Polarization Modes of Gravitational Waves

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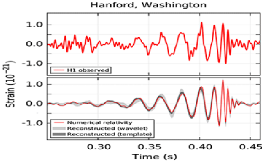
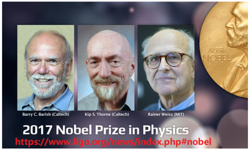
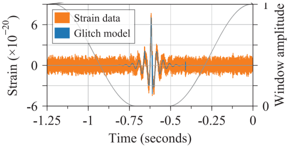
1. Introduction

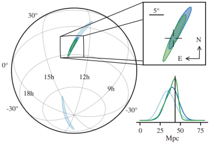
- On Sep 14, 2015, the LIGO team directly detected gravitational waves (GWs) produced by the merger of binary black holes (GW150914).





- On Aug 17, 2017, the GWs (GW170817) and electromagnetic signals produced by the merger of binary neutron stars were detected.

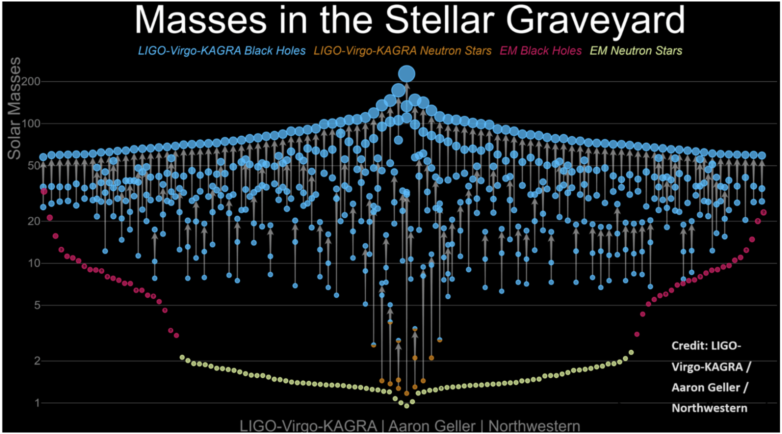




Phys.RevL. ett. 119, 161101 (2017)

1. Introduction

- LIGO-Virgo-KAGRA has detected approximately 300 GW events, opening a new era of multi-messenger astronomy.



1. Introduction

- Recently, NANOGrav[1] and CPTA[2] et al have independently announced compelling evidence for a stochastic signal in their latest data sets.

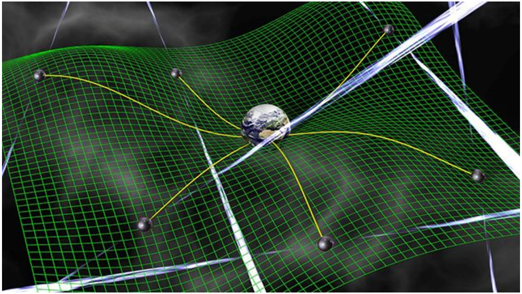


图 1: Pulsar Timing Array Artistic Imagination (Source: David Champion)

[1] G. Agazie et al. (NANOGrav), APJL 951, L9 (2023); APJL 951, L8 (2023).
[2] H. Xu et al., Res. Astron. Astrophys. 23, 075024 (2023).

1. Introduction

- Qing-Guo Huang et al searched for an isotropic non-tensorial gravitational-wave background in the NANOGrav 15-Year Data Set,

  and their analysis suggests that the scalar transverse correlations provide an explanation for the observed stochastic signal in the NANOGrav data [3] [4].

  They also studied spatial correlation between pulsars from interfering gravitational-wave sources in massive gravity[5].

- Lijing Shao et al analyzed the sensitivity of a next-generation ground-based detector network to nontensorial polarizations [6].

[3] Z. C. Chen, Y. M. Wu, Y. C. Bi, and Q. G. Huang, Phys.Rev.D 109 (2024) 084045.
[4] Z. C. Chen, C. Yuan, and Q. G. Huang, Sci. China Phys. Mech. Astron. 64, 120412 (2021).
[5] Yu-Mei Wu, Yan-Chen Bi and Qing-Guo Huang, Phys.Rev.D 112 (2025) 6, 064053
[6] J. Hu, D. Liang and L. Shao, Phys.Rev.D 109 (2024) 084023.

1. Introduction

- GWs are extremely weak:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{1}$$

- The geodesic deviation equation:

$$\frac{d^2\eta_i}{dt^2} = -R_{i0j0}\eta^j, \tag{2}$$

$$R_{i0j0} = AE_{ij}e^{ikx}, \quad \text{(plane GWs)} \tag{3}$$

where E_{ij} describes all the information about the polarizations of GWs:

$$E_{ij} = \begin{pmatrix} P_4 + P_6 & P_5 & P_2 \\ P_5 & -P_4 + P_6 & P_3 \\ P_2 & P_3 & P_1 \end{pmatrix}. \tag{4}$$

1. Introduction

- The polarization modes allowed by various modified gravity theories:

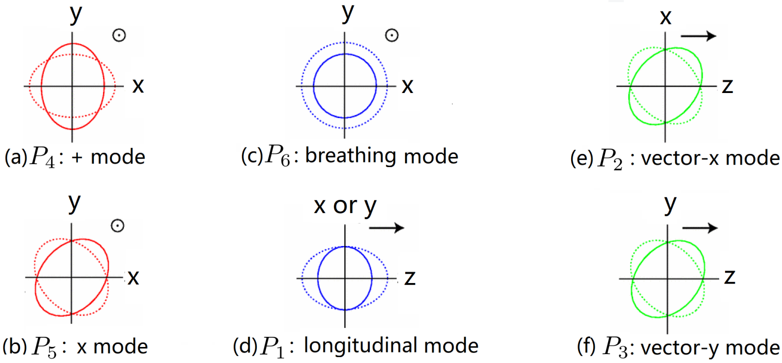


图 2: Six polarization modes of GWs. [7]

[7] D. M. Eardley, D. L. Lee, and A. P. Lightman, Phys. Rev. D 8, 3308 (1973).

1. Introduction

- There are numerous modified gravity theories, each with distinct predictions about GW effects.

- In $f(R)$ theory [8] and Horndeski theory[9]:
  - **Tensor modes**
    There are $+$ and $\times$ modes propagating at the speed of light.
  - **Vector modes**

$$P_2 = P_3 = 0. \tag{5}$$

  - **Scalar modes**
    A mixture of the longitudinal mode to the breathing mode with

$$|P_1/P_6| = m^2/\omega^2, \qquad v < c. \tag{6}$$

[8] D.C. Liang, Y.G. Gong, S.Q. Hou, and Y.Q. Liu, PRD 95, 104034 (2017).
[9] S.Q. Hou, Y.G. Gong, and Y.Q. Liu, EPJC 78, 378 (2018).

1. Introduction

In Palatini-Horndeski theory [10]:

- **Tensor modes**
  There are $+$ and $\times$ modes with the speed of light.
- **Vector modes**
  No vector modes.
- **Scalar modes**
  One or two scalar modes.

[10] Y.Q. Dong and Y.X. Liu, PRD 105, 064035 (2022).

1. Introduction

Polarization modes in generalized Proca theory:[11]

- Tensor modes

  There are $+$ and $\times$ modes with the speed

$$c_T{}^2 = \frac{\mathring{G}_4}{\mathring{G}_4 - \mathring{G}_{4,X} A^2}. \tag{7}$$

- Vector modes

  There are no vector modes when the speed of the tensor modes are the speed of light.

  There maybe two vector modes when the speed of tensor modes are not the speed of light.

- Scalar modes

  There is no or one scalar mode.

[11] Y.Q. Dong, Y.Q. Liu and Y.X. Liu, Phys.Rev.D 109, 024014 (2024).

1. Introduction

Polarization modes in other gravity theories:

- General polarization modes for the Rosen gravitational wave[12].
- Scalar-induced gravitational waves[13] [14].
- Nontensorial gravitational wave polarizations from the tensorial degrees of freedom[15].
- Measuring the ringdown scalar polarization of gravitational waves in Einstein-scalar-Gauss-Bonnet gravity[16].
- Polarizations of Gravitational Waves in the Bumblebee Gravity Model [17].
- Circularly polarized scalar induced gravitational waves[18].
- Torsionless metric–affine theories may allow two new polarizations.[19].

[12] B. Cropp and M. Visser, CQG 27, 165022 (2010), arXiv: 1004.2734.

[13] S. Wang et al, PRR 6, L012060 (2024); PRD 109, 083501 (2024); Sci. China-Phys. Mech. Astron. 68, 210412 (2025).

[14] X. Gao et al, PRD 110, 023537 (2024); EPJC 84, 736 (2024).

[15] S. Hou, X.L. Fan, T. Zhu, and Z.H. Zhu, PRD 109, 084011 (2024).

[16] T. Evstafyeva, M. Agathos and J.L. Ripley, PRD 107, 124010 (2023).

[17] D. Liang, R. Xu, X. Lu, and L. Shao, PRD 106, 124019 (2022).

[18] X. Gao et al, JCAP 10, 054 (2022).

[19] Y. Dong, X. Lai, Y. Fan, and Y. Liu, arXiv: 2504.09445 (2025).

Content

2. Can extra modes be detected?

2. Can extra modes be detected?

The extra polarization modes of GWs are not necessarily emitted by actual astrophysical source in practice, even if these modes are predicted in a theory[20].

- In Brans-Dicke theory, ϕ is directly coupled to R. $\Rightarrow \Box\phi \propto T$.
  - The scalar is not excited in globally vacuum spacetimes.
  - Dipole radiation is absent in **black hole** binaries ($T = 0$), but present in binaries involving at least one **neutron star** ($T \neq 0$).
- In more general scalar-tensor theories, ϕ is not only coupled to R, but also to more general curvature invariants such as the Gauss-Bonnet invariant or the Pontryagin density.
  - Dipole radiation is present in both **black hole** binaries and binaries involving at least one **neutron star** in general.

[20] E. Barausse, N. Yunes, and K. Chamberlain, Phys. Rev. Lett. 116, 241104 (2016).

2. Can extra modes be detected?

- In theories with vector fields (e.g., Lorentz violating gravity) or tensor fields (e.g., bimetric gravity), black holes may possess extra hairs.
  - Dipole radiation is present in both **black hole** binaries and binaries involving at least one **neutron star** in principle.

The presence of extra polarization modes is determined by not only by the astrophysical source but also by the specific modified gravity theory.

2. Can extra modes be detected?

LISA exhibits markedly enhanced sensitivity to non-tensor polarizations in some frequency regimes, providing a powerful probe of alternative metric theories[21].

- In the high part of its frequency band (larger than roughly 6×10^{-2} Hz), LISA is more than ten times sensitive to scalar-longitudinal and vector signals than to tensor and scalar-transverse waves.
- In the low part of its frequency band, LISA is equally sensitive to tensor and vector waves and somewhat less sensitive to scalar signals.

[21] M. Tinto and M. E. da Silva Alves, Phys. Rev. D 82, 122003 (2010).

2. Can extra modes be detected?

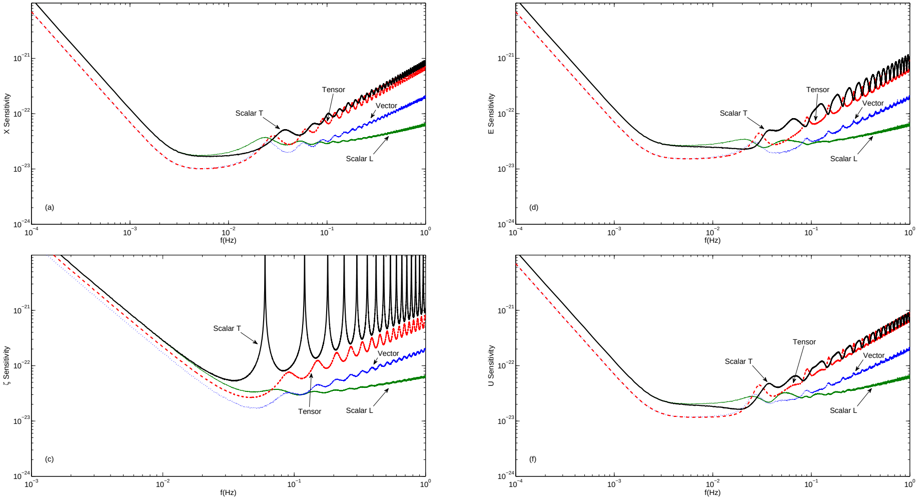


图 3: Sensitivities to polarization modes of GWs. [22]

[22] M. Tinto and M. E. da Silva Alves, Phys. Rev. D 82, 122003 (2010).

Content

3. Isaacson picture of gravitational waves

3. Isaacson picture of gravitational waves

- There are numerous modified gravity theories, each with distinct predictions about GW effects.

- The detection of gravitational waves (GWs) offers a powerful tool for testing various modified gravity theories.

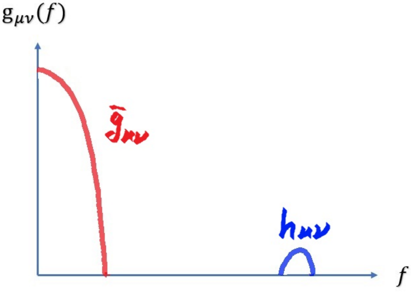
- The detectable physical quantities of GWs:
  - Polarizations of GWs [Y. Gong et al, PRD 95, 104034 (2017); EPJC 78, 378(2018); Universe 7, 9 (2021).] [P. Wagle et al, PRD 100, 124007 (2019).]
  - Dispersion relationships of various polarization modes
  - Radiated energy/angular momentum
  - Memory effect of GWs (nonlinear effects) [D. Christodoulou, PRL 67, 1486 (1991).] [L. Heisenberg et al, PRD 108, 024010 (2023).]
  - ......

- **Can a model-independent general framework be constructed to uniformly analyze the GW effects across various theories?**

3. Isaacson picture of gravitational waves

The Isaacson picture [Phys. Rev. 166, 1263 (1968); Phys. Rev. 166, 1272 (1968)]

- Definition of GWs



- Low-frequency background and high-frequency perturbation (GWs):

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{8}$$

3. Isaacson picture of gravitational waves

The Isaacson picture [Phys. Rev. 166, 1263 (1968); Phys. Rev. 166, 1272 (1968)]

- Expand Einstein field equation:

$$G_{\mu\nu} = G^{(0)}_{\mu\nu}\left[\bar{g}_{\mu\nu}\right] + G^{(1)}_{\mu\nu}\left[\bar{g}_{\mu\nu}, h_{\mu\nu}\right] + G^{(2)}_{\mu\nu}\left[\bar{g}_{\mu\nu}, h_{\mu\nu}\right] + ... \tag{9}$$

- High-frequency equation:

$$G^{(1)}_{\mu\nu} = 8\pi T^{H,(0)}_{\mu\nu}. \tag{10}$$

$\Rightarrow$ Polarization modes and dispersion relationships.

- Low-frequency equation:

$$G^{(0)}_{\mu\nu} = 8\pi\left(T^{L,(0)}_{\mu\nu} + t_{\mu\nu}\right), \quad t_{\mu\nu} = -\frac{1}{8\pi}\left\langle G^{(2)}_{\mu\nu}\right\rangle. \tag{11}$$

$\Rightarrow$ Effective energy-momentum tensor and memory effect.

- The Isaacson picture also applies to modified gravity theories with N-order derivatives, where $N < 19$.

3. Isaacson picture of gravitational waves

- The second-order perturbation action $S^{(2)}_{flat}$, with the Minkowski metric as the background, contains all necessary information to construct the Isaacson picture far from the source:

$$G^{(1)}_{\mu\nu}\left[\eta_{\mu\nu}, h_{\mu\nu}\right] \quad \text{can be derived from} \quad \frac{\delta S^{(2)}_{flat}}{\delta h^{\mu\nu}},$$

$$\left\langle G^{(2)}_{\mu\nu}\left[\eta_{\mu\nu}, h_{\mu\nu}\right]\right\rangle \quad \text{can be derived from} \quad \left\langle \frac{\delta S^{(2)}_{flat}}{\delta \eta^{\mu\nu}} \right\rangle.$$ [23] [24]

- Constructing a model-independent framework $\Rightarrow$ Constructing the most general $S^{(2)}_{flat}$.

[23] L. Heisenberg, N. Yunes, and J. Zosso, PRD **108**, 024010 (2023).

[24] Y.Q. Dong, X.B. Lai, Y.Q. Liu, and Y.X. Liu, EPJC 2025, arXiv: 2409.11838.

Content

4. Construction of most general 2nd-order actions

- The most general pure metric theory
- The most general scalar-tensor theory
- The most general vector-tensor theory

4. Construction of most general 2nd-order actions

- Lovelock's theorem [JMP 12, 498 (1971)]: 4D, GR.

Various methods for modifying general relativity

- Add additional fields
- Higher-order derivative theory
- n-dimensional spacetime
- Non-Riemannian geometry
- ......

4. Construction of most general 2nd-order actions

- We can construct the most general pure metric theory $S\left[g_{\mu\nu}\right]$ and scalar-tensor theory $S\left[g_{\mu\nu},\Psi\right]$ that satisfies the following assumptions in a Minkowski background:

Required assumptions

(1) Spacetime is a 4D Riemannian manifold

(2) The theory satisfies the least action principle

(3) The theory is generally covariant ($S^{(2)}_{flat}$ is gauge invariant)

(4) The action of a free particle is $\int ds = \int \sqrt{|g_{\mu\nu}dx^\mu dx^\nu|}$

4. Construction of most general 2nd-order actions

- Gravity is described by Riemannian geometry.

Elements required for constructing action

- The metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$
- The 4D totally antisymmetric tensor $E_{\mu\nu\lambda\rho}$
- The partial derivative ∂_μ
- The scalar field $\Psi = \Psi_0 + \psi$ (for scalar-tensor theory)

4. Construction of most general 2nd-order actions

- For pure metric theory

$$S^{(2)}_{flat} = S_1 + \sum_{I\geq 2}^{N} S_I, \tag{12}$$

$$\begin{aligned} S_1 &= \int d^4x\, a_1 h^{\mu\nu}\left[2\partial_\nu\partial^\lambda h_{\mu\lambda} - \Box h_{\mu\nu} - 2\partial_\mu\partial_\nu h + \eta_{\mu\nu}\Box h\right], &(13)\\ S_I &= \int d^4x\, h^{\mu\nu}\Big[(a_I + b_I)\,\Box^{I-2}\partial_\mu\partial_\nu\partial^\lambda\partial^\rho h_{\lambda\rho} - 2b_I\Box^{I-1}\partial_\nu\partial^\lambda h_{\mu\lambda} \\ &\quad + b_I\Box^I h_{\mu\nu} - 2a_I\Box^{I-1}\partial_\mu\partial_\nu h + a_I\eta_{\mu\nu}\Box^I h\Big]. &(14)\end{aligned}$$

- In order to reduce to general relativity, we should have $a_1 = 1/2$.

4. Construction of most general 2nd-order actions

- For scalar-tensor theory

$$S^{(2)}_{flat} = S_1 + \sum_{I\geq 2}^{N} S_I, \tag{15}$$

where

$$\begin{aligned} S_1 &= \int d^4x\, a_1 h^{\mu\nu}\Big[2\partial_\nu\partial^\lambda h_{\mu\lambda} - \Box h_{\mu\nu} - 2\partial_\mu\partial_\nu h + \eta_{\mu\nu}\Box h\Big] \\ &+ \int d^4x\, \psi\Big[c_0\psi + c_1\Box\psi + d_1\partial_\mu\partial_\nu h^{\mu\nu} - d_1\Box h\Big], \end{aligned} \tag{16}$$

$$\begin{aligned} S_I &= \int d^4x\, h^{\mu\nu}\Big[(a_I + b_I)\,\Box^{I-2}\partial_\mu\partial_\nu\partial^\lambda\partial^\rho h_{\lambda\rho} - 2b_I\Box^{I-1}\partial_\nu\partial^\lambda h_{\mu\lambda} \\ &\quad b_I + \Box^I h_{\mu\nu} - 2a_I\Box^{I-1}\partial_\mu\partial_\nu h + a_I\eta_{\mu\nu}\Box^I h\Big] \\ &+ \int d^4x\, \psi\Big[c_I\Box^I\psi + d_I\Box^{I-1}\partial_\mu\partial_\nu h^{\mu\nu} - d_I\Box^I h\Big]. \end{aligned} \tag{17}$$

4. Construction of most general 2nd-order actions

- We can construct the second-order action of the most general vector-tensor theory that satisfies the following assumptions in a Minkowski background:

Required assumptions

(1) Spacetime is a 4D Riemannian manifold

(2) The theory satisfies the least action principle

(3) The theory is generally covariant ($S^{(2)}_{flat}$ is gauge invariant)

(4) The field equations are second-order

(5) The action of a free particle is $\int ds = \int \sqrt{|g_{\mu\nu} dx^\mu dx^\nu|}$

4. Construction of most general 2nd-order actions

- For vector-tensor theory, the second-order ation is

$$S^{(2)}_{flat} = S^{(2)}_{flat}[\eta_{\mu\nu}, h_{\mu\nu}, A_\mu + B_\mu, E^{\mu\nu\lambda\rho}, \partial_\mu]. \tag{18}$$

$$S^{(2)}_{flat} = S^{(2)}_0 + S^{(2)}_1 + S^{(2)}_2 = \int d^4x\sqrt{-\eta}\,(\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2)\,, \tag{19}$$

$$\begin{aligned}
\mathcal{L}_0 &= A_{(0)}A^\mu A^\nu A^\lambda A^\rho h_{\mu\nu}h_{\lambda\rho} + 4A_{(0)}A^\mu A^\nu A^\lambda h_{\mu\nu}B_\lambda \\
&+ 4A_{(0)}A_\mu A_\nu B^\mu B^\nu, \qquad (20) \\
\mathcal{L}_1 &= A_{(1)}\left(E^{\mu\lambda\sigma\gamma}A^\nu A^\rho A_\sigma \partial_\gamma h_{\mu\nu}\right)h_{\lambda\rho} + B_{(1)}\left((A\cdot\partial)\,A_\mu A_\nu h\right)h^{\mu\nu} \\
&- 2B_{(1)}\left(A^\mu A^\nu \partial^\lambda h_{\mu\nu}\right)B_\lambda + 2A_{(1)}\left(E^{\mu\lambda\sigma\gamma}A_\sigma\partial_\gamma A^\nu h_{\mu\nu}\right)B_\lambda \\
&+ 2B_{(1)}\left((A\cdot\partial)\,A^\mu h\right)B_\mu \\
&- 4B_{(1)}\left(A_\mu \partial_\nu B^\mu\right)B^\nu - A_{(1)}\left(E^{\mu\nu\lambda\rho}\partial_\lambda A_\rho B_\mu\right)B_\nu. \qquad (21)
\end{aligned}$$

4. Construction of most general 2nd-order actions

$$
\begin{aligned}
\mathcal{L}_2 =\ & A_{(2)}\left(\Box A^\mu A^\nu A^\lambda A^\rho h_{\mu\nu}\right) h_{\lambda\rho} + B_{(2)}\left((A\cdot\partial)^2 A^\mu A^\nu A^\lambda A^\rho h_{\mu\nu}\right) h_{\lambda\rho} \\
& + C_{(2)}\left((A\cdot\partial) A^\mu A^\nu A^\lambda \partial^\rho h_{\mu\nu}\right) h_{\lambda\rho} + D_{(2)}\left(A^\mu A^\nu \partial^\lambda \partial^\rho h_{\mu\nu}\right) h_{\lambda\rho} \\
& + E_{(2)}\left(A^\mu A^\lambda \partial^\nu \partial^\rho h_{\mu\nu}\right) h_{\lambda\rho} - E_{(2)}\left(\Box A^\lambda A_\rho h_{\mu\lambda}\right) h^{\mu\rho} \\
& + F_{(2)}\left((A\cdot\partial)^2 A^\lambda A_\rho h_{\mu\lambda}\right) h^{\mu\rho} + G_{(2)}\left((A\cdot\partial) A^\lambda \partial_\rho h_{\mu\lambda}\right) h^{\mu\rho} \\
& - 2H_{(2)}\left(\partial^\lambda \partial_\rho h_{\mu\lambda}\right) h^{\mu\rho} + H_{(2)}\left(\Box h_{\mu\nu}\right) h^{\mu\nu} + 2H_{(2)}\left(\partial_\mu \partial_\nu h\right) h^{\mu\nu} - H_{(2)}\left(\Box h\right) h \\
& + I_{(2)}\left((A\cdot\partial)^2 h_{\mu\nu}\right) h^{\mu\nu} - D_{(2)}\left(\Box A_\mu A_\nu h\right) h^{\mu\nu} + J_{(2)}\left((A\cdot\partial)^2 A_\mu A_\nu h\right) h^{\mu\nu} \\
& - G_{(2)}\left((A\cdot\partial) A_\mu \partial_\nu h\right) h^{\mu\nu} + K_{(2)}\left((A\cdot\partial)^2 h\right) h \\
& + \left(4A_{(2)} + C_{(2)}\right)\left(\Box A^\mu A^\nu A^\lambda h_{\mu\nu}\right) B_\lambda + 4B_{(2)}\left((A\cdot\partial)^2 A^\mu A^\nu A^\lambda h_{\mu\nu}\right) B_\lambda \\
& + \left(C_{(2)} + 2J_{(2)}\right)\left((A\cdot\partial) A^\mu A^\nu \partial^\lambda h_{\mu\nu}\right) B_\lambda + 2\left(C_{(2)} + F_{(2)}\right)\left((A\cdot\partial) A^\mu \partial^\nu A^\lambda h_{\mu\nu}\right) B_\lambda \\
& + \left(2E_{(2)} - G_{(2)}\right)\left(A^\mu \partial^\nu \partial^\lambda h_{\mu\nu}\right) B_\lambda + \left(2D_{(2)} + G_{(2)}\right)\left(\partial^\mu \partial^\nu A^\lambda h_{\mu\nu}\right) B_\lambda \\
& + \left(-2E_{(2)} + G_{(2)}\right)\left(\Box A^\mu h_{\mu\lambda}\right) B^\lambda + 2F_{(2)}\left((A\cdot\partial)^2 A^\mu h_{\mu\lambda}\right) B^\lambda \\
& + \left(G_{(2)} + 4I_{(2)}\right)\left((A\cdot\partial) \partial^\mu h_{\mu\lambda}\right) B^\lambda + \left(-2D_{(2)} - G_{(2)}\right)\left(\Box A^\mu h\right) B_\mu \\
& + 2J_{(2)}\left((A\cdot\partial)^2 A^\mu h\right) B_\mu + \left(-G_{(2)} + 4K_{(2)}\right)\left((A\cdot\partial) \partial^\mu h\right) B_\mu \\
& + \left(4A_{(2)} + 2C_{(2)} + F_{(2)}\right)\left(\Box A_\mu A_\nu B^\mu\right) B^\nu + 4B_{(2)}\left((A\cdot\partial)^2 A_\mu A_\nu B^\mu\right) B^\nu \\
& + \left(2C_{(2)} + 2F_{(2)} + 4J_{(2)}\right)\left((A\cdot\partial) A_\mu \partial_\nu B^\mu\right) B^\nu \\
& + \left(E_{(2)} + 2I_{(2)} - G_{(2)} + 4K_{(2)}\right)\left(\partial_\mu \partial_\nu B^\mu\right) B^\nu \\
& + \left(-E_{(2)} + G_{(2)} + 2I_{(2)}\right)\left(\Box B_\mu\right) B^\mu + F_{(2)}\left((A\cdot\partial)^2 B_\mu\right) B^\mu,
\end{aligned}
$$

Content

5. Approaches for analysis

- Newman-Penrose formalism
- Bardeen's gauge-invariant formalism
- Decoupling conditions of scalar, vector, and tensor perturbations

5. Approaches for analysis

Newman-Penrose formalism

- Taking the null tetrad basis representation of Newman-Penrose(NP) formalism

$$k = \frac{1}{\sqrt{2}}(e_t + e_z); l = \frac{1}{\sqrt{2}}(e_t - e_z); m = \frac{1}{\sqrt{2}}(e_x + ie_y); \bar{m} = \frac{1}{\sqrt{2}}(e_x - ie_y). \tag{22}$$

- For plane GWs propagating at the speed of light (propagation direction as + z), the components of the Riemann tensor that may not be zero under the NP frame are [25]

$$\begin{aligned} P_1 &= R_{klkl}, & P_2 = \text{Re}(R_{klml}), \quad & P_3 = \text{Im}(R_{klml}), \\ P_4 &= \frac{1}{2}\text{Re}(R_{mlml}), & P_5 = \frac{1}{2}\text{Im}(R_{mlml}), \quad & P_6 = \frac{1}{2}R_{ml\bar{m}l}. \end{aligned} \tag{23}$$

[25] D. M. Eardley, D. L. Lee, and A. P. Lightman, Phys. Rev. D 8, 3308 (1973)

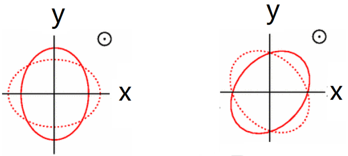
5. Approaches for analysis

- The components of the Ricci tensor that may not be zero under the NP frame are

$$R_{lk} = R_{lklk} = P_1; R_{ll} = 2R_{lml\bar{m}} = 2P_6;$$

$$R_{lm} = R_{lklm} = P_2 + iP_3; R_{l\bar{m}} = R_{lkl\bar{m}} = P_2 - iP_3. \tag{24}$$

- For GR, the Ricci tensor is zero, so only P_4 and P_5 can be non-zero. GR allows only $+$ and $\times$ modes.



(a) P_4 : + mode (b) P_5 : x mode

图 4: GR only has $+$ and $\times$ modes.

5. Approaches for analysis

- The NP formalism method can be extended to the case of plane GWs propagating at arbitrary speed[26].
- In this case, the components of the Riemann tensor that may not be zero under the NP frame are

$$
\begin{aligned}
P_1 &= R_{klkl}, \\
P_2 &= \mathrm{Re}(R_{klml}) - \mathrm{Re}(R_{mklk}), \\
P_3 &= \mathrm{Im}(R_{klml}) - \mathrm{Im}(R_{mklk}), \\
P_4 &= \frac{\mathrm{Re}(R_{mlml}) + \mathrm{Re}(R_{mkmk}) + 2\mathrm{Re}(R_{mkml})}{2}, \\
P_5 &= \frac{\mathrm{Im}(R_{mlml}) + \mathrm{Im}(R_{mkmk}) + 2\mathrm{Im}(R_{mkml})}{2}, \\
P_6 &= \frac{R_{ml\bar{m}l} + \mathrm{Re}(R_{mlkl}) + R_{mk\bar{m}k}}{2}.
\end{aligned}
\tag{25}
$$

[26] T. H. Hyum, Y. Kim, and S. Lee, Phys. Rev. D 99, 124002 (2019)

5. Approaches for analysis

Bardeen's gauge-invariant formalism

- We can decompose $h_{\mu\nu}$ and B^μ into irreducible pieces by STV decomposition [27] [28]:

$$\begin{aligned} B^0 &= B^0, \quad B^i = \partial^i \omega + \mu^i, \\ h_{00} &= h_{00}, \quad h_{0i} = \partial_i \gamma + \beta_i, \\ h_{ij} &= h_{ij}^{TT} + \partial_i \epsilon_j + \partial_j \epsilon_i + \frac{1}{3}\delta_{ij} H + (\partial_i \partial_j - \frac{1}{3}\delta_{ij}\Delta)\zeta. \end{aligned} \tag{26}$$

- The following combinations of these functions are gauge invariant:

$$\begin{aligned} & h_{ij}^{TT} = h_{ij}^{TT}, \qquad \Xi_i = \beta_i - \partial_0 \epsilon_i, \qquad \Sigma_i = \mu_i + A\partial_0 \epsilon_i, \\ & \phi = -\frac{1}{2} h_{00} + \partial_0 \gamma - \frac{1}{2}\partial_0 \partial_0 \zeta, \qquad \Theta = \frac{1}{3}\left(H - \Delta\zeta\right), \\ & \Omega = B^0 - A\partial_0 \gamma + \frac{1}{2} A \partial_0 \partial_0 \zeta, \qquad \Psi = \omega + \frac{1}{2} A \partial_0 \zeta, . \end{aligned} \tag{27}$$

[27] J. M. Bardeen, Phys. Rev. D 22, 1882–1905 (1980)
[28] R. Bluhm, S. H. Fung, and V. A. Kostelecky, Phys. Rev. D 77, 065020 (2008)

5. Approaches for analysis

- The specific form of $\overset{(1)}{R}_{i0j0}$ consisting of the gauge invariants is

$$\overset{(1)}{R}_{i0j0} = -\frac{1}{2}\partial_0\partial_0 h_{ij}^{TT} + \frac{1}{2}\partial_0\partial_i\Xi_j + \frac{1}{2}\partial_0\partial_j\Xi_i + \partial_i\partial_j\phi - \frac{1}{2}\delta_{ij}\partial_0\partial_0\Theta. \quad (28)$$

- For plane GWs propagating along $+z$ direction, the relationship between the polarization modes and the gauge invariants is [29]

$$\begin{aligned} P_1 &= \partial_z\partial_z\phi - \frac{1}{2}\partial_0\partial_0\Theta, \;\; P_2 = \frac{1}{2}\partial_0\partial_z\Xi_x, \quad P_3 = \frac{1}{2}\partial_0\partial_z\Xi_y, \\ P_4 &= -\frac{1}{2}\partial_0\partial_0 h_{xx}^{TT}, \qquad P_5 = -\frac{1}{2}\partial_0\partial_0 h_{xy}^{TT}, \;\; P_6 = -\frac{1}{2}\partial_0\partial_0\Theta. \end{aligned} \quad (29)$$

- For GR, only h_{ij}^{TT} is non-vanishing, which means only P_4 and P_5 can be non-zero. GR allows only $+$ and $\times$ modes.

[29] D. M. Eardley, D. L. Lee, A. P. Lightman, R. V. Wagoner, and C. M. Will. Phys. Rev. Lett., 30, 884–886 (1973).

5. Approaches for analysis

Decoupling conditions of scalar, vector, and tensor perturbations

- For decoupling case, each types of the scalar, vector, and tensor perturbations in the SVT decomposition evolve independently.

- Assumptions:
  (1) The cosmological background or Minkowski background,
  (2) SO(3) symmetry for the background fields.

- There are scalar, vector, and tensor perturbation equations:

$$
\begin{aligned}
Q_{\mu\nu} &= 0, && (30)\\
Q_{\mu} &= 0, && (31)\\
Q &= 0. && (32)
\end{aligned}
$$

5. Approaches for analysis

- After SVT decomposition, the perturbation equations can be rewritten as:

$$
\begin{aligned}
0 &= Q_{tt} = Q^{(s1)}(S), && (33)\\
0 &= Q_{ti} = Q_i^{(v1)}(V) + \partial_i Q^{(s2)}(S), && (34)\\
0 &= Q_{it} = Q_i^{(v2)}(V) + \partial_i Q^{(s3)}(S), && (35)\\
0 &= Q_{ij} = Q_{ij}^{(t1)}(T) + \partial_i Q_j^{(v3)}(V) + \partial_j Q_i^{(v4)}(V) \\
&\quad + Q^{(s4)}(S)\delta_{ij} + \partial_i\partial_j Q^{(s5)}(S), && (36)
\end{aligned}
$$

$$
\begin{aligned}
0 &= Q_t = Q^{(s6)}(S), && (37)\\
0 &= Q_i = Q_i^{(v5)}(V) + \partial_i Q^{(s7)}(S), && (38)
\end{aligned}
$$

$$
0 = Q = Q^{(s8)}(S). \qquad (39)
$$

Here, $\delta^{ij}Q_{ij}^{(t1)} = 0$, $\partial^i Q_{ij}^{(t1)} = \partial^i Q_{ji}^{(t1)} = 0$, and $\partial^i Q_i^{(v\bullet)} = 0$.

5. Approaches for analysis

- Using spatial divergence and taking trace, we can obtain

$$Q_{ij}^{(t1)}(T) = 0, \tag{40}$$

$$Q_i^{(v\bullet)}(V) = 0, \tag{41}$$

$$Q^{(s\bullet)} = 0. \tag{42}$$

Obviously, each types of the scalar, vector, and tensor perturbations in the SVT decomposition evolve independently.

- The decoupling allows for the independent analysis of the three types of perturbations.

Content

6. Polarization modes of gravitational waves

6. Polarization modes of gravitational waves

For pure metric theory and scalar-tensor theory:

- The equation describing the tensor mode GWs is

$$\Box\Big(-2a_1+\sum_{I\geq 2}^{N}2b_I\Box^{I-1}\Big)h_{ij}^{TT}=0. \tag{43}$$

- We can write the operator [H. Lu and C.N. Pope, PRL 106, 181302 (2011)]

$$-2a_1+\sum_{I\geq 2}^{N}2b_I\Box^{I-1}=\prod_{k=1}^{M}\left(\Box-m_k^2\right)^{n_k}. \tag{44}$$

- The equation of tensor modes can be rewritten as

$$\Box\prod_{k=1}^{M}\left(\Box-m_k^2\right)^{n_k}h_{ij}^{TT}=0. \tag{45}$$

- The equation of vector modes is

$$\prod_{k=1}^{M}\left(\Box-m_k^2\right)^{n_k}\Xi_i=0. \tag{46}$$

6. Polarization modes of gravitational waves

- The equations of the two scalar modes are

$$\prod_{k=1}^{M}\left(\Box - {m_k}^2\right)^{n_k}(2\phi-\Theta)=0, \tag{47}$$

$$\Lambda_2\prod_{k=1}^{P}\left(\Box-\tilde{m}_k^2\right)^{p_k}\Theta=\Lambda_3\Delta\prod_{k=1}^{Q}\left(\Box-{m'_k}^2\right)^{q_k}(2\phi-\Theta)\,. \tag{48}$$

6. Polarization modes of gravitational waves

- Scalar modes under various cases in most general pure metric theory.

Cases	**Conditions**	Breathing mode	Longitudinal mode	Dependent or not	$\mathcal{R} = m^2/\omega^2$
case 1.1	$m^2 \notin \{m_k^2\}, m^2 \notin \{\tilde{m}_k^2\}$	×	×	-	-
case 1.2	$m^2 \notin \{m_k^2\}, m^2 \in \{\tilde{m}_k^2\}$	✓	✓	✓	✓
case 2.1	$m^2 \in \{m_k^2\}, m^2 \notin \{\tilde{m}_k^2\}, m^2 \notin \left\{{m'_k}^2\right\}$	✓	✓	✓	×
case 2.2	$m^2 \in \{m_k^2\}, m^2 \notin \{\tilde{m}_k^2\}, m^2 \in \left\{{m'_k}^2\right\}$	×	✓	-	-
case 2.3	$m^2 \in \{m_k^2\}, m^2 \in \{\tilde{m}_k^2\}, m^2 \notin \left\{{m'_k}^2\right\}$	✓	✓	✓	✓
case 2.4	$m^2 \in \{m_k^2\}, m^2 \in \{\tilde{m}_k^2\}, m^2 \in \left\{{m'_k}^2\right\}$	✓	✓	×	-

6. Polarization modes of gravitational waves

- Scalar modes under various cases in most general scalar-tensor theory.

Cases	Conditions	Breathing mode	Longitudinal mode	Dependent or not	$\mathcal{R}=m^2/\omega^2$
case 1.1	$m^2 \notin \{m_k^2\}, det(\mathcal{A}) \neq 0$	×	×	-	-
case 1.2	$m^2 \notin \{m_k^2\}, det(\mathcal{A}) = 0, m^2 \in \{\tilde{m}_k^2\}, m^2 \in \{\hat{m}_k^2\}$	✓	✓	✓	✓
case 1.3	$m^2 \notin \{m_k^2\}, det(\mathcal{A}) = 0, m^2 \in \{\tilde{m}_k^2\}, m^2 \notin \{\hat{m}_k^2\}$	✓	✓	✓	✓
case 1.4	$m^2 \notin \{m_k^2\}, det(\mathcal{A}) = 0, m^2 \notin \{\tilde{m}_k^2\}, m^2 \in \{\hat{m}_k^2\}$	×	×	-	-
case 1.5	$m^2 \notin \{m_k^2\}, det(\mathcal{A}) = 0, m^2 \notin \{\tilde{m}_k^2\}, m^2 \notin \{\hat{m}_k^2\}$	✓	✓	✓	✓
case 2.1	$m^2 \in \{m_k^2\}, det(\mathcal{A}) \neq 0, \gamma = 0$	×	✓	-	-
case 2.2	$m^2 \in \{m_k^2\}, det(\mathcal{A}) \neq 0, \gamma \neq 0$	✓	✓	✓	×
case 3.1.1	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly independent, $m^2 \in \{\tilde{m}_k^2\}, m^2 \in \{\hat{m}_k^2\}$	✓	✓	✓	✓
case 3.1.2	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly independent, $m^2 \in \{\tilde{m}_k^2\}, m^2 \notin \{\hat{m}_k^2\}$	✓	✓	✓	✓
case 3.1.3	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly independent, $m^2 \notin \{\tilde{m}_k^2\}, m^2 \in \{\hat{m}_k^2\}$	×	×	-	-
case 3.1.4	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly independent, $m^2 \notin \{\tilde{m}_k^2\}, m^2 \notin \{\hat{m}_k^2\}$	✓	✓	✓	✓
case 3.2.1	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly dependent, $m^2 \in \{\tilde{m}_k^2\}, m^2 \in \{\hat{m}_k^2\}, m^2 \in \{{m'_k}^2\}$	✓	✓	×	-
case 3.2.2	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly dependent, $m^2 \notin \{\tilde{m}_k^2\}, m^2 \in \{\hat{m}_k^2\}, m^2 \in \{{m'_k}^2\}$	×	✓	-	-
case 3.2.3	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly dependent, $m^2 \in \{\tilde{m}_k^2\}, m^2 \notin \{\hat{m}_k^2\}, m^2 \in \{{m'_k}^2\}$	✓	✓	×	-
case 3.2.4	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly dependent, $m^2 \in \{\tilde{m}_k^2\}, m^2 \in \{\hat{m}_k^2\}, m^2 \notin \{{m'_k}^2\}$	✓	✓	✓	✓
case 3.2.5	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly dependent, $m^2 \notin \{\tilde{m}_k^2\}, m^2 \notin \{\hat{m}_k^2\}, m^2 \in \{{m'_k}^2\}$	✓	✓	×	-
case 3.2.6	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly dependent, $m^2 \notin \{\tilde{m}_k^2\}, m^2 \in \{\hat{m}_k^2\}, m^2 \notin \{{m'_k}^2\}$	✓	✓	✓	×
case 3.2.7	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly dependent, $m^2 \in \{\tilde{m}_k^2\}, m^2 \notin \{\hat{m}_k^2\}, m^2 \notin \{{m'_k}^2\}$	✓	✓	×	-
case 3.2.8	$m^2 \in \{m_k^2\}, det(\mathcal{A}) = 0,\ \alpha,\ \beta$ linearly dependent, $m^2 \notin \{\tilde{m}_k^2\}, m^2 \notin \{\hat{m}_k^2\}, m^2 \notin \{{m'_k}^2\}$	✓	✓	×	-

6. Polarization modes of gravitational waves

Some universal inferences in pure metric theory and scalar-tensor theory:

Inferences

- Tensor modes $(v = c)$ ✓ Vector modes $(v = c)$ $\times$
  Scalar modes $(v = c)$ must be breathing modes
- Tensor modes $(m \neq 0)$ $\Leftrightarrow$ Vector modes $(m \neq 0)$
- Only tensor modes $(v = c)$ $\Rightarrow$ No vector modes
  $\Rightarrow$ All scalar modes $\left(\left|\frac{P_1}{P_6}\right| = \frac{m^2}{\omega^2}\right)$
- A scalar mode $\left(\left|\frac{P_1}{P_6}\right| \neq \frac{m^2}{\omega^2}\right)$ $\Rightarrow$ Tensor, vector modes $(m \neq 0)$ exist
- A scalar mode $(m \neq 0)$ $\wedge$ no tensor mode $(m \neq 0)$
  $\Rightarrow$ This scalar mode $\left(\left|\frac{P_1}{P_6}\right| = \frac{m^2}{\omega^2}\right)$

6. Polarization modes of gravitational waves

Some universal inferences in vector-tensor theory:

Inferences

- The properties of polarization modes are different from pure metric theory and scalar-tensor theory.
- Tensor modes generally deviate from the speed of light.
- The $E^{\mu\nu\lambda\rho}$ term will break the parity symmetry of vector modes.
  In some cases, there may even be only one of the left-handed or right-handed waves.
  However, $E^{\mu\nu\lambda\rho}$ can also lead to superluminal problem.
- Scalar modes satisfy one of the following three cases:
  - (1) no scalar mode;
  - (2) two independent modes: breathing and longitudinal ;
  - (3) a mixture of two modes.

More details can be found in [arXiv: 2409.11838].

Content

7. Constraints from observations

6. Constraints from observations

- On August 17, 2017, a binary neutron star coalescence candidate (GW170817) was observed through GWs by Advanced LIGO and Virgo [30].
- About 1.7 seconds later, the Fermi Gamma-ray Burst Monitor independently detected a gamma-ray burst (GRB170817A) [31].
- These observations placed a tight constraint on the speed of tensor modes of GWs [32]:

$$-3 \times 10^{-15} \leq \frac{c_t}{c} - 1 \leq 7 \times 10^{-16}. \tag{49}$$

[30] B. P. Abbott et al. Phys. Rev. Lett., 119, 161101 (2017)
[31] V. Savchenko et al. Astrophys. J. Lett., 848, L15 (2017)
[32] B. P. Abbott et al. Astrophys. J. Lett., 848, L13 (2017)

7. Constraints from observations

Horndeski theory

- The constraint (49) is so tight, that we can say that the tensor gravitational wave speed in the Universe is equal to the basic constant c (speed of light).

- The condition that the tensor gravitational wave speed equals the speed of light is [33]

$$2G_{4,X} - 2G_{5,\phi} + (H\dot{\phi} - \ddot{\phi})G_{5,X} = 0. \tag{50}$$

- If we do not allow any tuning among functions, the Horndeski Lagrangian is restricted to be of the form [34] [35]

$$S(g, \Gamma, \phi) = \int d^4x \sqrt{g} \left[G_2(\phi, X) + G_3(\phi, X)\Box\phi + G_4(\phi)R\right]. \tag{51}$$

[33] R. Kase and S. Tsujikawa. Int. J. Mod. Phys. D 28, 1942005 (2019)

[34] A. Goldstein et al., Astrophys. J. 848, no. 2, L14 (2017)

[35] B. P. Abbott et al. Astrophys. J. 848, no. 2, L13 (2017)

7. Constraints from observations

Palatini-Horndeski theory

- Consider that the tensor gravitational wave speed in the Universe is equal to the basic constant c.

- In the case of being free from the Ostrogradsky instability, the condition that the tensor gravitational wave speed equals the speed of light on any spatially flat cosmological background is only [36]

$$G_{4,X} = 0, \quad G_5 = 0. \tag{52}$$

- Then, the action is simplified as

$$S(g, \Gamma, \phi) = \int d^4x \sqrt{g} \left[K(\phi, X) - G_3(\phi, X) \tilde{\Box}\phi + G_4(\phi) \tilde{R} \right]. \tag{53}$$

[36] Y.-Q. Dong, Y.-Q. Liu, and Y.-X. Liu. Eur. Phys. J. C 83, 702 (2023)

7. Constraints from observations

Bumblebee model

- The action of the Bumblebee model is given by [37]

$$S = \int d^4x \left[\frac{1}{2\kappa}(R - 2\Lambda + \xi B^\mu B^\nu R_{\mu\nu}) - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - V(B^\mu B_\mu \pm b^2)\right] + S_m. \quad (54)$$

- In cosmological background, the constraint in Eq. (49), combined with the requirements for the absence of ghost, Laplacian, and tachyonic instabilities, leads to the following constraints [38]:
  (1) The Lorentz-violating parameter: $-6 \times 10^{-15} \lesssim \xi b^2 \leq 0$.
  (2) Two tensor modes: $c_t/c < 1$.
  (3) Two vector modes: $c_v/c > 1$.
  (4) One mixed scalar mode: $c_s/c > 1$.

[37] V. A. Kostelecky. Phys. Rev. D, 69, 105009 (2004)

[38] X.-B. Lai, Y.-Q. Dong, Y.-Z. Fan, and Y.-X. Liu. arXiv:2509.13958 [gr-qc]

7. Constraints from observations

- While the speed of GWs is strongly constrained at the frequencies of ground-based detectors, deviations are possible at the smaller (mHz) LISA frequencies.
- In some Lorentz-violation theories, the Lorentz invariance is broken at low energies ($c_T < 1$ at low frequencies) and recovered at high energies ($c_T = 1$ at high frequencies).
- LISA's sources and low-frequency band excel in constraining GW dispersion relations, tightening graviton mass bounds by 2–3 orders of magnitude over ground-based GW experiments[39].
- Even very small differences in speed could result in large differences in arrival times between modes,so any beyond GR polarizations would be outside the standard detection window[40].

[39] M. Colpi et al., arXiv:2402.07571 (2024).

[40] K. Schumacher, N. Yunes, and K. Yagi, Phys. Rev. D 108, 104038 (2023).

8. Conclusion

- We have established a model-independent framework for analyzing gravitational wave polarization modes across various gravity theories.
- Our analysis reveals some universal relationships between tensor, vector, and scalar modes.
- Future detectors like LISA/Taiji/Tianqin, with enhanced sensitivity to non-tensor polarizations in certain bands, will provide powerful tests for these theories, offering profound insights into gravity beyond General Relativity.

Thanks for your listening!