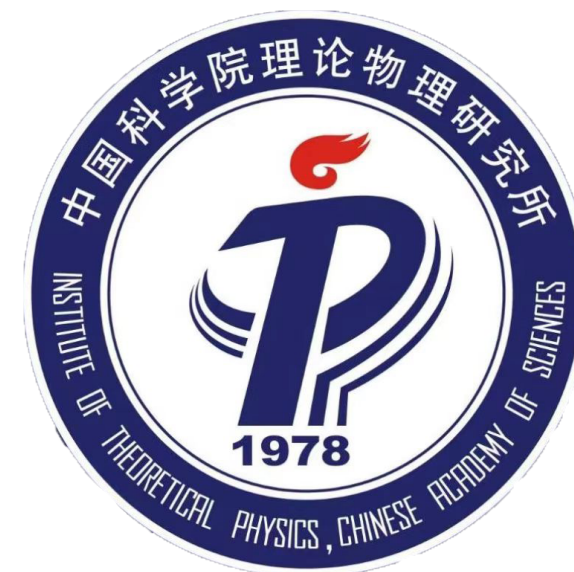

Simulating scalar-induced gravitational waves

Xiang-Xi Zeng

- Scalar-induced gravitational waves with non-Gaussianity up to all orders, arXiv: 2508.10812, **X-X Zeng**, Z Ning, R-G Cai, and S-J Wang.
- Scalar-Induced Gravitational Waves Including Isocurvature Perturbations with Lattice Simulations, arXiv: 2510.02106, **X-X Zeng**.

2025/10/19

彭桓武理论物理创新研究中心“2025引力波物理研讨会”



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- Introduction
- Simulation setup
- Full-order vs finite order
- Isocurvature case and the general case
- Summary



➤ Introduction

➤ Simulation setup

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➤ Isocurvature case and the general case

➤ Summary

1 Introduction



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What is SIGW?

Starting from the conformal Newtonian gauge

Up to second order, this is all we need

$$ds^2 = a(\tau)^2 \left\{ -(1 + 2\Phi)d\tau^2 + \left[(1 - 2\Phi)\delta_{ij} + \frac{1}{2}h_{ij} \right] dx^i dx^j \right\}$$



$$h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = -4S_{ij}^{\text{TT}}$$

$$S_{ij} = 4\Phi\partial_i\partial_j\Phi + 2\partial_i\Phi\partial_j\Phi - \frac{4}{3(1+\omega)\mathcal{H}^2}\partial_i(\Phi' + \mathcal{H}\Phi)\partial_j(\Phi' + \mathcal{H}\Phi)$$

$$\Phi'' + 3\mathcal{H}(1 + c_s^2)\Phi' + (2\mathcal{H}' + (1 + 3c_s^2)\mathcal{H}^2 - c_s^2\nabla^2)\Phi = 0$$

for adiabatic perturbation

1 Introduction



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After doing the Fourier transformation, you will find

$$\langle h_{ij} h_{ij} \rangle \sim \iint d^3 q d^3 k \langle \zeta \zeta \zeta \zeta \rangle$$

For adiabatic initial condition

$$\Phi_i = \frac{3 + 3\omega}{5 + 3\omega} \zeta_i$$

How to deal with the four-point correlation function?

Typically

$$\langle \zeta \zeta \zeta \zeta \rangle = \int D\zeta \mathcal{P}[\zeta] \zeta \zeta \zeta \zeta$$

For local type field



$$\zeta = f(\zeta_g)$$

$$\zeta = \zeta_g + F_{\text{NL}} \zeta_g^2 + G_{\text{NL}} \zeta_g^3 + H_{\text{NL}} \zeta_g^4 + \dots$$



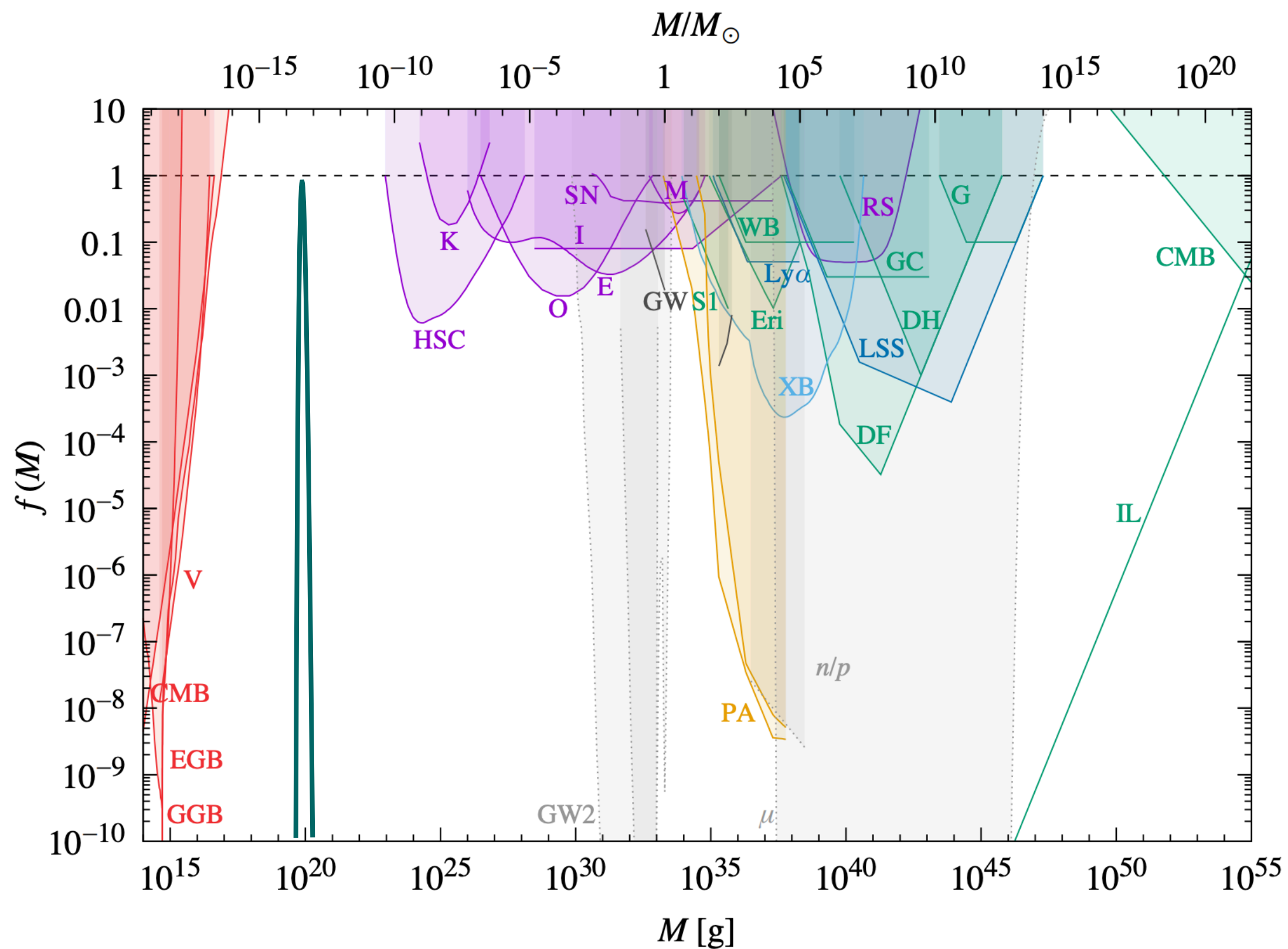
$$\langle \zeta \zeta \zeta \zeta \rangle = \langle (\zeta_g + F_{\text{NL}} \zeta_g^2 + \dots)^4 \rangle$$

1 Introduction



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Here is a problem



For a peak-like (monochromatic PBH) power spectrum to explain all the dark matter

$$\mathcal{P}_{\zeta_g}(k) = A \frac{(k/k_*)^3}{\sqrt{2\pi e}} \exp \left[-\frac{(k/k_* - 1)^2}{2e^2} \right]$$



$$A \sim O(0.01)$$



$$\zeta_g \sim O(0.1)$$



Seems not good

$$\zeta = 0.1 + 5 \times 0.01 + 3 \times 0.001 + \dots$$

B. Carr, K. Kohri, Y. Sendouda and J. Yokoyama, Rept. Prog. Phys. 2002.12778

1 Introduction



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For F_{NL} order, you need to evaluate 8-multiple integrals.

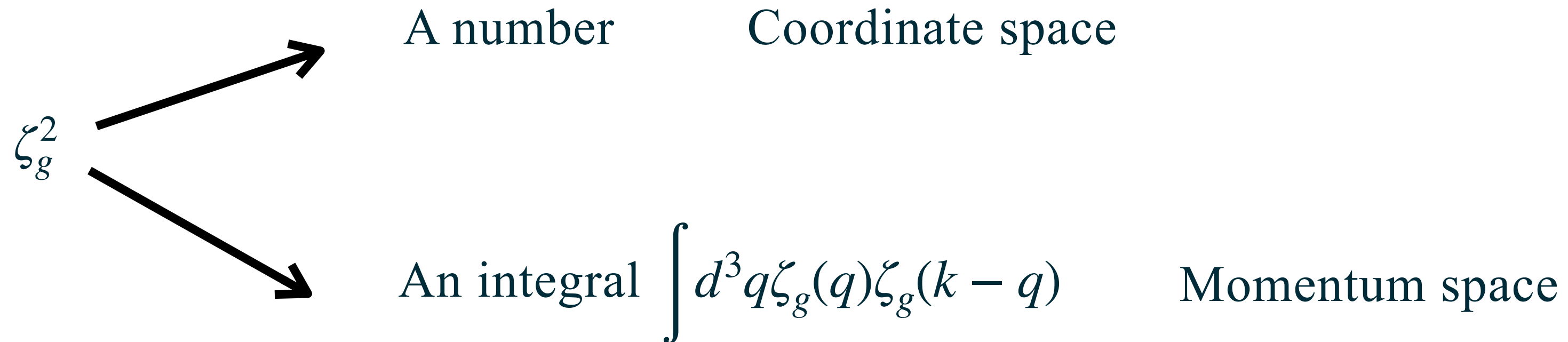
$$\zeta = \zeta_g + F_{\text{NL}}\zeta_g^2 + G_{\text{NL}}\zeta_g^3 + H_{\text{NL}}\zeta_g^4 + \dots$$

For G_{NL} order, you need to evaluate 14-multiple integrals.

...



Because we are calculating in momentum space



1 Introduction



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For F_{NL} order, you need to evaluate 8-multiple integrals.

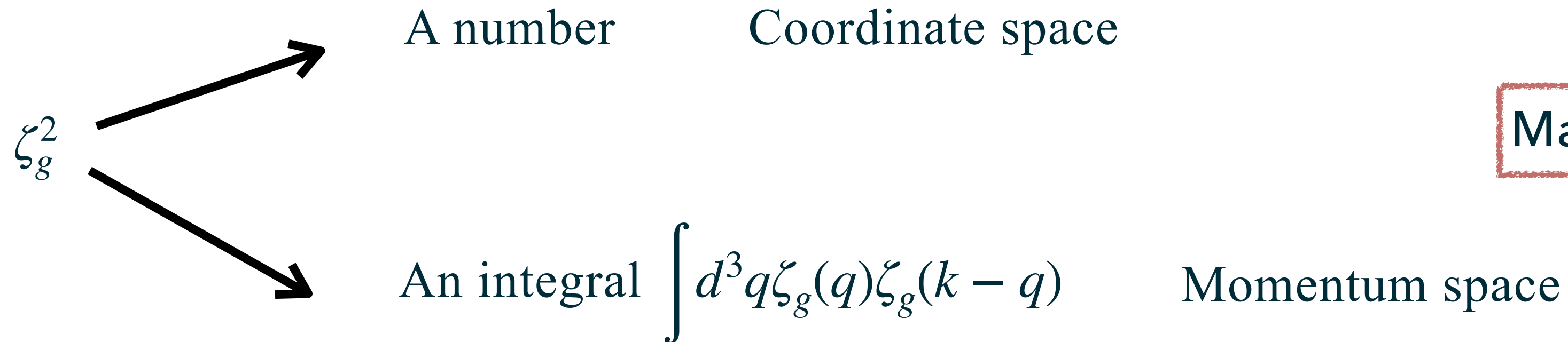
$$\zeta = \zeta_g + F_{\text{NL}}\zeta_g^2 + G_{\text{NL}}\zeta_g^3 + H_{\text{NL}}\zeta_g^4 + \dots$$

For G_{NL} order, you need to evaluate 14-multiple integrals.

...



Because we are calculating in momentum space



Maybe we need a lattice simulation!

➤ Introduction

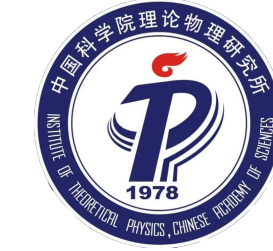
➤ Simulation setup

➤ Full-order vs finite order

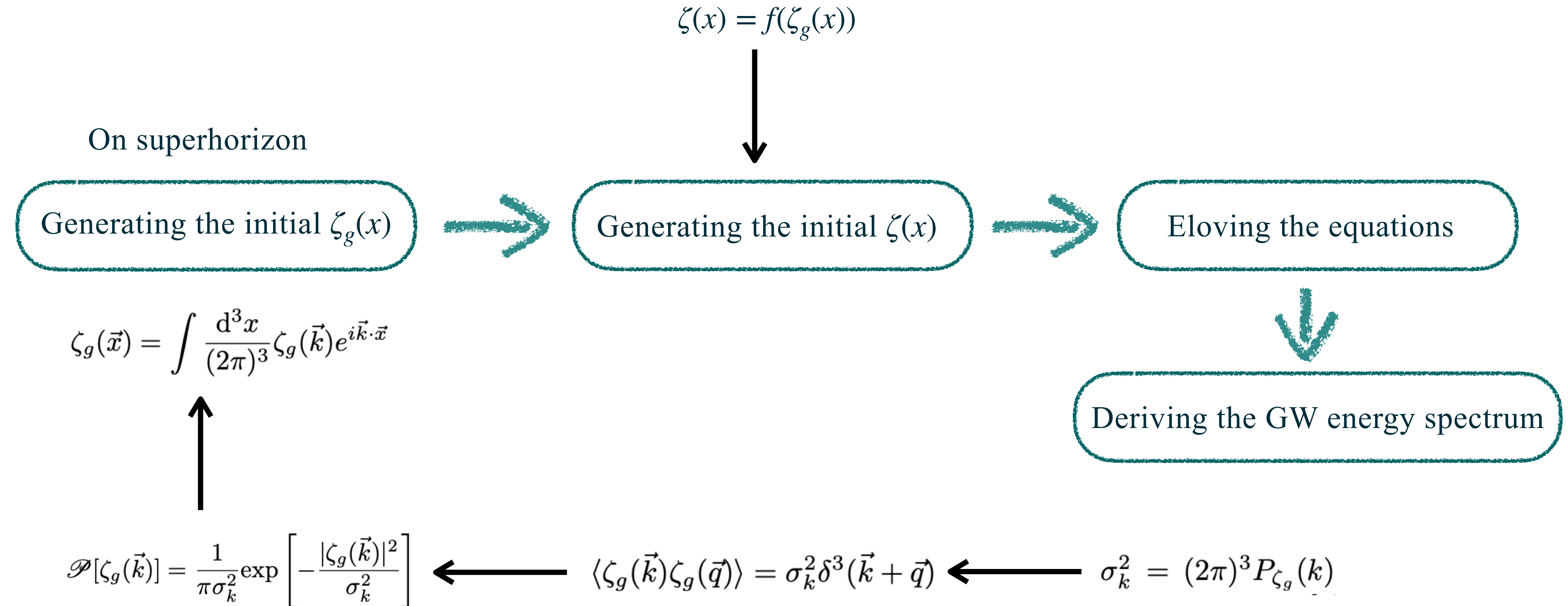
➤ Isocurvature case and the general case

➤ Summary

2 Simulation setup



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2 Simulation setup



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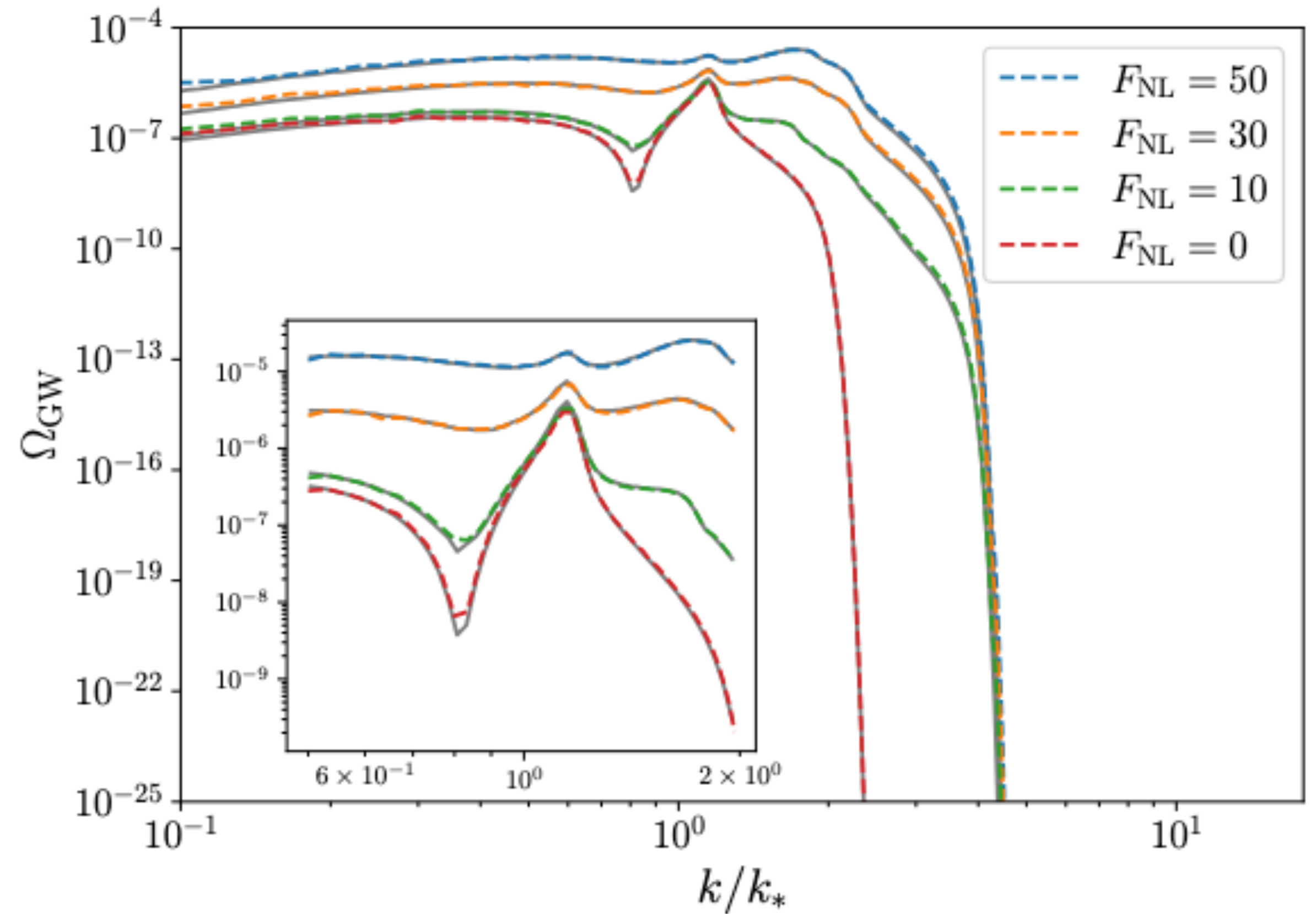
Benchmark with the semi-analytical results

$$\mathcal{P}_{\zeta_g}(k) = A \frac{(k/k_*)^3}{\sqrt{2\pi e}} \exp \left[-\frac{(k/k_* - 1)^2}{2e^2} \right]$$

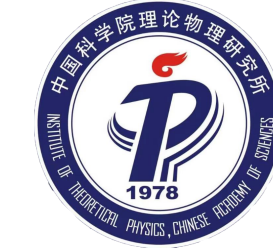
$$A = 0.01, e = 1/30$$

— Semi-analytical results
- - - Simulation results

Two results are consistent with each other!



2 Simulation setup



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The efficiency of the lattice simulation

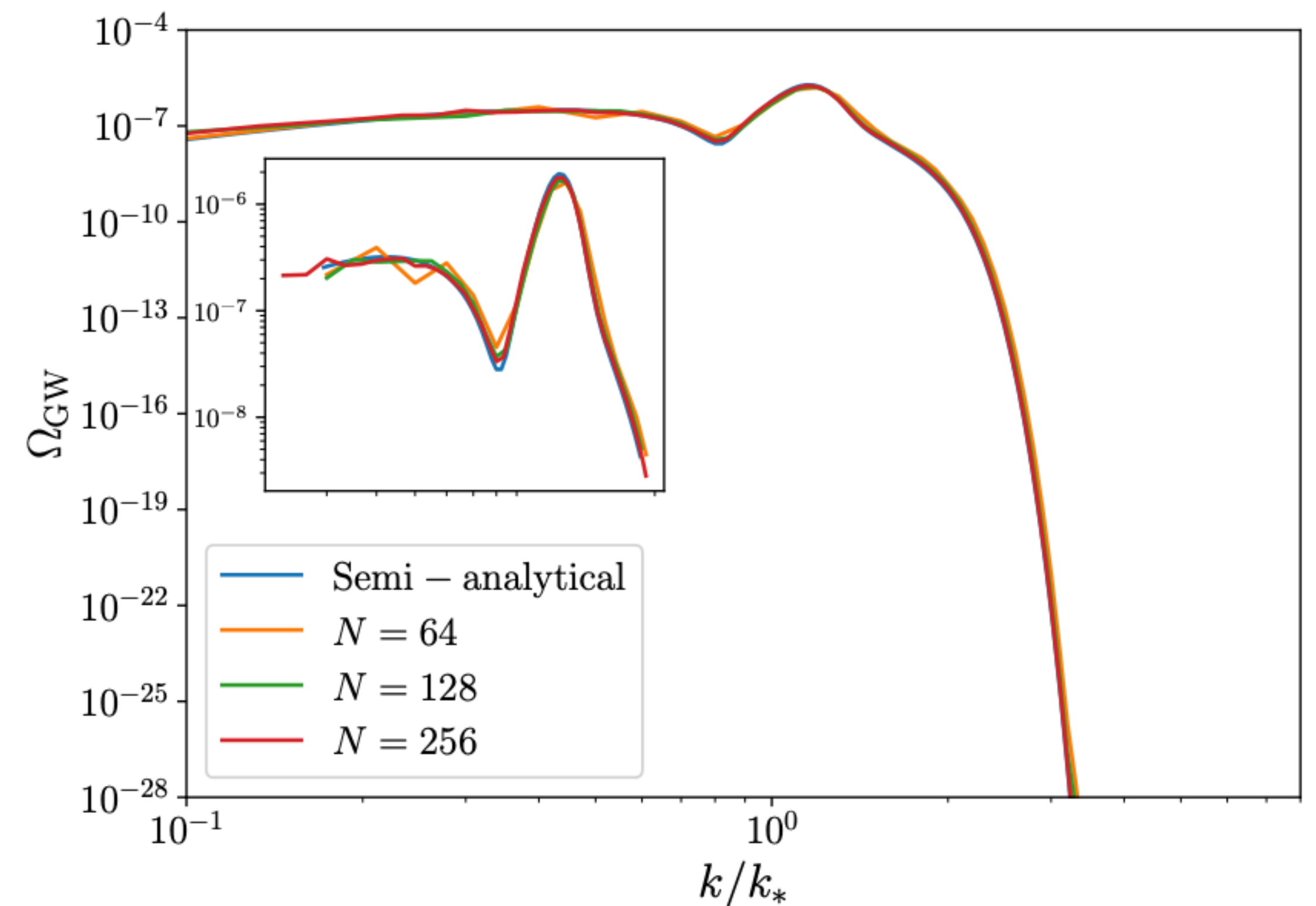
$$\langle h_{ij} h_{ij} \rangle \sim \int \int d^3 q d^3 k d^3 l \langle (\zeta_g + F_{\text{NL}} \zeta_g^2)(\zeta_g + F_{\text{NL}} \zeta_g^2)(\zeta_g + F_{\text{NL}} \zeta_g^2)(\zeta_g + F_{\text{NL}} \zeta_g^2) \rangle$$



Need to do 8-multiple integral

Semi-analytical results 14-core i9-10th 200 points 1 day

Simulation results RTX-4060 N=128 6 mins



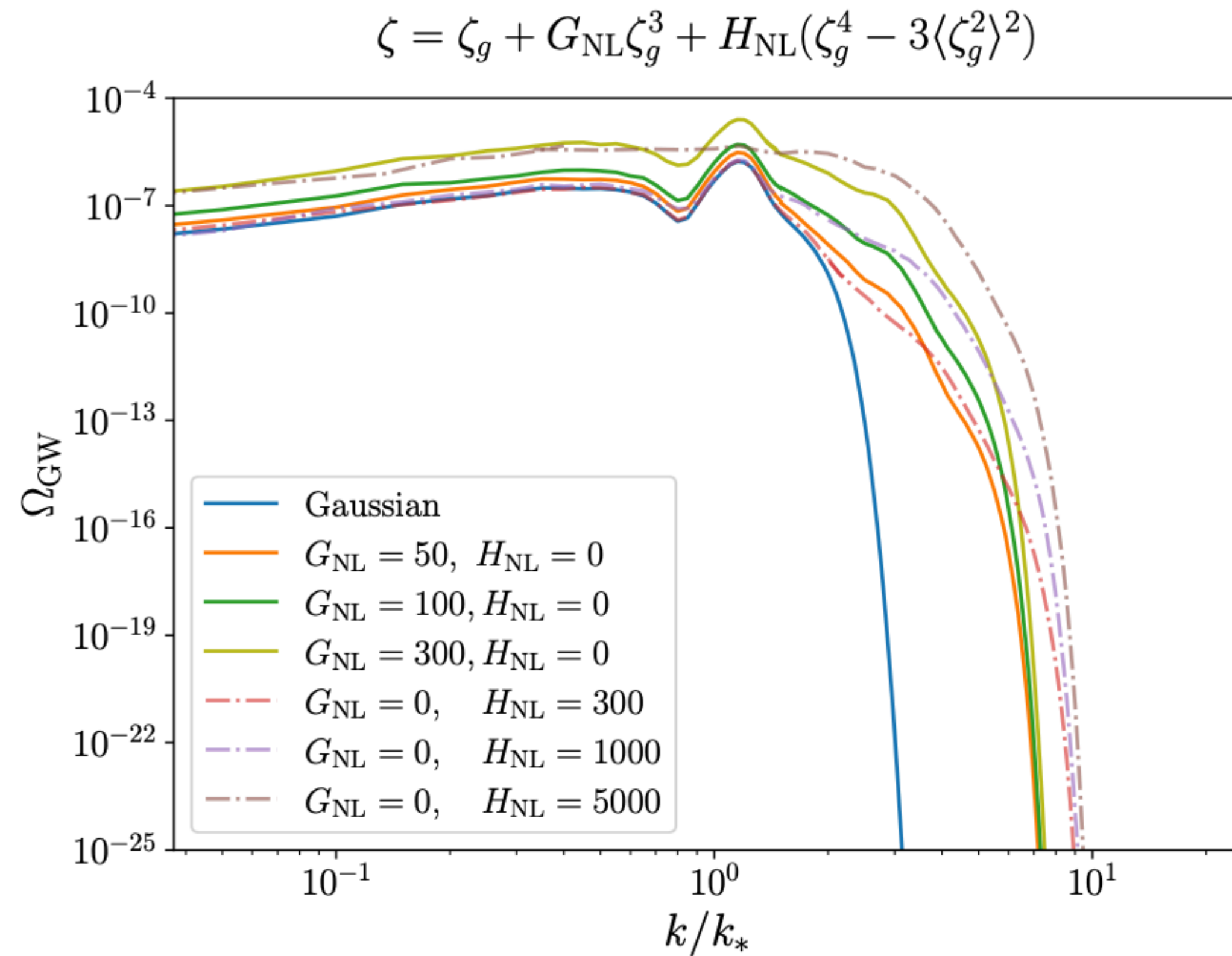
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3 Full-order vs finite order

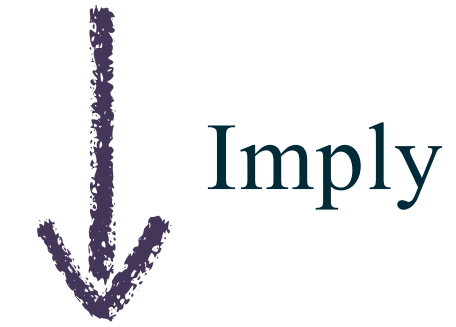


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Higher order effects



The higher order usually contributes to a higher cutoff



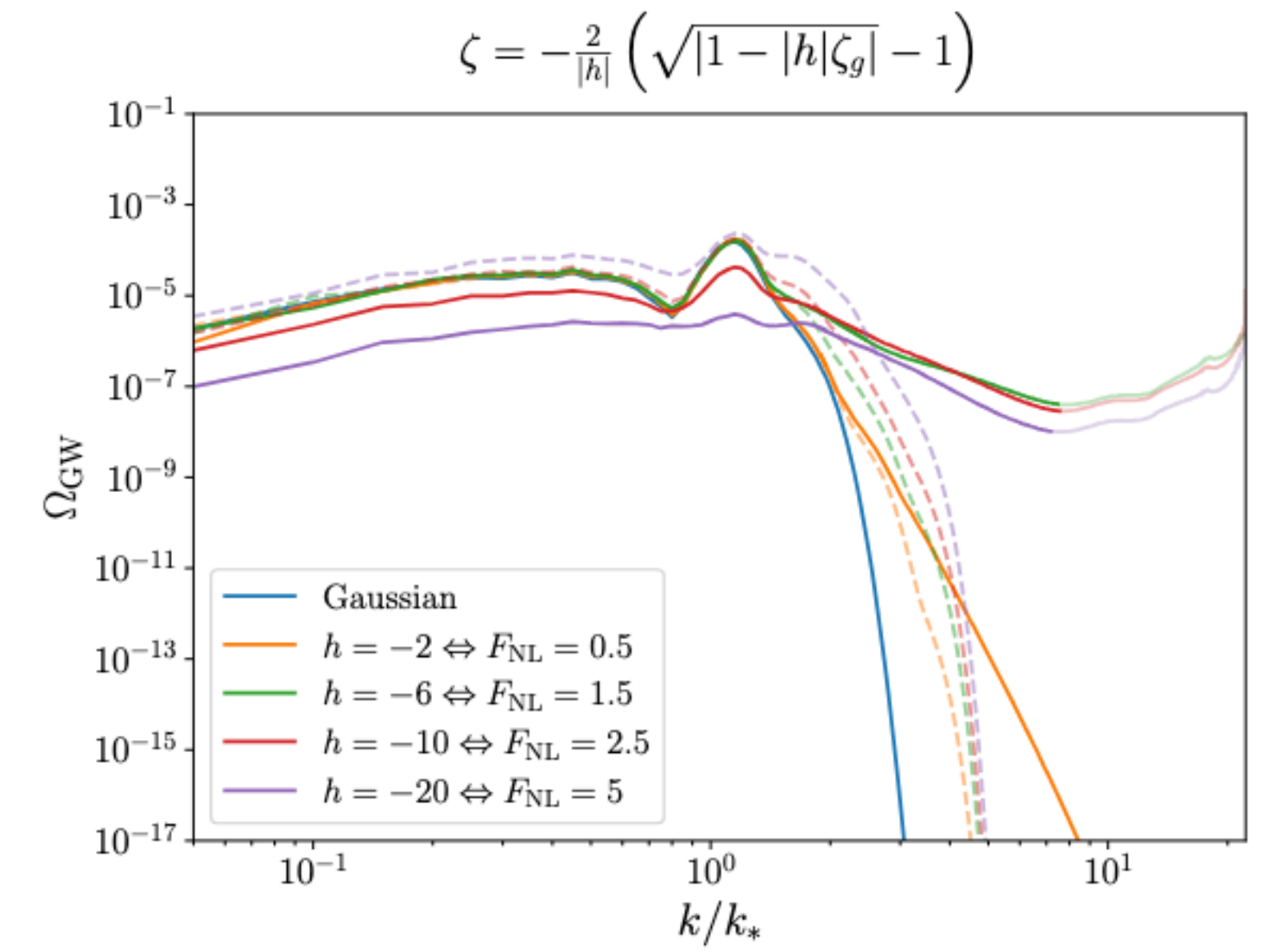
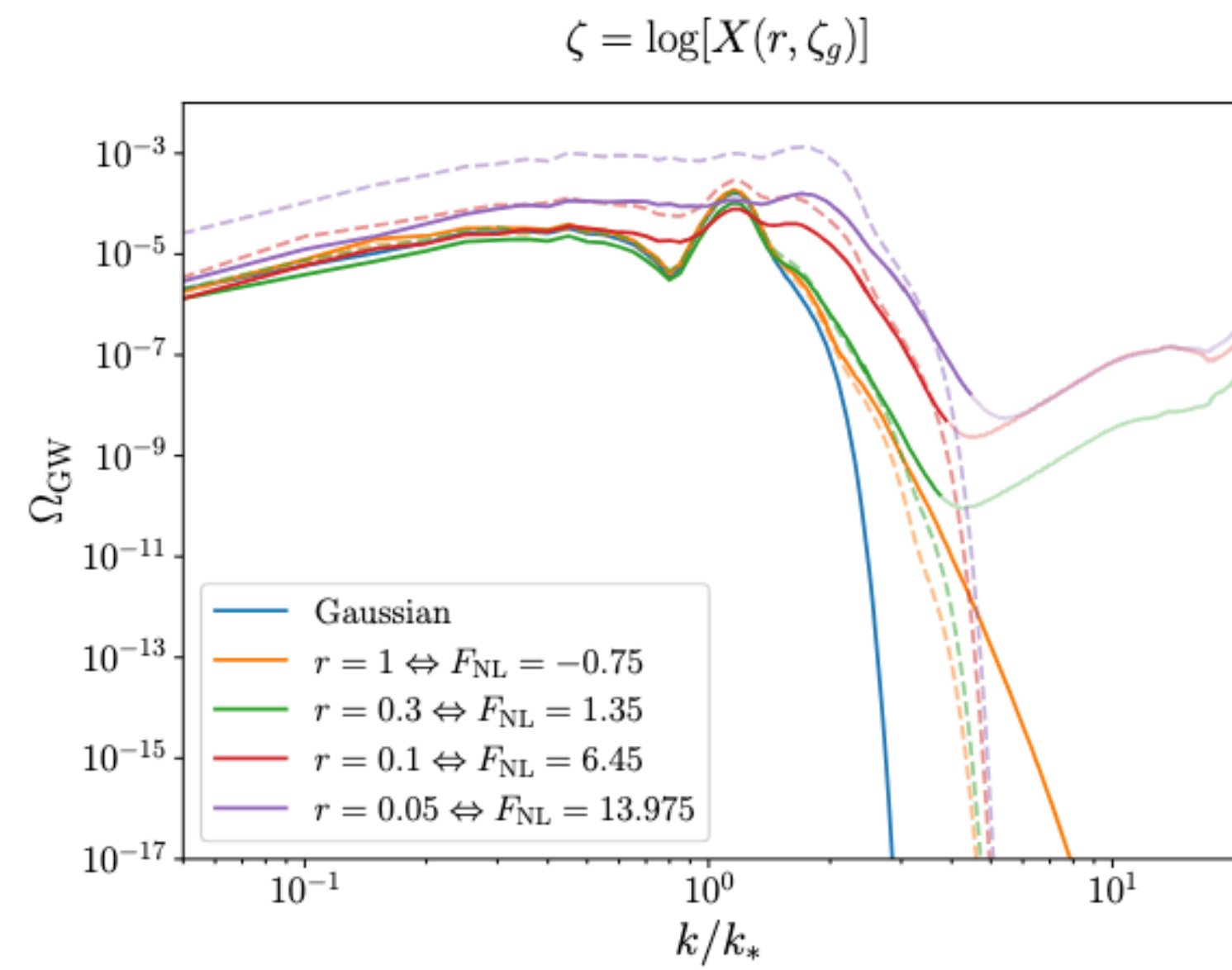
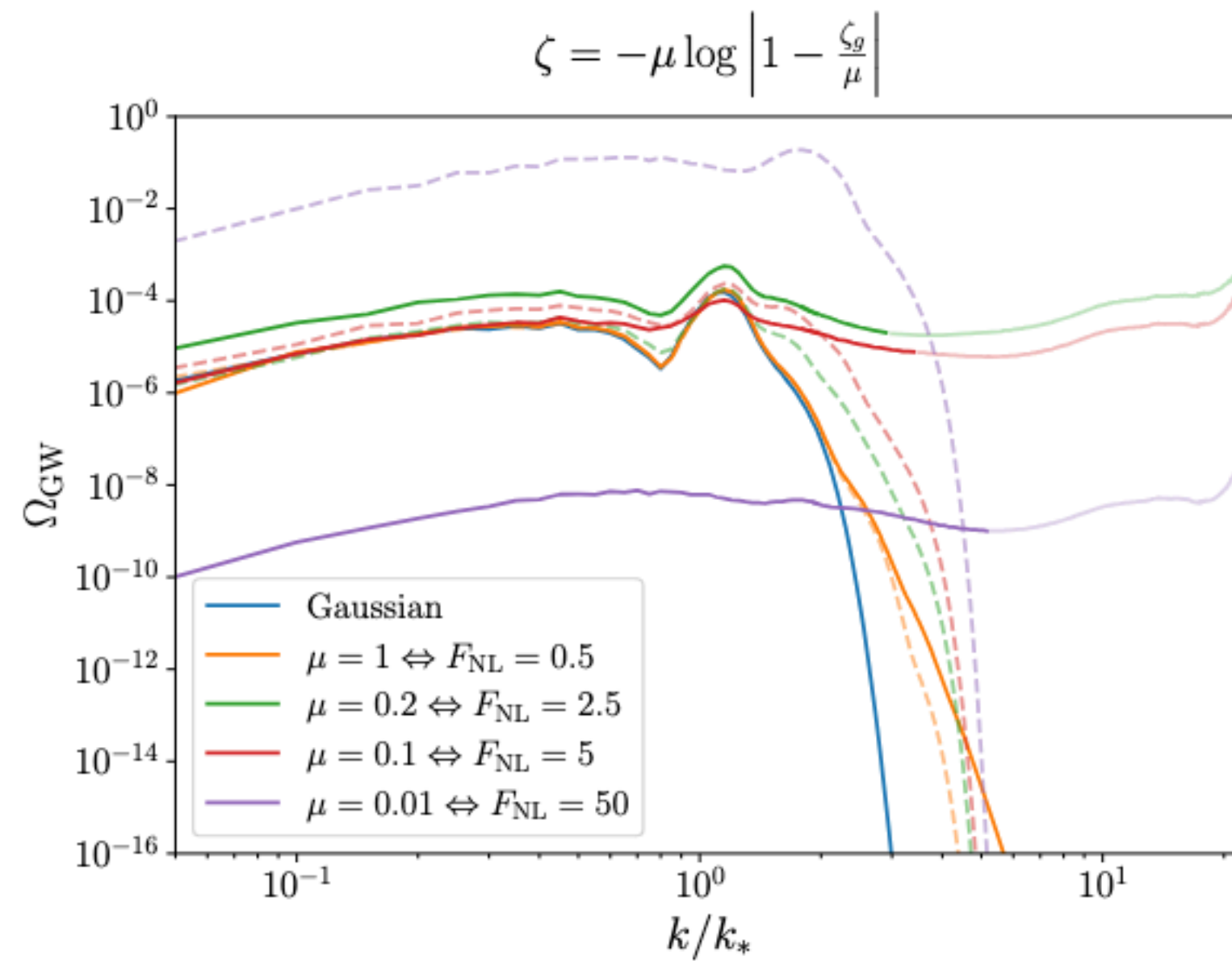
Full-order will not have such a cutoff

3 Full-order vs finite order



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Specific Models



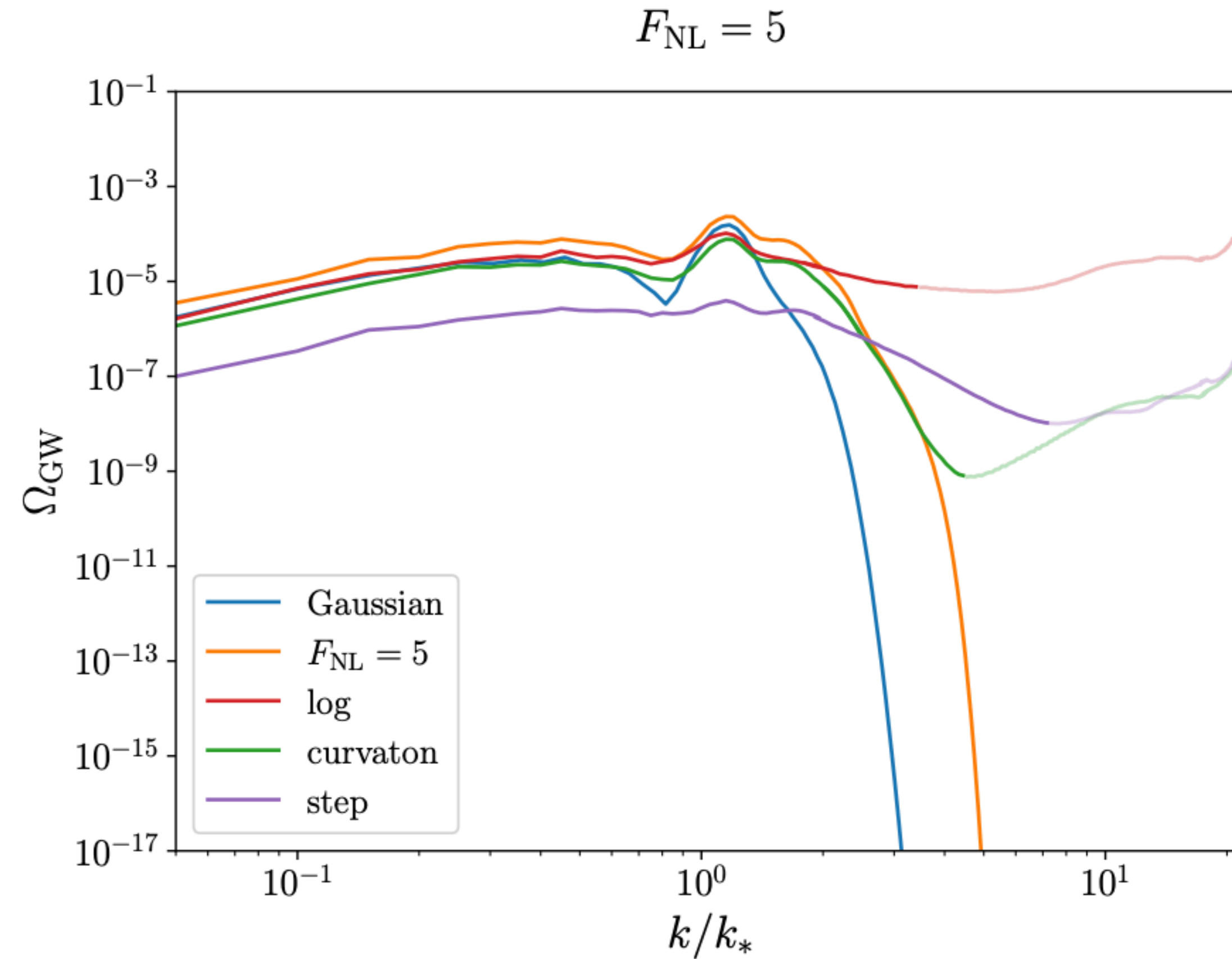
Curvaton model

———— Full-order
----- Finite order (F_{NL})

3 Full-order vs finite order



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Maybe it's possible to distinguish inflation models by SIGW!

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4 The isocurvature case and the general case



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Equations

At background level

$$\begin{aligned}3\mathcal{H}^2 &= a^2(\rho_m + \rho_r), \\3(\mathcal{H}^2 + 2\mathcal{H}') &= -a^2\rho_r, \\ \rho'_m + 3\mathcal{H}\rho_m &= -aQ, \\ \rho'_r + 4\mathcal{H}\rho_r &= aQ. \\ aQ &\equiv \rho_m a\Gamma, \\ \Gamma &= \frac{n}{t_{\text{eva}} - t},\end{aligned}$$

At second-order

$$\begin{aligned}h''_{ij} + 2\mathcal{H}h'_{ij} - \Delta h_{ij} &= T_{ij}{}^{lm} S_{lm}, \\ S_{ij} &= 4\partial_i\Phi\partial_j\Phi + 2a^2\left(\rho_m\partial_i v_m\partial_j v_m + \frac{4}{3}\rho_r\partial_i v_r\partial_j v_r\right)\end{aligned}$$

At first-order

$$\begin{aligned}6\mathcal{H}\Phi' + 6\mathcal{H}^2\Phi - 2\Delta\Phi &= a^2(\delta\rho_m + \delta\rho_r) \equiv a^2\delta\rho, \\ \Phi' + \mathcal{H}\Phi &= \frac{1}{2}a^2\left(\rho_m v_m + \frac{4}{3}\rho_r v_r\right) \equiv \frac{1}{2}a^2\rho V, \\ \Phi'' + 3\mathcal{H}\Phi' + (\mathcal{H}^2 + 2\mathcal{H}')\Phi &= -\frac{1}{6}a^2\delta\rho_r, \\ \delta\rho'_m + 3\mathcal{H}\delta\rho_m + \rho_m(3\Phi' + \Delta v_m) &= -a\delta Q + a\Phi Q, \\ \delta\rho'_r + 4\mathcal{H}\delta\rho_r + \frac{4}{3}\rho_r(3\Phi' + \Delta v_r) &= a\delta Q - a\Phi Q, \\ v'_m + \mathcal{H}v_m - \Phi &= 0, \\ v'_r + \frac{1}{4}\frac{\delta\rho_r}{\rho_r} - \Phi &= \frac{aQ}{\rho_r}\left(\frac{3}{4}v_m - v_r\right).\end{aligned}$$

4 The isocurvature case and the general case



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Theoretically, it's convenient to define the isocurvature perturbation

$$S \equiv \frac{\delta\rho_m}{\rho_m} - \frac{\delta\rho_r}{\rho_r + p_r} = \frac{\delta\rho_m}{\rho_m} - \frac{3}{4} \frac{\delta\rho_r}{\rho_r}$$

Then, one can get the familiar equations

$$V'_{\text{rel}} + 3c_s^2 \mathcal{H} V_{\text{rel}} + \frac{3}{2a^2 \rho_r} c_s^2 \Delta \Phi + \frac{3\rho_m}{4\rho_r} c_s^2 S - \frac{aQ}{4\rho_r} \frac{\rho V - 4(\rho_m + \rho_r) V_{\text{rel}}}{\rho_m + 4\rho_r/3} = 0$$

$$\Phi'' + 3\mathcal{H}(1 + c_s^2)\Phi' + (\mathcal{H}^2(1 + 3c_s^2) + 2\mathcal{H}')\Phi - c_s^2 \Delta \Phi = \frac{a^2}{2} \rho_m c_s^2 S,$$

$$S' = -\Delta V_{\text{rel}} - a(\delta Q - Q\Phi) \frac{3\rho_m + 4\rho_r/3}{4\rho_m \rho_r} + aQ \left(\frac{\delta\rho_m}{\rho_m^2} + \frac{3}{4} \frac{\delta\rho_r}{\rho_r^2} \right)$$

$$c_s^2 = \frac{1}{3} \left(1 + \frac{3\rho_m}{4\rho_r} \right)^{-1} = \frac{4}{9} \frac{\rho_r}{\rho_m + 4\rho_r/3}, \quad V_{\text{rel}} \equiv v_m - v_r$$

4 The isocurvature case and the general case



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When there is no energy-transfer, an analytical solution on superhorizon exists

$$S_k(\xi) = S_i(k),$$
$$\Phi_k(\xi) = \Phi_i(k) \left(\frac{8}{5\xi^3} (\sqrt{1+\xi} - 1) - \frac{4}{5\xi^2} + \frac{1}{5\xi} + \frac{9}{10} \right) \\ + S_i(k) \left(\frac{16}{5\xi^3} (1 - \sqrt{1+\xi}) + \frac{8}{5\xi^2} - \frac{2}{5\xi} + \frac{1}{5} \right),$$

$$\xi \equiv a/a_{\text{eq}}$$

$$\frac{a(\tau)}{a_{\text{eq}}} = 2 \left(\frac{\tau}{\tau_*} \right) + \left(\frac{\tau}{\tau_*} \right)^2 \quad \tau_{\text{eq}} = (\sqrt{2}-1)\tau_*$$

Then, you just need to give the initial value of S and Φ

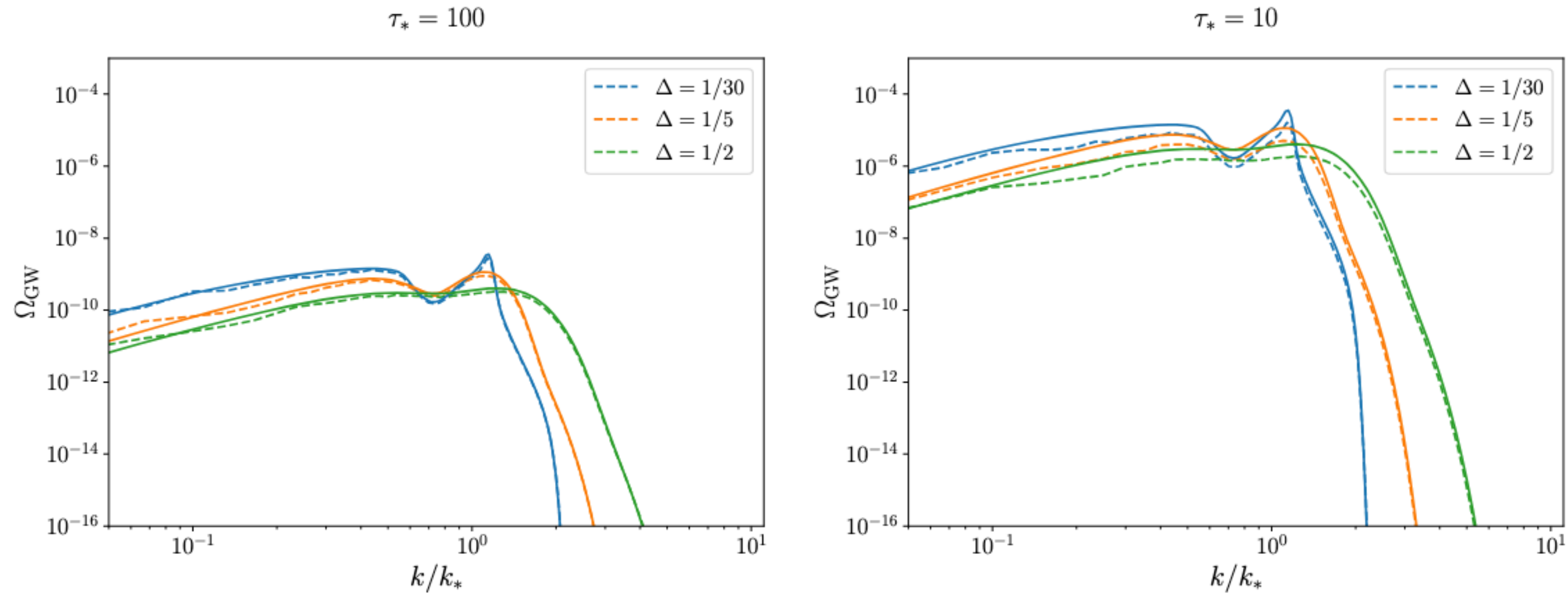
4 The isocurvature case and the general case



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Comparing with the semi-analytical results

$$\mathcal{P}_S(k) = B \frac{(k/k_{*2})^3}{\sqrt{2\pi}\Delta_{\text{iso}}} \exp\left[-\frac{(k/k_{*2} - 1)^2}{2\Delta_{\text{iso}}^2}\right] \quad B = 100$$



G. Domenech, S. Passaglia, S. Renaux-Petel, JCAP 03 (2022) 023, [2112.10163]

4 The isocurvature case and the general case

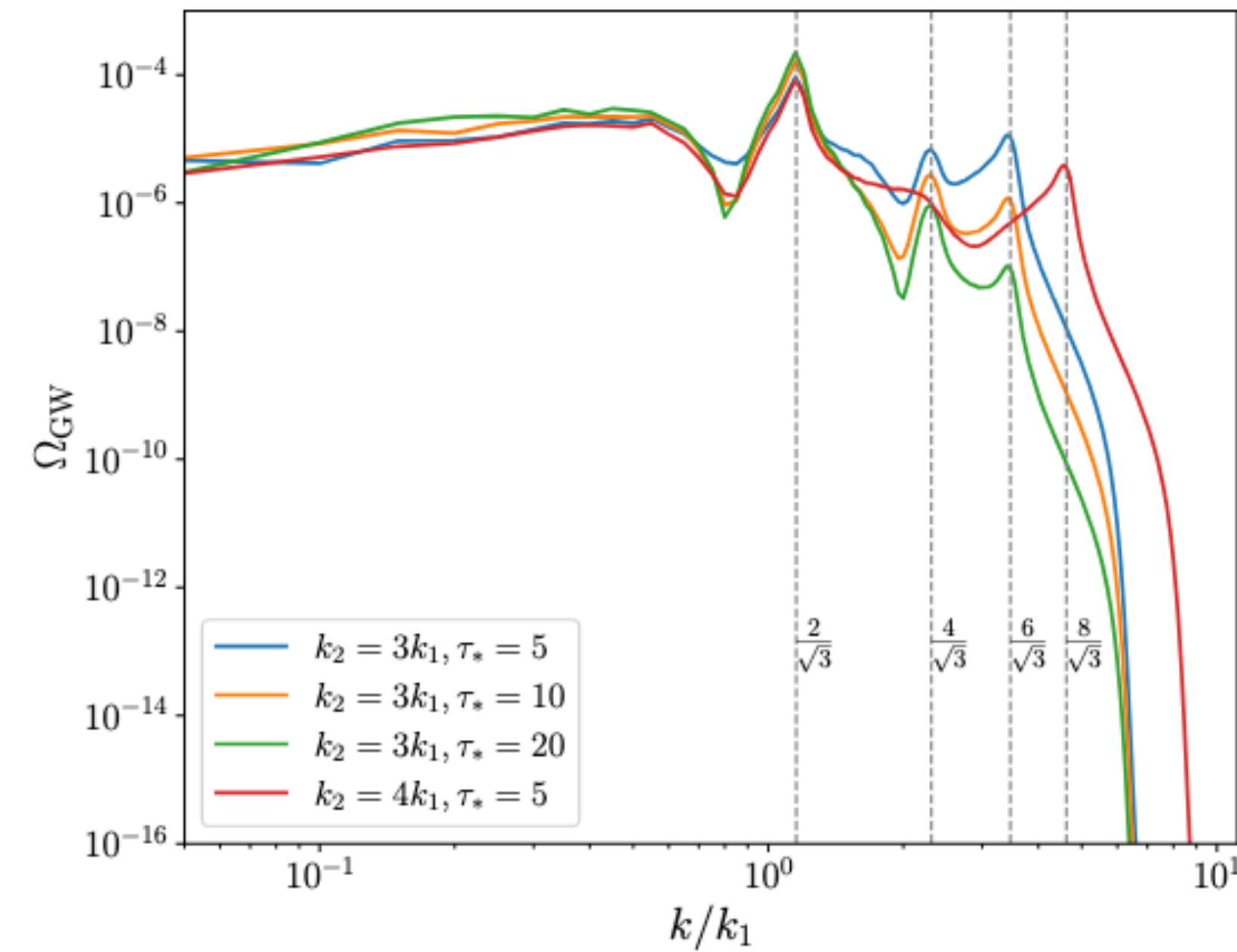
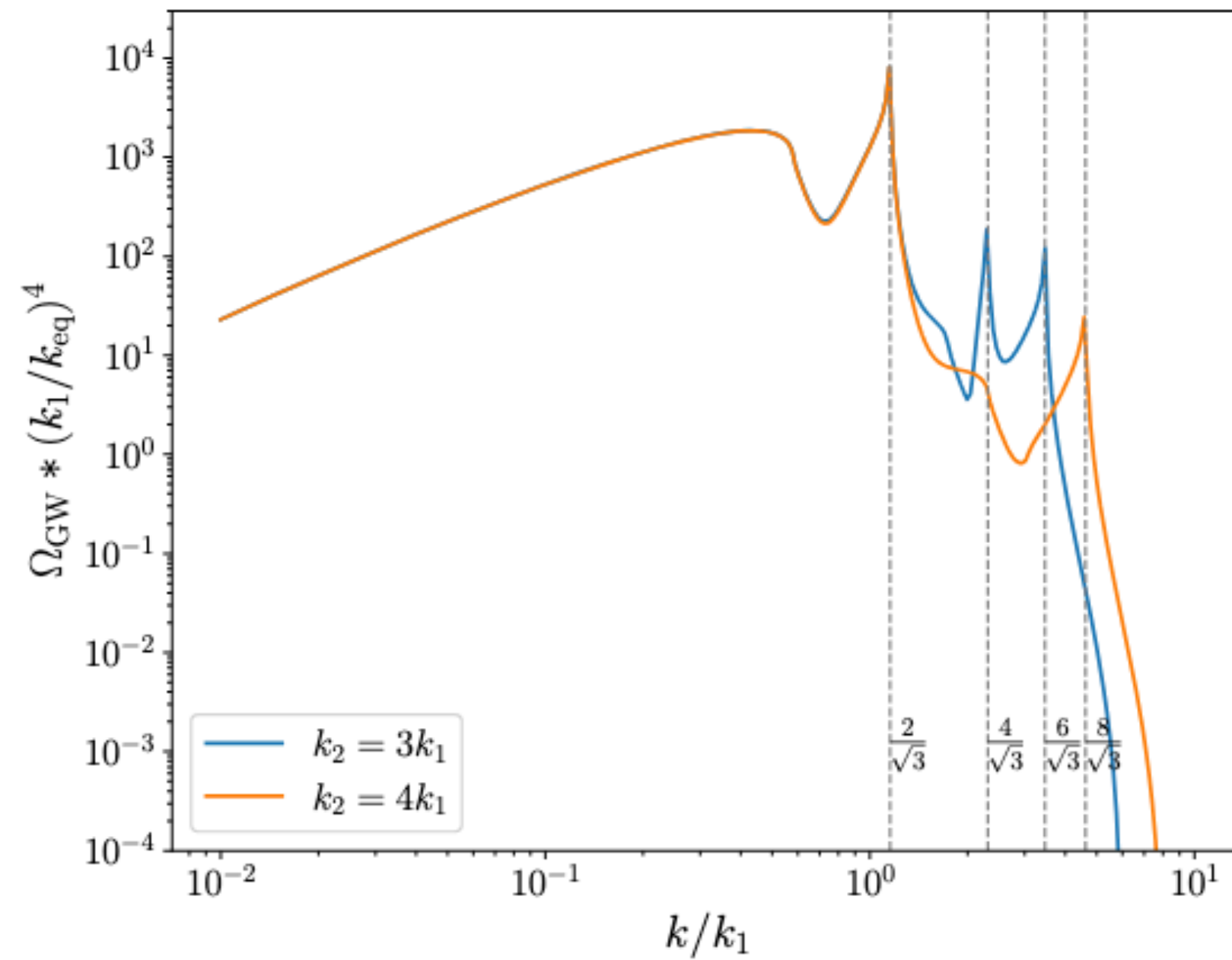


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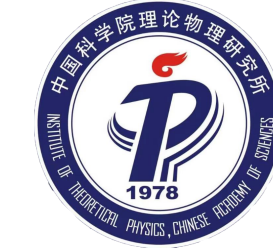
Multi-peaks structures

$$\Omega_{\text{GW,iso}} = \frac{2}{3} \sum_{i,j} B_i B_j \tilde{k}_i^{-1} \tilde{k}_j^{-1} \left(\frac{4\tilde{k}_i^{-2} - (1 - \tilde{k}_j^{-2} + \tilde{k}_i^{-2})^2}{4\tilde{k}_i^{-1} \tilde{k}_j^{-1}} \right)^2 \frac{I^2(k, \tau_c \rightarrow \infty, \tilde{k}_i^{-1}, \tilde{k}_j^{-1})}{\Theta(k_i + k_j - k) \Theta(k - |k_i - k_j|)},$$

$$\left(\frac{k_i}{k} + \frac{k_j}{k} \right) = \sqrt{3}.$$



4 The isocurvature case and the general case

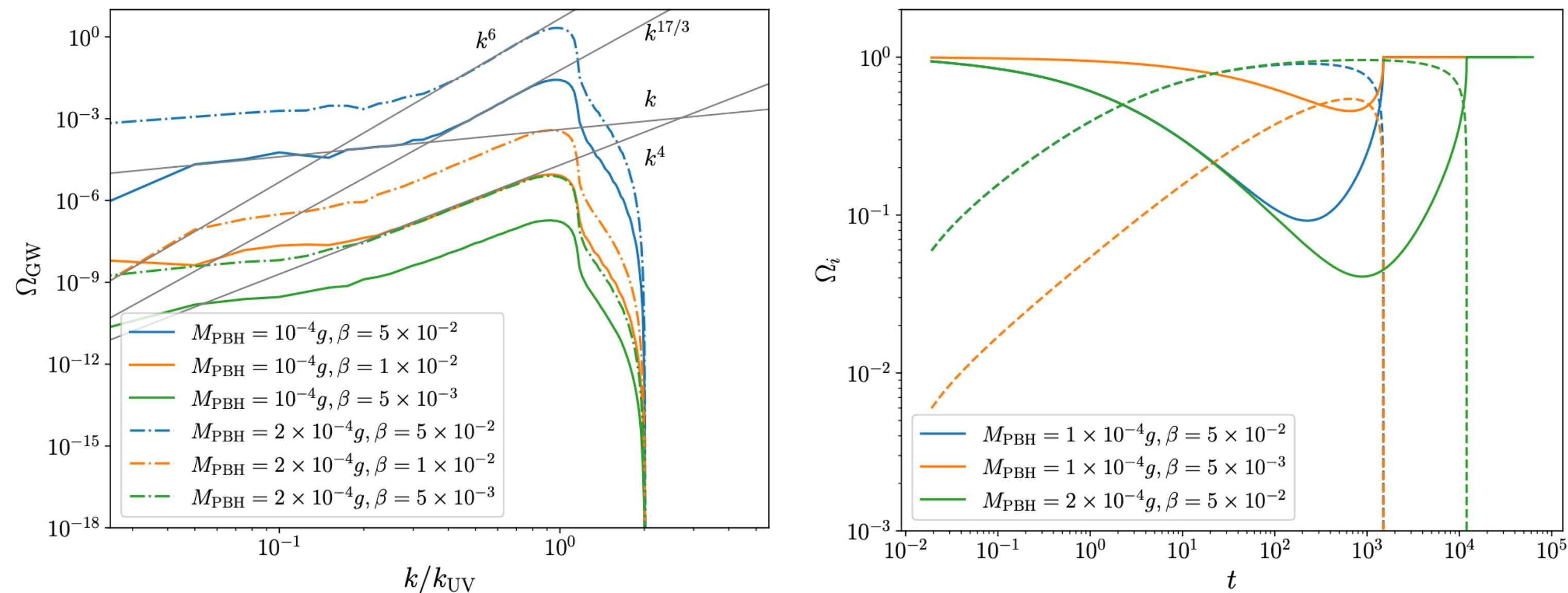


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PBH-dominated era

T. Papanikolaou, V. Vennin, D. Langlois, JCAP 03 (2021) 053, [2010.11573]

G. Domenech, C. Lin, M. Sasaki, JCAP 04 (2021) 062, [2012.08151]



$$\mathcal{P}_S(k) = \frac{2}{3\pi} \left(\frac{k}{k_{\text{UV}}} \right)^3 \Theta(k_{\text{UV}} - k)$$

Figure 3. *Left:* The GWs energy spectra sourced by PBH isocurvature perturbations. We have set $k_{\text{UV}} = 40$ in our simulations. *Right:* The energy density fraction $\Omega_i = \rho_{i=m,r}/(\rho_m + \rho_r)$ of PBH and radiation. The dashed lines denote Ω_m , while the solid line describes Ω_r . t is the cosmic time.

4 The isocurvature case and the general case

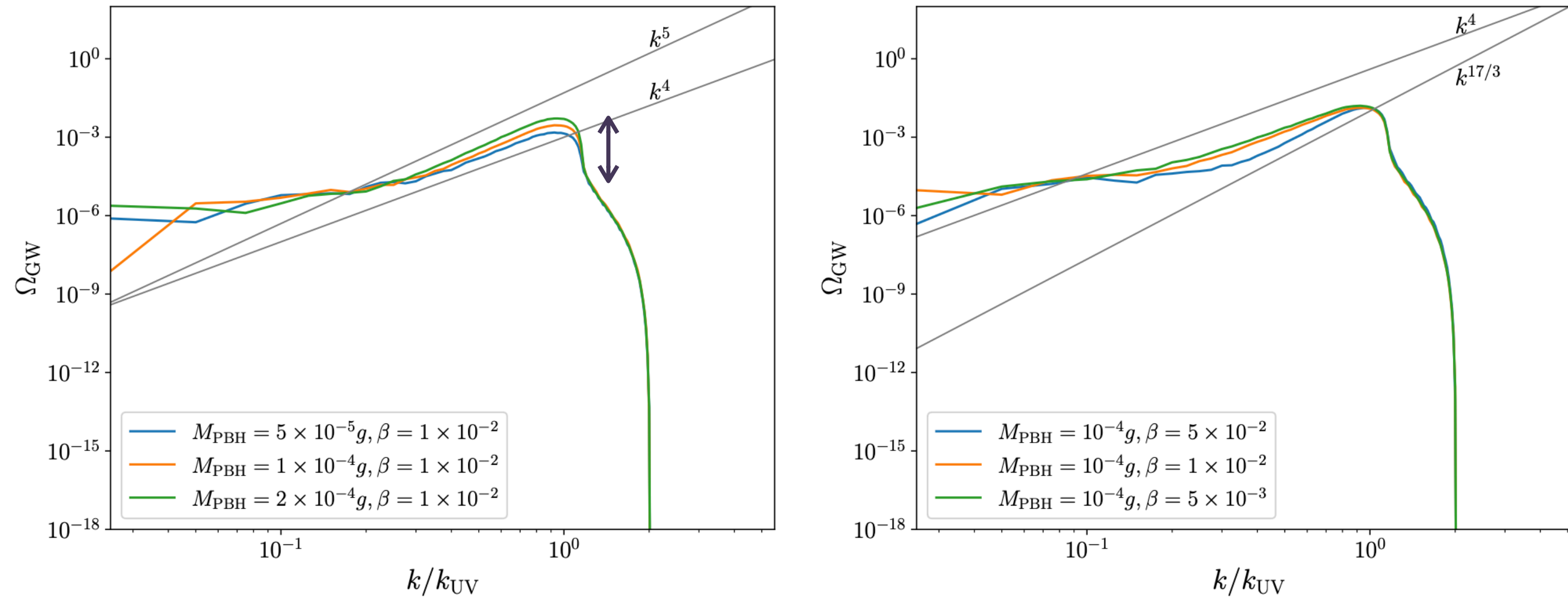


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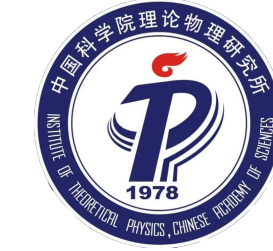
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$$\mathcal{P}_S(k) = \frac{2}{3\pi} \left(\frac{k}{k_{\text{UV}}} \right)^3 \Theta(k_{\text{UV}} - k)$$

Figure 4. *Left:* The GWs energy spectra sourced by PBH isocurvature perturbations for different PBH masses. *Right:* The GWs energy spectra sourced by PBH isocurvature perturbations for different PBH abundances. In both figures, we have multiplied a constant to make their ultraviolet part the same.

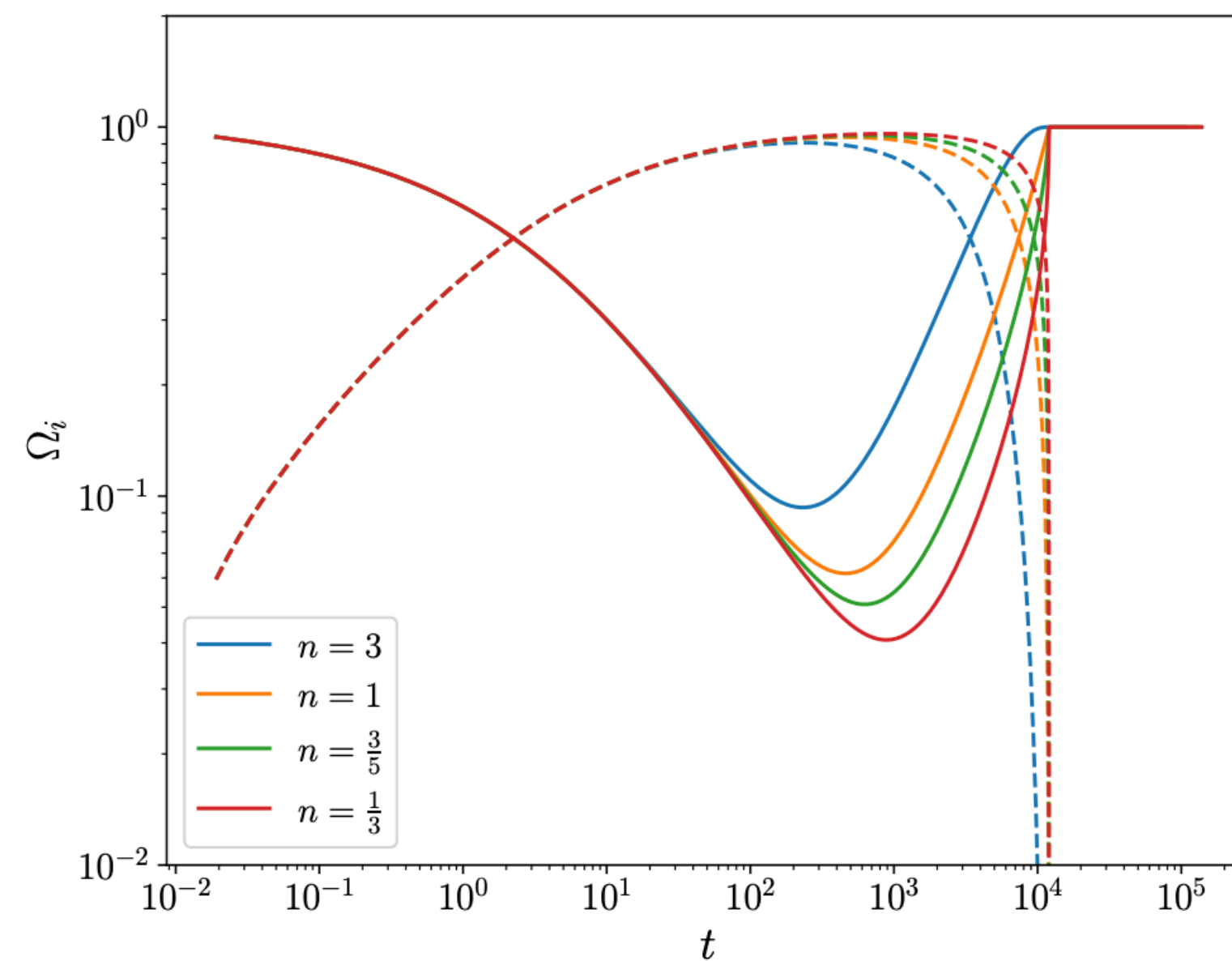
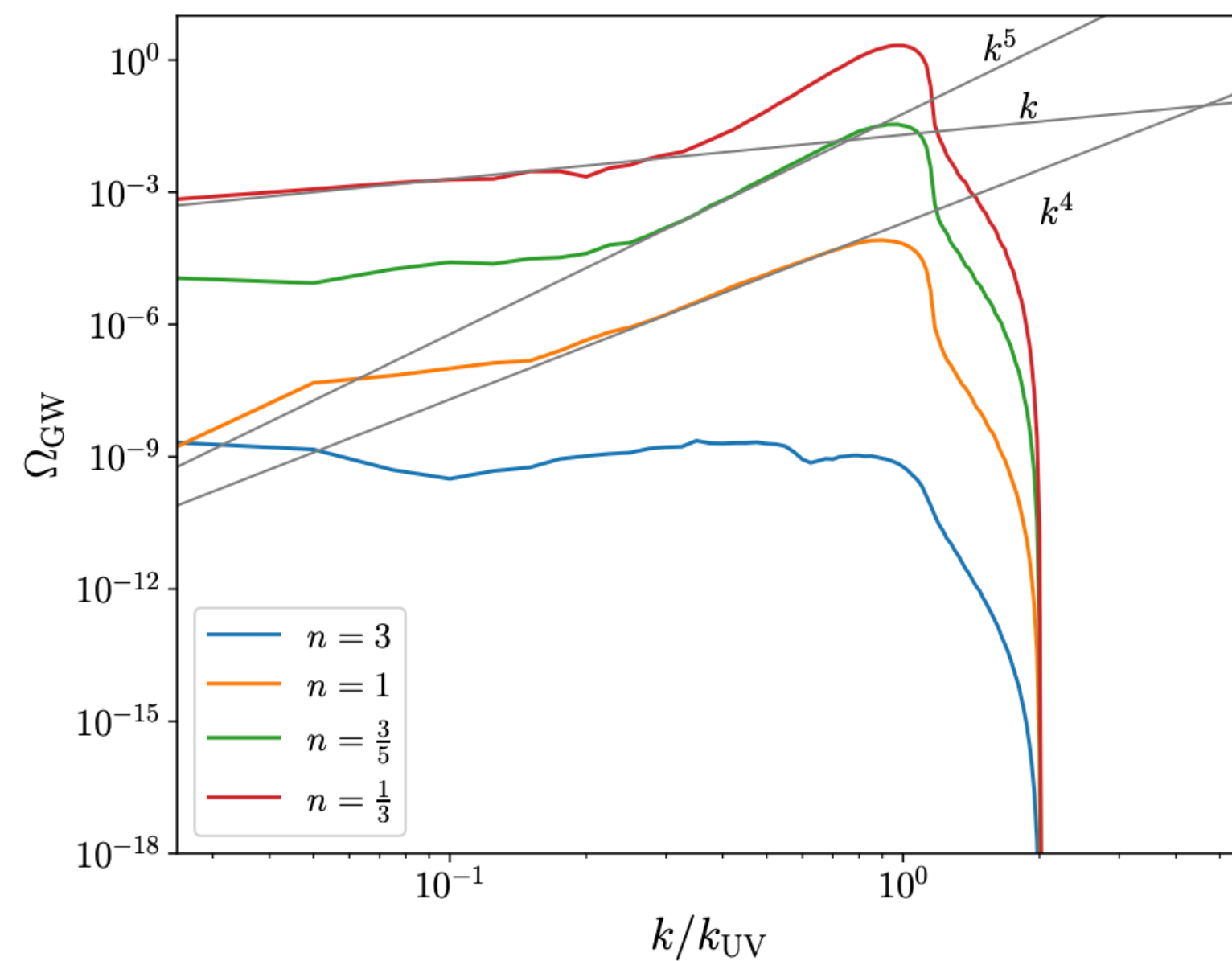
4 The isocurvature case and the general case



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Q-Balls-dominated era

M. Pearce, L. Pearce, G. White, C. Balazs, [2503.03101]



$$aQ \equiv \rho_m a \Gamma,$$

$$\Gamma = \frac{n}{t_{\text{eva}} - t},$$

$$\mathcal{P}_S(k) = \frac{2}{3\pi} \left(\frac{k}{k_{\text{UV}}} \right)^3 \Theta(k_{\text{UV}} - k)$$

Figure 5. *Left:* The GWs energy spectra sourced by Q-balls isocurvature perturbations. We have set $k_{\text{UV}} = 40$ in our simulations. *Right:* The energy density fraction $\Omega_i = \rho_{i=m,r}/(\rho_m + \rho_r)$ of Q-balls and radiation. The dashed lines denote Ω_m , while the solid line describes Ω_r . t is the cosmic time. The initial abundance is set as $\beta = 5 \times 10^{-2}$.

4 The isocurvature case and the general case

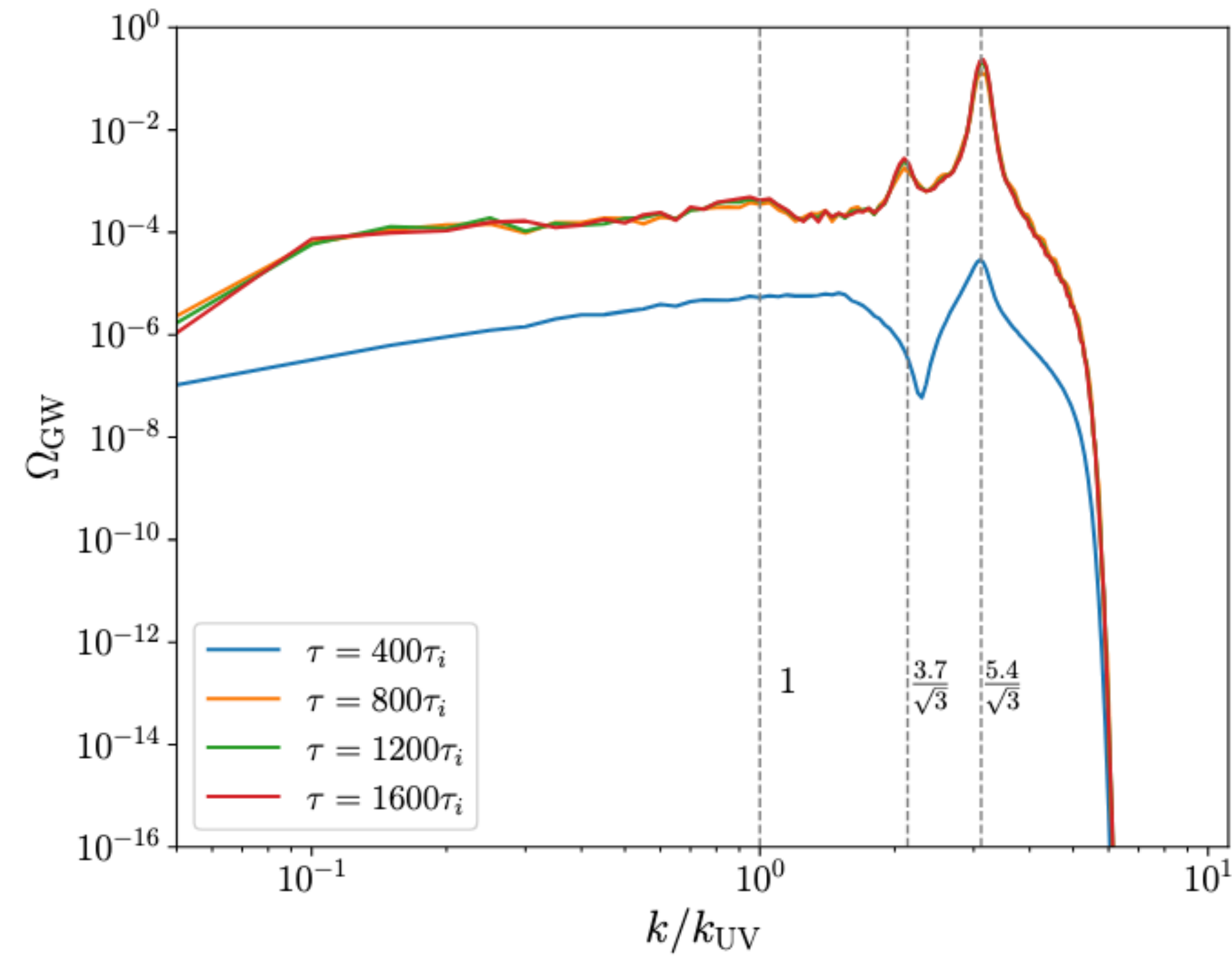


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PBH-dominated era

Mixed initial condition

$$k_{\text{UV}} = \gamma^{-\frac{1}{3}} \beta^{\frac{1}{3}} k_*$$



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When you consider a curvature perturbation with a sizeable amplitude $O(10^{-4} - 10^{-2})$

- SIGW with full-order non-Gaussianity will have a very different ultra-violet behavior
- The peak frequency may have a change
- It might give you a very different amplitude

Therefore, special care should be taken when using SIGW to constrain the abundance of PBHs.

- Don't be afraid of doing lattice simulation, it's quick! It can be done in your laptop!
- We can treat the general initial condition now, while there is no specific semi-analytical formulas.



Thanks for your attention!

