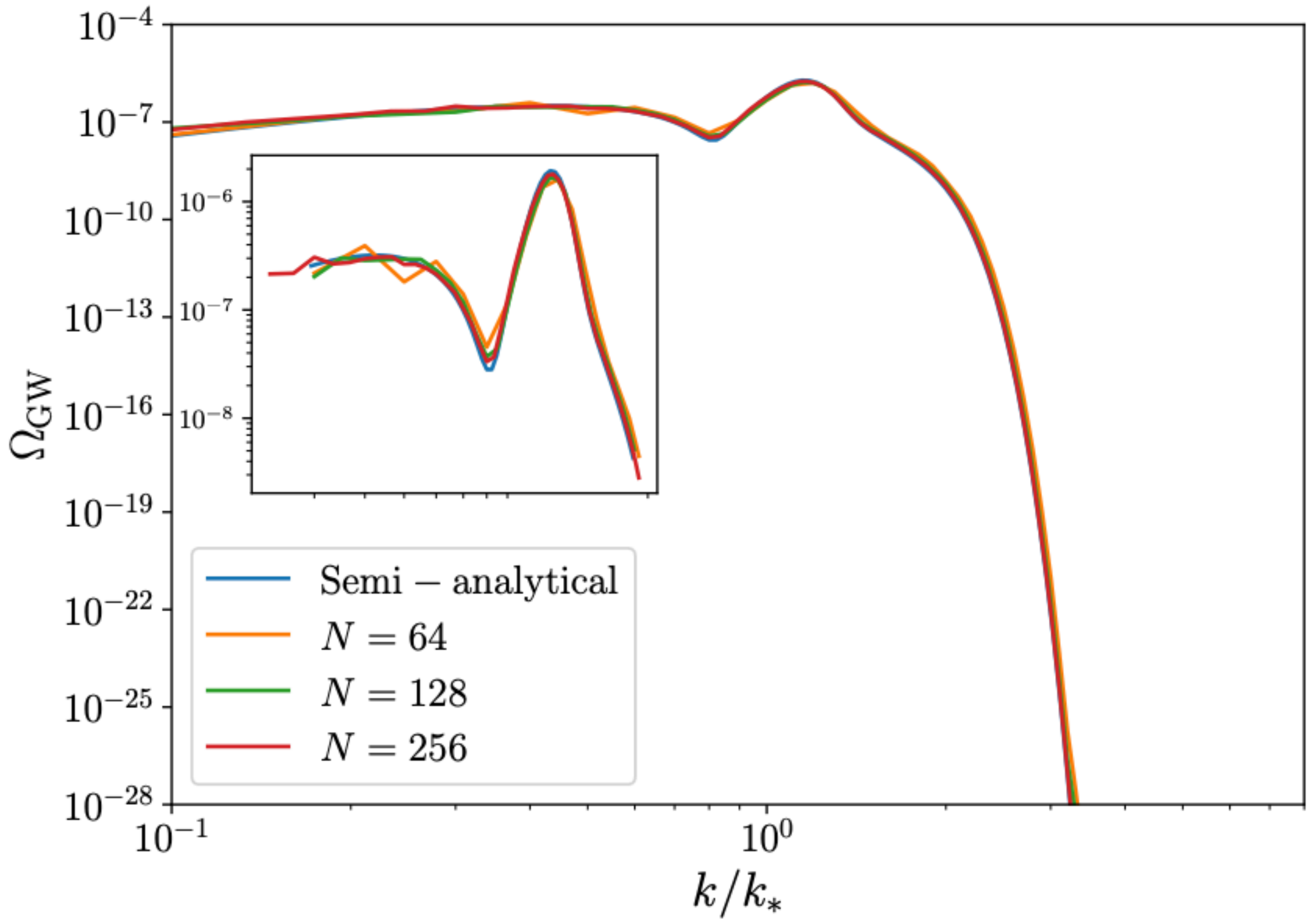
$$\langle h_{ij} h_{ij} \rangle \sim \int\int d^3q d^3k d^3l \langle (\zeta_g + F_{\rm NL}\zeta_g^2)(\zeta_g + F_{\rm NL}\zeta_g^2)(\zeta_g + F_{\rm NL}\zeta_g^2)(\zeta_g + F_{\rm NL}\zeta_g^2) \rangle$$

↓

Need to do 8-multiple integral

Semi-analytical results 14-core i9-10th 200 points 1day

Simulation results RTX-4060 N=128 6 mins



Contents

3 Full-order vs finite order

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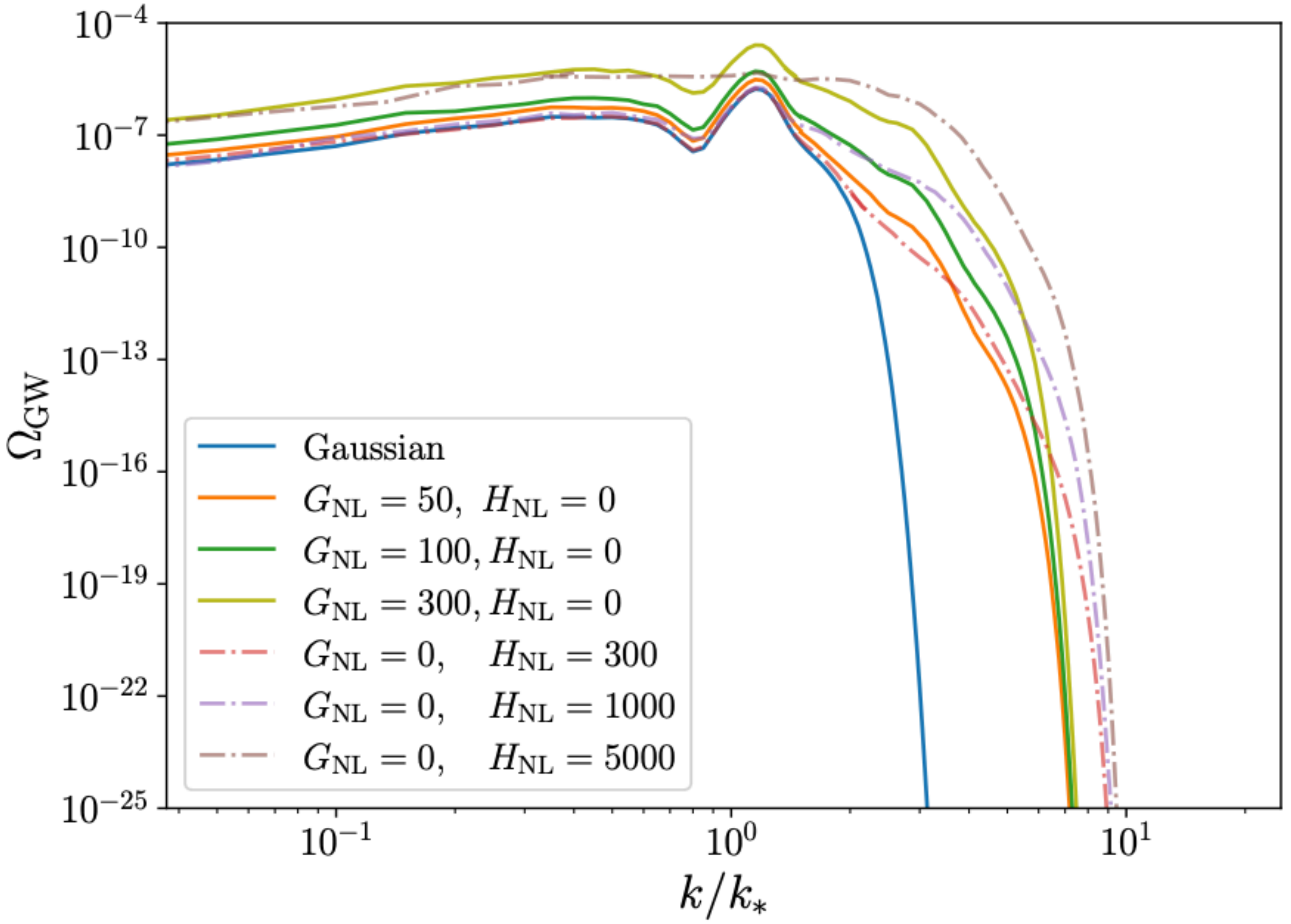
Higher order effects

$$\zeta = \zeta_g + G_{\rm NL}\zeta_g^3 + H_{\rm NL}(\zeta_g^4 - 3\langle\zeta_g^2\rangle^2)$$



The higher order usually contributes to a higher cutoff

↓ Imply

Full-order will not have such a cutoff

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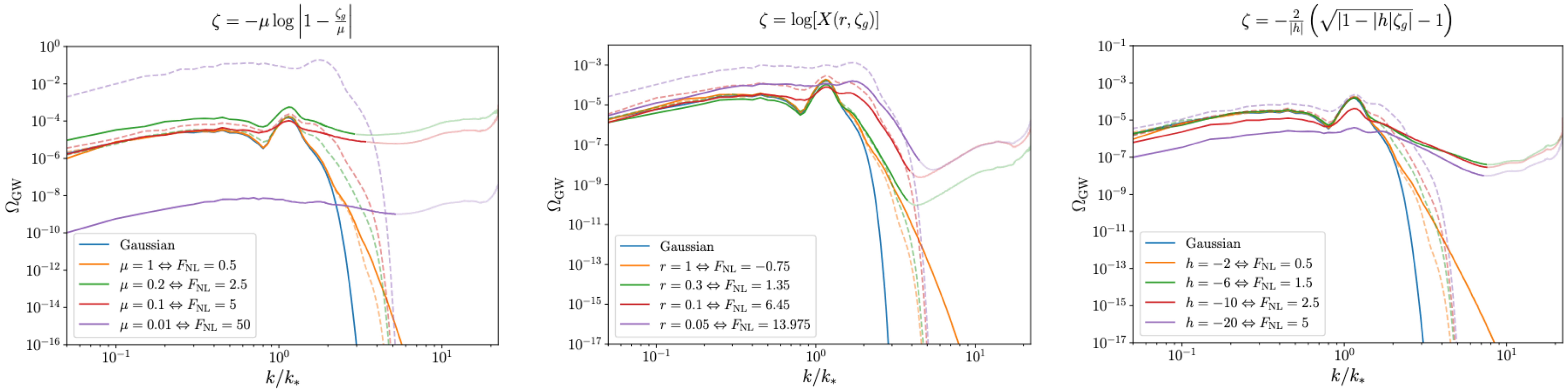
Specific Models

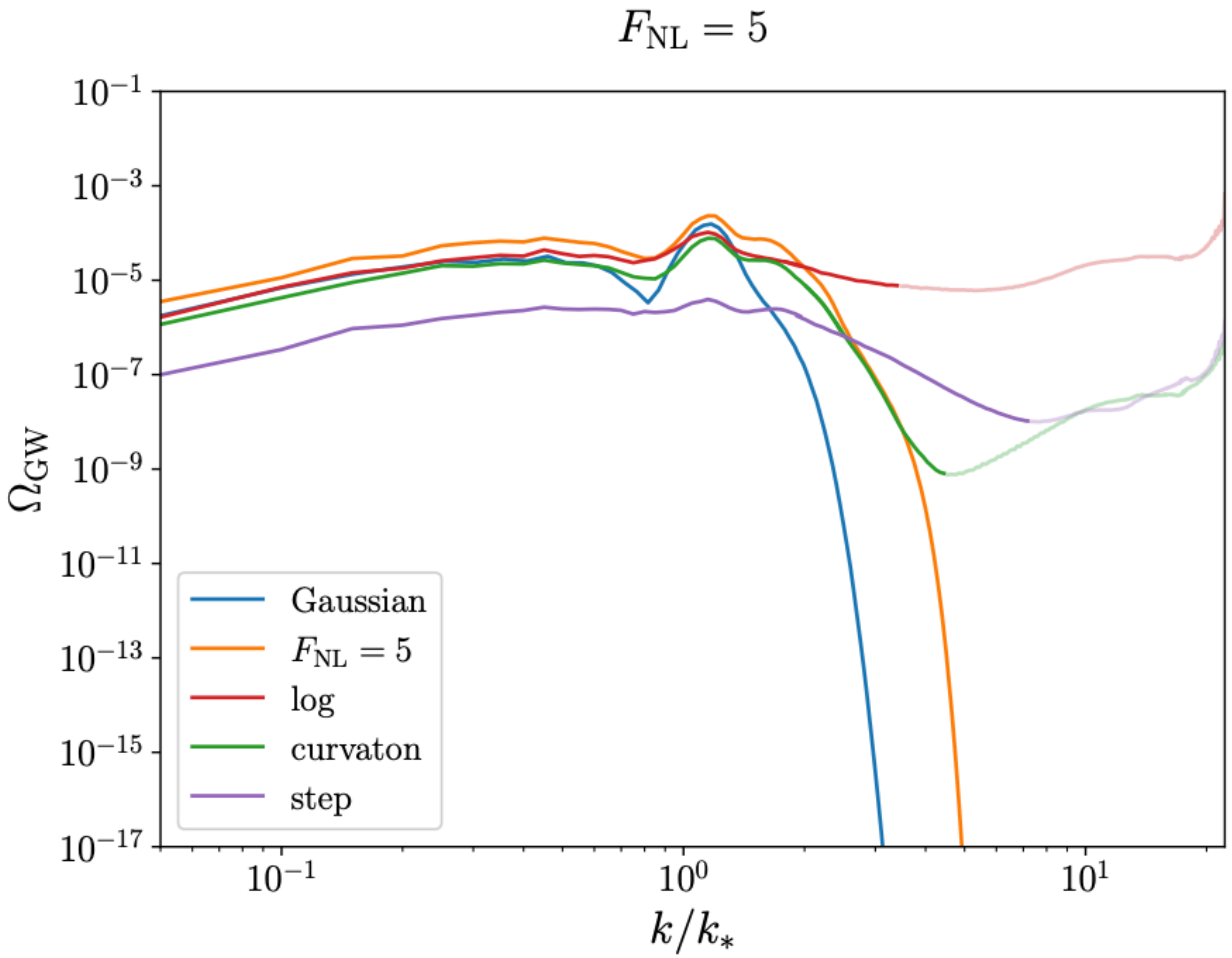


Curvaton model

Full-order

Finite order ($F_{\rm NL}$)

3 Full-order vs finite order



Maybe it's possible to distinguish inflation models by SIGW!

Contents

Equations

At background level

$$3\mathcal{H}^2 = a^2(\rho_m + \rho_r),$$
$$3(\mathcal{H}^2 + 2\mathcal{H}') = -a^2\rho_r,$$
$$\rho_m' + 3\mathcal{H}\rho_m = -aQ,$$
$$\rho_r' + 4\mathcal{H}\rho_r = aQ.$$
$$aQ \equiv \rho_m a\Gamma,$$
$$\Gamma = \frac{n}{t_{\rm eva} - t},$$

At second-order

$$h_{ij}'' + 2\mathcal{H}h_{ij}' - \Delta h_{ij} = T_{ij}{}^{lm}S_{lm},$$
$$S_{ij} = 4\partial_i\Phi\partial_j\Phi + 2a^2\left(\rho_m\partial_i v_m\partial_j v_m + \frac{4}{3}\rho_r\partial_i v_r\partial_j v_r\right)$$

At first-order

$$6\mathcal{H}\Phi' + 6\mathcal{H}^2\Phi - 2\Delta\Phi = a^2(\delta\rho_m + \delta\rho_r) \equiv a^2\delta\rho,$$
$$\Phi' + \mathcal{H}\Phi = \frac{1}{2}a^2\left(\rho_m v_m + \frac{4}{3}\rho_r v_r\right) \equiv \frac{1}{2}a^2\rho V,$$
$$\Phi'' + 3\mathcal{H}\Phi' + (\mathcal{H}^2 + 2\mathcal{H}')\Phi = -\frac{1}{6}a^2\delta\rho_r,$$

$$\delta\rho_m' + 3\mathcal{H}\delta\rho_m + \rho_m(3\Phi' + \Delta v_m) = -a\delta Q + a\Phi Q,$$
$$\delta\rho_r' + 4\mathcal{H}\delta\rho_r + \frac{4}{3}\rho_r(3\Phi' + \Delta v_r) = a\delta Q - a\Phi Q,$$
$$v_m' + \mathcal{H}v_m - \Phi = 0,$$
$$v_r' + \frac{1}{4}\frac{\delta\rho_r}{\rho_r} - \Phi = \frac{aQ}{\rho_r}\left(\frac{3}{4}v_m - v_r\right).$$

4 The isocurvature case and the general case

Theoreticlly, it's convenient to define the isocurvature perturbation

$$S \equiv \frac{\delta\rho_m}{\rho_m} - \frac{\delta\rho_r}{\rho_r + p_r} = \frac{\delta\rho_m}{\rho_m} - \frac{3}{4}\frac{\delta\rho_r}{\rho_r}$$

Then, one can get the familiar equations

$$V'_{\rm rel} + 3c_s^2\mathcal{H}V_{\rm rel} + \frac{3}{2a^2\rho_r}c_s^2\Delta\Phi + \frac{3\rho_m}{4\rho_r}c_s^2 S - \frac{aQ}{4\rho_r}\frac{\rho V - 4(\rho_m+\rho_r)V_{\rm rel}}{\rho_m + 4\rho_r/3} = 0$$

$$\Phi'' + 3\mathcal{H}(1+c_s^2)\Phi' + \left(\mathcal{H}^2(1+3c_s^2) + 2\mathcal{H}'\right)\Phi - c_s^2\Delta\Phi = \frac{a^2}{2}\rho_m c_s^2 S,$$

$$S' = -\Delta V_{\rm rel} - a(\delta Q - Q\Phi)\frac{3}{4}\frac{\rho_m + 4\rho_r/3}{\rho_m\rho_r} + aQ\left(\frac{\delta\rho_m}{\rho_m^2} + \frac{3}{4}\frac{\delta\rho_r}{\rho_r^2}\right)$$

$$c_s^2 = \frac{1}{3}\left(1 + \frac{3\rho_m}{4\rho_r}\right)^{-1} = \frac{4}{9}\frac{\rho_r}{\rho_m + 4\rho_r/3} \qquad V_{\rm rel} \equiv v_m - v_r$$

When there is no energy-transfer, an analytical solution on superhorizon exists

$$S_k(\xi) = S_i(k),$$

$$\Phi_k(\xi) = \Phi_i(k)\left(\frac{8}{5\xi^3}(\sqrt{1+\xi}-1) - \frac{4}{5\xi^2} + \frac{1}{5\xi} + \frac{9}{10}\right) + S_i(k)\left(\frac{16}{5\xi^3}(1-\sqrt{1+\xi}) + \frac{8}{5\xi^2} - \frac{2}{5\xi} + \frac{1}{5}\right),$$

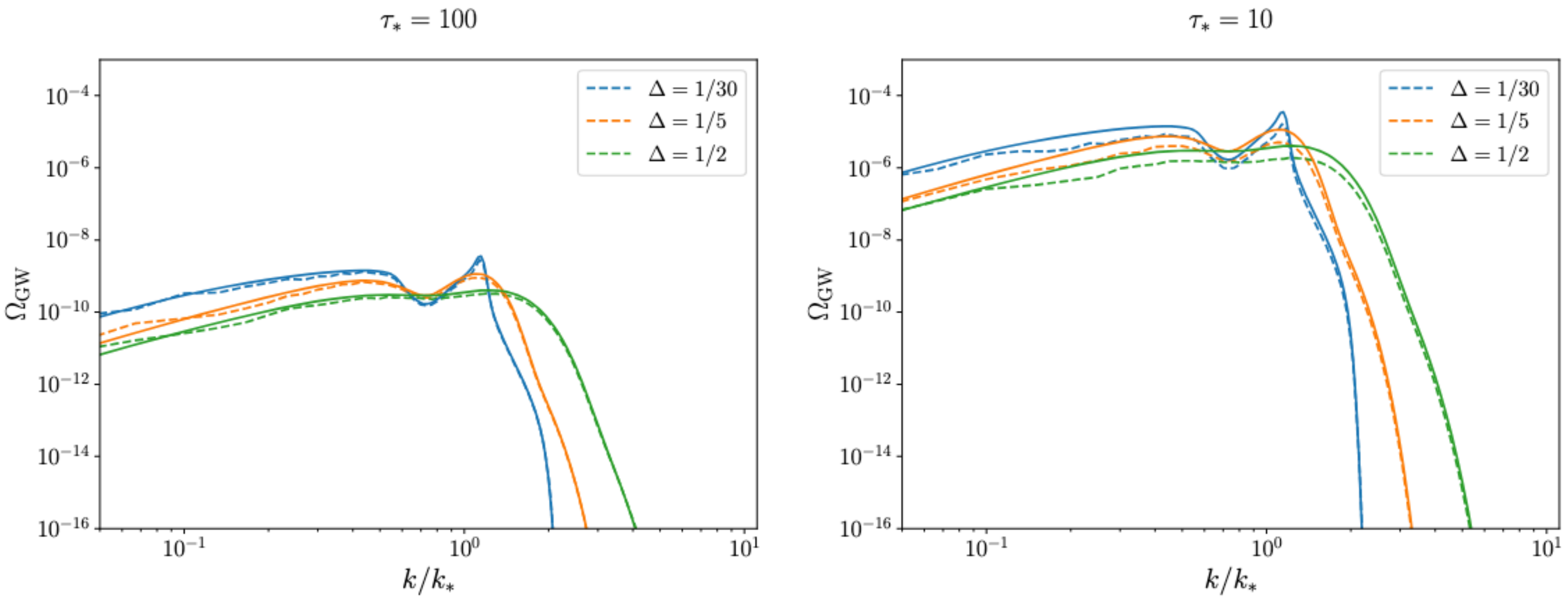
$$\xi \equiv a/a_{\rm eq}$$

$$\frac{a(\tau)}{a_{\rm eq}} = 2\left(\frac{\tau}{\tau_*}\right) + \left(\frac{\tau}{\tau_*}\right)^2 \qquad \tau_{\rm eq} = (\sqrt{2}-1)\tau_*$$

Then, you just need to give the initial value of S and Φ

Comparing with the semi-analytical results

$$\mathcal{P}_S(k) = B\frac{(k/k_{*2})^3}{\sqrt{2\pi}\Delta_{\rm iso}}\exp\left[-\frac{(k/k_{*2}-1)^2}{2\Delta_{\rm iso}^2}\right] \qquad B = 100$$



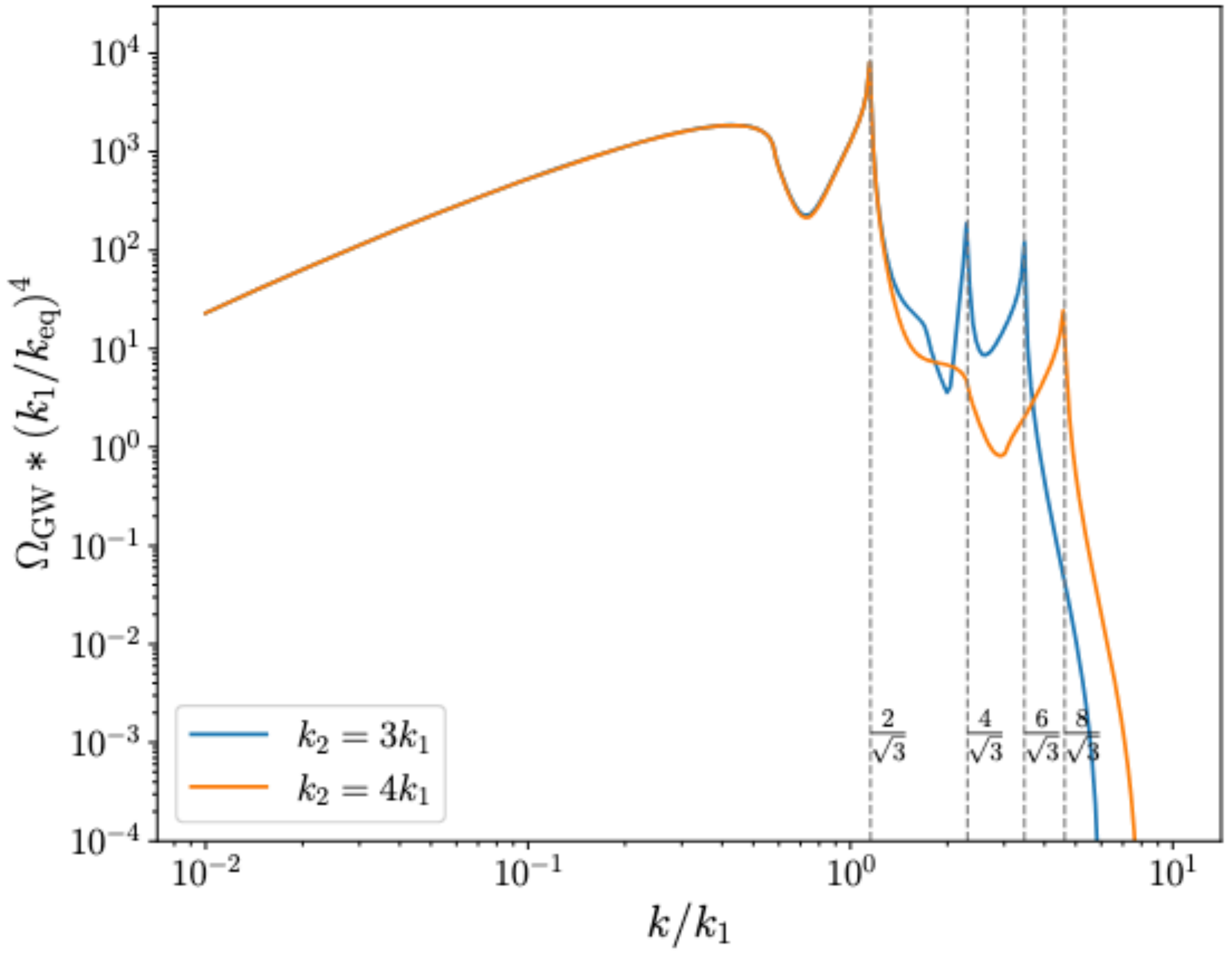
JCAP 03 (2022) 023, [2112.10163]

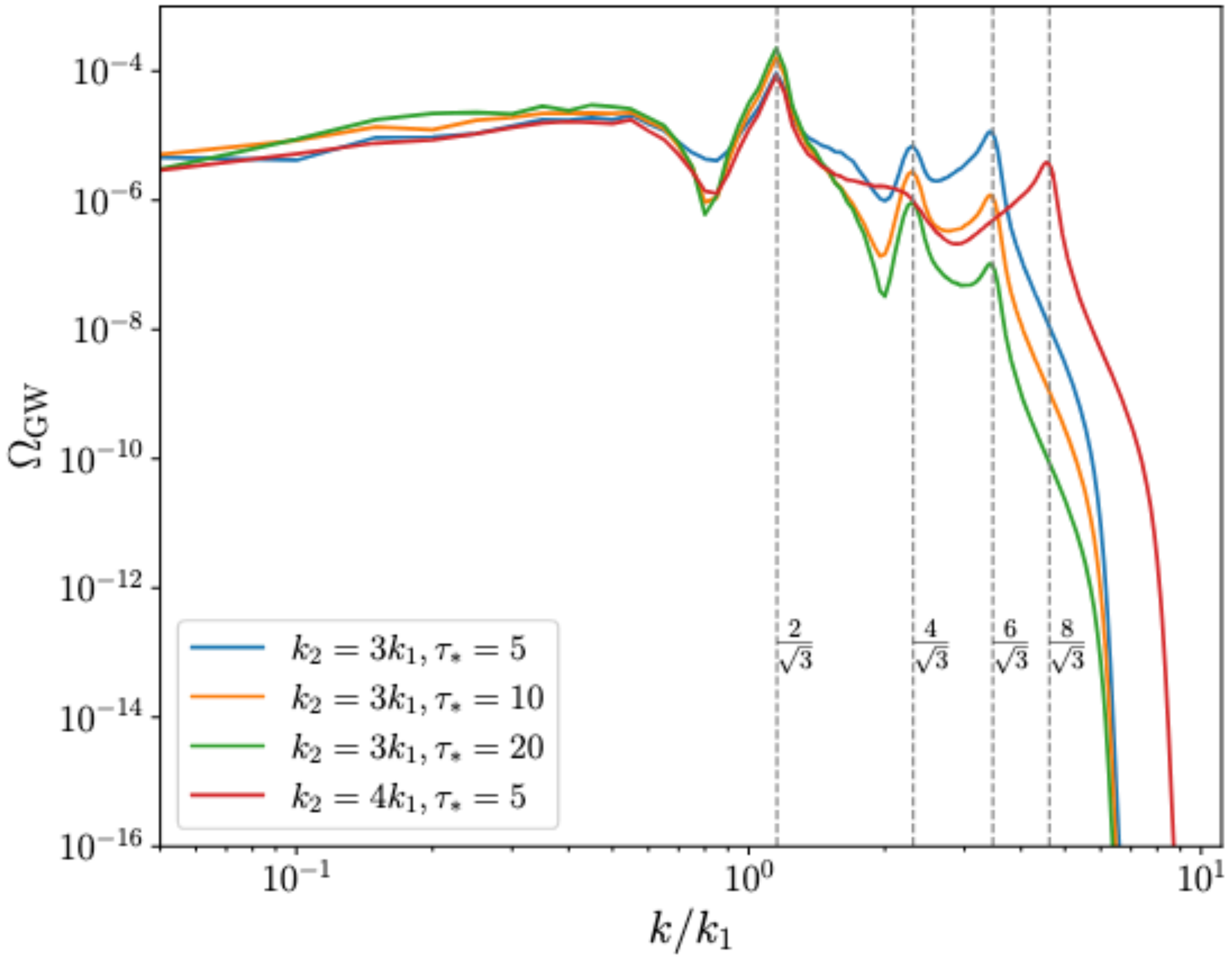
4 The isocurvature case and the general case

Multi-peaks structures

$$\Omega_{\mathrm{GW,iso}} = \frac{2}{3}\sum_{i,j}^{n} B_i B_j \tilde{k}_i^{-1}\tilde{k}_j^{-1}\left(\frac{4\tilde{k}_i^{-2} - (1-\tilde{k}_j^{-2}+\tilde{k}_i^{-2})^2}{4\tilde{k}_i^{-1}\tilde{k}_j^{-1}}\right)^2 \overline{I^2(k,\tau_c\to\infty,\tilde{k}_i^{-1},\tilde{k}_j^{-1})}$$
$$\Theta(k_i+k_j-k)\Theta(k-|k_i-k_j|),$$

$$\left(\frac{k_i}{k}+\frac{k_j}{k}\right) = \sqrt{3}.$$





PBH-dominated era

$$\mathcal{P}_S(k) = \frac{2}{3\pi}\left(\frac{k}{k_{\mathrm{UV}}}\right)^3 \Theta(k_{\mathrm{UV}} - k)$$

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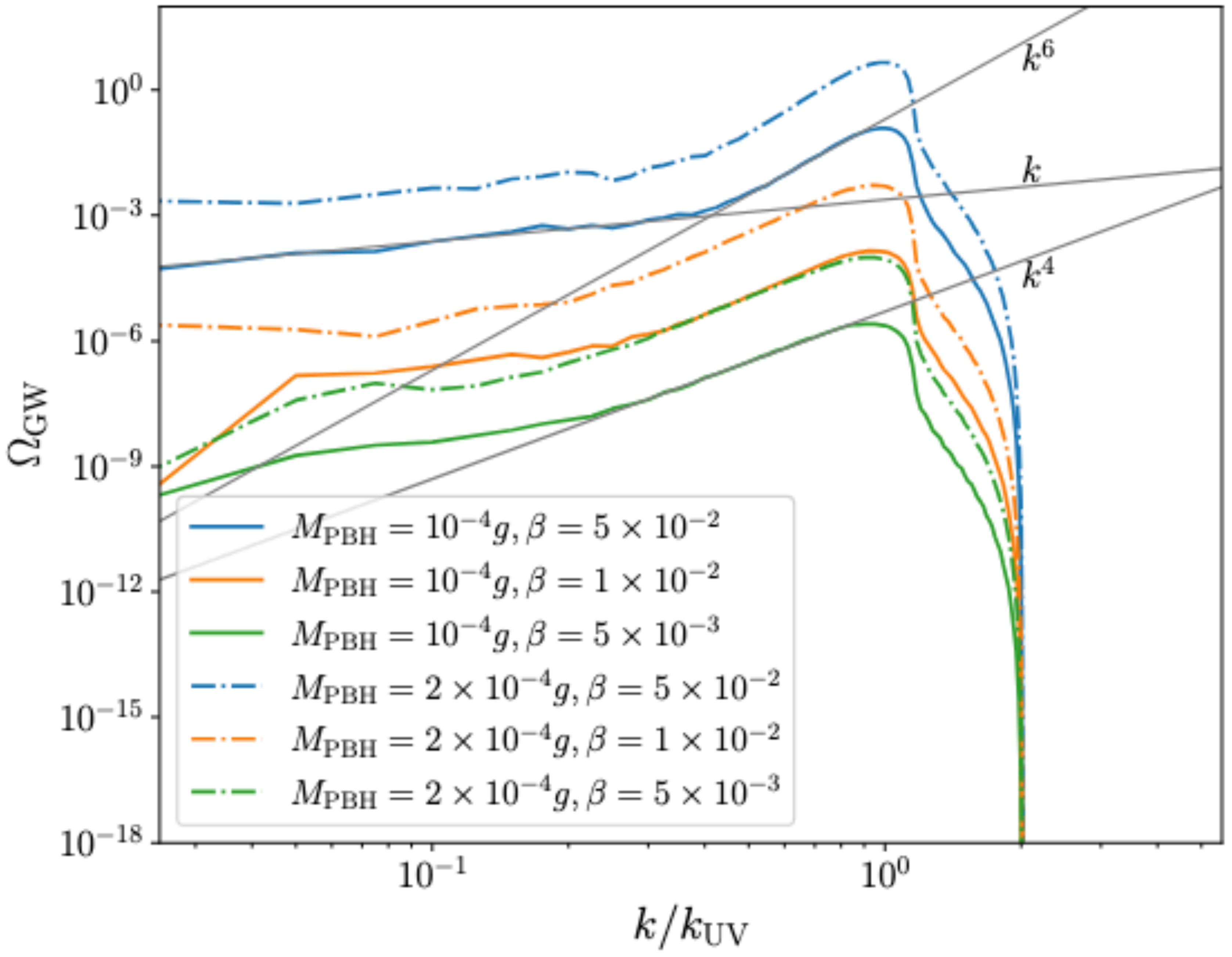
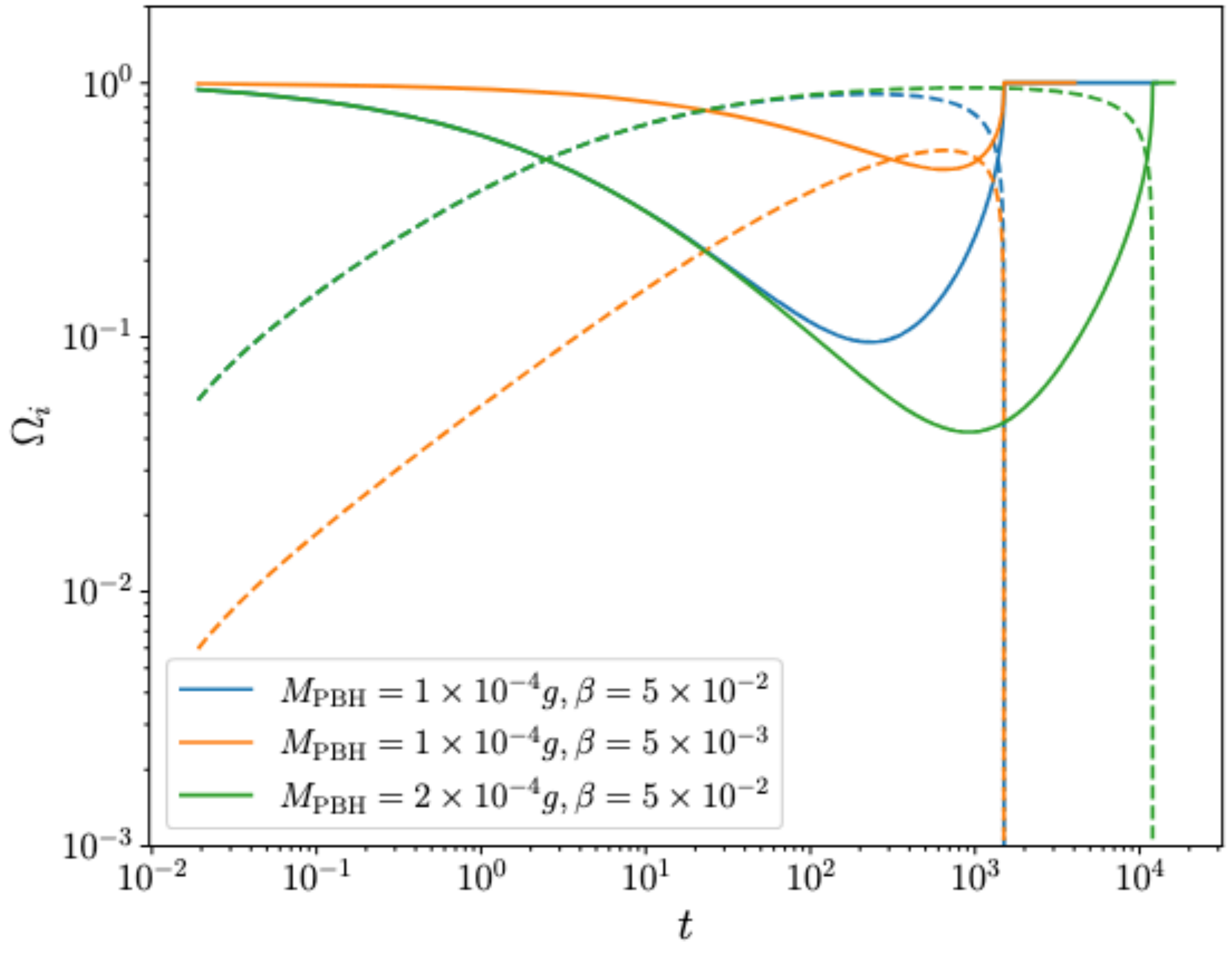
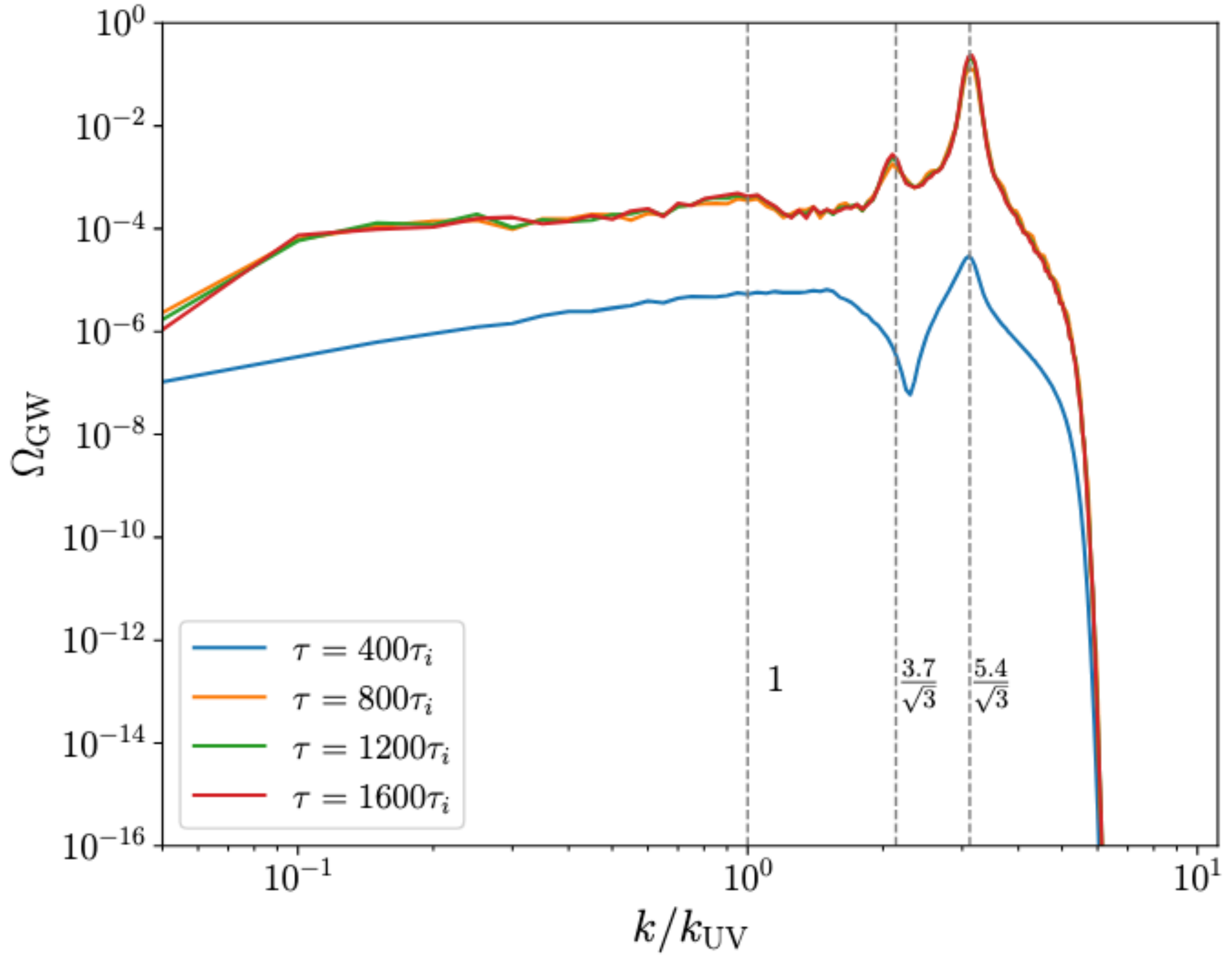




PBH-dominated era

Mixed initial condition

$$k_{\rm UV} = \gamma^{-\frac{1}{3}} \beta^{\frac{1}{3}} k_*$$



Contents

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5 Summary

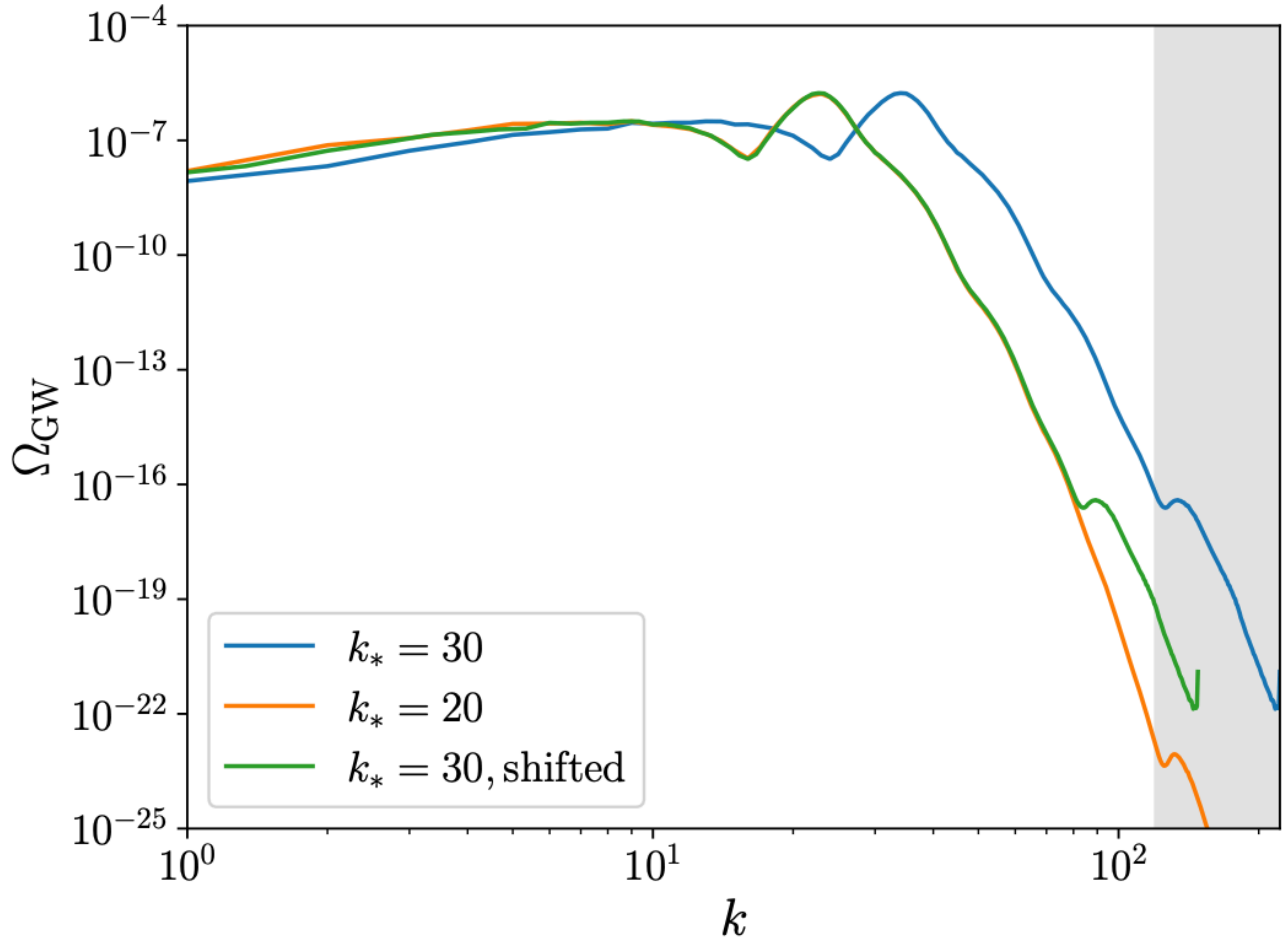
When you consider a curvature perturbation with a sizeable amplitude $O(10^{-4} - 10^{-2})$

- SIGW with full-order non-Gaussianity will have a very different ultra-violet behavior
- The peak frequency may have a change
- It might give you a very different amplitude

Therefore, special care should be taken when using SIGW to constrain the abandunce of PBHs.

- Don’t be afraid of doing lattice simulation, it’s quick! It can be done in your laptop!
- We can treat the general initial condition now, while there is no specific semi-analytical formulas.

Thanks for your attention!


10^{-4}
10^{-7}
10^{-10}
10^{-13}
10^{-16}
10^{-19}
10^{-22}
10^{-25}
$\Omega_{\rm GW}$
10^0
10^1
10^2
k
$k_* = 30$
$k_* = 20$
$k_* = 30$, shifted