Bubble wall velocity from Kadanoff-Baym equations: fluid dynamics and microscopic interactions

Jiang Zhu
Based on: arXiv:2504.13724
in collaboration with professor Michael Ramsey Musolf

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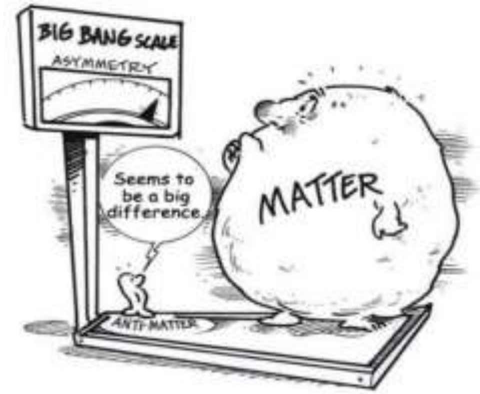
Introduction



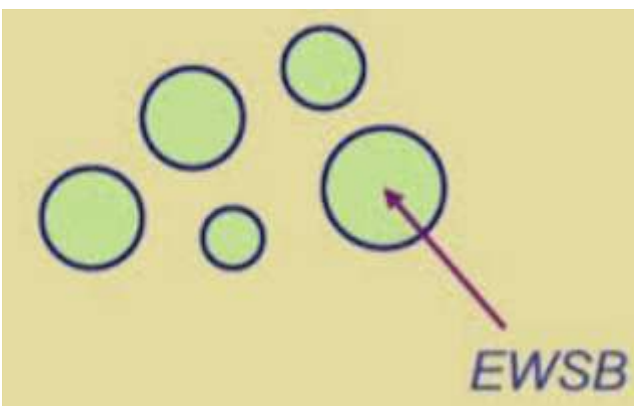
- **Introduction to the Bubble wall velocity**

One of the biggest problems of our universe

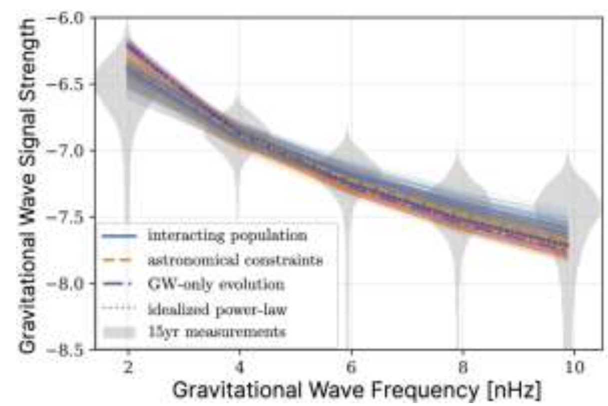
Matter and Antimatter asymmetry

One solution:

- Electroweak Baryogenesis
- Many BSM achieve it through FOEWPT
- Predict Stochastic GW signal as observable



Collision



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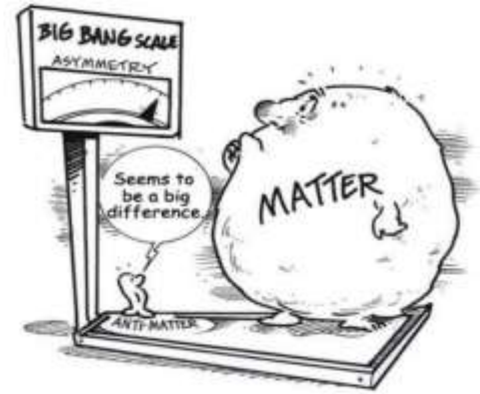

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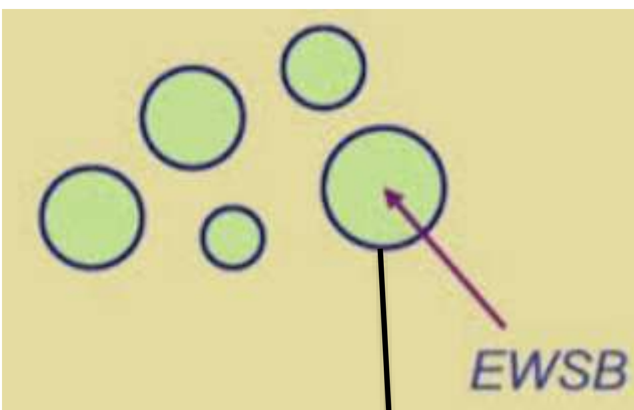

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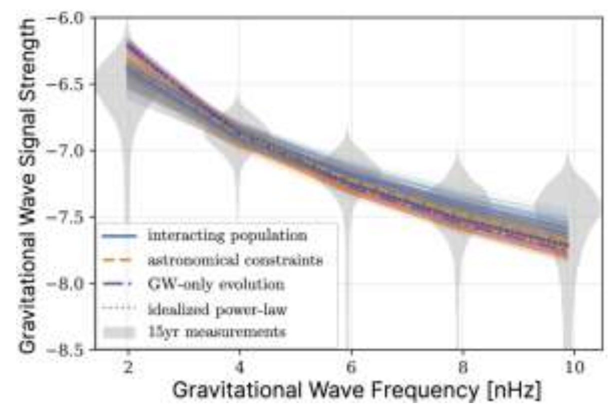

Key input parameter: bubble wall velocity to decribe how fast did the bubble growth

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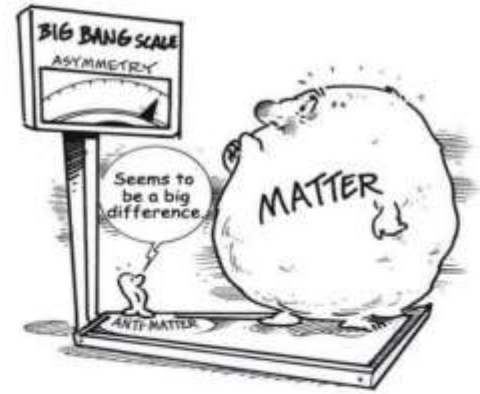

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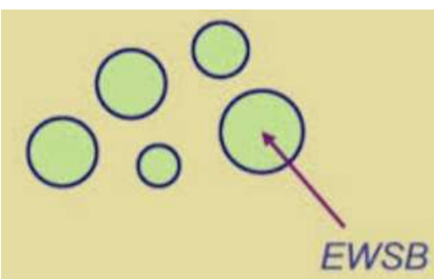

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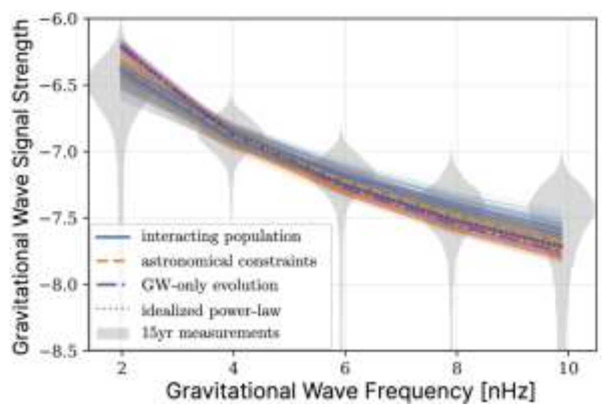

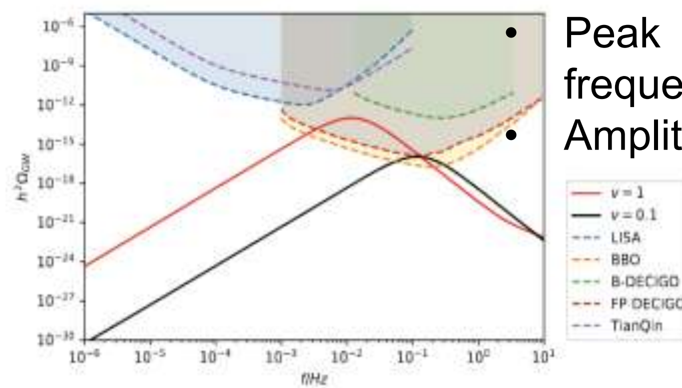

- Peak frequency
- Amplitude

1. Will there a terminal velocity(bubble expand stably)?
2. If is, how fast the terminal velocity is?

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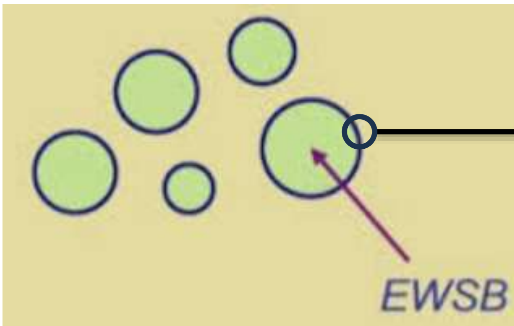

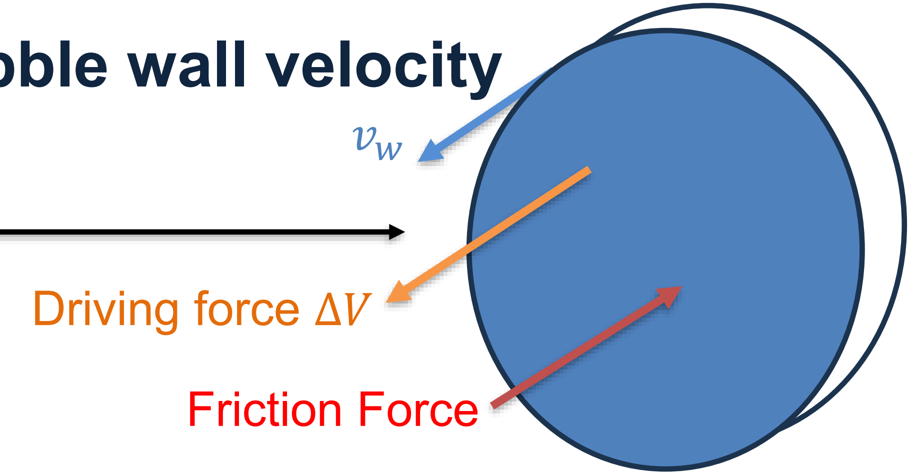

- **First method**

Force is momentum transport thus consider the ***Boltzmann equation***

$$\frac{p^\mu}{E_p}\partial_\mu f_a(x,p,t) + \partial_{p^i}[F^i f_a(x,p,t)] = \mathcal{C}[f_a],$$

Benoit Laurent and James M. Cline. Phys. Rev. D, 106(2):023501, 2022.

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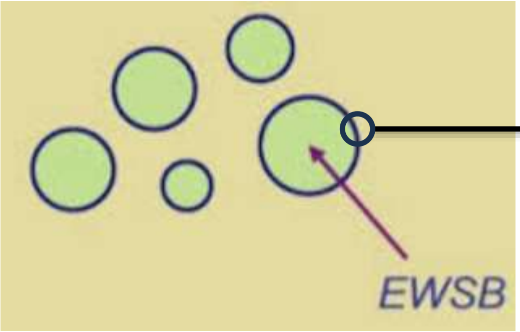

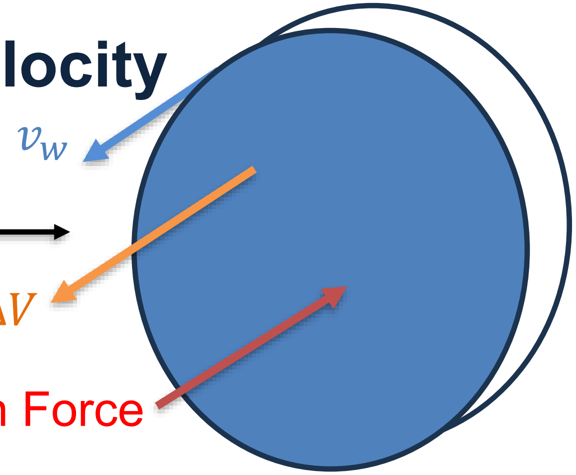

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$$\int dz \int \frac{d^3\mathbf{p}}{(2\pi)^3} p^\nu \left[\frac{p^\mu}{E_p}\partial_\mu f_a(x,p,t) + \partial_{p^i}[F^i f_a(x,p,t)] = \mathcal{C}[f_a], \right]$$

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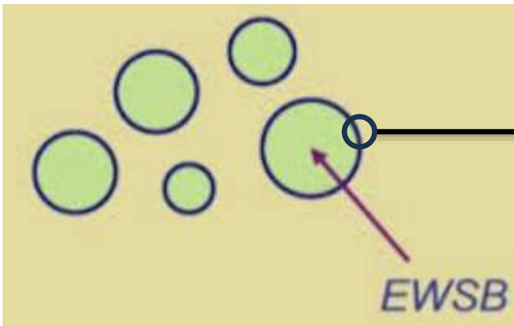

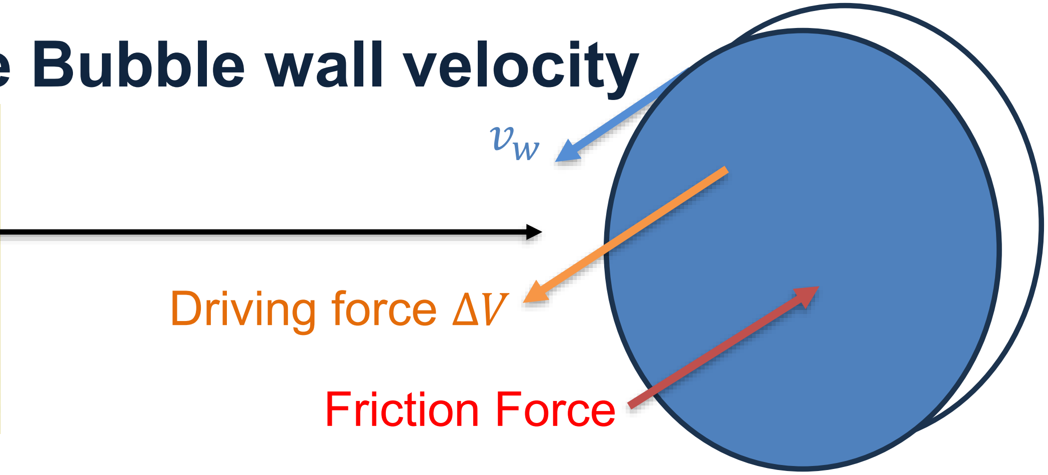

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$$P_{\text{tot}} = \sum_a \int dz \int \frac{d^3\mathbf{p}}{(2\pi)^3} F^z f_a + \sum_a \int dz \int \frac{d^3\mathbf{p}}{(2\pi)^3} p^z \mathcal{C}[f_a] - \Delta[V(\phi)]_{\phi_f}^{\phi_t}$$

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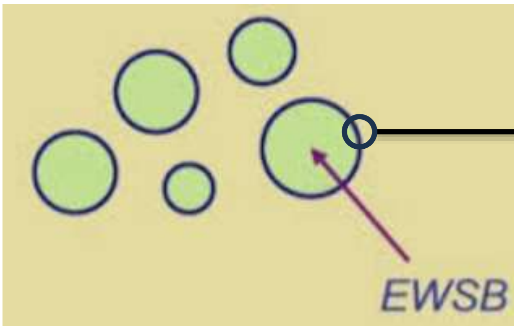

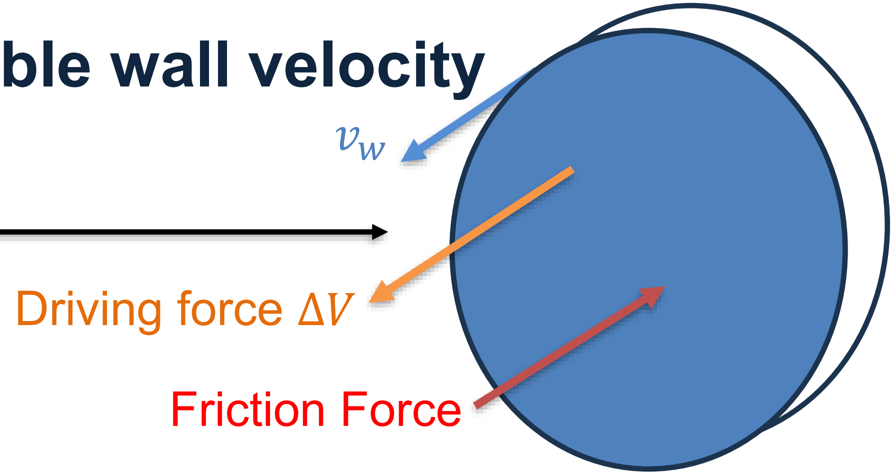

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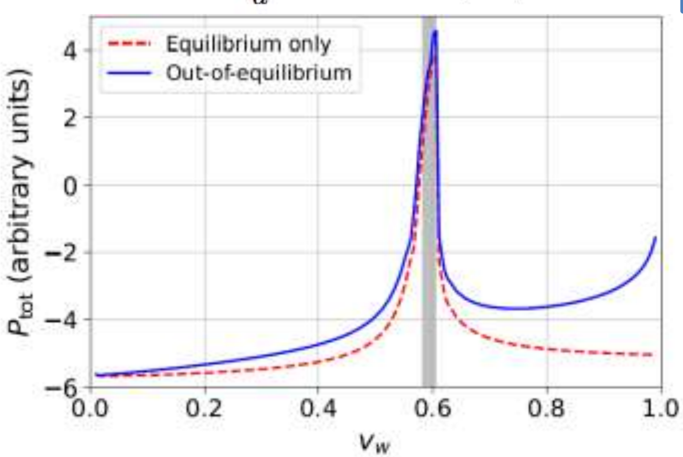
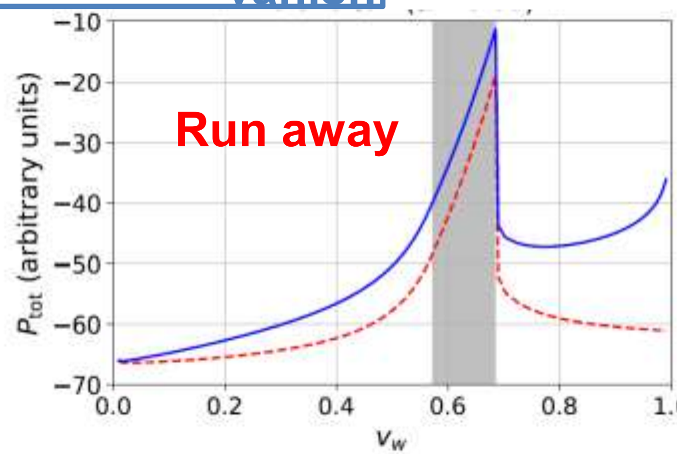
vanish





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- **How we study the Bubble wall velocity**
- **Second method**

However, what if the we considering the interaction between particle and background field(VEV) **Dietrich Bodeker and Guy D. Moore. JCAP, 05:025,2017.**

$$\frac{F_{fric}^{a\to bc}}{A} = \int \frac{d^3\mathbf{p}_a}{(2\pi)^3}\frac{p_a^z}{E_a} f_a \int dP_{a\to bc}(p_a^z - k_b^z - k_c^z),$$

$$flux \times momentum\ transfer$$

[1] Bubble wall dynamics from nonequilibrium quantum field theory. arXiv:2504.13725

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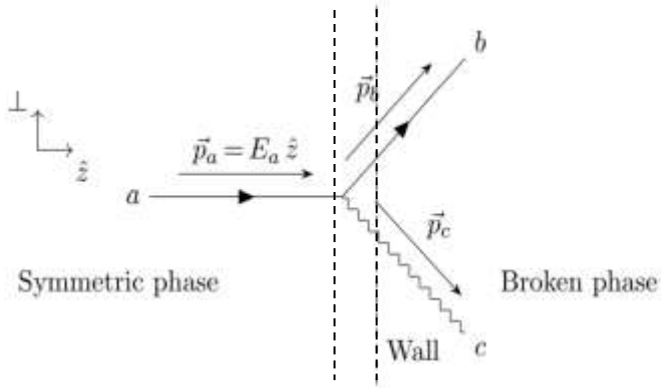
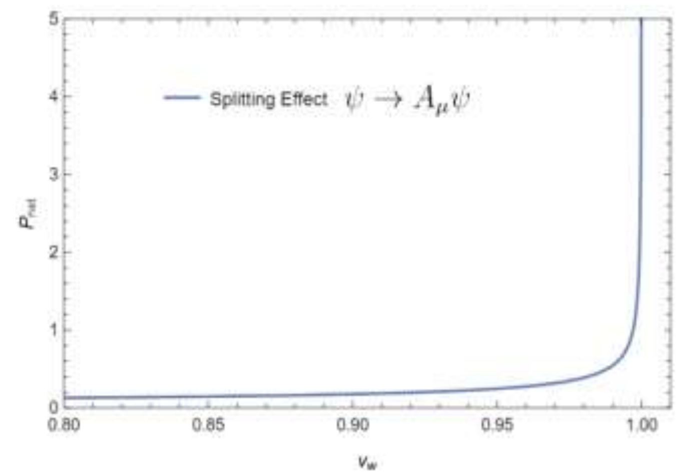
Non-conservation Of momentum

$flux \times momentum\ transfer$

Momentum conservation broken between particle interaction, because the existence of background field [1]





Not exist in ordinary Boltzmann equation which stop the run-away bubble

[1] Bubble wall dynamics from nonequilibrium quantum field theory. arXiv:2504.13725

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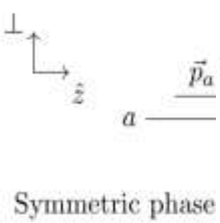
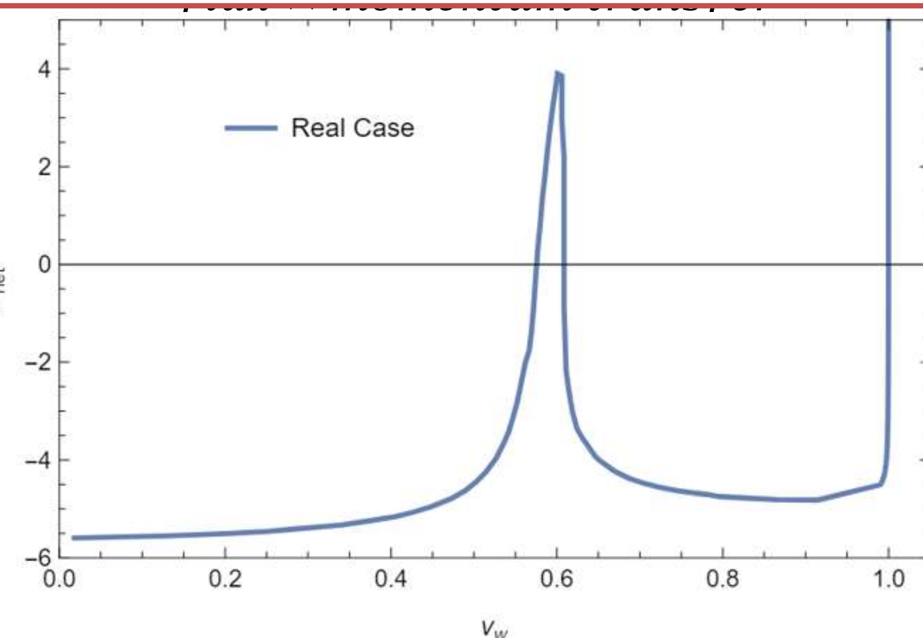
However, what if the we considering the interaction between particle and background field(VEV)

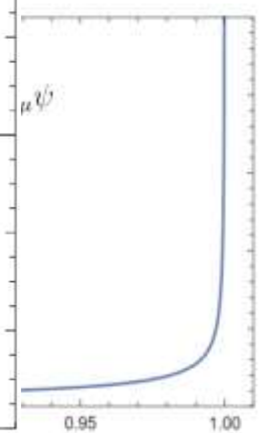
Dietrich Bodeker and Guy D. Moore, JCAP, 05:025 2017

Which Method is correct? Or Is there a consistent method?

Momentum c ... on, because the existence







Not exist in ... un-away bubble

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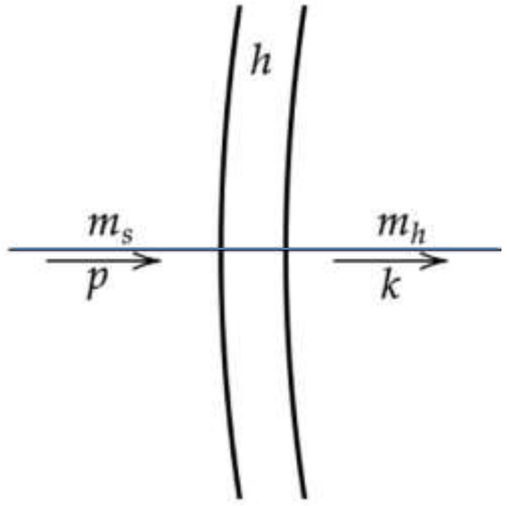
Modified Boltzmann equation from the K-B equation

- **Field description from classical mass variation**

Now we start to analyze the problem in more illustrating ways. Let us considering the field description of the particles and bubble interaction

To build this frameworks, let us consider the classical mass variation of the particles bring by the background field



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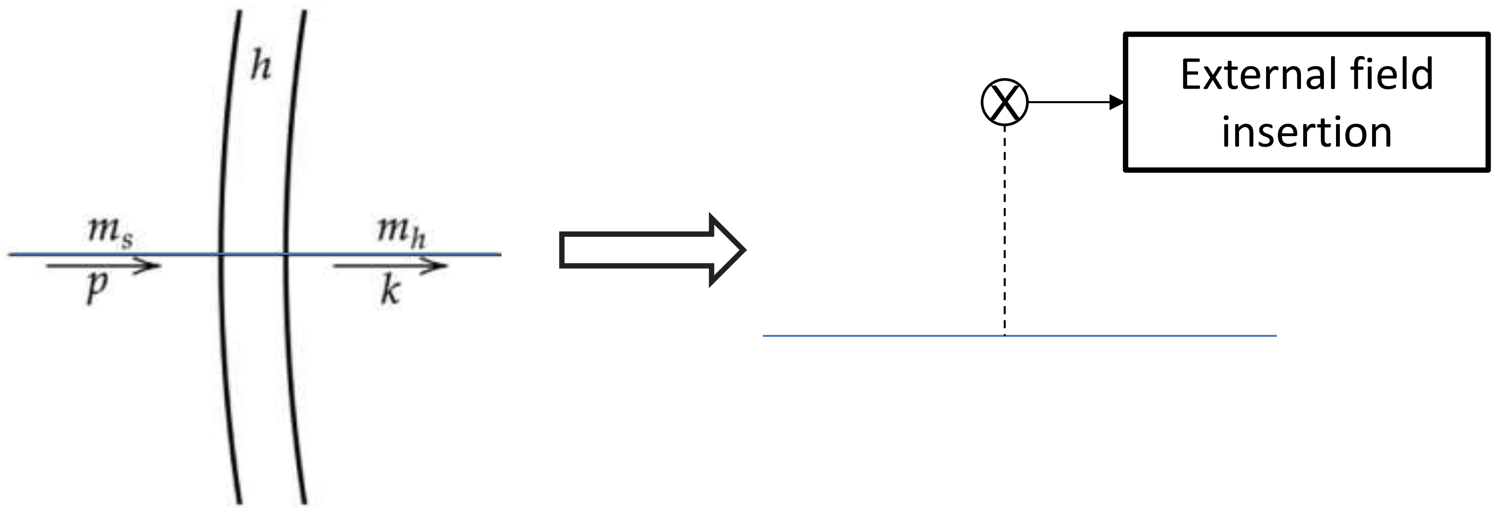

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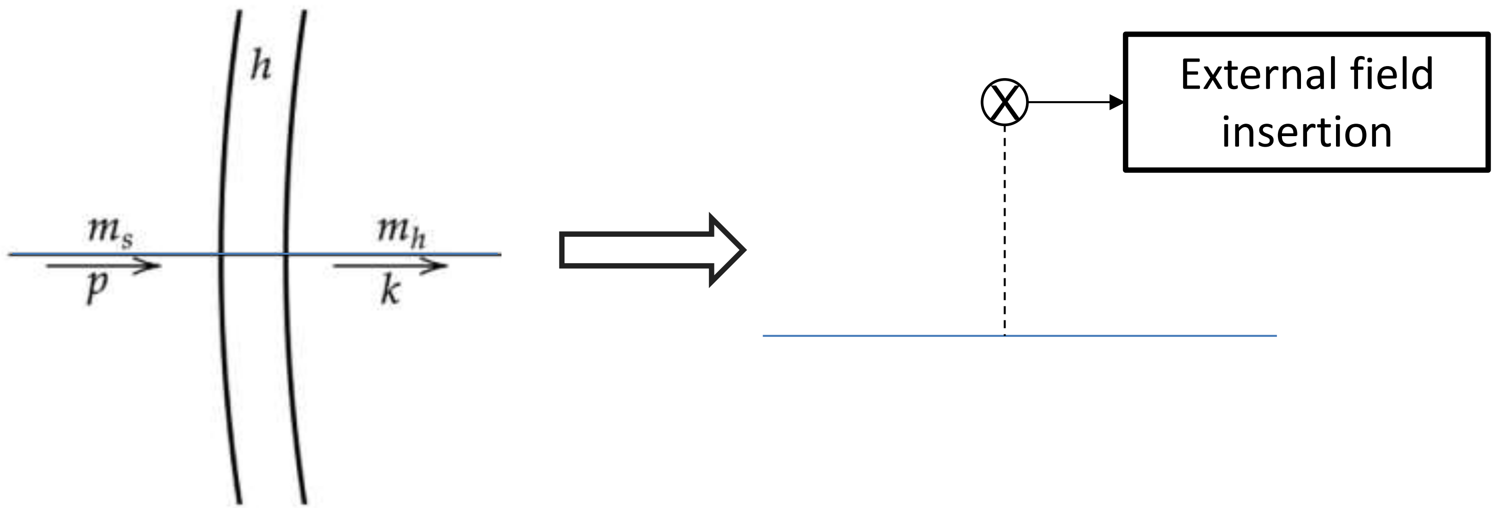

This process can be computed in QFT and the contribution is the classical force term in BE

$$\partial_\mu \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{p^\mu p^\nu}{E_p} f_a = \boxed{\int \frac{d^3\mathbf{p}}{(2\pi)^3} F^\nu f_a} + \int \frac{d^3\mathbf{p}}{(2\pi)^3} p^\nu \mathcal{C}[f_a]$$

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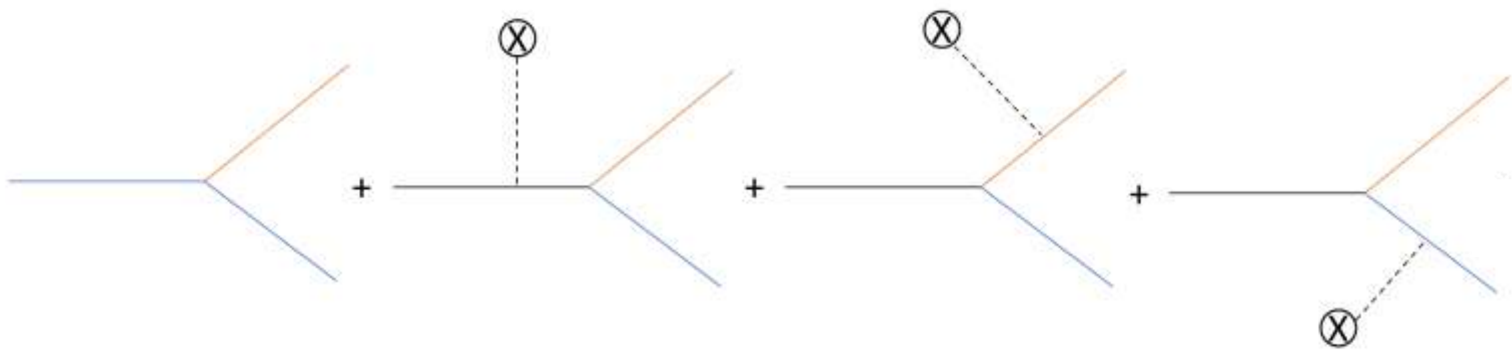
- **Momentum non-conservation(Background) effect**

Now, let us consider the collision contribution.

$$\partial_\mu \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{p^\mu p^\nu}{E_p} f_a = \int \frac{d^3\mathbf{p}}{(2\pi)^3} F^\nu f_a + \boxed{\int \frac{d^3\mathbf{p}}{(2\pi)^3} p^\nu \mathcal{C}[f_a]}$$

Naively thinking, one may describe the collision effect by the Feynman diagrams with the external field insertion as

$\mathcal{C}[f_a]$ = 

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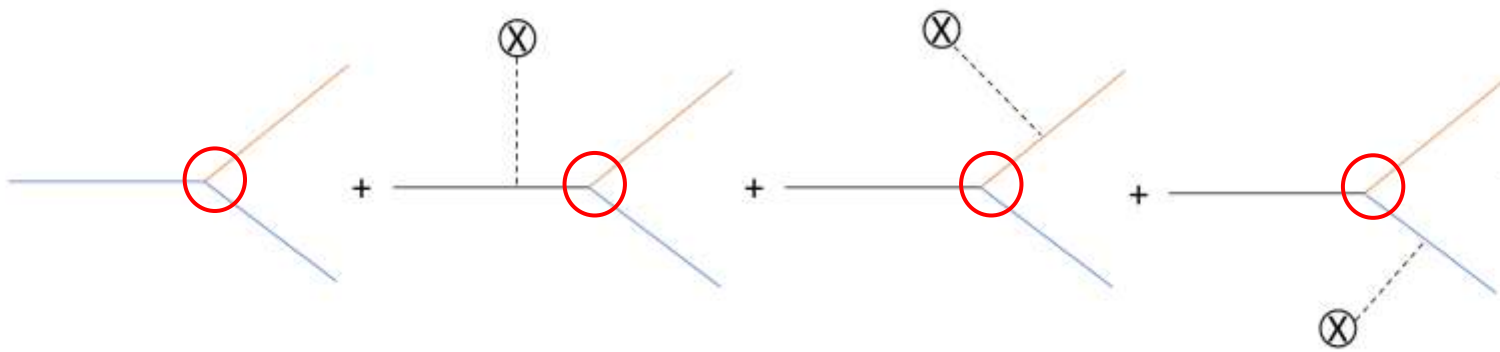


Since the external field only insertion at the initial and final leg, which do not including the **collision point**. One may expect that the momentum conservation would not be affected by the background field

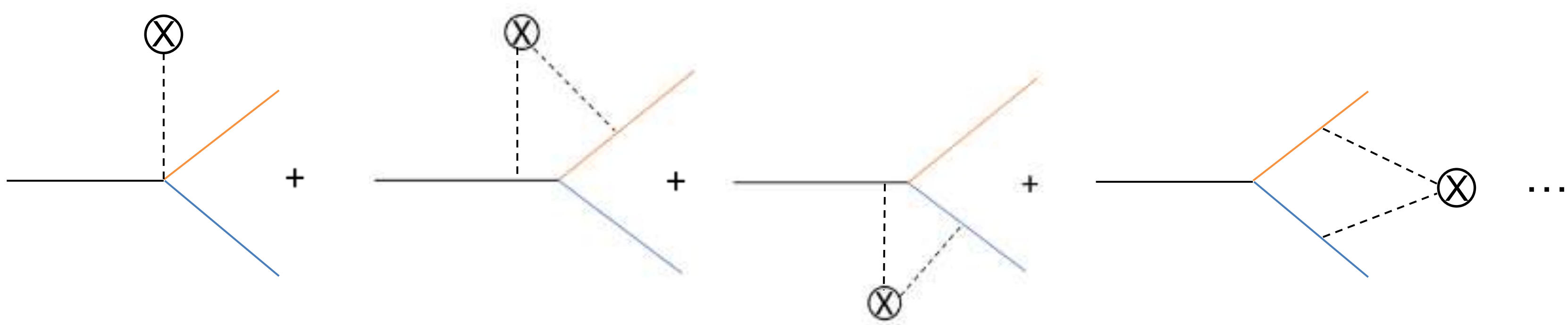
Collison term contribution will vanish since the collision terms maintain the momentum conservation

Modified Boltzmann equation from the K-B equation

- **Momentum non-conservation(Background) effect**

However, the problems is more complicated than the naive think, in the about diagram we have ignored some important terms which is the interference term between many particle and background field

From the Feynman diagram, those effect would be insertion the same external field between different particles



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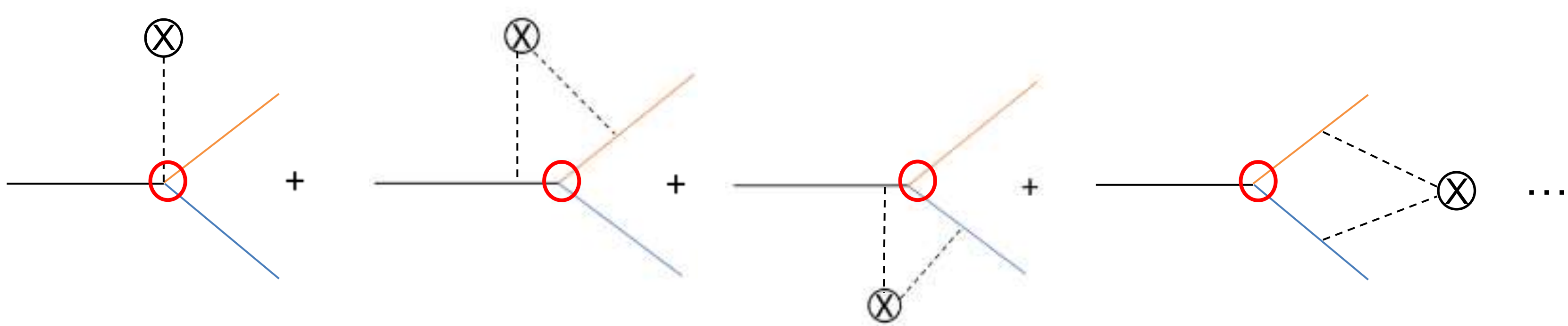

Now the **collision point** is inside the field insertion. As a consequence, one may expect that the collision term may not maintain the momentum conservation, since in the scattering process the particle need to pass momentum into background field

Modified Boltzmann equation from the K-B equation

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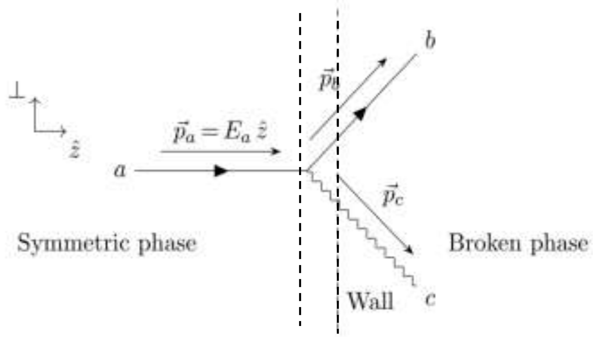

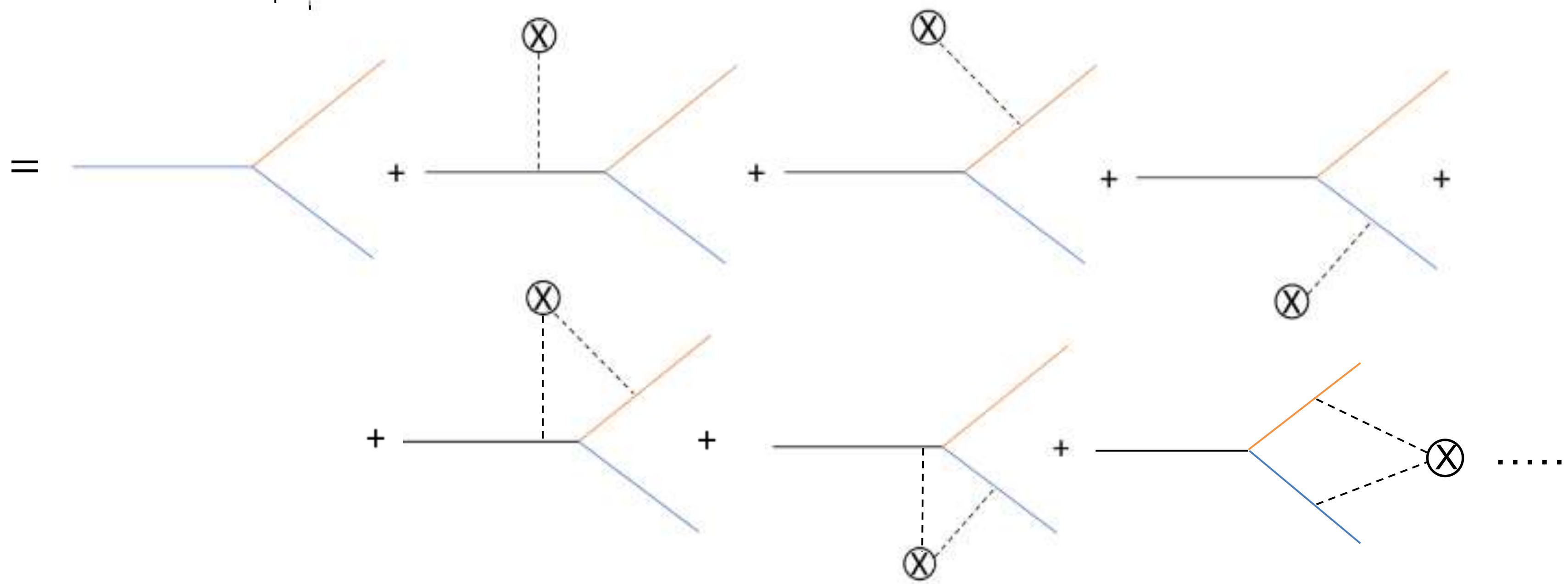

WHICH IS NOT IN THE ORDINARY BOLTZMANN EQUATION

Modified Boltzmann equation from the K-B equation

- **Can we get those Result from the first principle?**

After we understand the, let us now start to build a equation to describe those effect.

Idea is obtain the quantum kinetic equation from **the Dyson-Schwinger Equation for two point function + The Closed Time Path formalism**[1,2]

$$\widetilde{G}(k;X) = \int d^4r\, e^{i\,k\cdot r}\, \widetilde{G}(x,y) \qquad x = X + \tfrac{r}{2},\ y = X - \tfrac{r}{2}$$

Kadanoff-Baym equation:

[1] G. Z. Zhou, Z. B. Su, B. I. Hao and L. Yu, Phys. Rev. B 22, 3385-3407 (1980)
[2] V. Cirigliano, C. Lee, M. J. Ramsey-Musolf and S. Tulin, Phys. Rev. D 2010

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$$\left(\frac{1}{2}\partial_X^2 - 2k^2\right)\boxed{\widetilde{G}(k,X)} + e^{-i\diamond}\{m^2(X), \widetilde{G}(k,X)\} = -2i\widetilde{I} - ie^{-i\diamond}\boxed{\widetilde{\Pi}(k,X)}, \widetilde{G}(k,X)$$

$$A(k + \frac{i}{2}\partial_X, X - \frac{i}{2}\partial_k)B(k,X) = e^{-i\diamond}(A(k,X)B(k,X))$$

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$$-2ik\cdot\partial_X\widetilde{G}(k,X) + e^{-i\diamond}[m^2(X),\widetilde{G}(k,X)] = -ie^{-i\diamond}[\widetilde{\Pi}(k,X),\widetilde{G}(k,X)]$$

Implication: Include the Non-local effect bring by the Non-trivial VEV field

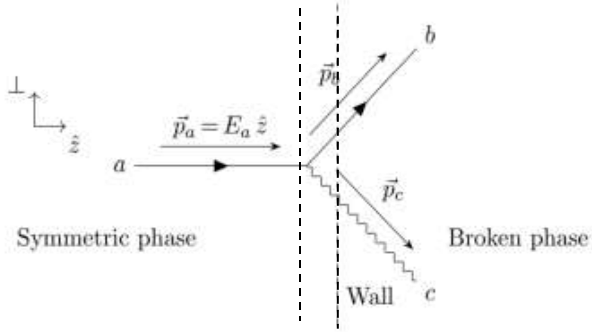
[1] G. Z. Zhou, Z. B. Su, B. I. Hao and L. Yu, Phys. Rev. B 22, 3385-3407 (1980)
[2] V. Cirigliano, C. Lee, M. J. Ramsey-Musolf and S. Tulin, Phys. Rev. D 2010

Modified Boltzmann equation from the K-B equation

- **Can we get those Result from the first principle?**

Time Scale Expansion

Those equation are complicated operator equation. There is no way to directly solve it even through numerical method.



Time scale in the systems

- The intrinsic time scale of the particles τ_{int}
- The bubble wall time scale τ_{wall}
- The Collision time scale τ_{coll}

Approximation:

If there is a hierarchy between those time scale, one can use it to expand those equation!

$$\epsilon \sim \frac{\tau_{int}}{\tau_{wall}} \sim \frac{\tau_{int}}{\tau_{coll}} \ll 1$$

[1] G. Z. Zhou, Z. B. Su, B. I. Hao and L. Yu, Phys. Rev. B 22, 3385-3407 (1980)
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Modified Boltzmann equation from the K-B equation

• Focus on Non-local effect of K-B equations

With out dive too much into mathematic detail, let us list the conclusion of this approximation

$$-2ik\cdot\partial_X G^{\gtrless} + e^{-i\diamond}[m^2, G^{\gtrless}] = \boxed{-ie^{-i\diamond}}([\Pi^h, G^{\gtrless}] + [\Pi^{\gtrless}, G^h] + \frac{1}{2}\{\Pi^>, G^<\} - \frac{1}{2}\{G^>, \Pi^<\}).$$

Any **diamond operator** would generate terms with ϵ. So, we can expand the complicated K-B equation as long as coefficient not contain poles

There is two type of terms from the diamond operator

- First terms: It operating in the Green's function

$$e^{-i\diamond_{k,x}}G(p,x)G(k,x)$$

Since the function is analytical for both coordinate, one can do such a expansion to get a simpler equation

Modified Boltzmann equation from the K-B equation

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There is two type of terms from the diamond operator

- First terms: It operating in the Green's function

$$e^{-i\diamond_{k,x}} G(p,x)G(k,x)$$

- Second terms: It operating in the Dirac delta function

$$e^{-i\diamond_{k,x}} \delta^{(4)}(p+k)G(k,x)$$

For this terms, pole is evidently existed. As a consequences, one can not trust the result from the expansion and ϵ must be resumed.

Modified Boltzmann equation from the K-B equation

- **New aspect: The modified BE from KBE**

In order to counting those effect, we start from more general kinetic equation: Kadanoff-Baym equation in Close Path Formalism

And derived the modified version of Boltzmann equation

$$\left[2k_z\frac{\partial}{\partial z}-\frac{dm^2(z)}{dz}\frac{\partial}{\partial k_z}\right]\frac{f_\phi(k,z)}{E_k}=\mathcal{C}[f_\phi]$$

But now the momentum non-conservation induced by background field exist in collision terms

$$\mathcal{C}[f]=-\int\frac{d^3\mathbf{p}}{(2\pi)^3}\int\frac{d^3\mathbf{p}'}{(2\pi)^3}F(k,z)\frac{1+f_\Phi(p,z)}{2E_p}\frac{1+f_\Phi(p',z)}{2E_{p'}}$$
$$\times(2\pi)^3\delta(E_k-E_p-E_{p'})\delta^2(\mathbf{k}_\perp-\mathbf{p}_\perp-\mathbf{p}'_\perp)+(\Delta p_z\leftrightarrow-\Delta p_z)$$
$$+\,\text{InverseProcess},$$

Momentum non-conservation between the particles

The F is the non-local interaction matrix element related dependent on specific interaction for $1\rightarrow 2$ process

$$F(k,z)=\int dz' f_\phi(k,z')Y(z')Y(2z-z')e^{-2i\Delta p_z(z-z')}.$$

Modified Boltzmann equation from the K-B equation

- **The friction force from the modified BE**

Let us discussed the friction force come from this modified Boltzmann equation, to find the physics

$$P_{\text{tot}} = \sum_a \int dz \int \frac{d^3\mathbf{p}}{(2\pi)^3} F^z f_a + \sum_a \int dz \int \frac{d^3\mathbf{p}}{(2\pi)^3} p^z \mathcal{C}[f_a] - \Delta[V(\phi)]_{\phi_f}^{\phi_t}$$

After some algebra and ballistic approximation , one can prove

$$\sum_a \int dz \int \frac{d^3\mathbf{p}}{(2\pi)^3} p^z \mathcal{C}[f_a] = -\int \frac{d^3\mathbf{p}_a}{(2\pi)^3} f_a(p_a) \int \frac{d^3\mathbf{p}_b}{2E_b(2\pi)^3} \frac{d^3\mathbf{p}_c}{2E_c(2\pi)^3} \frac{|M_{a\to bc}|^2}{2E_a} (2\pi)^3$$
$$\times \delta(E_a - E_b - E_c)\delta^{(2)}(\mathbf{p}_a^\perp - \mathbf{k}_b^\perp - \mathbf{k}_c^\perp)(p_a^z - p_b^z - p_c^z)$$
$$+ \text{Inverse Process}$$

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$$\times \delta(E_a - E_b - E_c)\delta^{(2)}(\mathbf{p}_a^\perp - \mathbf{k}_b^\perp - \mathbf{k}_c^\perp)(p_a^z - p_b^z - p_c^z)$$
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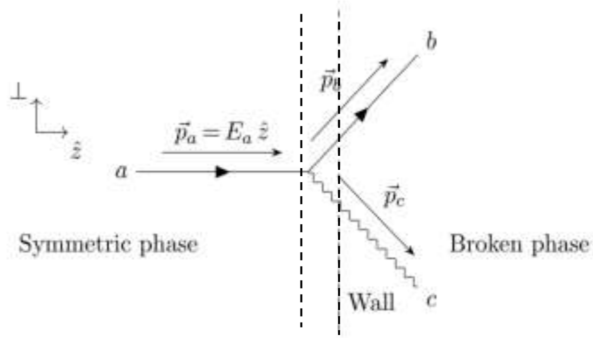
$$\sum_a \int dz \int \frac{d^3\mathbf{p}}{(2\pi)^3} p^z \mathcal{C}[f_a] = \int \frac{d^3\mathbf{p}_a}{(2\pi)^3} \frac{p_a^z}{E_a} f_a \int dP_{a\to bc}(p_a^z - k_b^z - k_c^z),$$

$flux \times momentum\ transfer$

It is momentum transfer rate between bubble and all the particle inside the 3-particle interaction



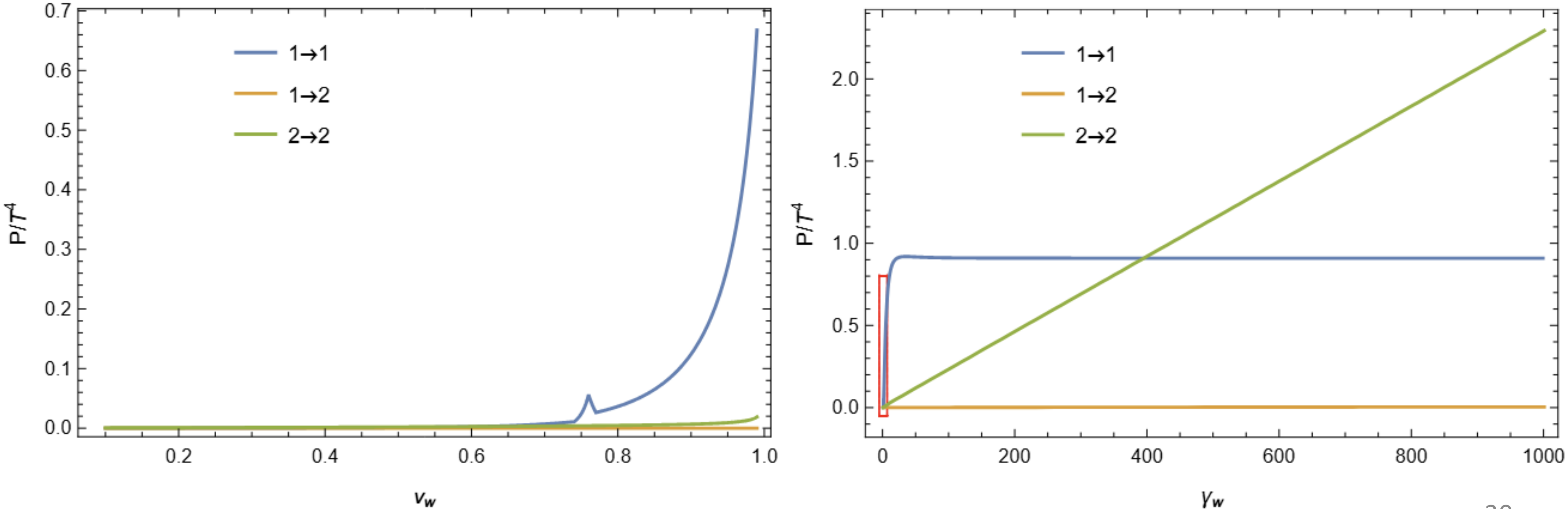
Modified Boltzmann equation from the K-B equation

- **Example**

Toy model:

$$\mathcal{L}_{int} = \lambda\phi^2\Phi^2$$

where ϕ is a light scalar field, Φ is a heavy scalar field with $m_\phi < m_\Phi$. This model is prevalent in the BSM model, also called as the Higgs portal model, which generates the FOEWPT

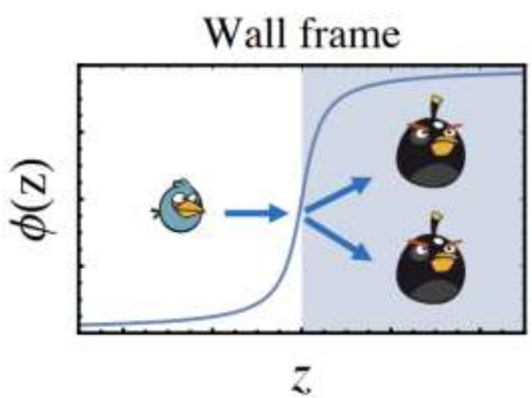


Summary

- **Summary**
  1. We build a systematic frameworks to compute v_w by the modified BE which include the information do not exit in ordinary one.

  2. We find, in principle, those effect can be found by solving the K-B equation, and discuss v_w for any $i \rightarrow f$ interaction.

  3. Those $2 \rightarrow 2$ in $\phi^2\Phi^2$ theory will induce a friction force $\propto \gamma_w$ and eliminate run-aways bubble configuration.

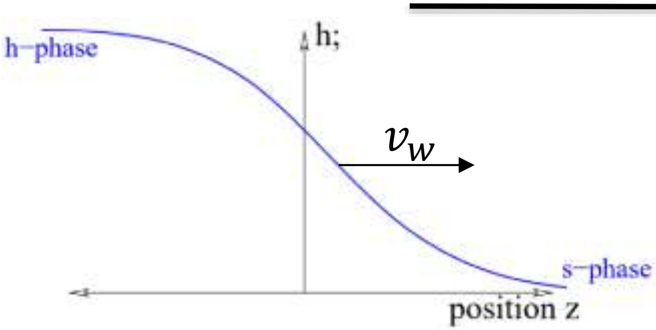


Backup

- **Where did this inconsistency come from?**

The key equation to study the force is momentum transport between the bubble and particle, let us focus on the Boltzmann equation(BE)

$$(p^\mu \partial_\mu + mF^\mu \partial_{p^\mu})\theta(p^0)\delta(p^2 - m^2)f = C[f],$$



Suppose bubble is bigger enough and moving in z-direction

$$\int dz \int \frac{d^3p}{(2\pi)^3} p^\mu$$

If the bubble reach the stable state, we can stablish the relation between microscopic force and fluid energy momentum tensor

$$\Delta\left[\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{p_z^2}{E} f(p,z)\right]_{z=b}^{z=s} = \int dz \int \frac{d^3\vec{p}}{(2\pi)^3} F_{sp} f(p,z) + \int dz \int \frac{d^4p}{(2\pi)^4} p_z C[f] = 0$$

T_f^{zz} component of fluid EMT — Classical force — Particle interaction

So, the BE in ordinary QFT can never solve the inconsistency

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Backup

- **General Boltzmann Equation**

$$\frac{p^\mu}{E_p}\partial_\mu f_a(x,p,t) + \partial_{p^i}[F^i f_a(x,p,t)] = \mathcal{C}[f_a],$$

Normal collision terms

$$\mathcal{C}[f_a] = -f_a(x,p,t)\prod_{i\neq a}\int \frac{d^3\mathbf{p}_i}{(2\pi)^3 2E_i}\prod_f \int \frac{d^3\mathbf{p}_f}{2E_f(2\pi)^3}|\mathcal{M}|^2_{i\to f}\delta^4(\sum_i p_i - \sum_f p_f) + \text{Inverse Process.}$$

For $1 \to 2$ decay process the pressure contributed by sum of all particle form as

$$\sum_{i=a}^{b,c}\int dz \int \frac{d^3 p_i}{(2\pi)^3} p_i^z \mathcal{C}[f_i] \propto (\sum_i p_i^z - \sum_f p_f^z)\delta^{(4)}(\sum_i p_i - \sum_f p_f) = 0$$

Backup

- **Can we get those Result from the first principle?**

The Kadanoff-Baym Equation is the Wigner transformation $G(k,X) = \int d^4 r G(x,y) e^{ik\cdot r}$ of CTP DSE $X = \frac{x+y}{2}$, $r = x - y$

$$(\frac{1}{2}\partial_X^2 - 2k^2)G^{\gtrless} + e^{-i\diamond}\{m^2, G^{\gtrless}\} = -ie^{-i\diamond}(\{\Pi^h, G^{\gtrless}\} + \{\Pi^{\gtrless}, G^h\} + \frac{1}{2}[\Pi^>, G^<] - \frac{1}{2}[G^>, \Pi^<]),$$
$$-2ik\cdot\partial_X G^{\gtrless} + e^{-i\diamond}[m^2, G^{\gtrless}] = -ie^{-i\diamond}([\Pi^h, G^{\gtrless}] + [\Pi^{\gtrless}, G^h] + \frac{1}{2}\{\Pi^>, G^<\} - \frac{1}{2}\{G^>, \Pi^<\}),$$

One can prove that in the approximation $1 \gg \epsilon_{wall} = \frac{\tau_{int}}{\tau_{wall}}$ above two equation will give us the Boltzmann equation

V. Cirigliano, C. Lee, M. J. Ramsey-Musolf and S. Tulin, Phys. Rev. D 81, 103503 (2010)

Solution for constrain equation

$$G^>(k,X) = 2\pi\delta[k^2 - m(X)^2](\Theta(k^0)[1 + f(k,X)] + \Theta(-k^0)\bar{f}(-k,X))$$
$$G^<(k,X) = 2\pi\delta[k^2 - m(X)^2](\Theta(k^0)f(k,X) + \Theta(-k^0)[1 + \bar{f}(-k,X)])$$

Solution for kinetic equation

$$\left[2k\cdot\partial_X - \nabla_X m^2(X)\cdot\frac{\partial}{\partial\mathbf{k}}\right]\frac{f(k,X)}{E_k} = \int_0^\infty \frac{dk^0}{2\pi}\frac{1}{2}(\{\Pi^>, G^<\} - \{\Pi^<, G^>\}),$$

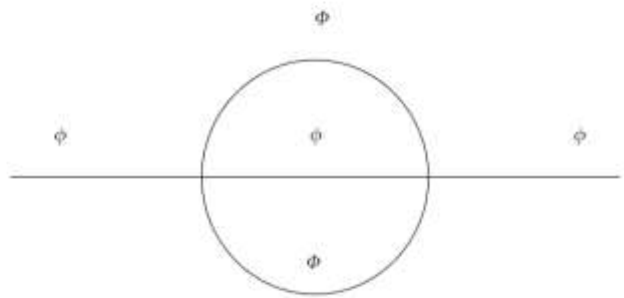
Backup

- **Focus on Non-local effect of K-B equations**

However when deriving the Boltzmann equation, we have ignore some important terms when discussing the non-local effect

$$-2ik\cdot\partial_X G^{\gtrless} + e^{-i\diamond}[m^2, G^{\gtrless}] = -ie^{-i\diamond}\left([\Pi^h, G^{\gtrless}] + [\Pi^{\gtrless}, G^h] + \frac{1}{2}\{\Pi^>, G^<\} - \frac{1}{2}\{G^>, \Pi^<\}\right),$$

Consider $\phi^2\Phi^2$ interaction



$$\Longrightarrow \quad \Pi^a(x,y) = -\frac{\lambda^2}{2} G^a_\Phi(x,y) G^a_\Phi(x,y) G^a_\phi(x,y),$$

The collision terms will be

$$e^{-i\diamond}\{\Pi^a, G^b\} = -\frac{\lambda^2}{2}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4p'}{(2\pi)^4}\int\frac{d^4k'}{(2\pi)^4}(2\pi)^4\delta^4\left(k+\boxed{\frac{i}{2}d_z}-k'-p-p'\right)$$

Non-local term inside δ we should not expand it

$$d_{z/k_z} = (0,0,0,\partial_{z/k_z}) \qquad G^a_\Phi\left(p, z-\frac{i}{2}\partial_{k_z}\right)G^a_\Phi\left(p', z-\frac{i}{2}\partial_{k_z}\right)G^a_\phi\left(k', z-\frac{i}{2}\partial_{k_z}\right)G^b_\phi(k, z)$$

We ignored all derive inside this terms since each of term will give order of

Consider $\phi^2\Phi^2$ interaction

Backup

- **Focus on Non-local effect of K-B equations**

To recover the non-local effect in K-B equation, we apply the Wigner transform in the average coordinate $X = \frac{x+y}{2}$

$$G^a(k,l) = \int d^4X e^{il\cdot X} G^a(k,X) \qquad G^a(k,X) = \int d^4l e^{-il\cdot X} G^a(k,l).$$

The we can counter the non-local effect inside δ-function by substituting it into collision terms in kinetic equation

$$e^{-i\diamond}\{\Pi^a, G^b\} = -\frac{\lambda^2}{2}\int \frac{d^4p}{(2\pi)^4}\int \frac{d^4p'}{(2\pi)^4}\int \frac{d^4k'}{(2\pi)^4}(2\pi)^4\delta^4(k+\frac{i}{2}d_z - k' - p - p')$$
$$G^a_\Phi(p, z - \frac{i}{2}\partial_{k_z})G^a_\Phi(p', z - \frac{i}{2}\partial_{k_z})G^a_\phi(k', z - \frac{i}{2}\partial_{k_z})G^b_\phi(k,z)$$

$$\Downarrow \quad \Delta p_z = k_z - k'_z - p_z - p'_z$$

$$e^{-i\diamond}\{\Pi^a, G^b\} = -\frac{\lambda^2}{2}\int \frac{d^4k'}{(2\pi)^4}\int \frac{d^4p}{(2\pi)^4}\int \frac{d^4p'}{(2\pi)^4}(2\pi)^3\delta(k_0 - k'_0 - p_0 - p'_0)$$
$$\delta^2(\mathbf{k}_\perp - \mathbf{k}'_\perp - \mathbf{p}_\perp - \mathbf{p}'_\perp)G^a_\Phi(p,z)G^a_\Phi(p',z)G^a_\phi(k',z)G^b_\phi(k, 2\Delta p'_z)e^{-2i\Delta p'_z z}$$

Backup

- **Deriving the effect in Modified BE from KBE**

Assuming, the general solution of Green's function is still hold

$$G^>(k,X) = 2\pi\delta[k^2 - m(X)^2](\Theta(k^0)[1+f(k,X)] + \Theta(-k^0)\bar{f}(-k,X))$$
$$G^<(k,X) = 2\pi\delta[k^2 - m(X)^2](\Theta(k^0)f(k,X) + \Theta(-k^0)[1+\bar{f}(-k,X)])$$

Deriving the modified non-local modified Boltzmann equation from KBE

$$\begin{aligned}&\left[2k_z\frac{\partial}{\partial z} - \frac{dm^2(z)}{dz}\frac{\partial}{\partial k_z}\right]\frac{f(k,z)}{E_k}\\ &= -\int\frac{d^3\mathbf{k}'}{(2\pi)^3}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\int\frac{d^3\mathbf{p}'}{(2\pi)^3}F(k,z)\frac{f_\phi(k',z)}{2E_{k'}}\frac{1+f_\Phi(p,z)}{2E_p}\frac{1+f_\Phi(p',z)}{2E_{p'}}\\ &\quad\times(2\pi)^3\delta(E_k+E_{k'}-E_p-E_{p'})\delta^2(\mathbf{k}_\perp+\mathbf{k}'_\perp-\mathbf{p}_\perp-\mathbf{p}'_\perp) + (\Delta p_z \leftrightarrow -\Delta p_z)\\ &+ \text{InverseProcess},\end{aligned}$$

The collision term do not have full momentum conservations anymore

$$F(k,z) = \int dz' f_\phi(k,z')Y(z')Y(2z-z')e^{-2i\Delta p_z(z-z')}.$$

Backup

• Friction force from KBE

Integral the collision terms in K-B kinetic equation by $\int dz \int \frac{d^4k}{(2\pi)^4} k_z$ to extract the corresponding friction force

$$\frac{F_{fric}^{KB}}{A} = -\int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{f_\phi^{in}(k)}{2E_k} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \frac{f_\phi^{in}(k')}{2E_{k'}} \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} \int \frac{d^3\mathbf{p}'}{(2\pi)^3 2E_{p'}} |A|^2 (k_z + k'_z - p_z - p'_z)$$
$$\times (2\pi)^3 \delta(E_k + E_{k'} - E_p - E_{p'}) \delta^2(\mathbf{k}_\perp + \mathbf{k}'_\perp - \mathbf{p}_\perp - \mathbf{p}'_\perp)$$

We have assumed that $f(k, X)$ is equilibriums distribution function

$$F(k,z) = \int dz' f_\phi(k,z') Y(z') Y(2z - z') e^{-2i\Delta p_z (z-z')}. \Longrightarrow |A|^2 = |\int dz \lambda(z) e^{-i\Delta p_z z}|^2 \approx |M|^2_{\phi^2 \to \Phi^2}$$

The friction force from BE in BQFT

$$\frac{F_{fric}^{2\to2}}{A} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{f(p)}{2E_p} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \frac{f(p')}{2E'_p} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2E_k} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \frac{1}{2E'_k} |M|^2_{2\to2} (p_z + p'_z - k_z - k'_z)$$
$$\times (2\pi^3) \delta(E_p + E'_p - E_k - E'_k) \delta^{(2)}(\mathbf{p}_\perp + \mathbf{p}'_\perp - \mathbf{k}_\perp - \mathbf{k}'_\perp)$$

the effect in BQFT BE can be derived from KBE

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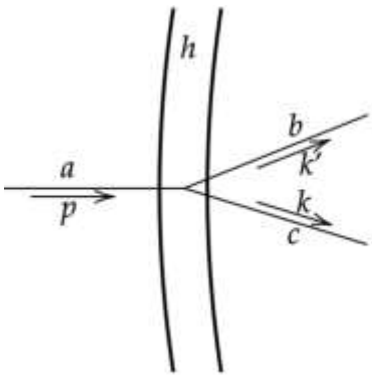
Backup

- **The QFT in Background field and Boltzmann Equations**

$\int dz \int \frac{d^4p}{(2\pi)^4} p_z \mathcal{C}[f] \neq 0$ Indicate we need non-local effect to break the $\Delta p = 0$



We find the natural way is to introducing the effect of the background Field(the bubble) as source for $\Delta p \neq 0$,

$$[\partial^2 + m^2(z)]\phi(x) = 0$$

$$\phi_p(x) = \chi_p(z) e^{-i(Et - \mathbf{p}_\perp \cdot \mathbf{x}_\perp)}$$

Then quantized the field by the eigenfunction of EoMs with z-dependent mass

$$\hat{\phi} \approx \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} [\hat{a}_k \chi_k(z) e^{-i(Et - k_\perp \cdot x_\perp)} + \hat{a}_k^\dagger \chi_k^*(z) e^{i(Et - k_\perp \cdot x_\perp)}]$$

One can derive the corresponding Boltzmann equation in BQFT as

$$\frac{p^\mu}{E_p}\partial_\mu f_a(x,p,t) + \partial_{p^i}[F^i f_a(x,p,t)] = - f_a(x,p,t) \prod_{i \neq a} n_i \int \frac{\langle f|iT|i\rangle\langle i|iT|f\rangle}{T\prod_i (2E_i) V} \prod_f \frac{d^3\mathbf{p}_f}{2E_f(2\pi)^3} \equiv \mathcal{C}[f]$$

$$\langle i|iT|f\rangle = iM_{i\to f}(2\pi)^3 \delta(E_i - E_f)\delta^2(\mathbf{p}_i^\perp - \mathbf{p}_f^\perp) \quad |M|_{i\to j} \approx |\int dz V(z) e^{-i\Delta p_z z}|^2$$

Backup

- **Collision terms**

$$\frac{F^{a\to bc}_{fric}}{A} = \int \frac{d^3\mathbf{p}_a}{(2\pi)^3}\frac{p^z_a}{E_a} f_a \int dP_{a\to bc}(p^z_a - k^z_b - k^z_c),$$

$$\int dP_{a\to bc} = \int \frac{d^3\mathbf{k}_b}{(2\pi)^3}\frac{1}{2E_b}\int \frac{d^3\mathbf{k}_c}{(2\pi)^3}\frac{1}{2E_c}\langle\phi|iT|k_b,k_c\rangle\langle k_b,k_c|iT|\phi\rangle,$$

$$|\phi\rangle = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{\phi(p)}{2E_p}|p\rangle, \quad \text{with} \quad \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{|\phi(p)|^2}{2E_p} = 1,$$

$$\langle p|iT|k_b,k_c\rangle = \int d^4x \chi^*_p(z)V(z)\chi_b(z)\chi_c(z)e^{i(E_a-E_b-E_c)t}e^{-i(\mathbf{p}_a-\mathbf{k}_b-\mathbf{k}_c)_\perp\cdot\mathbf{x}_\perp},$$

$$\langle p|iT|k_b,k_c\rangle = M_{a\to bc}(2\pi)^3\delta(E_a-E_b-E_c)\delta^2(\mathbf{p}^\perp-\mathbf{k}^\perp_b-\mathbf{k}^\perp_c),$$

Backup

- **Collision terms**

$$\int dP_{a\to bc} = \int \frac{d^3\mathbf{k}_b}{(2\pi)^3}\frac{1}{2E_b}\int \frac{d^3\mathbf{k}_c}{(2\pi)^3}\frac{1}{2E_c}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{\phi(p)}{2E_p}\int \frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{\phi^*(p')}{2E_{p'}}|M|^2_{a\to bc}$$
$$(2\pi)^3\delta(E_p - E_b - E_c)\delta^2(\mathbf{p}^\perp - \mathbf{k}_b^\perp - \mathbf{k}_c^\perp)(2\pi)^3\delta(E_{p'} - E_b - E_c)\delta^2(\mathbf{p}'^\perp - \mathbf{k}_b^\perp - \mathbf{k}_c^\perp)$$
$$= \int \frac{d^3\mathbf{k}_b}{(2\pi)^3}\frac{1}{2E_b}\int \frac{d^3\mathbf{k}_c}{(2\pi)^3}\frac{1}{2E_c}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{|\phi(p)|^2}{2E_p}\frac{1}{2p^z}|M|^2_{a\to bc}$$
$$(2\pi)^3\delta(E_p - E_b - E_c)\delta^2(\mathbf{p}^\perp - \mathbf{k}_b^\perp - \mathbf{k}_c^\perp)$$
$$= \int \frac{d^3\mathbf{k}_b}{(2\pi)^3}\frac{1}{2E_b}\int \frac{d^3\mathbf{k}_c}{(2\pi)^3}\frac{1}{2E_c}\frac{1}{2p_a^z}|M|^2_{a\to bc}(2\pi)^3\delta(E_a - E_b - E_c)\delta^2(\mathbf{p}_a^\perp - \mathbf{k}_b^\perp - \mathbf{k}_c^\perp),$$

$\Downarrow$

$$\frac{F^{a\to bc}_{fric}}{A} = \int \frac{d^3\mathbf{p}_a}{(2\pi)^3}\frac{f_a}{2E_a}\int \frac{d^3\mathbf{k}_b}{(2\pi)^3}\frac{d^3\mathbf{k}_c}{(2\pi)^3}\frac{|M|^2_{a\to bc}}{2E_b 2E_c}(p_a^z - k_b^z - k_c^z)$$
$$\times (2\pi)^3\delta(E_a - E_b - E_c)\delta^2(\mathbf{p}_a^\perp - \mathbf{k}_b^\perp - \mathbf{k}_c^\perp),$$