Effective Operator Construction and its Applications

Hao-Lin Li 李浩林

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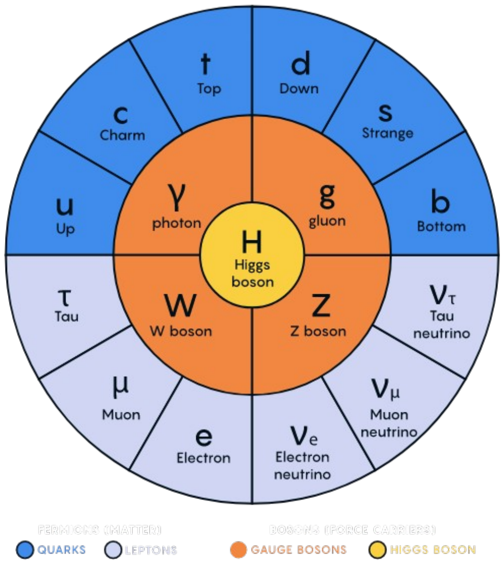
Beijing

SM is not complete

Standard Model

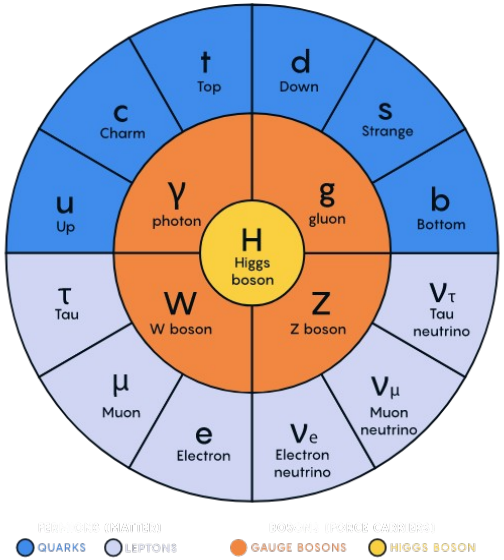


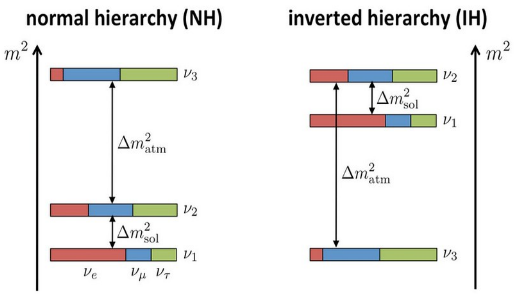
- In 1979, Sheldon Glashow, Abdus Salam, and Steven Weinberg shared the Nobel prize for their contributions to the unification of the electromagnetic and weak forces [1] .
- In 1984, Carlo Rubbia and Simon van der Meer shared the Nobel prize for their decisive contributions to the discovery of the W and Z bosons, the carriers of the weak force [1] .
- In 1999, Gerard 't Hooft and Martinus Veltman shared the Nobel prize for their elucidation of the quantum structure of the electroweak interactions [1] .
- In 2004, David Gross, Hugh David Politzer, and Frank Wilczek shared the Nobel prize for their discovery of asymptotic freedom, the property that explains the behavior of the strong force [1] .
- In 2008, Yoichiro Nambu, Makoto Kobayashi, and Toshihide Maskawa shared the Nobel prize for their discoveries of the mechanisms of spontaneous symmetry breaking and CP violation in the Standard Model [1] .
- In 2013, François Englert and Peter Higgs shared the Nobel prize for their theoretical discovery of the Higgs mechanism, which gives mass to the particles in the Standard Model [2] [1] .

Nobel prices related to the Standard Model

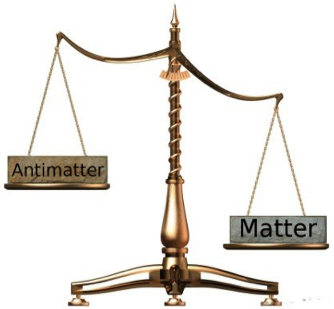
SM is not complete

Standard Model

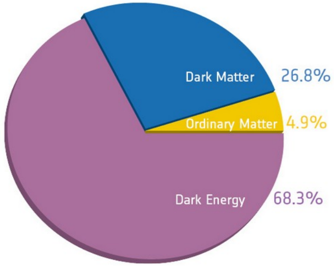




Neutrino Mass



Matter-Antimatter Asymmetry

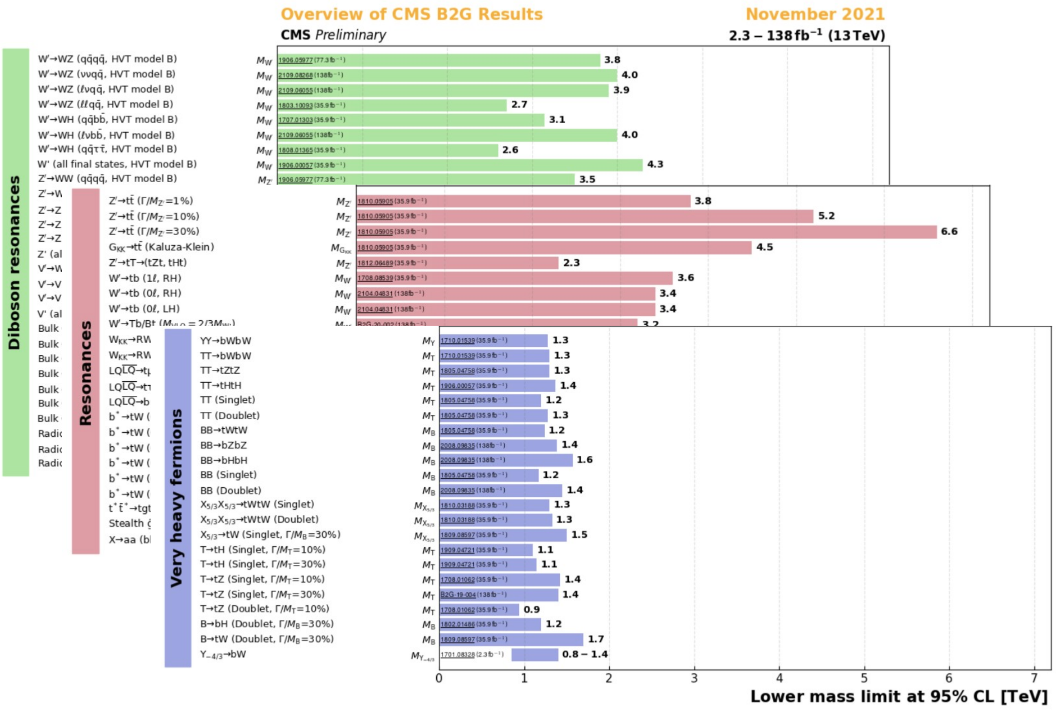


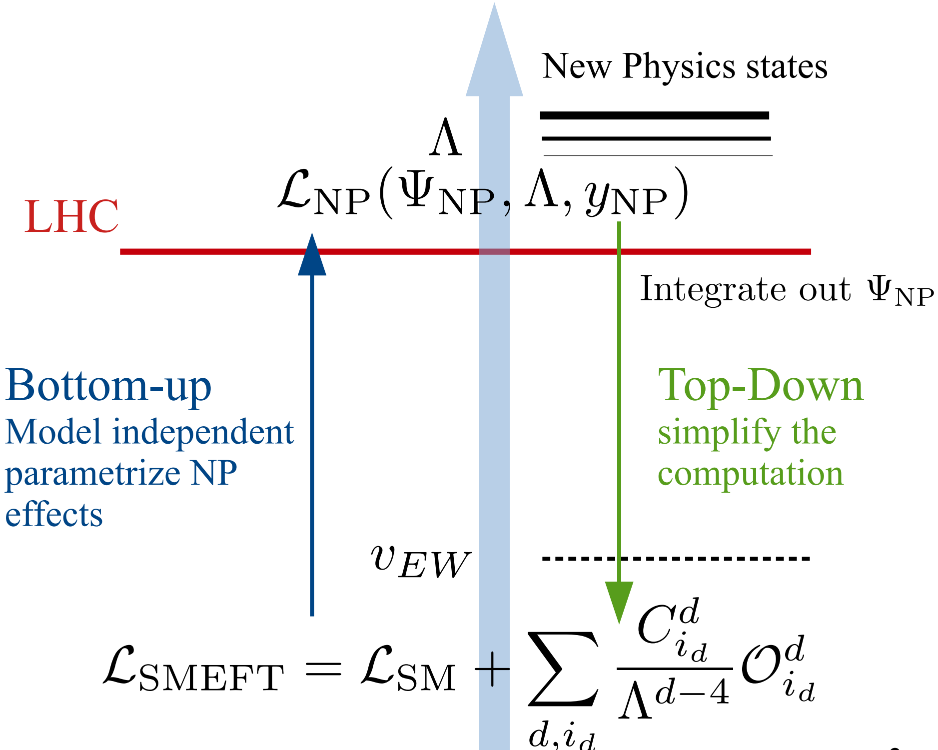
Dark Matter and Dark Energy

New Physics Must Exist

Why EFT

- New Physics scale might be large compared to the SM Electroweak scale
- EFT provides a Universal way to parameterize the all kinds of new physics effects
- EFT simplifies and better organizes the theoretical calculation





Why EFT

- New Physics scale might be large compared to the SM Electroweak scale
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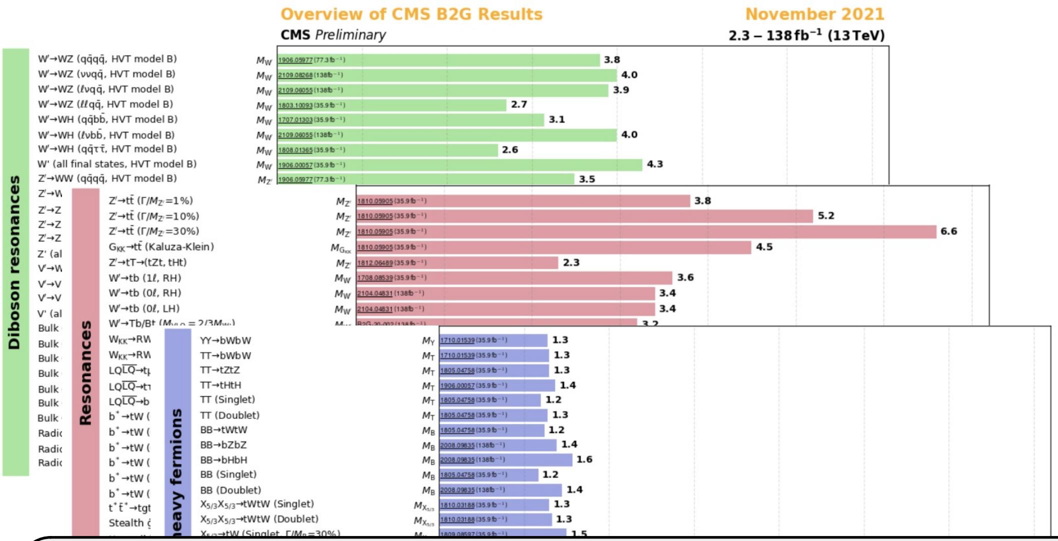

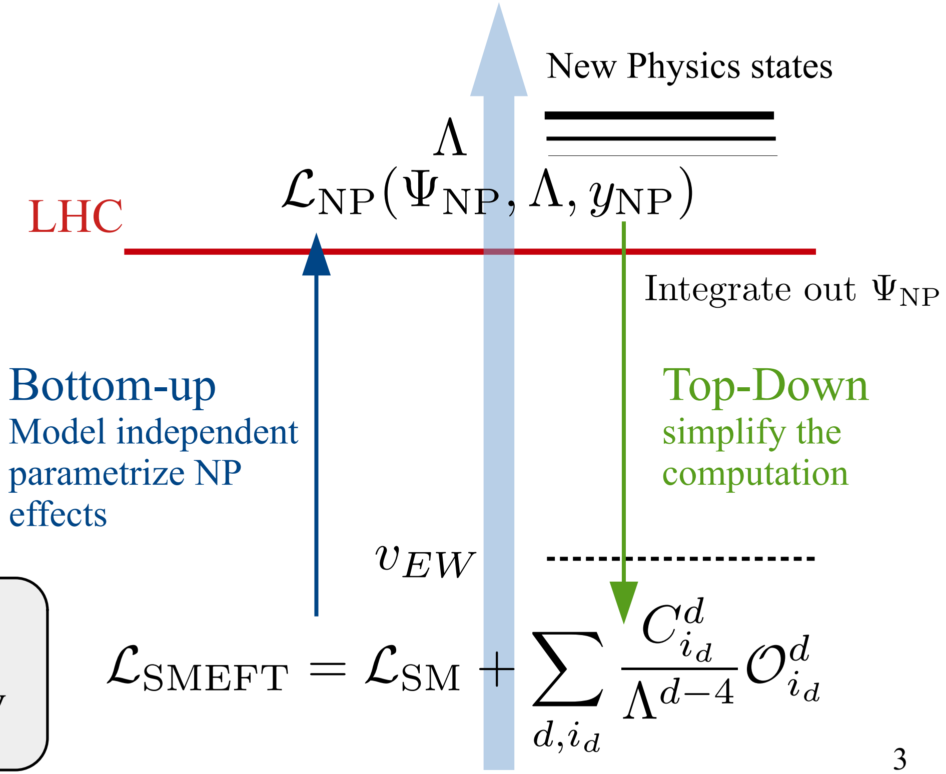

First Task: Define the EFT – Construct Operator Basis
non-trivial: # of operator is large, subject to redundancy

Higher Dimensional Operators

Dim-5 Weinberg, 1979

$$\epsilon_{ij}\epsilon_{mn}(L^i C L^m) H^j H^n$$

Dim-6

~30 years

$$O^{(1)}_{\varphi\ell} = i(\varphi^\dagger D_\mu \varphi)(\bar\ell\gamma^\mu\ell)\,,\quad O^{(3)}_{\varphi\ell} = i(\varphi^\dagger D_\mu \tau^I\varphi)(\bar\ell\gamma^\mu\tau^I\ell)\,,\quad O_{\varphi e} = i(\varphi^\dagger D_\mu \varphi)(\bar e\gamma^\mu e)\,,$$
$$O^{(1)}_{\varphi q} = i(\varphi^\dagger D_\mu \varphi)(\bar q\gamma^\mu q)\,,\quad O^{(3)}_{\varphi q} = i(\varphi^\dagger D_\mu \tau^I\varphi)(\bar q\gamma^\mu\tau^I q)\,,\quad O_{\varphi_u} = i(\varphi^\dagger D_\mu \varphi)(\bar u\gamma^\mu u)\,,$$
$$O_{\varphi_d} = i(\varphi^\dagger D_\mu \varphi)(\bar d\gamma^\mu d)\,,\quad O_{\varphi\varphi} = i(\varphi^\dagger \varepsilon D_\mu \varphi)(\bar u\gamma^\mu d)\,.$$

$$O^{(1)}_{\ell\ell} = \tfrac12(\bar\ell\gamma_\mu\ell)(\bar\ell\gamma^\mu\ell)\,,\quad O^{(3)}_{\ell\ell} = \tfrac12(\bar\ell\gamma_\mu\tau^I\ell)(\bar\ell\gamma^\mu\tau^I\ell)\,,$$
$$O^{(1,1)}_{qq} = \tfrac12(\bar q\gamma_\mu q)(\bar q\gamma^\mu q)\,,\quad O^{(8,1)}_{qq} = \tfrac12(\bar q\gamma_\mu\lambda^A q)(\bar q\gamma^\mu\lambda^A q)\,,$$
$$O^{(1,3)}_{qq} = \tfrac12(\bar q\gamma_\mu\tau^I q)(\bar q\gamma^\mu\tau^I q)\,,\quad O^{(8,3)}_{qq} = \tfrac12(\bar q\gamma_\mu\lambda^A\tau^I q)(\bar q\gamma^\mu\lambda^A\tau^I q)\,,$$
$$O^{(1)}_{\ell q} = (\bar\ell\gamma_\mu\ell)(\bar q\gamma^\mu q)\,,\quad O^{(3)}_{\ell q} = (\bar\ell\gamma_\mu\tau^I\ell)(\bar q\gamma^\mu\tau^I q)\,.$$

$$O_{e\varphi} = (\varphi^\dagger\varphi)(\bar\ell e\varphi)\,,\quad O_{u\varphi} = (\varphi^\dagger\varphi)(\bar q u\tilde\varphi)\,,\quad O_{d\varphi} = (\varphi^\dagger\varphi)(\bar q d\varphi)\,,$$

$$O_{\varphi G} = \tfrac12(\varphi^\dagger\varphi)G^A_{\mu\nu}G^{A\mu\nu}\,,\quad O_{\varphi\tilde G} = (\varphi^\dagger\varphi)\tilde G^A_{\mu\nu}G^{A\mu\nu}\,,$$
$$O_{\varphi W} = \tfrac12(\varphi^\dagger\varphi)W^I_{\mu\nu}W^{I\mu\nu}\,,\quad O_{\varphi\tilde W} = (\varphi^\dagger\varphi)\tilde W^I_{\mu\nu}W^{I\mu\nu}\,,$$
$$O_{\varphi B} = \tfrac12(\varphi^\dagger\varphi)B_{\mu\nu}B^{\mu\nu}\,,\quad O_{\varphi\tilde B} = (\varphi^\dagger\varphi)\tilde B_{\mu\nu}B^{\mu\nu}\,,$$
$$O_{WB} = (\varphi^\dagger\tau^I\varphi)W^I_{\mu\nu}B^{\mu\nu}\,,\quad O_{\tilde WB} = (\varphi^\dagger\tau^I\varphi)\tilde W^I_{\mu\nu}B^{\mu\nu}\,,$$
$$O^{(1)}_{\varphi} = (\varphi^\dagger\varphi)(D_\mu\varphi^\dagger D^\mu\varphi)\,,\quad O^{(3)}_{\varphi} = (\varphi^\dagger D^\mu\varphi)(D_\mu\varphi^\dagger\varphi)\,.$$

$$O_{ee} = \tfrac12(\bar e\gamma_\mu e)(\bar e\gamma^\mu e)\,,$$
$$O^{(1)}_{uu} = \tfrac12(\bar u\gamma_\mu u)(\bar u\gamma^\mu u)\,,\quad O^{(8)}_{uu} = \tfrac12(\bar u\gamma_\mu\lambda^A u)(\bar u\gamma^\mu\lambda^A u)\,,$$
$$O^{(1)}_{dd} = \tfrac12(\bar d\gamma_\mu d)(\bar d\gamma^\mu d)\,,\quad O^{(8)}_{dd} = \tfrac12(\bar d\gamma_\mu\lambda^A d)(\bar d\gamma^\mu\lambda^A d)\,,$$
$$O_{eu} = (\bar e\gamma_\mu e)(\bar u\gamma^\mu u)\,,\quad O_{ed} = (\bar e\gamma_\mu e)(\bar d\gamma^\mu d)\,,$$
$$O^{(1)}_{ud} = (\bar u\gamma_\mu u)(\bar d\gamma^\mu d)\,,\quad O^{(8)}_{ud} = (\bar u\gamma_\mu\lambda^A u)(\bar d\gamma^\mu\lambda^A d)\,.$$

$$O_{\ell e} = (\bar\ell e)(\bar e\ell)\,,\quad O_{\ell u} = (\bar\ell u)(\bar u\ell)\,,\quad O_{\ell d} = (\bar\ell d)(\bar d\ell)\,,\quad O_{qe} = (\bar q e)(\bar e q)\,,$$
$$O^{(1)}_{qu} = (\bar q u)(\bar u q)\,,\quad O^{(8)}_{qu} = (\bar q\lambda^A u)(\bar u\lambda^A q)\,,$$
$$O^{(1)}_{qd} = (\bar q d)(\bar d q)\,,\quad O^{(8)}_{qd} = (\bar q\lambda^A d)(\bar d\lambda^A q)\,,$$
$$O_{qde} = (\bar\ell e)(\bar d q)\,.$$

$$O_{\varphi} = \tfrac13(\varphi^\dagger\varphi)^3\,,\quad O_{\partial\varphi} = \tfrac12\partial_\mu(\varphi^\dagger\varphi)\,\partial^\mu(\varphi^\dagger\varphi)\,.$$

$$O_{D_e} = (\bar\ell D_\mu e)D^\mu\varphi\,,\quad O_{\bar D_e} = (D_\mu\bar\ell e)D^\mu\varphi\,,$$
$$O_{Du} = (\bar q D_\mu u)D^\mu\tilde\varphi\,,\quad O_{\bar Du} = (D_\mu\bar q u)D^\mu\tilde\varphi\,,$$
$$O_{Dd} = (\bar q D_\mu d)D^\mu\varphi\,,\quad O_{\bar Dd} = (D_\mu\bar q d)D^\mu\varphi\,,$$
$$O_{eW} = (\bar\ell\sigma^{\mu\nu}\tau^I e)\varphi W^I_{\mu\nu}\,,\quad O_{eB} = (\bar\ell\sigma^{\mu\nu}e)\varphi B_{\mu\nu}\,,$$
$$O_{uG} = (\bar q\sigma^{\mu\nu}\lambda^A u)\tilde\varphi G^A_{\mu\nu}\,,$$
$$O_{uW} = (\bar q\sigma^{\mu\nu}\tau^I u)\tilde\varphi W^I_{\mu\nu}\,,\quad O_{uB} = (\bar q\sigma^{\mu\nu}u)\tilde\varphi B_{\mu\nu}\,,$$
$$O_{dG} = (\bar q\sigma^{\mu\nu}\lambda^A d)\varphi G^A_{\mu\nu}\,,$$
$$O_{dW} = (\bar q\sigma^{\mu\nu}\tau^I d)\varphi W^I_{\mu\nu}\,,\quad O_{dW} = (\bar q\sigma^{\mu\nu}d)\varphi B_{\mu\nu}\,.$$

$$O_G = f_{ABC}G^{A\nu}_\mu G^{B\lambda}_\nu G^{C\mu}_\lambda\,,\quad O_{\tilde G} = f_{ABC}\tilde G^{A\nu}_\mu G^{B\lambda}_\nu G^{C\mu}_\lambda\,,$$
$$O_W = \varepsilon_{IJK}W^{I\nu}_\mu W^{J\lambda}_\nu W^{K\mu}_\lambda\,,\quad O_{\tilde W} = \varepsilon_{IJK}\tilde W^{I\nu}_\mu W^{J\lambda}_\nu W^{K\mu}_\lambda\,.$$

$$O^{(1)}_{qq} = (\bar q u)(\bar q d)\,,\quad O^{(8)}_{qq} = (\bar q\lambda^A u)(\bar q\lambda^A d)\,,\quad O_{\ell q} = (\bar\ell e)(\bar q u)\,.$$

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Buchmuller, Wyler, 1986

Grzadkowski, Iskrzynski, Misiak, Rosiek, 2010

X^3		φ^6 and $\varphi^4 D^2$		$\psi^2\varphi^3$	
Q_G	$f^{ABC}G^{A\nu}_\mu G^{B\rho}_\nu G^{C\mu}_\rho$	Q_φ	$(\varphi^\dagger\varphi)^3$	$Q_{e\varphi}$	$(\varphi^\dagger\varphi)(\bar l_p e_r\varphi)$
$Q_{\widetilde G}$	$f^{ABC}\widetilde G^{A\nu}_\mu G^{B\rho}_\nu G^{C\mu}_\rho$	$Q_{\varphi\Box}$	$(\varphi^\dagger\varphi)\Box(\varphi^\dagger\varphi)$	$Q_{u\varphi}$	$(\varphi^\dagger\varphi)(\bar q_p u_r\widetilde\varphi)$
Q_W	$\varepsilon^{IJK}W^{I\nu}_\mu W^{J\rho}_\nu W^{K\mu}_\rho$	$Q_{\varphi D}$	$\left(\varphi^\dagger D^\mu\varphi\right)^\star\left(\varphi^\dagger D_\mu\varphi\right)$	$Q_{d\varphi}$	$(\varphi^\dagger\varphi)(\bar q_p d_r\varphi)$
$Q_{\widetilde W}$	$\varepsilon^{IJK}\widetilde W^{I\nu}_\mu W^{J\rho}_\nu W^{K\mu}_\rho$				

$Q_{\varphi G}$	
$Q_{\varphi\widetilde G}$	
$Q_{\varphi W}$	
$Q_{\varphi\widetilde W}$	
$Q_{\varphi B}$	
$Q_{\varphi\widetilde B}$	
$Q_{\varphi WB}$	
$Q_{\varphi\widetilde WB}$	

$(\bar LL)(\bar LL)$		$(\bar RR)(\bar RR)$		$(\bar LL)(\bar RR)$	
Q_{ll}	$(\bar l_p\gamma_\mu l_r)(\bar l_s\gamma^\mu l_t)$	Q_{ee}	$(\bar e_p\gamma_\mu e_r)(\bar e_s\gamma^\mu e_t)$	Q_{le}	$(\bar l_p\gamma_\mu l_r)(\bar e_s\gamma^\mu e_t)$
$Q^{(1)}_{qq}$	$(\bar q_p\gamma_\mu q_r)(\bar q_s\gamma^\mu q_t)$	Q_{uu}	$(\bar u_p\gamma_\mu u_r)(\bar u_s\gamma^\mu u_t)$	Q_{lu}	$(\bar l_p\gamma_\mu l_r)(\bar u_s\gamma^\mu u_t)$
$Q^{(3)}_{qq}$	$(\bar q_p\gamma_\mu\tau^I q_r)(\bar q_s\gamma^\mu\tau^I q_t)$	Q_{dd}	$(\bar d_p\gamma_\mu d_r)(\bar d_s\gamma^\mu d_t)$	Q_{ld}	$(\bar l_p\gamma_\mu l_r)(\bar d_s\gamma^\mu d_t)$
$Q^{(1)}_{lq}$	$(\bar l_p\gamma_\mu l_r)(\bar q_s\gamma^\mu q_t)$	Q_{eu}	$(\bar e_p\gamma_\mu e_r)(\bar u_s\gamma^\mu u_t)$	Q_{qe}	$(\bar q_p\gamma_\mu q_r)(\bar e_s\gamma^\mu e_t)$
$Q^{(3)}_{lq}$	$(\bar l_p\gamma_\mu\tau^I l_r)(\bar q_s\gamma^\mu\tau^I q_t)$	Q_{ed}	$(\bar e_p\gamma_\mu e_r)(\bar d_s\gamma^\mu d_t)$	$Q^{(1)}_{qu}$	$(\bar q_p\gamma_\mu q_r)(\bar u_s\gamma^\mu u_t)$
		$Q^{(1)}_{ud}$	$(\bar u_p\gamma_\mu u_r)(\bar d_s\gamma^\mu d_t)$	$Q^{(8)}_{qu}$	$(\bar q_p\gamma_\mu T^A q_r)(\bar u_s\gamma^\mu T^A u_t)$
		$Q^{(8)}_{ud}$	$(\bar u_p\gamma_\mu T^A u_r)(\bar d_s\gamma^\mu T^A d_t)$	$Q^{(1)}_{qd}$	$(\bar q_p\gamma_\mu q_r)(\bar d_s\gamma^\mu d_t)$
				$Q^{(8)}_{qd}$	$(\bar q_p\gamma_\mu T^A q_r)(\bar d_s\gamma^\mu T^A d_t)$

$(\bar LR)(\bar RL)$ and $(\bar LR)(\bar LR)$		B-violating	
Q_{ledq}	$(\bar l^j_p e_r)(\bar d_s q^j_t)$	Q_{duq}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\left[(d^\alpha_p)^T C u^\beta_r\right]\left[(q^{\gamma j}_s)^T C l^k_t\right]$
$Q^{(1)}_{quqd}$	$(\bar q^j_p u_r)\varepsilon_{jk}(\bar q^k_s d_t)$	Q_{qqu}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\left[(q^{\alpha j}_p)^T C q^{\beta k}_r\right]\left[(u^\gamma_s)^T C e_t\right]$
$Q^{(8)}_{quqd}$	$(\bar q^j_p T^A u_r)\varepsilon_{jk}(\bar q^k_s T^A d_t)$	Q_{qqq}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jn}\varepsilon_{km}\left[(q^{\alpha j}_p)^T C q^{\beta k}_r\right]\left[(q^{\gamma m}_s)^T C l^n_t\right]$
$Q^{(1)}_{lequ}$	$(\bar l^j_p e_r)\varepsilon_{jk}(\bar q^k_s u_t)$	Q_{duu}	$\varepsilon^{\alpha\beta\gamma}\left[(d^\alpha_p)^T C u^\beta_r\right]\left[(u^\gamma_s)^T C e_t\right]$
$Q^{(3)}_{lequ}$	$(\bar l^j_p\sigma_{\mu\nu}e_r)\varepsilon_{jk}(\bar q^k_s\sigma^{\mu\nu}u_t)$		

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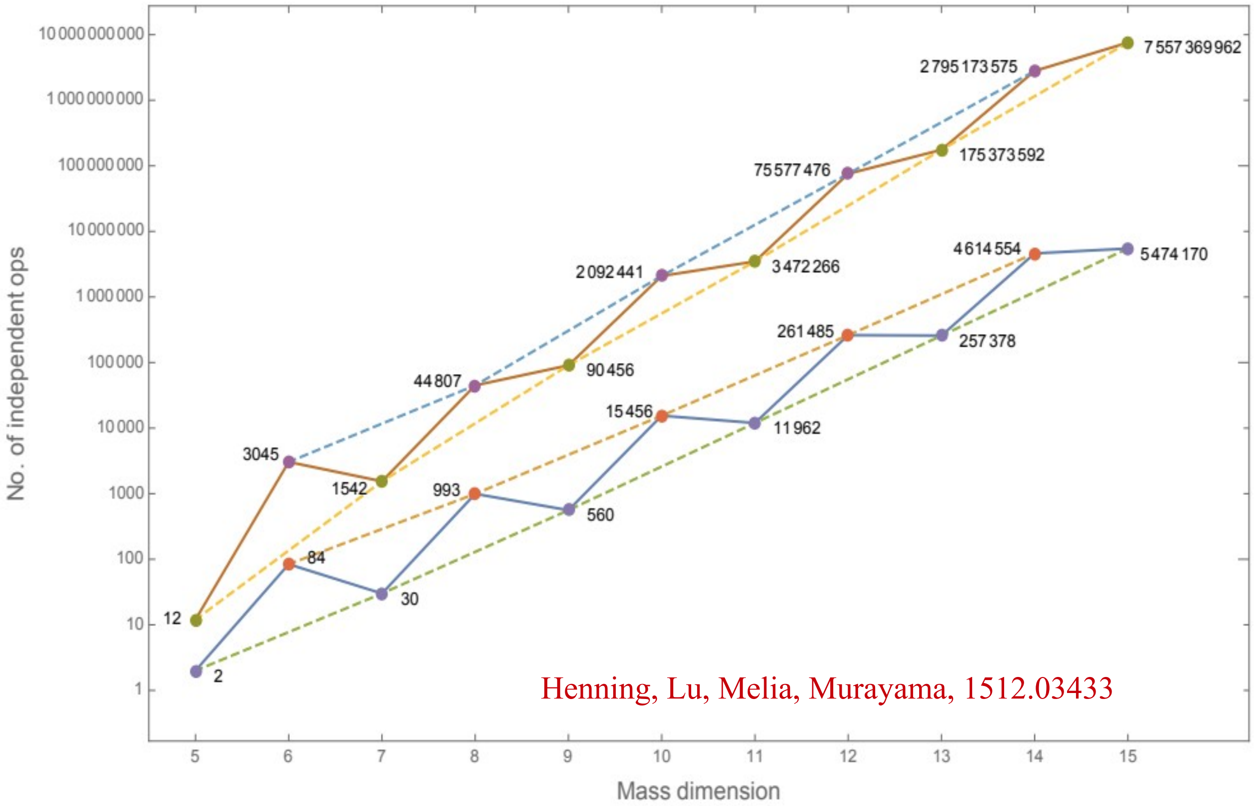
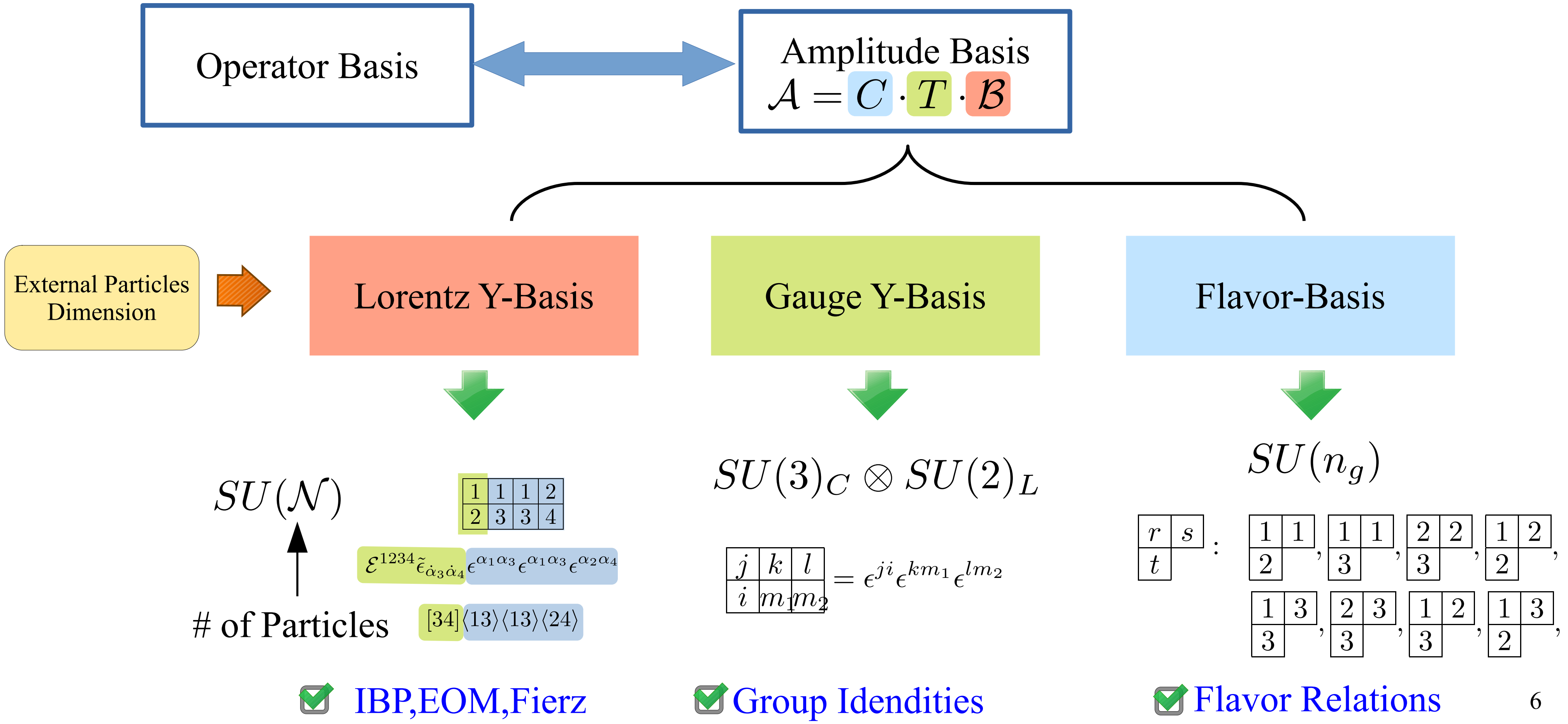
Higher Dimensional Operators



Large number of operators, difficult to track the redundancy relations: EOM, IBP, group identity, flavor relations

Young Tensor Method

[**HLL**, et.al. 2005.00008, 2201.04639]

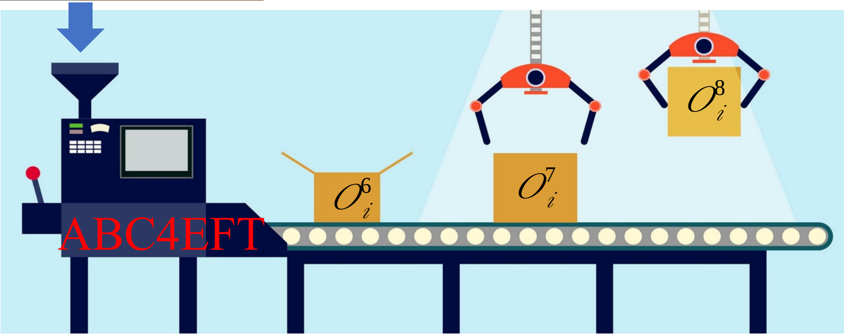


Young Tensor Method

[HLL, et.al. 2005.00008, 2201.04639]

Mathemtaica program ABC4EFT:
automated the basis construction

Quantum Fields	Symmetry



SMEFT dim-8	Phys. Rev. D 104, 015026
SMEFT dim-9	Phys. Rev. D 104, 015025
LEFT dim<=9	JHEP 06 (2021)
LEFT dim<=9	JHEP 11 (2021)
GRSMEFT dim<=9	JHEP 10 (2023)

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============================
ABC4EFT 1.1.0
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A Mathematica Package for
Amplitude Basis Construction for Effective Field Theories

Authors: Hao-Lin Li, lihaolin1991@gmail.com
         Zhe Ren, renzhe@itp.ac.cn
         Ming-Lei Xiao, minglei.xiao@northwestern.edu
         Jiang-Hao Yu, jhyu@itp.ac.cn
         Yu-Hui Zheng, zhengyuhui@itp.ac.cn

The package is available at hepforge
For the latest version, see the GitHub
If you use this package in your research,
Please cite: arXiv: 2201.04639, 2005.00008, 2007.07899
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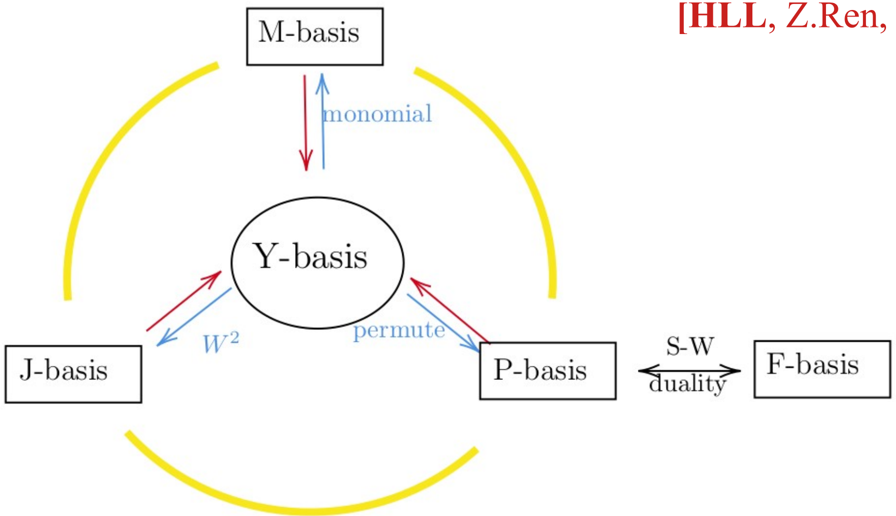
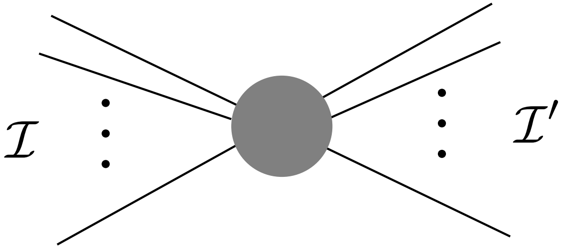
Different Operator/Amplitude Basis

Operator Type: Fixed field contents and the number of derivative $W_L W_L H H^\dagger D,\ Q^3 L$

[**HLL**, Z.Ren, M.-L.Xiao, J.-H.Yu, Y.-H. Zheng, 2201.04639]

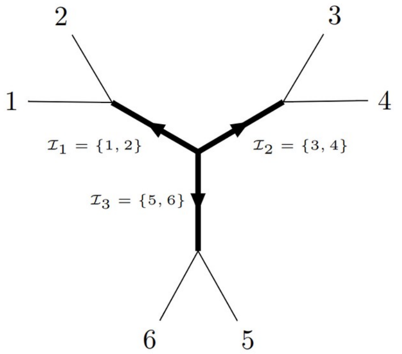


- **Y-basis**: obtained with Young tensor method and amplitude operator correspondence.
- **M-basis**: independent monomial operators
- **P-basis**: irrep of symmetric group of repeated fields—also irrep of $SU(n_f)$
- **F-basis**: independnent flavor tensor spaces – eliminate the improper and redundant flavor tensors
- **J-basis: Eigen-basis of Casimirs**



$\mathcal{O}^{J,\mathbf{R}}_{\mathcal{I}\to\mathcal{I}'} \sim \mathcal{T}(\mathbf{R})\bar{B}^J\ (\mathcal{I}\to\mathcal{I}')$

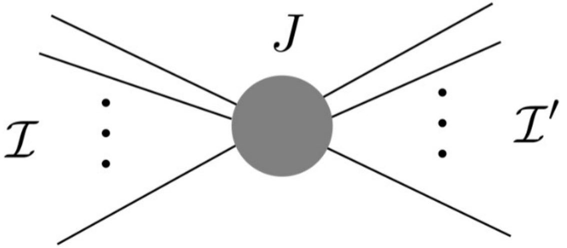
$$\begin{cases} W^2_{\mathcal{I}}\bar{B}^J = -s_{\mathcal{I}}J(J+1)\bar{B}^J \\ \mathbb{C}_{\mathcal{I}}\mathcal{T}(\mathbf{R}) = C(\mathbf{R})\mathcal{T}(\mathbf{R}) \end{cases}$$



$$\mathcal{O}^{J_1\mathbf{R}_1, J_2\mathbf{R}_2, J_3\mathbf{R}_3} \sim \mathcal{T}(\mathbf{R}_1,\mathbf{R}_2,\mathbf{R}_3)B^{J_1,J_2,J_3}$$

J-Basis as Generalized Partial-Wave Basis

Systematic way to construct the J-basis: [M. Jiang, J. Shu, M.-L. Xiao, Y.-H Zheng, 2001.0448]



$$W_{\mathcal{I}}^2 \bar{B}^J(\mathcal{I} \to \mathcal{I}') = -s_{\mathcal{I}} J(J+1) \bar{B}^J(\mathcal{I} \to \mathcal{I}') \qquad s_{\mathcal{I}} = (\sum_{i \in \mathcal{I}} p_i)^2$$

$$W_{\mathcal{I}}^2 = \frac{1}{8} P^2 \left(\mathrm{Tr}\left[M_{\mathcal{I}}{}^2 \right] + \mathrm{Tr}\left[\tilde{M}_{\mathcal{I}}{}^2 \right] \right) - \frac{1}{4} \mathrm{Tr}\left[P^\top M_{\mathcal{I}} P \widetilde{M}_{\mathcal{I}} \right]$$

$$M_{\mathcal{I},\alpha\beta} = i \sum_{i \in \mathcal{I}} \left(\lambda_{i\alpha} \frac{\partial}{\partial \lambda_i^\beta} + \lambda_{i\beta} \frac{\partial}{\partial \lambda_i^\alpha} \right)$$

$$\tilde{M}_{\mathcal{I},\dot{\alpha}\dot{\beta}} = i \sum_{i \in \mathcal{I}} \left(\tilde{\lambda}_{i\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\beta}}} + \tilde{\lambda}_{i\dot{\beta}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \right)$$

$W_{\mathcal{I}}$ Pauli-Lubanski operator

Given an amplitude basis [**Y-basis**], one can find the **representation matrix** of the Casimir operator and therefore find eigen-basis

Dim[amplitudes] $\{h_1, h_2 \ldots, h_N\}$ → $\{\mathcal{B}_i^y\}$ → $W_{\mathcal{I}}^2 \mathcal{B}_i^y = -s_{\mathcal{I}} \mathcal{W}_i^{2\,j} \mathcal{B}_j^y$

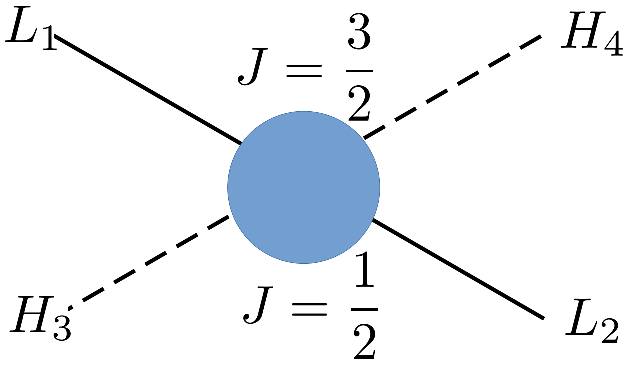
J-Basis as Generalized Partial-Wave Basis

Take $L_1 L_2 H_3 H_4 D^2$ as an example:

$\mathcal{W}^2$

$$\mathcal{B}^y_{\psi^2\phi^2 D^2} = \begin{pmatrix} s_{34}\langle 12\rangle \\ [34]\langle 13\rangle\langle 24\rangle \end{pmatrix}, \quad W^2_{\{13\}}\mathcal{B}^y = s_{13}\begin{pmatrix} -\frac{15}{4} & 2 \\ 0 & -\frac{3}{4} \end{pmatrix}\mathcal{B}^y, \quad \mathcal{K}^{jy}_{\mathcal{B}} = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \mathcal{B}^j = \mathcal{K}^{jy}_{\mathcal{B}}\mathcal{B}^y = \begin{cases} 3s_{34}\langle 12\rangle + 2[34]\langle 13\rangle\langle 24\rangle & J=\frac{3}{2} \\ \langle 13\rangle\langle 24\rangle & J=\frac{1}{2} \end{cases}$$



J-Basis as Generalized Partial-Wave Basis

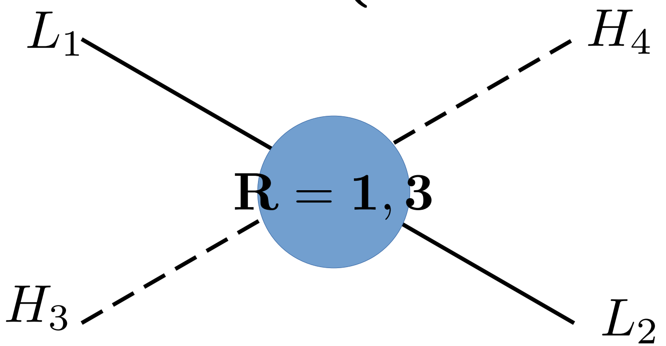
Take $L_1 L_2 H_3 H_4 D^2$ as an example:

$$\mathcal{T}^m_{LLHH} = \begin{pmatrix} \epsilon^{ik}\epsilon^{jl} \\ \epsilon^{ij}\epsilon^{kl} \end{pmatrix}, \quad \underset{\{13\}}{\mathbb{C}_2} \circ \mathcal{T}^m = (\underset{\{13\}}{C_2})^{\mathrm{T}}.\mathcal{T}^m = \begin{pmatrix} 0 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \epsilon^{ik}\epsilon^{jl} \\ \epsilon^{ij}\epsilon^{kl} \end{pmatrix}.$$

$C_2(\mathbf{1})$ $\quad$ $C_2(\mathbf{3})$

$$\mathcal{K}^{jm}_G.(\underset{\{13\}}{C_2})^{\mathrm{T}}(\mathcal{K}^{jm}_G)^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix} \text{ with } \mathcal{K}^{jm}_G = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}$$

$$\Rightarrow \mathcal{T}^j = \mathcal{K}^{jm}_G \mathcal{T}^m = \begin{cases} \epsilon^{ik}\epsilon^{jl} & \mathbf{R} = \mathbf{1} \\ \epsilon^{ik}\epsilon^{jl} - 2\epsilon^{ij}\epsilon^{kl} & \mathbf{R} = \mathbf{3} \end{cases}$$



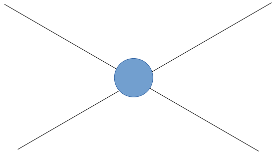
Application 1: Partial-Wave Unitarity Bound

[C. Degrande **HLL**, L.-X. Xu. To appear]

$\mathcal{A}_{EFT} \sim cE^n$ unitarity violation at high energy

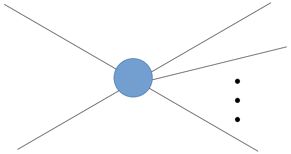
- Perturbative PWU bound on Wilson coefficients depend on E

- Traditional 2→2 process,
  - J-basis unique for fixed J: Wigner D-functions
  - Good for operator with #field ≤ 4



$\mathcal{A}^J_{2\to 2} \sim D^J_{mm'}$

- When dim > 6, operator with #field > 4
  - Needs PWU bounds for 2→ n process
  - J-basis degenerate for fixed J: obtained by our YT-method
- - 2→ n Bounds can be extract from 2→ 2 process with expansion of



$B^{J,a}_{2\to n}$

$$M_{i\to X} = \sum_{J,a} C^{Ja}_{i\to X} B^{Ja}_{i\to X}$$

$$\int d\Pi_X B^{Ja}_{i\to X} (B^{J'a'}_{i\to X})^* (2\pi)^4 \delta^4(p_X - p_i) = g^{Ja}_{i\to X}(s) \delta_{aa'} \delta_{JJ'}$$

$$\frac{\sum_{a,X\neq i} g^{Ja}_{i\to X}(s) |C^{Ja}_{i\to X}|^2}{16\pi(J+1/2)} \leq 1$$

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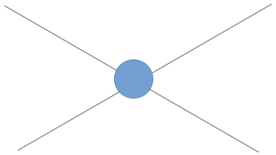

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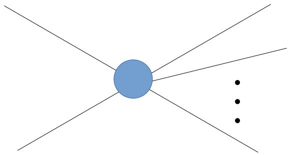

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$$\frac{\sum_{a,X\neq i} g^{Ja}_{i\to X}(s) |C^{Ja}_{i\to X}|^2}{16\pi(J+1/2)} \leq 1$$

Outstanding problem: computing $g(s)$ analytically

Why? Numerical hard; Keep s-dependence explict; verify Exact zero

Application 1: Partial-Wave Unitarity Bound

N-body massless phase-space integral

$$\prod_{i=1}^{N} \frac{d^3 p_i}{(2\pi)^3 2E_i} \, (2\pi)^4 \delta^4 \left(k_1 + k_2 - \sum_{i=1}^{N} p_i \right)$$

Parameterize the final momenta with spinor helicity variables: $\lambda^\alpha(p_i) = u_i \lambda^\alpha(k_1) + v_i \lambda^\alpha(k_2)$

$$\begin{pmatrix} \lambda^\alpha(p_1) & \lambda^\alpha(p_2) & \cdots & \lambda^\alpha(p_N) \end{pmatrix}^T = \begin{pmatrix} u_1 & u_2 & \cdots & u_N \\ v_1 & v_2 & \cdots & v_N \end{pmatrix}^T \begin{pmatrix} \lambda^\alpha(k_1) \\ \lambda^\alpha(k_2) \end{pmatrix}$$

Spinor variables for two Initial state particles

u, *v* are two complex variables

$$d\Pi_N = (2\pi)^{4-3N} s^{N-2} \frac{d^N u d^N v}{U(1)^N} \delta(1 - |\vec{u}|^2) \delta(1 - |\vec{v}|^2) \delta^2(\vec{u}^\dagger \vec{v})$$

Momentum conservation

Little group redundancy for an overall phase of the spinor variables can be used to fix the phase of *u* to zero

$p^\mu (\sigma_\mu)^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}$ Invariant under $\lambda \to e^{i\phi}\lambda, \tilde{\lambda} = \lambda^*$

Application 1: Partial-Wave Unitarity Bound

Processing u integral: $u_i = r_i e^{-i\phi_i}$

$$\frac{d^N u}{U(1)^N}\delta(1-|\vec{u}|^2) = \frac{\prod_{i=1}^{N} r_i dr_i d\phi_i}{U(1)^N}\delta\left(1-\sum_{i=1}^{N} r_i^2\right) = \int \prod_{i=1}^{N}\left[r_i dr_i d\phi_i \delta(\phi_i)\right]\delta\left(1-\sum_{i=1}^{N} r_i^2\right)$$

r_i can be parameterized on the spherical coordinate of S^{N-1}

$$u_i = r_i$$

$$\begin{aligned}
r_N &= \cos\theta_{N-1} \ , \\
r_{N-1} &= \sin\theta_{N-1}\cos\theta_{N-2} \ , \\
r_{N-2} &= \sin\theta_{N-1}\sin\theta_{N-2}\cos\theta_{N-3} \ , \\
&\vdots \\
r_2 &= \sin\theta_{N-1}\ldots\sin\theta_2\cos\theta_1 \ , \\
r_1 &= \sin\theta_{N-1}\ldots\sin\theta_2\sin\theta_1
\end{aligned}$$

$$\frac{d^N u}{U(1)^N}\delta(1-|\vec{u}|^2) = \frac{1}{2}\left(\prod_{i=1}^{N-1}\sin^{2i-1}\theta_i\cos\theta_i\right)d\theta_1\ldots d\theta_{N-1}$$

Application 1: Partial-Wave Unitarity Bound

Processing *v* integral:

$$d^N v\delta(1-|\vec{v}|^2)\boxed{\delta^2(\vec{u}^\dagger\vec{v})} = \frac{d^{N-1}v}{|u_N|^2}\ \delta\left(1-\sum_{i=1}^{N-1}|v_i|^2-|v_N|^2\right)$$

$$\boxed{v_N = -\frac{\sum_{i=1}^{N-1} u_i^* v_i}{u_N^*} = -\frac{\sum_{i=1}^{N-1} r_i v_i}{r_N}}$$

Change integral variables $v = Ov'$ $\quad (O^{-1})^2 = I + \frac{1}{r_N^2}\mathbf{r}^T\mathbf{r} \quad \mathbf{r} = (r_1, r_2, \ldots, r_{N-1})$

$$d^N v\delta(1-|\vec{v}|^2)\delta^2(\vec{u}^\dagger\vec{v}) = d^{N-1}v'\ \delta\left(1-\sum_{i=1}^{N-1}|v_i'|^2\right)$$

Embedding of S^{2N-3} in $\mathbb{C}^{N-1}$

$$\begin{aligned}
v_1' &= e^{-i\xi_1}\cos\eta_1\\
v_2' &= e^{-i\xi_2}\sin\eta_1\cos\eta_2\\
&\vdots\\
v_{N-2}' &= e^{-i\xi_{N-2}}\sin\eta_1\ldots\sin\eta_{N-3}\cos\eta_{N-2}\\
v_{N-1}' &= e^{-i\xi_{N-1}}\sin\eta_1\ldots\sin\eta_{N-3}\sin\eta_{N-2}.
\end{aligned}$$

$$d^{N-1}v'\ \delta\left(1-\sum_{i=1}^{N-1}|v_i'|^2\right) = \left(\prod_{k=1}^{N-2}\cos\eta_k\sin^{2(N-2-k)+1}\eta_k\right)d\xi_i\ldots d\xi_{N-1}d\eta_i\ldots d\eta_{N-2}$$

Application 1: Partial-Wave Unitarity Bound

Processing v integral:

$$d^N v\delta(1-|\vec{v}|^2)\delta^2(\vec{u}^\dagger\vec{v}) = \frac{d^{N-1}v}{|u_N|^2}\ \delta\left(1-\sum_{i=1}^{N-1}|v_i|^2-|v_N|^2\right)$$

$$v_N = -\frac{\sum_{i=1}^{N-1} u_i^* v_i}{u_N^*} = -\frac{\sum_{i=1}^{N-1} r_i v_i}{r_N}$$

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\end{aligned}$$

Key point: *O* is also analytically solvable

$(O^{-1})^2$ is rank-1 update of identity matrix using Sherman–Morrison Formula:

$$O = I - \left(\frac{\sqrt{1+\beta}-1}{\beta\sqrt{1+\beta}}\right)\frac{\mathbf{r}^T\mathbf{r}}{r_N^2}, \quad \beta = \frac{|\mathbf{r}|^2}{r_N^2}$$

$$d^{N-1}v'\ \delta\left(1-\sum_{i=1}^{N-1}|v_i'|^2\right) = \left(\prod_{k=1}^{N-2}\cos\eta_k\sin^{2(N-2-k)+1}\eta_k\right)d\xi_i\ldots d\xi_{N-1}d\eta_i\ldots d\eta_{N-2}$$

Application 1: Partial-Wave Unitarity Bound

u and *v* completely expressed with angular and phase parameters, and all the delta functions are resolved

$$v_i = O_{ij} v'_j \ (i, j \in 1, 2, \ldots, N-1) \qquad v_N = -\frac{\sum_{i=1}^{N-1} u_i^* v_i}{u_N^*} = -\frac{\sum_{i=1}^{N-1} r_i v_i}{r_N}$$

$$O = I - \left(\frac{\sqrt{1+\beta} - 1}{\beta\sqrt{1+\beta}} \right) \frac{\mathbf{r}^T \mathbf{r}}{r_N^2}, \quad \beta = \frac{|\mathbf{r}|^2}{r_N^2}$$

Sherman–Morrison Formula

$$u_i = r_i$$

$$\begin{aligned} r_N &= \cos\theta_{N-1} \ , \\ r_{N-1} &= \sin\theta_{N-1} \cos\theta_{N-2} \ , \\ r_{N-2} &= \sin\theta_{N-1} \sin\theta_{N-2} \cos\theta_{N-3} \ , \\ &\vdots \\ r_2 &= \sin\theta_{N-1} \ldots \sin\theta_2 \cos\theta_1 \ , \\ r_1 &= \sin\theta_{N-1} \ldots \sin\theta_2 \sin\theta_1 \end{aligned}$$

$$\begin{aligned} v'_1 &= e^{-i\xi_1} \cos\eta_1 \\ v'_2 &= e^{-i\xi_2} \sin\eta_1 \cos\eta_2 \\ &\vdots \\ v'_{N-2} &= e^{-i\xi_{N-2}} \sin\eta_1 \ldots \sin\eta_{N-3} \cos\eta_{N-2} \\ v'_{N-1} &= e^{-i\xi_{N-1}} \sin\eta_1 \ldots \sin\eta_{N-3} \sin\eta_{N-2}. \end{aligned}$$

Application 1: Partial-Wave Unitarity Bound

For 3-body final state:

$$u_1 = \sin\theta_2 \sin\theta_1, \quad u_2 = \cos\theta_1 \sin\theta_2, \quad u_3 = \cos\theta_2.$$

An equivalent parameterization [J.E.Miro, et.al. 2005.06983]

$$v_1 = e^{-i\xi_1}\cos\eta_1\left(\cos^2\theta_1 + \cos\theta_2\sin^2\theta_1\right) + e^{-i\xi_2}\sin\eta_1(\cos\theta_2 - 1)\cos\theta_1\sin\theta_1\ ,$$
$$v_2 = e^{-i\xi_1}\cos\eta_1(\cos\theta_2 - 1)\cos\theta_1\sin\theta_1 + e^{-i\xi_2}\sin\eta_1\left(\cos\theta_2\cos^2\theta_1 + \sin^2\theta_1\right)\ ,$$
$$v_3 = -\sin\theta_2(e^{-i\xi_1}\cos\eta_1\sin\theta_1 + e^{-i\xi_2}\cos\theta_1\sin\eta_1).$$

For 4-body final state:
Our new result

$$\begin{aligned} u_1 &= \sin\theta_3\sin\theta_2\sin\theta_1, \\ u_2 &= \sin\theta_3\sin\theta_2\cos\theta_1, \\ u_3 &= \sin\theta_3\cos\theta_2, \\ u_4 &= \cos\theta_3, \end{aligned}$$

$$\begin{aligned} v_1 &= e^{-i\xi_2}\sin\eta_1\cos\eta_2(\cos\theta_3 - 1)\sin^2\theta_2\sin\theta_1\cos\theta_1 \\ &\quad + e^{-i\xi_3}\sin\eta_1\sin\eta_2(\cos\theta_3 - 1)\sin\theta_2\cos\theta_2\sin\theta_1 \\ &\quad + e^{-i\xi_1}\cos\eta_1\left(\sin^2\theta_2\left(\cos\theta_3\sin^2\theta_1 + \cos^2\theta_1\right) + \cos^2\theta_2\right), \\ v_2 &= e^{-i\xi_2}\sin\eta_1\cos\eta_2\left(\sin^2\theta_2\left(\cos\theta_3\cos^2\theta_1 + \sin^2\theta_1\right) + \cos^2\theta_2\right) \\ &\quad + e^{-i\xi_3}\sin\eta_1\sin\eta_2(\cos\theta_3 - 1)\sin\theta_2\cos\theta_2\cos\theta_1 \\ &\quad + e^{-i\xi_1}\cos\eta_1(\cos\theta_3 - 1)\sin^2\theta_2\sin\theta_1\cos\theta_1, \\ v_3 &= e^{-i\xi_2}\sin\eta_1\cos\eta_2(\cos\theta_3 - 1)\sin\theta_2\cos\theta_2\cos\theta_1 \\ &\quad + e^{-i\xi_3}\sin\eta_1\sin\eta_2\left(\cos\theta_3\cos^2\theta_2 + \sin^2\theta_2\right) \\ &\quad + e^{-i\xi_1}\cos\eta_1(\cos\theta_3 - 1)\sin\theta_2\cos\theta_2\sin\theta_1, \\ v_4 &= -\sin\theta_3\left[\sin\theta_2\left(e^{-i\xi_2}\sin\eta_1\cos\eta_2\cos\theta_1 + e^{-i\xi_1}\cos\eta_1\sin\theta_1\right)\right. \\ &\quad \left. + e^{-i\xi_3}\sin\eta_1\sin\eta_2\cos\theta_2\right]. \end{aligned}$$

Generalization to N body is straightforward!

Application 1: Partial-Wave Unitarity Bound

Example: $M = \langle 14\rangle[45]$

$$|4\rangle = u_2(\theta_1,\theta_2)|1\rangle + v_2(\theta_1,\theta_2,\xi_1,\xi_2,\eta_2)|2\rangle$$
$$|5\rangle = u_3(\theta_1,\theta_2)|1\rangle + v_3(\theta_1,\theta_2,\xi_1,\xi_2,\eta_2)|2\rangle$$

$$|4] = |4\rangle^*,\ \ |5] = |5\rangle^*$$

$$M = (|v_2|^2 u_3^* - v_2 u_2^* v_3^*)\boxed{\langle 12\rangle[21]}$$

← Center of mass energy square s

For on-shell local amplitudes the integral factorize, thus can always be done analytically

$$\int |M|^2 \mathrm{dPS}_3 = \int f_1(\theta_1)d\theta_1 \int f_2(\theta_2)d\theta_2 \int f_3(\eta_1)d\eta_1 \int f_4(\xi_1)d\xi_1 \int f_5(\xi_2)d\xi_2$$

We provide the Mathematica code to compute the integral for 3- and 4-body final state

M_1 M_2 $\#FS$

```
PSIntAMPUser[ab[1, 4] × sb[4, 5], ab[1, 4] × sb[4, 5], 3, {2, 3}]
```

incoming label

$$\frac{s^3}{3072\,\pi^3}$$

$$\int d\Pi_{k\notin\{i,j\}}(2\pi)^4\delta^4(p_i+p_j-\sum_{k\notin\{i,j\}}p_k)\mathrm{M}_1^*\mathrm{M}_2$$

Application 1: Partial-Wave Unitarity Bound

A SMEFT dim-8 example: $C_{f1_{f}6}|H|^2H^\dagger \overleftrightarrow{D}_\mu H(\overline{e_R}_{f_6}\gamma^\mu e_{Rf_1})$

1. The corresponding local on-shell amplitude is:

$$\begin{aligned} M^{f_1f_6}_{i_2i_3i_4i_5} = & \quad C_{f_1f_6}\Big\{\big(\delta^{i_2}_{i_4}\delta^{i_3}_{i_5}15[56]+\text{sym}(45)\big)+\text{sym}(23) \\ - & \quad \big(\delta^{i_2}_{i_4}\delta^{i_3}_{i_5}13[36]+\text{sym}(23)\big)+\text{sym}(45)\Big\}, \end{aligned}$$

2. For the channel: $H_{i_2}(p_2)H_{i_3}(p_3)\to e^+(-p_1)e^-(-p_6)H^{\dagger i_4}(-p_4)H^{\dagger i_5}(-p_5)$
Derive the J-basis and normalization factors

$$\begin{aligned} B^{J=1} &= 2\langle 13\rangle[36]+\langle 14\rangle[46]+\langle 15\rangle[56], \quad g^{J=1}=\frac{s^4}{184320\pi^5} \\ B^{J=0}_1 &= \frac{\langle 14\rangle[46]+\langle 15\rangle[56]}{\sqrt{2}}, \quad g^{J=1}_1=\frac{s^4}{737280\pi^5}, \\ B^{J=0}_2 &= \frac{-\langle 14\rangle[46]+\langle 15\rangle[56]}{\sqrt{2}}, \quad g^{J=1}_2=\frac{s^4}{1474560\pi^5} \end{aligned}$$

$$\Rightarrow \quad \frac{\sum_{f_1f_6}|C_{f_1f_6}|^2 s^4}{737280\pi^6}\leq 1$$

3. Iterate over all possible scattering channels and find out the strongest bound

Application 2: Studying Chiral Gauge Theory with fRG

[**HHL**, A. P.-Gutierres, S. Vatani, L.-X. Xu, 2507.21208]

- Functional Renormalization Group:
  a non-perturbative method widely used in QCD

See Dupuis,et.al [2006.04853] for a reivew

- Basic idea: effective average action: $\Gamma_k[\phi]$

$$\int [\mathcal{D}\phi]_{p>k} = \int \mathcal{D}\phi \, \exp\left(-\Delta S_k[\phi]\right)$$

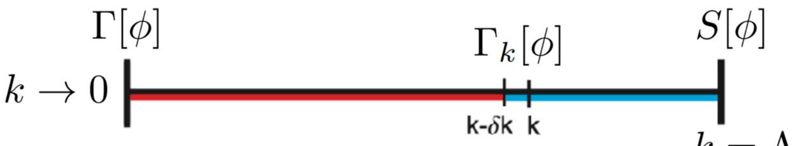
$$\Delta S_k[\phi] = \int_p \phi(p) R_k \phi(-p)$$

$$\Gamma_k[\phi] = \int_x J(x)\phi(x) - \mathcal{W}_k[J] - \Delta S_k[\phi]$$

Wetterich '93

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr}\left[\frac{1}{\Gamma_k^{(2)} + R_k} \partial_t R_k\right]$$

$\partial_t \equiv k\partial_k$



IR cutoff scale: $k \sim T \sim \langle p \rangle$

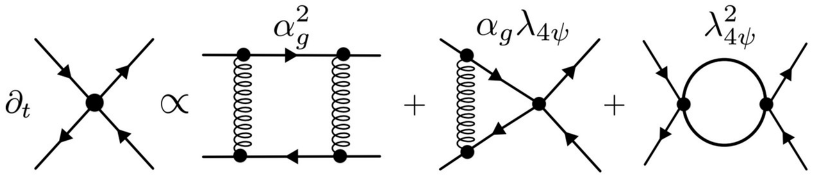
- Average action of fields over a k^{-d} space-time volume
- Kadanoff's block-spinning idea in continuum limit

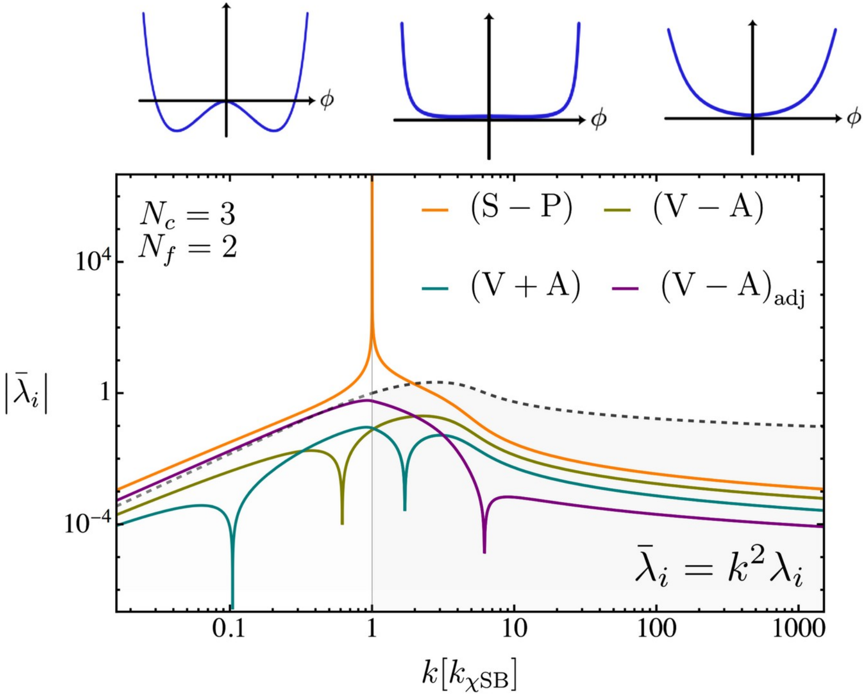
Application 2: Studying Chiral Gauge Theory with fRG

Used to study the dynamical chiral symmetry breaking in QCD-like theories

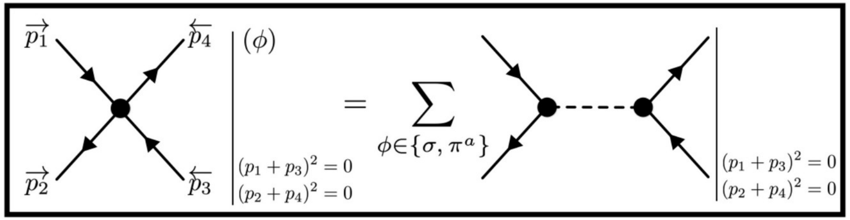
$$\Gamma = \int_x \frac{1}{4}F^2 + i\bar{\psi}D\psi + \mathcal{L}_{gf} + \mathcal{L}_{gh} + \lambda_i(\bar{\psi}\mathcal{T}_i\psi)^2 + \kappa_i(\bar{\psi}\mathcal{T}_i\psi)^3 + \ldots$$

Vertex expansion





Exact field redefinition Stratonovich'57 Hubbard'59



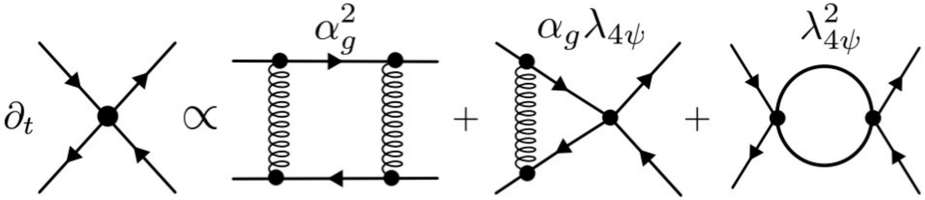
$\lambda_i \propto 1/m_{\phi_i}^2$ $\qquad m_{\phi_i}^2 = \partial_{\phi_i^2}V(\phi_1)$

$\lambda_i \to \infty$ $\qquad m_{\phi_i}^2 \to 0$ $\qquad \xi \to \infty$

$\langle\bar{\psi}\psi\rangle \neq 0$

Infinite correlation length and phase transition

Application 2: Studying Chiral Gauge Theory with fRG

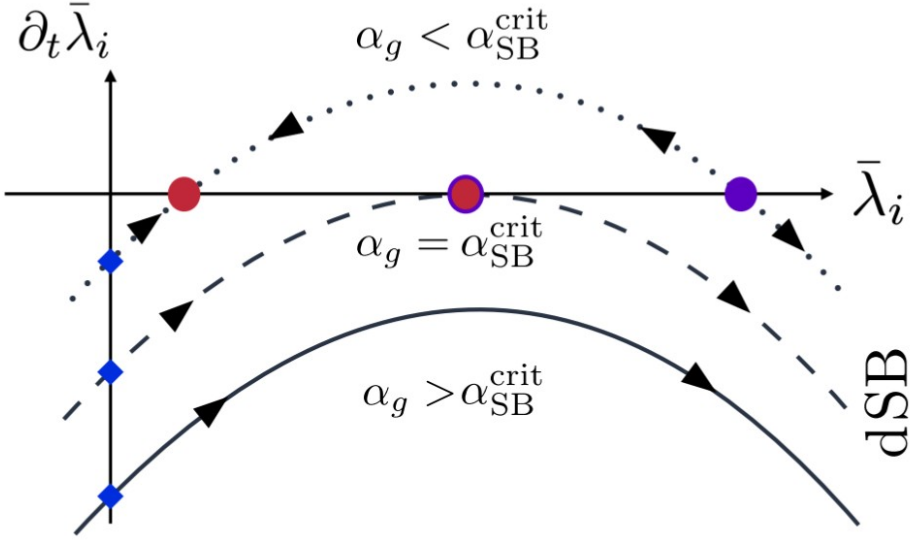


$\lambda_i \to \infty \quad m^2_{\phi_i} = 0 \quad \xi \to \infty \quad \text{PT}$

Goertz,A.P.-Gutierres,Pawlowski[2412.12254]

$$\partial_t \bar{\lambda}_i \propto 2\,\bar{\lambda}_i + \boldsymbol{c}_{\mathrm{A},i} \cdot \alpha_g^2 + \boldsymbol{c}_{\mathrm{B},ij} \cdot \alpha_g\, \bar{\lambda}_j + \boldsymbol{c}_{\mathrm{C},ijk} \cdot \bar{\lambda}_j \bar{\lambda}_k + \ldots$$

- Necessary conditions for dSB:
  1. Resonant structure (<u>naive</u>):
     $$\frac{\boldsymbol{c}_{\mathrm{A},i}}{\boldsymbol{c}_{\mathrm{C},iii}} > 0$$
  2. Critical strength of gauge dynamics is reached:
     $$\alpha_g > \alpha_{\mathrm{SB}}^{\mathrm{crit}}$$



$$\alpha_{\mathrm{SB}}^{\mathrm{crit}} = \inf \left\{ \alpha_g \,\middle|\, \exists i : \ \left(\partial_t \bar{\lambda}_i \le 0 \quad \forall \bar{\lambda}_j \right) \right\}$$

Application 2: Studying Chiral Gauge Theory with fRG

- **Georgi-Glashow model**

  - Traditional GUT

  - Conjectured rich dynamics, multi-scales and condensates, tumbling

	Gauge $SU(N_c)$	Global $SU(N_c-4)$	$U(1)$
ψ	$\overline{\square}$	$\square$	$-(N_c-2)$
χ	$\begin{array}{c}\square\\[-4pt]\square\end{array}$	1	N_c-4

State of the art:

- Most attractive channel

Raby,Dimopoulos,Susskind ’79
Dimopoulos,Raby,Susskind ’80
Eichten,Feinberg ‘82

-Anomaly mediated SUSY

Bai,Stolarski[2111.11214]
Csáki,Murayama,Telem[2104.10171]

- Anomaly matching and higer forms

Bolognesi,Konishi,Luzio[2101.02601]

- Lattice no applicable (Nielsen-Ninomiya Theorem)

Application 2: Studying Chiral Gauge Theory with fRG

➢ **Generalized Georgi-Glashow model**

- Generalization to arbitirity generations N_{gen} copys of G-G model
- Anomaly free and still purely chiral

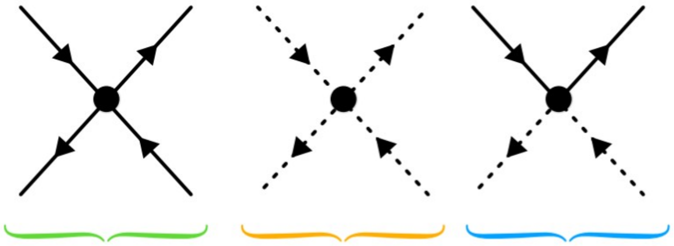
	Gauge $SU(N_c)$	Global $SU(N_{\text{gen}}(N_c-4))$	$SU(N_{\text{gen}})$	$U(1)$
ψ	$\overline{\square}$	$\square$	1	$-(N_c-2)$
χ	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	1	$\square$	N_c-4

- Define the IR phase landscape in N_{gen} vs N_c plane
- Contact with perturbative limit: loss of asymptotic freedom
- Conformal limit: IR FP numerically small but not parametrically

Application 2: Studying Chiral Gauge Theory with fRG

Truncate effective action to four fermion interaction (vertex expansion)

$$\Gamma_{4\mathrm{F}}[\bar{\psi},\psi,\bar{\chi},\chi] = -\int_x Z_\psi^2 \underbrace{\sum_{i=1}^{2} \lambda_i \mathcal{O}_i}_{} + Z_\chi^2 \underbrace{\sum_{i=3}^{5} \lambda_i \mathcal{O}_i}_{} + Z_\psi Z_\chi \underbrace{\sum_{i=6}^{7} \lambda_i \mathcal{O}_i}_{}$$



$$\begin{cases} \mathcal{O}_1 = (\psi^\dagger \bar{\sigma}^\mu \psi)(\psi^\dagger \bar{\sigma}^\mu \psi) \\ \mathcal{O}_2 = (\psi^{\dagger f_1} \bar{\sigma}^\mu \psi_{f_2})(\psi^{\dagger f_2} \bar{\sigma}^\mu \psi_{f_1}) \end{cases}$$

$$\begin{cases} \mathcal{O}_3 = (\chi^{\dagger f_1} \bar{\sigma}^\mu \chi_{f_2})(\chi^{\dagger f_2} \bar{\sigma}^\mu \chi_{f_1}) \\ \mathcal{O}_4 = (\chi^\dagger \bar{\sigma}^\mu \chi)(\chi^\dagger \bar{\sigma}^\mu \chi) \\ \mathcal{O}_5 = (\chi^\dagger \bar{\sigma}^\mu T_{\mathrm{anti}}\, \chi)(\chi^\dagger \bar{\sigma}^\mu T_{\mathrm{anti}}\, \chi) \end{cases}$$

$$\begin{cases} \mathcal{O}_6 = (\psi^\dagger \bar{\sigma}^\mu \psi)(\chi^\dagger \bar{\sigma}^\mu \chi) \\ \mathcal{O}_7 = (\psi^\dagger \bar{\sigma}^\mu T_{\mathrm{a-fund}}\, \psi)(\chi^\dagger \bar{\sigma}^\mu T_{\mathrm{anti}}\, \chi) \end{cases}$$

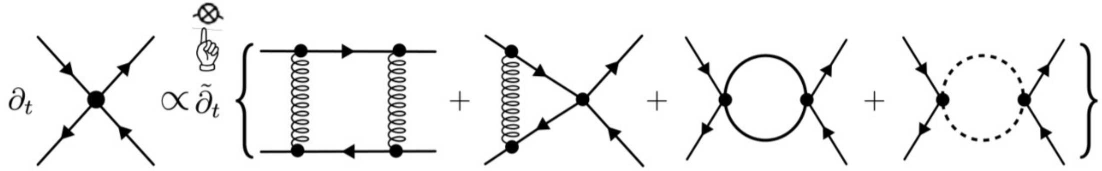
Complete four fermion basis is derived for arbitrary *Ngen* and *Nc*.
Except for *Ngen*=1 and *Nc*=5, where one additional operator is needed:

$$\mathcal{O}_8 = \epsilon_{i_1 i_2 i_3 i_4 i_5} \left(\chi^{i_1 i_2} \chi^{i_3 i_4}\right) \left(\chi^{i_5 j} \psi_j\right)$$

Application 2: Studying Chiral Gauge Theory with fRG

- Flow of four-point functions

$$\partial_t \Gamma_k^{(\bar\psi_a \psi_b \bar\psi_c \psi_d)} = -\partial_t \left(Z_\psi^2 \, \lambda_1 \right) \mathcal{T}_{1,L}^{abcd} - \partial_t \left(Z_\psi^2 \lambda_2 \right) \mathcal{T}_{2,L}^{abcd}$$



$$\partial_t \bar\lambda_i \propto 2\,\bar\lambda_i + \boldsymbol{c}_{\mathrm{A},i} \cdot \alpha_g^2 + \boldsymbol{c}_{\mathrm{B},ij} \cdot \alpha_g \, \bar\lambda_j + \boldsymbol{c}_{\mathrm{C},ijk} \cdot \bar\lambda_j \bar\lambda_k + \ldots$$

$$\partial_t \bar\lambda_1 = (2 + 2\eta_\psi)\bar\lambda_1 + \frac{k^2}{Z_\psi^2 \mathcal{N}} \left\{ \partial_t \left[\mathcal{P}_{1,R}^{abcd} \Gamma_k^{(\bar\psi_a \psi_b \bar\psi_c \psi_d)} \right] (N_c N_\psi + 1) - \partial_t \left[\mathcal{P}_{2,R}^{abcd} \Gamma_k^{(\bar\psi_a \psi_b \bar\psi_c \psi_d)} \right] (N_c + N_\psi) \right\}$$

$$\partial_t \bar\lambda_2 = (2 + 2\eta_\psi)\bar\lambda_2 + \frac{k^2}{Z_\psi^2 \mathcal{N}} \left\{ \partial_t \left[\mathcal{P}_{2,R}^{abcd} \Gamma_k^{(\bar\psi_a \psi_b \bar\psi_c \psi_d)} \right] (N_c N_\psi + 1) - \partial_t \left[\mathcal{P}_{1,R}^{abcd} \Gamma_k^{(\bar\psi_a \psi_b \bar\psi_c \psi_d)} \right] (N_c + N_\psi) \right\}$$

Anomalous dimensions:

$$\eta_i = -\frac{\partial_t Z_i}{Z_i}$$

- System of flows:

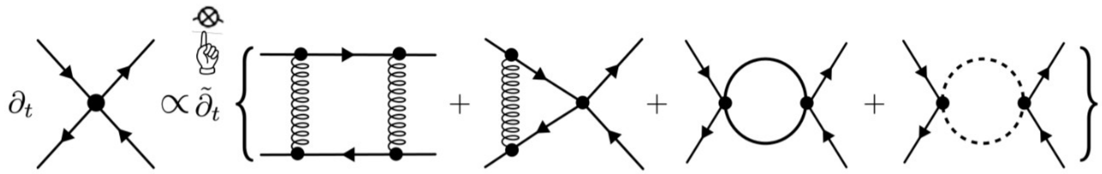
$$\partial_t \bar\lambda_1 = (2 + 2\,\eta_\psi)\bar\lambda_1 + \frac{1}{16\pi^2} \Bigg[\frac{g^4 \left(3N_c^2 + 4\right)(5\eta_A + 3\eta_\psi - 45)}{160 N_c^2} + \frac{g^2 (5\eta_A + 6\eta_\psi - 60)(\bar\lambda_1 - \bar\lambda_2 N_c)}{10 N_c}$$
$$- \frac{1}{20}(\eta_\chi - 5) N_\chi \left(\bar\lambda_6^2 (N_c - 1) N_c + \frac{(N_c - 2)}{2N_c} \bar\lambda_7^2 \right) + \frac{4}{5}(\eta_\psi - 5) \left(\bar\lambda_2^2 - \bar\lambda_1 \bar\lambda_2 (N_c + N_\psi) - \bar\lambda_1^2 \frac{(N_c N_\psi - 1)}{2} \right) \Bigg]$$

$$\partial_t \bar\lambda_2 = (2 + 2\,\eta_\psi)\bar\lambda_2 + \frac{1}{16\pi^2} \Bigg[\frac{g^4 \left(N_c^2 - 8\right)(5\eta_A + 3\eta_\psi - 45)}{160 N_c} + \frac{g^2 (5\eta_A + 6\eta_\psi - 60)(\bar\lambda_2 - \bar\lambda_1 N_c)}{10 N_c}$$
$$- \frac{1}{40}(\eta_\chi - 5) \bar\lambda_7^2 (N_c - 2) N_\chi + \frac{2}{5}(\eta_\psi - 5) \left(4\bar\lambda_1 \bar\lambda_2 - \bar\lambda_2^2 (N_c + N_\psi) \right) \Bigg]$$

New technology is developed for the analytical tracing involving anti-symmetric representation.

Application 2: Studying Chiral Gauge Theory with fRG

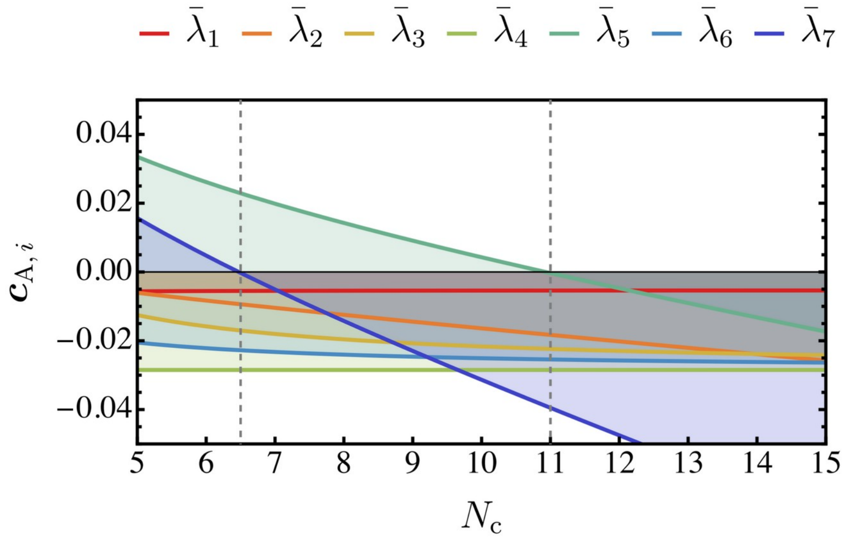
- Finding the resonant structure



$$\partial_t \bar{\lambda}_i \propto 2\,\bar{\lambda}_i + \boldsymbol{c}_{\mathrm{A},i} \cdot \alpha_g^2 + \boldsymbol{c}_{\mathrm{B},ij} \cdot \alpha_g\,\bar{\lambda}_j + \boldsymbol{c}_{\mathrm{C},ijk} \cdot \bar{\lambda}_j \bar{\lambda}_k + \ldots$$

-Necessary condition: $\dfrac{\boldsymbol{c}_{A,i}}{\boldsymbol{c}_{C,iii}} > 0$

-In generalized G-G models: $\boldsymbol{c}_{C,iii} > 0 \quad \forall i$

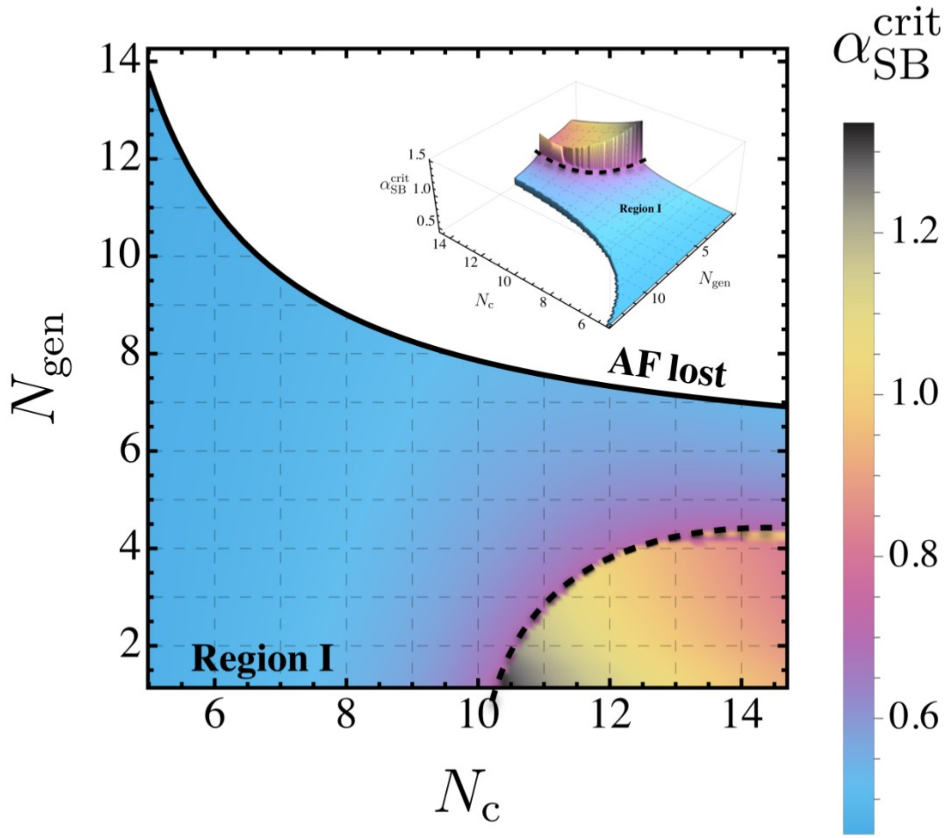


$$\boldsymbol{c}_{\mathrm{A},\,i} = \frac{9}{2}\left\{ -\frac{1}{4N_c^2} - \frac{3}{16},\ -\frac{N_c^2-8}{16N_c},\ -1,\ \boxed{-1+\frac{4+2N_c}{N_c^2}},\ 1-\frac{N_c}{8}+\frac{4}{N_c},\ \boxed{-1+\frac{N_c+2}{N_c^2}},\ 1-\frac{N_c}{4}+\frac{4}{N_c} \right\}.$$

$$\mathcal{O}_7 = \left(\psi^\dagger \bar{\sigma}^\mu\, T_{\mathrm{a-fund}}\, \psi\right)\left(\chi^\dagger \bar{\sigma}^\mu\, T_{\mathrm{anti}}\, \chi\right)$$

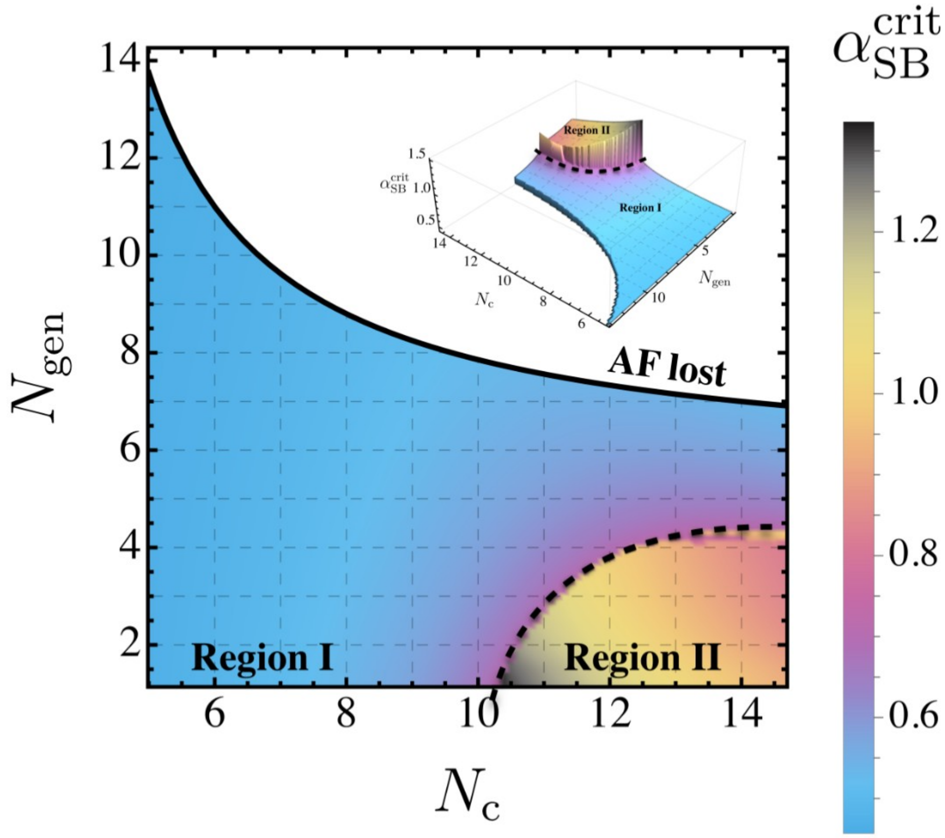
$$\mathcal{O}_5 = \left(\chi^\dagger \bar{\sigma}^\mu\, T_{\mathrm{anti}}\, \chi\right)\left(\chi^\dagger \bar{\sigma}^\mu\, T_{\mathrm{anti}}\, \chi\right)$$

Application 2: Studying Chiral Gauge Theory with fRG



- **Region I:**
  - Weak $\alpha_g^{\rm crit}$: dynamics derivable within perturbation theory
  - Clear dominance of $\mathcal{O}_5 = (\chi^\dagger \bar{\sigma}^\mu T_{\rm anti}\,\chi)(\chi^\dagger \bar{\sigma}^\mu T_{\rm anti}\,\chi)$
  - Condensate $\langle \chi\chi \rangle \neq 0$

Application 2: Studying Chiral Gauge Theory with fRG



- **Region I:**
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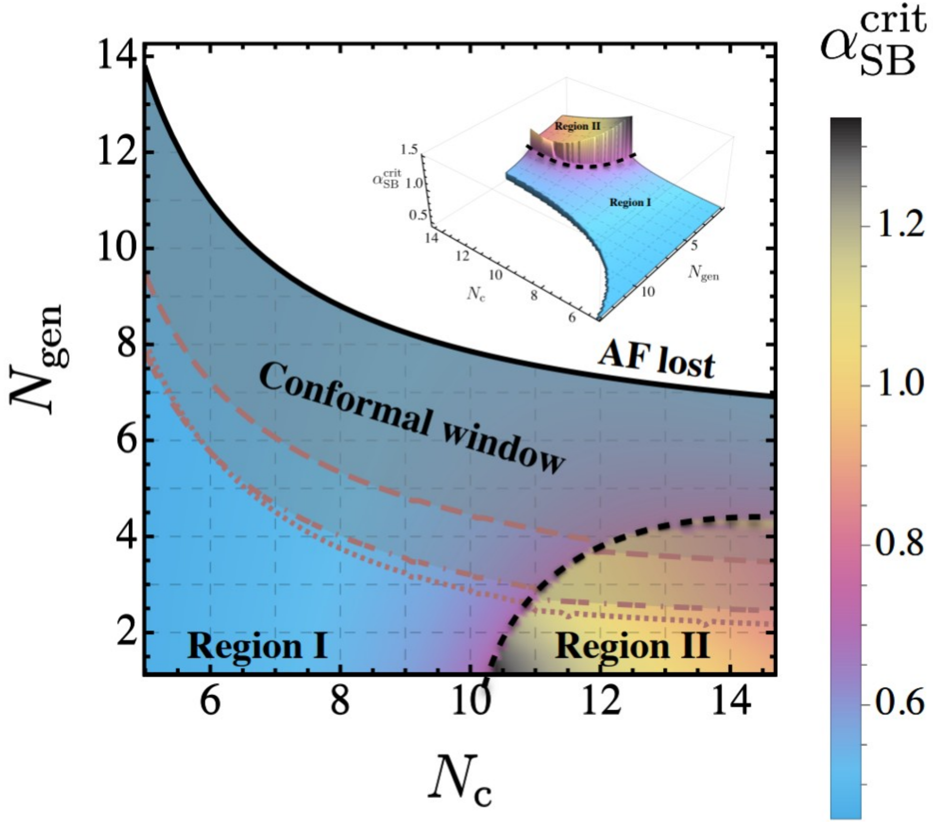
  - Condensate $\langle \chi\chi \rangle \neq 0$

- **Region II:**
  -strong α_g^{crit} non-perturbative, higer-order effects relevant
  -cannot resolve a clear single resonant channel.

Application 2: Studying Chiral Gauge Theory with fRG



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  - Condensate $\langle \chi\chi \rangle \neq 0$

- **Region II:**
  -strong $\alpha_g^{\rm crit}$ non-perturbative, higer-order effects relevant
  -cannot resolve a clear single resonant channel.

- **Conformal window:**
  $\alpha_g^{\rm crit} = \alpha_g^*$

$$\begin{cases} \alpha_g^*\big|_{\overline{\rm MS}\,2-\rm loop} & \text{- - - -} \\ \alpha_g^*\big|_{\overline{\rm MS}\,3-\rm loop} & \text{— · —} \\ \alpha_g^*\big|_{\overline{\rm MS}\,4-\rm loop} & \cdots\cdots \end{cases}$$

New!

Summary:

- J-basis as generalized partial-wave basis can be derived systematically with Casimir operator method

- N-body massless phase space is completely solvable using the spinor variable technique.

- Basis construction can help advance the study of non-perturbative method like fRG.

- In generalized G-G model, we find the evidence of condensate in the anti-symmetric representation .

- Future work: obtain the exact direction of condensate, gauge J-basis will help to diagnose.

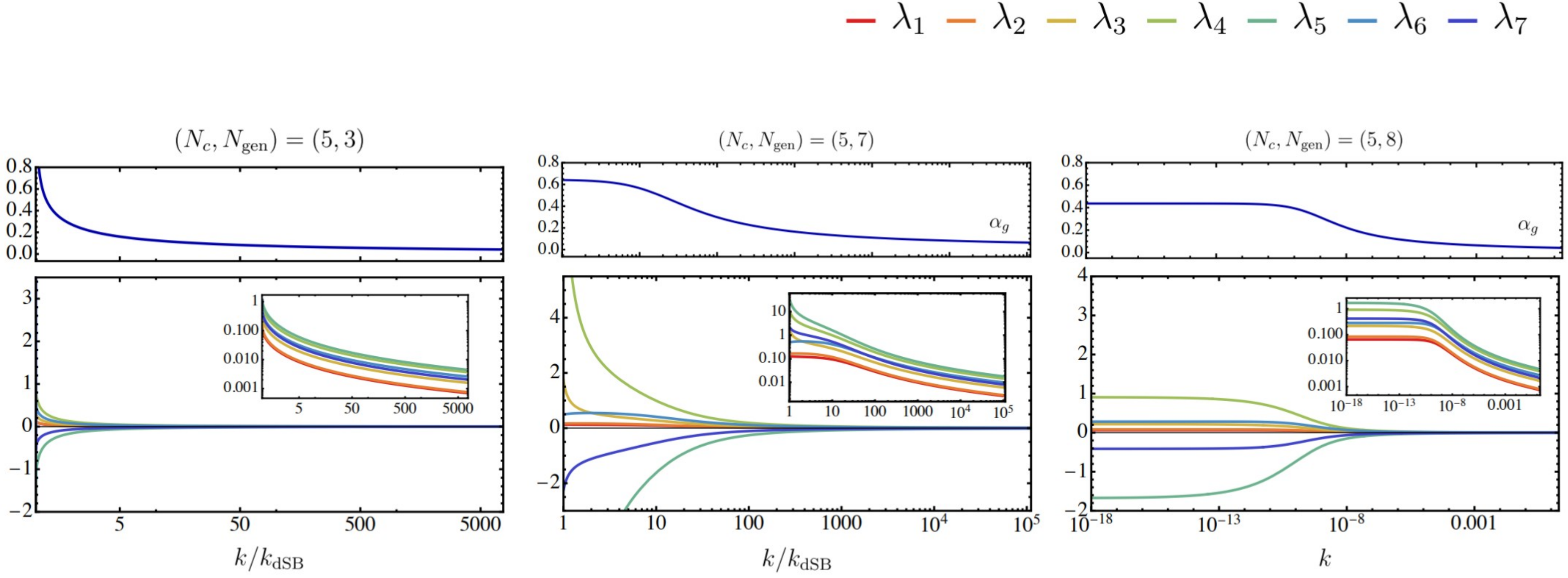
Comparison Ngen=1

MAC: $\langle \chi\psi \rangle$ $N=5, N\geq 7$ $\langle \chi\chi \rangle$ $\langle \chi\psi \rangle$ $N=6$

Anomaly matching: $\langle \chi\chi \rangle$ $\langle \chi\psi \rangle$

Anomaly mediated SUSY breaking $\langle \chi\psi \rangle$ $\langle \psi\psi \rangle$

Some result



λ_1 λ_2 λ_3 λ_4 λ_5 λ_6 λ_7
$(N_c, N_{\rm gen}) = (5,3)$
$(N_c, N_{\rm gen}) = (5,7)$
$(N_c, N_{\rm gen}) = (5,8)$
α_g
$k/k_{\rm dSB}$
k

Sherman–Morrison Formula

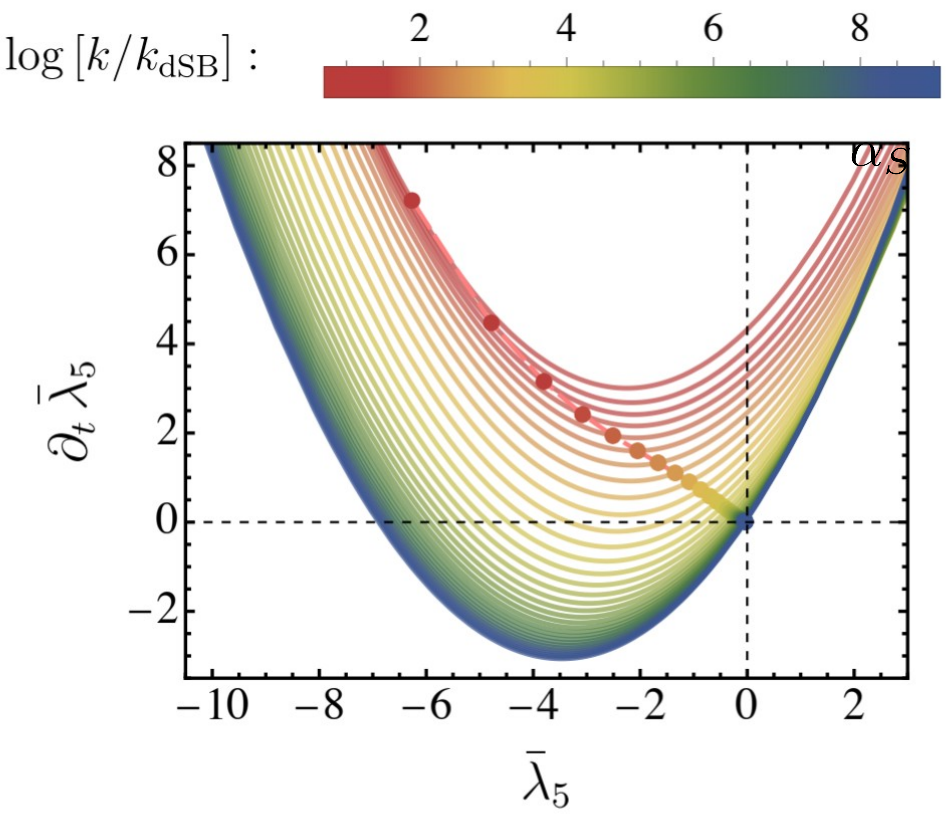
$$\sqrt{1+\mathbf{u}^T\mathbf{u}} = 1 + \left(\frac{\sqrt{1+|\mathbf{u}|^2}-1}{|\mathbf{u}|^2}\right)\mathbf{u}^T\mathbf{u},$$

$$\left(1+\mathbf{u}^T\mathbf{u}\right)^{-1} = 1 - \frac{1}{1+|\mathbf{u}|^2}\mathbf{u}^T\mathbf{u},$$

$$\begin{aligned} \mathbb{C}_2 &= \mathbb{T}^a\mathbb{T}^a, \text{ for both } SU(2) \text{ and } SU(3), \\ \mathbb{C}_3 &= d^{abc}\mathbb{T}^a\mathbb{T}^b\mathbb{T}^c, \text{ for } SU(3) \text{ only,} \end{aligned}$$

$$\mathbb{T}^A_{\otimes\{\mathbf{r}_i\}} = \sum_{i=1}^{N} E_{\mathbf{r}_1} \times E_{\mathbf{r}_2} \times \cdots \times T^A_{\mathbf{r}_i} \times \ldots E_{\mathbf{r}_N}$$

$$\underset{\mathbb{S}}{\mathbb{T}}^A \circ \Theta_{I_1 I_2 \ldots I_N} = \sum_{i\in\mathbb{S}}^{N} (T^A_{r_i})^Z_{I_i}\Theta_{I_1\ldots I_{i-1} Z I_{i+1} I_N}.$$


$\log [k/k_{\mathrm{dSB}}]$:
2
4
6
8
α_S
8
6
4
2
0
−2
$\partial_t \bar{\lambda}_5$
−10
−8
−6
−4
−2
0
2
$\bar{\lambda}_5$

Nc=5, Ngen=7

$$S = \int \bar{\psi} \partial\!\!\!/ \psi - \frac{\lambda}{2} \left(\left(\bar{\psi}\psi \right)^2 - \left(\bar{\psi}\gamma^5\psi \right)^2 \right)$$

$$Z \propto \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{S} = \mathcal{N} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\phi \;\; e^{\frac{m^2}{2}\phi^2} \; e^{S}$$

$$\phi = (\sigma, \pi)$$

$$\sigma \to \sigma + y \frac{\bar{\psi}\psi}{\sqrt{2}m^2} \qquad \pi \to \pi + i\, y \frac{\bar{\psi}\gamma^5\psi}{\sqrt{2}m^2}$$

$$Z \propto \mathcal{N} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\phi e^{S_{BF}}$$

$$S_{FB} = \int \bar{\psi}\partial\psi + \frac{y}{\sqrt{2}} \bar{\psi} \left(\sigma + i\pi\gamma^5 \right) \psi + \frac{m^2}{2} \left(\sigma^2 + \pi^2 \right) \qquad \lambda \equiv \frac{y^2}{2m^2}$$

$$\sigma = \frac{-y}{\sqrt{2}m^2} \bar{\psi}\psi \qquad \pi = \frac{-iy}{\sqrt{2}m^2} \bar{\psi}\gamma^5\psi$$

$$S_{\mathrm{B}} = \int \mathrm{d}^4 x \left\{ m^2 \phi^* \phi - \ln \det \left[\mathrm{i}\partial\!\!\!/ + h \left(P_{\mathrm{L}}\phi - P_{\mathrm{R}}\phi^* \right) \right] \right\}$$

$$U(1)_A \qquad \begin{pmatrix} \sigma \\ \pi \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{pmatrix} \begin{pmatrix} \sigma \\ \pi \end{pmatrix}$$

$$\partial_t \Gamma[\bar{\phi}] = \frac{1}{2} \mathrm{Tr}\, G_k\, \partial_t R_k\,,$$

$$\partial_t \Gamma^{(1)}[\bar{\phi}] = -\frac{1}{2} \mathrm{Tr}\, \Gamma_k^{(3)} \left(G_k\, \partial_t R_k\, G_k \right),$$

$$\partial_t \Gamma^{(2)}[\bar{\phi}] = -\frac{1}{2} \mathrm{Tr} \left[\Gamma_k^{(4)} - 2\, \Gamma_k^{(3)}\, G_k\, \Gamma_k^{(3)} \right] \left(G_k\, \partial_t R_k\, G_k \right),$$

$$\partial_t \Gamma^{(3)}[\bar{\phi}] = -\frac{1}{2} \mathrm{Tr} \left[\Gamma_k^{(5)} - 6\, \Gamma_k^{(4)}\, G_k\, \Gamma_k^{(3)} + 6\, \Gamma_k^{(3)}\, G_k\, \Gamma_k^{(3)}\, G_k\, \Gamma_k^{(3)} \right] \left(G_k\, \partial_t R_k\, G_k \right),$$

$$\begin{aligned}\partial_t \Gamma^{(4)}[\bar{\phi}] = -\frac{1}{2} \mathrm{Tr} \Big[& \Gamma_k^{(6)} - 8\, \Gamma_k^{(5)}\, G_k\, \Gamma_k^{(3)} - 6\, \Gamma_k^{(4)}\, G_k\, \Gamma_k^{(4)} + 18\, \Gamma_k^{(4)}\, G_k\, \Gamma_k^{(3)}\, G_k\, \Gamma_k^{(3)} \\ & + 12\, \Gamma_k^{(3)}\, G_k\, \Gamma_k^{(4)}\, G_k\, \Gamma_k^{(3)} - 24\, G_k\, \Gamma_k^{(3)}\, G_k\, \Gamma_k^{(3)}\, G_k\, \Gamma_k^{(3)} \cdot G_k\, \Gamma_k^{(3)} \Big] \left(G_k\, \partial_t R_k\, G_k \right),\end{aligned}$$

Repeated Field: Y-Basis to P-Basis

Once obtained the complete and independent Lorentz and Gauge Y-Basis for a given type,
Then we obtain a basis of independent Flavor-Blind Y-Basis operator

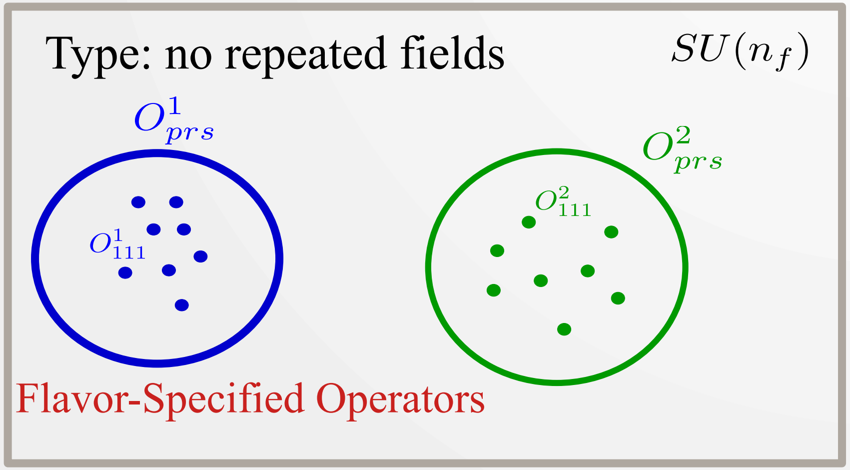
Gauge Y-Basis
Dim d_G

$\otimes$

Lorentz Y-Basis
Dim: d_B

↓

Flavor-Blind $\{ \mathcal{O}_i^{y} \}$ Y-Basis
$i = 1,2,\ldots, d_G \cdot d_B$



Example:

$$d_{\mathcal{C}} u_{\mathcal{C}} d_{\mathcal{C}}^{\dagger} u_{\mathcal{C}}^{\dagger}$$

$$(d_{\mathcal{C}p})^{a}_{\alpha_1} (u_{\mathcal{C}r})^{b}_{\alpha 2} (d_{\mathcal{C}}{}^{\dagger}_{s})^{\dot\alpha_3}_{c} (u_{\mathcal{C}}{}^{\dagger}_{t})^{\dot\alpha_4}_{d}$$

$$T_G = \delta^c_a \delta^d_b, \quad \delta^d_a \delta^c_b \quad \bigotimes \quad \mathcal{M}^{y} = (d^a_{\mathcal{C}p} u^b_{\mathcal{C}r})(d_{\mathcal{C}c}{}^{\dagger}_{s} u_{\mathcal{C}d}{}^{\dagger}_{t})$$

↓

$$\mathcal{O}^1_{prst} = (d_{\mathcal{C}}{}^{a}_{p} u_{\mathcal{C}}{}^{b}_{r})(d_{\mathcal{C}}{}^{\dagger}_{s a} u_{\mathcal{C}}{}^{\dagger}_{t b})$$

$$\mathcal{O}^2_{prst} = (d_{\mathcal{C}}{}^{a}_{p} u_{\mathcal{C}}{}^{b}_{r})(d_{\mathcal{C}}{}^{\dagger}_{s b} u_{\mathcal{C}}{}^{\dagger}_{t a})$$

Each can be viewed as
independnet generic flavor tensor.

Repeated Field: Y-Basis to P-Basis

The repeated fields: fields with the same quantum numbers

L^{f_1} And L^{f_2} are repeated fields, L and $L^\dagger$ are not.

It is well-known that : $O^{f_1 f_2}_{LLHH} = O^{f_2 f_1}_{LLHH}$

Flavor relation are not simple to derive

$$
\begin{aligned}
L^{f_1} L^{f_2} H H &= \epsilon^{i_1 j_1} \epsilon^{i_2 j_2} \epsilon^{\alpha_1 \alpha_2} L^{f_1}_{\alpha_1, i_1} L^{f_2}_{\alpha_2, i_2} H_{j_1} H_{j_2} \\
&= \epsilon^{i_2 j_1} \epsilon^{i_1 j_2} \epsilon^{\alpha_2 \alpha_1} L^{f_1}_{\alpha_2, i_2} L^{f_2}_{\alpha_1, i_1} H_{j_1} H_{j_2} \\
&= \epsilon^{i_2 j_1} \epsilon^{i_1 j_2} \epsilon^{\alpha_1 \alpha_2} L^{f_2}_{\alpha_1, i_1} L^{f_1}_{\alpha_2, i_2} H_{j_1} H_{j_2} \\
&= \epsilon^{i_1 j_1} \epsilon^{i_2 j_2} \epsilon^{\alpha_1 \alpha_2} L^{f_2}_{\alpha_1, i_1} L^{f_1}_{\alpha_2, i_2} H_{j_2} H_{j_1} \\
&= L^{f_2} L^{f_1} H H
\end{aligned}
$$

rename i's and α's

Exchange α in ϵ
and swap L (anticommute)

rename j's
two H's are symmetric

Repeated Field: Y-Basis to P-Basis

What about Q^3L?

Oringally in Ref.[1,2]:

$$Q_{prst}^{qqq\ell\,(1)} = \epsilon_{\alpha\beta\gamma}\epsilon_{ij}\epsilon_{kl}(q_p^{i\alpha}Cq_r^{j\beta})(q_s^{\gamma k}Cl_t^l)\,,$$
$$Q_{prst}^{qqq\ell\,(3)} = \epsilon_{\alpha\beta\gamma}(\tau^I\epsilon)_{ij}(\tau^I\epsilon)_{kl}(q_p^{i\alpha}Cq_r^{j\beta})(q_s^{\gamma k}Cl_t^l)$$

	B-violating
Q_{duq}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\left[(d_p^\alpha)^T Cu_r^\beta\right]\left[(q_s^{\gamma j})^T Cl_t^k\right]$
Q_{qqu}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\left[(q_p^{\alpha j})^T Cq_r^{\beta k}\right]\left[(u_s^\gamma)^T Ce_t\right]$
$Q_{qqq}^{(1)}$	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\varepsilon_{mn}\left[(q_p^{\alpha j})^T Cq_r^{\beta k}\right]\left[(q_s^{\gamma m})^T Cl_t^n\right]$
$Q_{qqq}^{(3)}$	$\varepsilon^{\alpha\beta\gamma}(\tau^I\varepsilon)_{jk}(\tau^I\varepsilon)_{mn}\left[(q_p^{\alpha j})^T Cq_r^{\beta k}\right]\left[(q_s^{\gamma m})^T Cl_t^n\right]$
Q_{duu}	$\varepsilon^{\alpha\beta\gamma}\left[(d_p^\alpha)^T Cu_r^\beta\right]\left[(u_s^\gamma)^T Ce_t\right]$

Latter it is found in Ref.[3] that:

$$Q_{prst}^{qqq\ell} = \epsilon_{\alpha\beta\gamma}\epsilon_{il}\epsilon_{jk}(q_p^{i\alpha}Cq_r^{j\beta})(q_s^{k\gamma}C\ell_t^l)$$

$$Q_{prst}^{qqq\ell} + Q_{rpst}^{qqq\ell} = Q_{sprt}^{qqq\ell} + Q_{srpt}^{qqq\ell}$$

$$Q_{prst}^{qqq\ell\,(1)} = -(Q_{prst}^{qqq\ell} + Q_{rpst}^{qqq\ell})\,,$$
$$Q_{prst}^{qqq\ell\,(3)} = -(Q_{prst}^{qqq\ell} - Q_{rpst}^{qqq\ell})\,,$$

Flavor relations

Disadvantage:

1.It's an agony to find these relations by hand with increasing dimension.

2. It's hard to tell how to find independent entries—Flavor Specified Operaotrs (or equivalently how to parameterize the wilson coefficients)

[1]L. F. Abbott and Mark B. Wise, Phys. Rev. D 22, 2208
[2]B. Grzadkowski, M. Iskrzyński, M. Misiak & J. Rosiek, JHEP 10 (2010) 085
[3]R. Alonso, H.-M. Chang, E. E. Jenkins, A. V. Manohar, B. Shotwell, Physics Letters B 734 (2014) 302

Repeated Field: Y-Basis to P-Basis

Go back to *LLHH*:
What do we really mean when we say something is totally symmetric or antisymmetric?

It actually means
the objects is a 1-dim irreducible representation of the corresponding S_m group

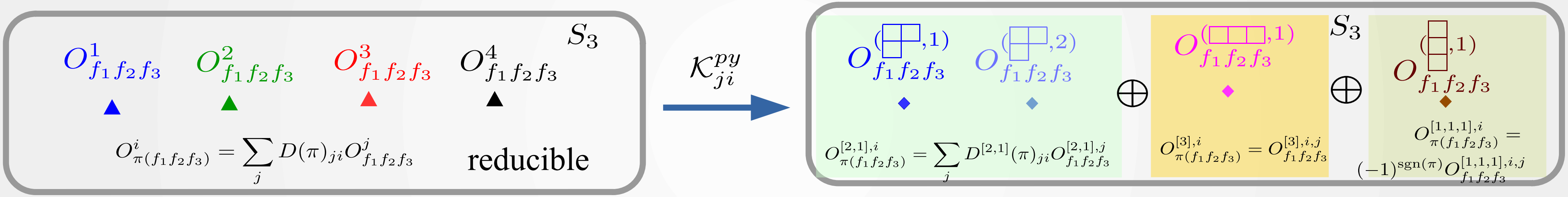
$$\underbrace{\square \cdots \square}_{m} = [m] \qquad m\left\{\begin{array}{c}\square\\ \square\\ \vdots\\ \square\end{array}\right. = [1,1,\ldots,1] = [1^m]$$

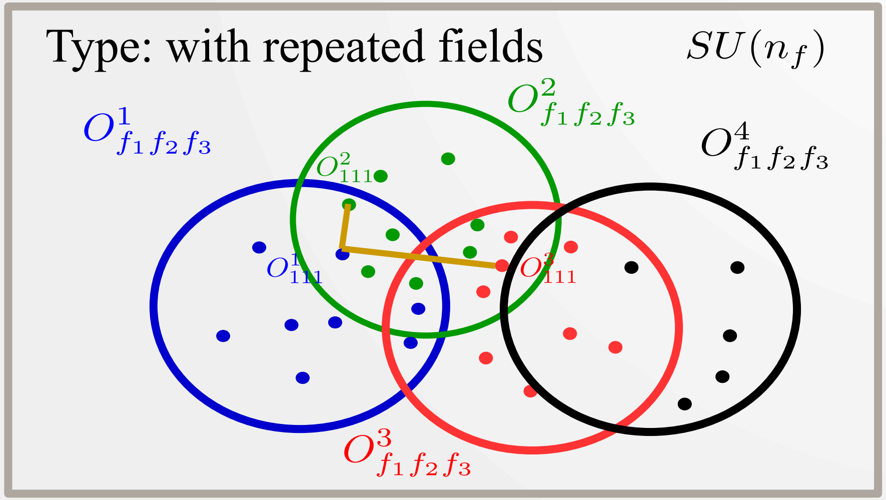
Independent
Flavor Specifed operators :

Repeated Field: Y-Basis to P-Basis

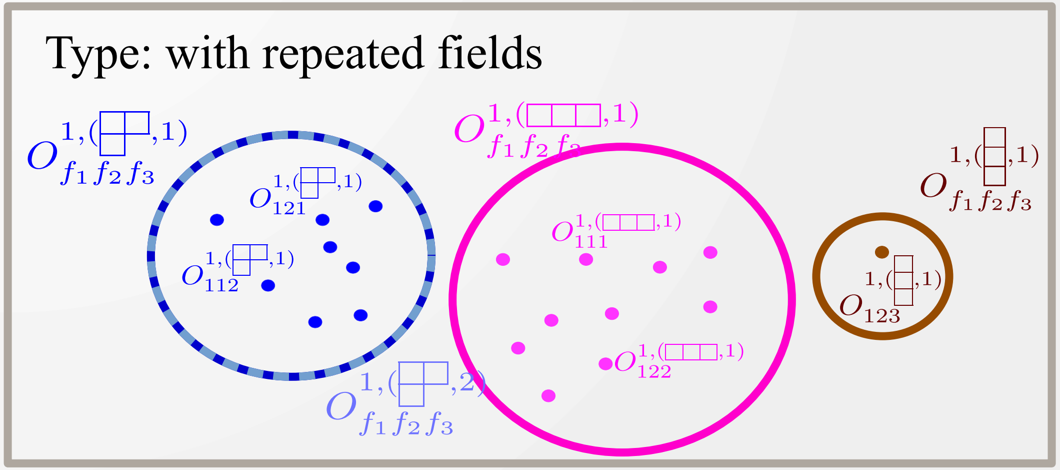
Flavor-Blind $\{\ O_i^{\mathrm{y}}\ \}$ Y-Basis
$i = 1,2,\ldots,\ d_G \cdot d_B$

Flavor-Blind $\{O_j^{\mathrm{p}}\ \}$ P-Basis
$i = 1,2,\ldots,\ d_G \cdot d_B$





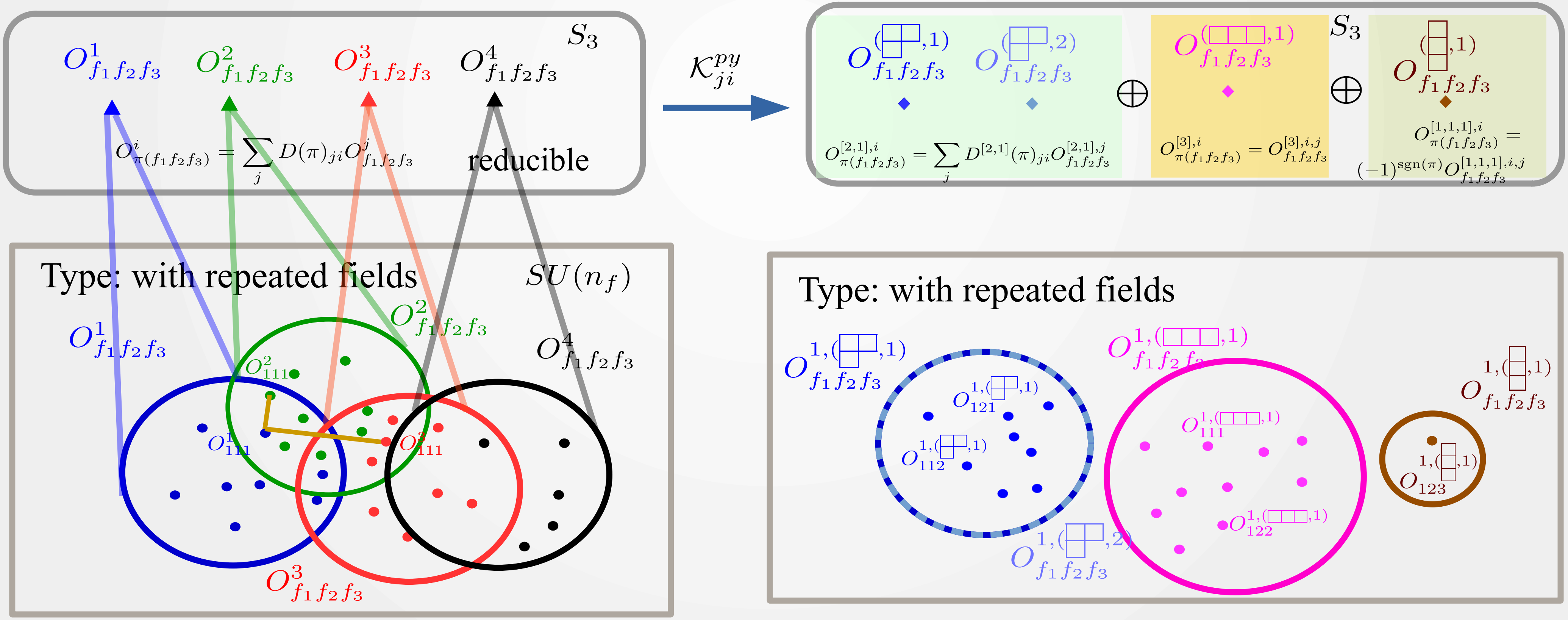
$$O^1_{111} = xO^2_{111} + yO^3_{111}$$



Repeated Field: Y-Basis to P-Basis

Flavor-Blind $\{ O^{y}_{i} \}$ Y-Basis
$i = 1,2,\ldots, d_G \cdot d_B$

Flavor-Blind $\{ O^{p}_{j} \}$ P-Basis
$i = 1,2,\ldots, d_G \cdot d_B$

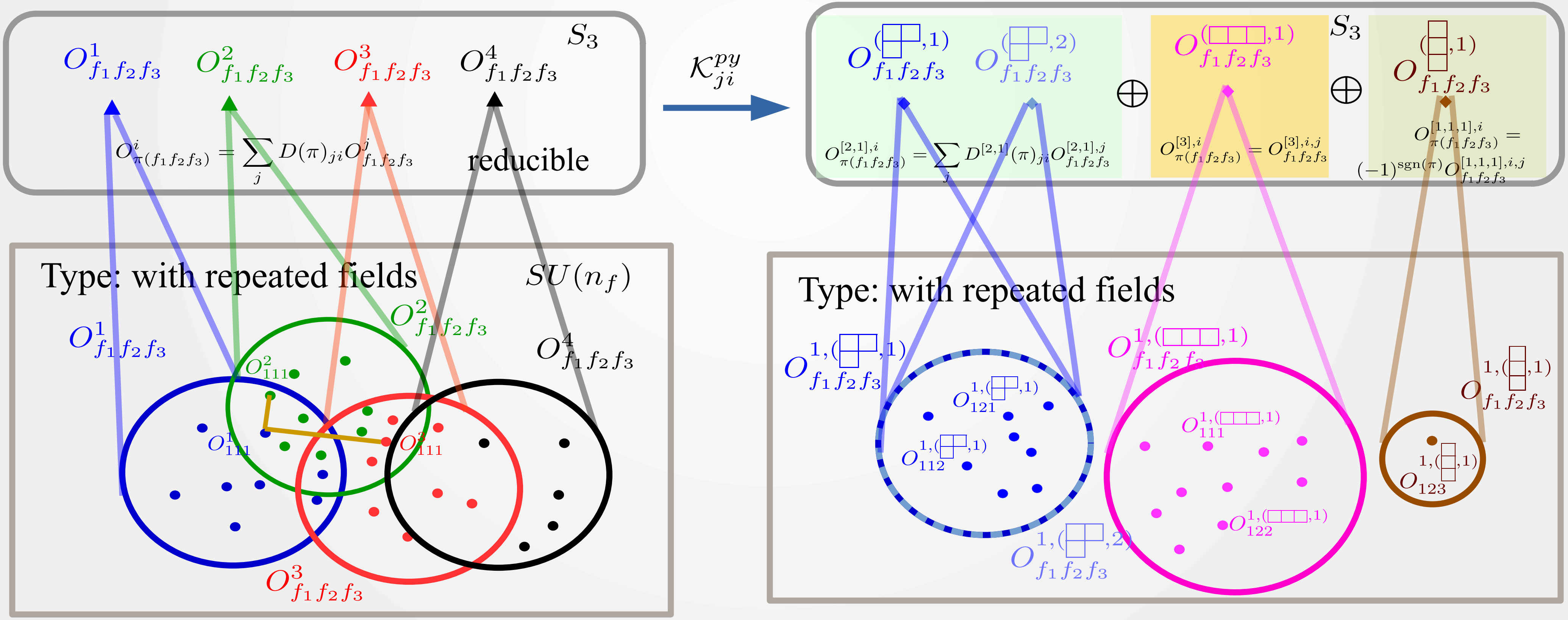


$O^1_{111} = xO^2_{111} + yO^3_{111}$

Repeated Field: Y-Basis to P-Basis

Flavor-Blind $\{ \mathcal{O}^{y}_{i} \}$ Y-Basis
$i = 1,2,\ldots, d_G \cdot d_B$

Flavor-Blind $\{ \mathcal{O}^{p}_{j} \}$ P-Basis
$i = 1,2,\ldots, d_G \cdot d_B$



$$\mathcal{O}^1_{111} = x\mathcal{O}^2_{111} + y\mathcal{O}^3_{111}$$

Two neccceary ingredients

1. What kind of S_m or SU(n_f) irreps that one operator type can have.

Sym2int & GroupMath Renato M. Fonseca, Phys. Rev. D 101, 035040 (2020)
ABC4EFT **H.-L. Li**, Z. Ren, J. Shu, M.-L. Xiao, J.-H.Yu, Y.-H. Zheng, arXiv: 2005.00008
H.-L. Li, Z. Ren, M.-L. Xiao, J.-H.Yu, Y.-H. Zheng, arXiv: 2007.07899

2. Known the possible irreps how to obtain the form of the corresponding operators

ABC4EFT **H.-L. Li**, Z. Ren, J. Shu, M.-L. Xiao, J.-H.Yu, Y.-H. Zheng, arXiv: 2005.00008
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Resolution to the 1st

An operator point of view

$$\underbrace{\pi \circ \mathcal{O}^{\{f_k,\dots\}}}_{\text{permute flavor}} = T_{\text{SU3}}^{\{g_k,\dots\}} T_{\text{SU2}}^{\{h_k,\dots\}} \mathcal{M}^{\{f_{\pi(k)},\dots\}}_{\{g_k,\dots\},\{h_k,\dots\}}$$

$$= T_{\text{SU3}}^{\{g_{\pi(k)},\dots\}} T_{\text{SU2}}^{\{h_{\pi(k)},\dots\}} \mathcal{M}^{\{f_{\pi(k)},\dots\}}_{\{g_{\pi(k)},\dots\},\{h_{\pi(k)},\dots\}}$$

$$= \underbrace{\left(\pi \circ T_{\text{SU3}}^{\{g_k,\dots\}}\right)\left(\pi \circ T_{\text{SU2}}^{\{h_k,\dots\}}\right)}_{\text{permute gauge}} \underbrace{\left(\pi \circ \mathcal{M}^{\{f_k,\dots\}}_{\{g_k,\dots\},\{h_k,\dots\}}\right)}_{\text{permute Lorentz}}$$

Rename the dummy indices

Example: $LLHH$

$$T_{SU(2)}^{i_1 i_2, j_1 j_2} = (\epsilon^{i_1 j_1}\epsilon^{i_2 j_2} + \epsilon^{i_2 j_1}\epsilon^{i_1 j_2})$$

$$\begin{aligned}
(12) \circ \mathcal{O}^{f_1 f_2} &= \mathcal{O}^{f_2 f_1} \\
&= T_{SU2}^{i_1 i_2, j_1 j_2} \epsilon^{\alpha_1 \alpha_2} L^{f_2}_{\alpha_1, i_1} L^{f_1}_{\alpha_2, i_2} H_{j_1} H_{j_2} \\
&= T_{SU(2)}^{i_2 i_1, j_1 j_2} \epsilon^{\alpha_2 \alpha_1} L^{f_1}_{\alpha_2, i_1} L^{f_2}_{\alpha_1, i_2} H_{j_1} H_{j_2} \\
&= \left(\pi \circ T_{SU2}^{i_1 i_2, j_1 j_2}\right)\left(\pi \circ \mathcal{M}^{\{f_1 f_2, 11\}}_{\{i_1 i_2, j_1 j_2\}}\right) \\
&= \mathcal{O}^{f_1 f_2, 11}
\end{aligned}$$

Allowed irreps of flavor is determined by irreps of gauge and Lorentz

$$\lambda_f = \lambda_G \odot \lambda_{\mathcal{M}}$$

Resolution to the 2nd

Known the possible S_m or $SU(n_f)$ irreps how to obtain the operators with that permutation symmetry for flavor indices?

$$T^{abc} \longrightarrow T^{abc}_{\begin{array}{|c|c|}\hline \, & \, \\ \hline \, & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}} = \mathcal{Y}\left[\begin{array}{|c|c|}\hline a & b \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right] \circ T^{abc}$$

Antisymmetrize the Columns, then Symmetrize the rows

Antisym *ac* first: $T^{abc} - T^{cba}$

Then symm *ab*: $(T^{abc}) + T^{bac} - (T^{cba} + T^{cab})$

What we have: Flavor-Blind Y-Basis,

The remaining problem is how to find the set of Flavor-Blind Y-Basis operators, such that acting on the Young symmetrizer they become different space

Potential problem:

$$\mathcal{Y}\left[\begin{array}{|c|c|}\hline a & b \\ \hline\end{array}\right] \circ O^{ab}_{(1)} = \mathcal{Y}\left[\begin{array}{|c|c|}\hline a & b \\ \hline\end{array}\right] \circ O^{ab}_{(2)}$$

$$\mathcal{Y}\left[\begin{array}{|c|c|}\hline a & b \\ \hline\end{array}\right] \circ O^{ab}_{(1)} = 0$$

Resolution to the 2nd

We can obtain the represenation matrix of S_m in the Gauge and Lorentz Y-Basis :

$$\pi \circ T_i^{\mathrm{y}} = \sum_i D_G[\pi]_{ij} T_j^{\mathrm{y}} \qquad \pi \circ \mathcal{M}_i^{\mathrm{y}} = \sum_i D_L[\pi]_{ij} \mathcal{M}_j^{\mathrm{y}}$$

Then

$$\pi \circ \mathcal{O}_{(ij)}^{\mathrm{y}} = \sum_{kl} D_G[\pi]_{ik} D_G[\pi]_{jl} T_k^{\mathrm{y}} \mathcal{M}_l^{\mathrm{y}} \qquad D_{\mathcal{O}}[\pi] = D_G[\pi] \otimes D_L[\pi]$$

$$\mathcal{Y} = \sum_i c_i \pi_i \longrightarrow \mathcal{Y} \circ O_i^{\mathrm{y}} = \sum_j D_{\mathcal{O}}[\mathcal{Y}]_{ij} O_j^{\mathrm{y}} \qquad D_{\mathcal{O}}[\mathcal{Y}] = \sum_i c_i D_{\mathcal{O}}[\pi]$$

Choose independent rows in matrix $D_{\mathcal{O}}[\mathcal{Y}]_{ij}$ as P-Basis operators -- Terms

Example of LQQQ

$$T^{\mathrm{y}}_{\mathrm{SU(2)},1} = \epsilon^{ik}\epsilon^{jl},\ T^{\mathrm{y}}_{\mathrm{SU(2)},2} = \epsilon^{ij}\epsilon^{kl}$$

$$(i \to 1, j \to 2, k \to 3, l \to 4)$$

$$D_{\mathrm{SU(2)}}[(12)] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$D_{\mathrm{SU(2)}}[(123)] = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$B^{\mathrm{y}}_1 = \epsilon^{\alpha\beta}\epsilon^{\gamma\delta} L_{pi\alpha} Q_{raj\beta} Q_{sbk\gamma} Q_{tcl\delta}$$
$$B^{\mathrm{y}}_2 = \epsilon^{\alpha\gamma}\epsilon^{\beta\delta} L_{pi\alpha} Q_{sbk\gamma} Q_{raj\beta} Q_{tcl\delta}$$

$$(\beta \to 1, \gamma \to 2, \delta \to 3)$$

$$D_{\mathrm{L}}[(12)] = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$D_{\mathrm{L}}[(123)] = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

$$T^{\mathrm{y}}_{\mathrm{SU(3)},1} = \epsilon^{abc}$$

$$(a \to 1, b \to 2, c \to 3)$$

$$D_{\mathrm{SU(3)}}[(12)] = -1$$

$$D_{\mathrm{SU(3)}}[(123)] = 1$$

$$\mathcal{O}^{\mathrm{y}}_i = \{\epsilon^{abc}\epsilon^{ik}\epsilon^{jl}\,(L_{pi}\,Q_{raj})\,(Q_{sbk}\,Q_{tcl}),\ \epsilon^{abc}\epsilon^{ik}\epsilon^{jl}\,(L_{pi}\,Q_{sbk})\,(Q_{raj}\,Q_{tcl}),\ \epsilon^{abc}\epsilon^{ij}\epsilon^{kl}\,(L_{pi}\,Q_{raj})\,(Q_{sbk}\,Q_{tcl}),\ \epsilon^{abc}\epsilon^{ij}\epsilon^{kl}\,(L_{pi}\,Q_{sbk})\,(Q_{raj}\,Q_{tcl})\}$$

$$D_{\mathcal{O}}[(12)] = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad D_{\mathcal{O}}[(123)] = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$$

Example of LQQQ

$$\mathcal{Y}\left[\begin{array}{|c|}\hline r\\\hline s\\\hline t\\\hline\end{array}\right] O_1^{\mathrm{y}} = 1/2 O_1^{\mathrm{y}} - 1/2 O_4^{\mathrm{y}}$$

$$\mathcal{Y}\left[\begin{array}{|c|c|c|}\hline r & s & t\\\hline\end{array}\right] = \begin{pmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} \end{pmatrix}$$

$$\mathcal{Y}\left[\begin{array}{|c|}\hline r\\\hline s\\\hline t\\\hline\end{array}\right] = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathcal{Y}\left[\begin{array}{|c|c|}\hline r & s\\\hline t & \multicolumn{1}{c}{}\\\cline{1-1}\end{array}\right] = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$$\mathcal{O}_1^{\mathrm{y}} = \epsilon^{abc}\epsilon^{ik}\epsilon^{jl}(L_{pi}Q_{raj})(Q_{sbk}Q_{tcl})$$

Final result:

$$\mathcal{Y}\left[\begin{array}{|c|c|c|}\hline r & s & t\\\hline\end{array}\right] \circ \mathcal{O}_1^{\mathrm{y}}$$

$$\mathcal{Y}\left[\begin{array}{|c|c|}\hline r & s\\\hline t & \multicolumn{1}{c}{}\\\cline{1-1}\end{array}\right] \circ \mathcal{O}_1^{\mathrm{y}}$$

$$\mathcal{Y}\left[\begin{array}{|c|}\hline r\\\hline s\\\hline t\\\hline\end{array}\right] \circ \mathcal{O}_1^{\mathrm{y}}$$

$$\langle \{\boldsymbol{p}_i \lambda_i\} | s, J; P, \sigma, a \rangle = C^{s,J}_{P,\sigma,a}(\{\boldsymbol{p}_i \lambda_i\}) \delta^4 (P - \sum_{i=1}^{N} p_i),$$

$$C^{s,J}_{\Lambda P,\sigma,a}\left(\{\Lambda \boldsymbol{p}_i, \lambda_i\}\right) = \prod_i e^{-i\lambda_i w_i(\Lambda, p_i)} \sum_{\sigma'} C^{s,J}_{P,\sigma',a}\left(\{\boldsymbol{p}_i, \lambda_i\}\right) D^{\dagger}[R(\Lambda, p_i)]_{\sigma'\sigma}.$$