Classical Gravitational Dynamics from Quantum Field Theory

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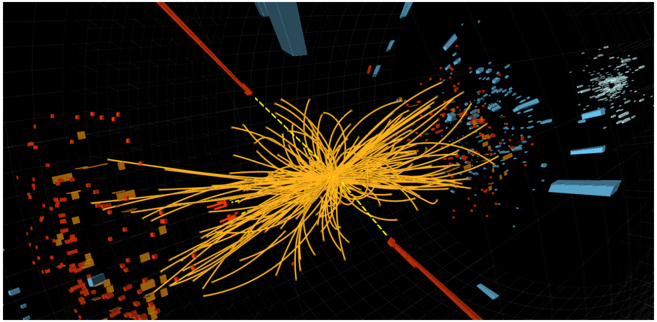
PLB 831 (2022) 137203 PRL 125 (2020) 261103 PRD 102 (2020) 124025 JHEP 06 (2021) 012

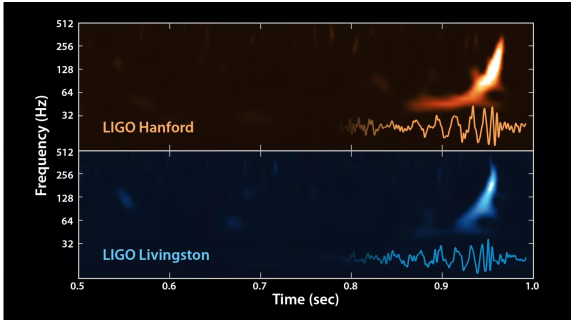
第十四届新物理研讨会

济南 2025.07.24

Precision era of fundamental physics

止於至善





Two historic breakthroughs in science:

- Higgs bosons at the LHC (2012)
- Gravitational waves by the LIGO (2016)
- High-energy and gravitational physics entered a precision era!

Opened a new window on the Universe!

Precision era of fundamental physics

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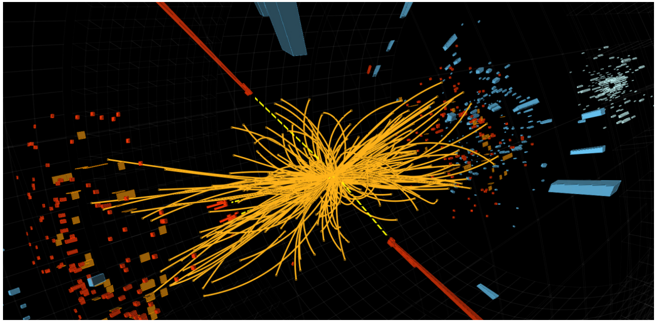

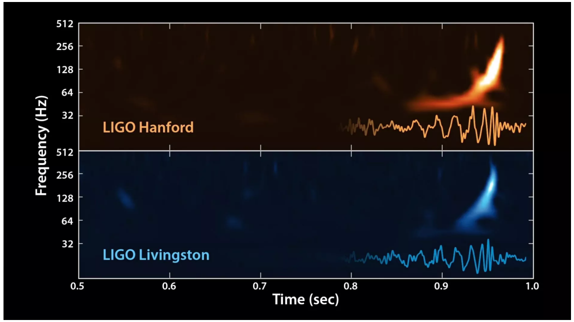

Two historic breakthroughs in science:

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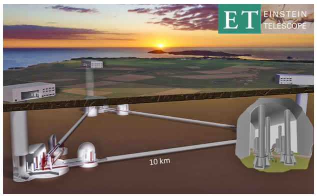
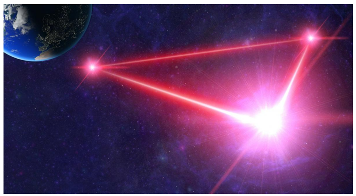
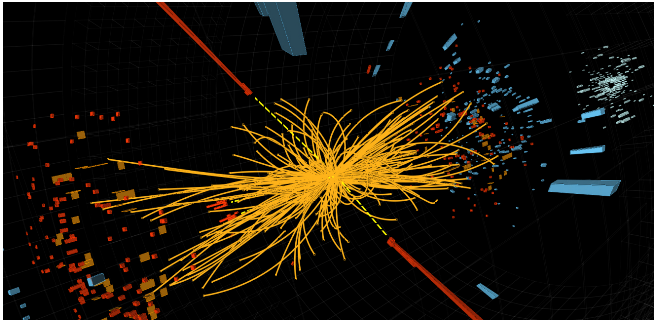
Opened a new window on the Universe! discovery potential = precise theoretical predictions!

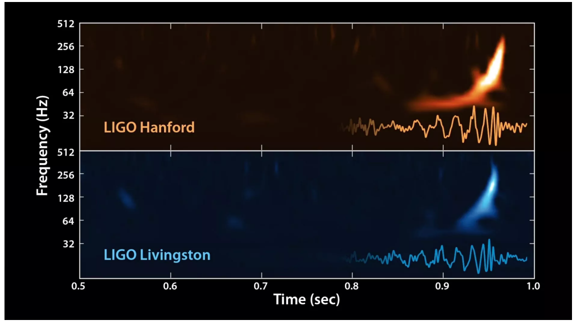
LIGO VIRGO KAGRA





Precision era of fundamental physics





Two historic breakthroughs in science:

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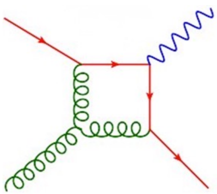
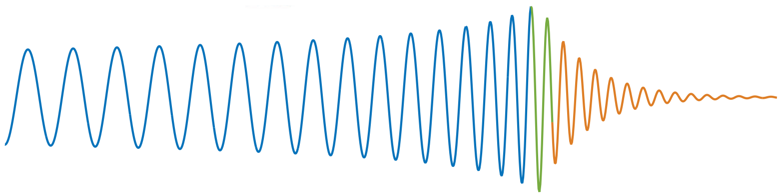
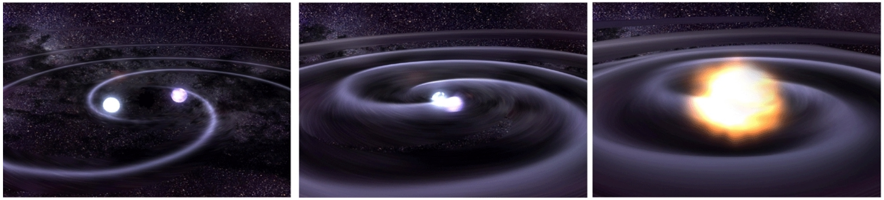
Modern techniques from LHC physics are playing a crucial role in precision GW physics!



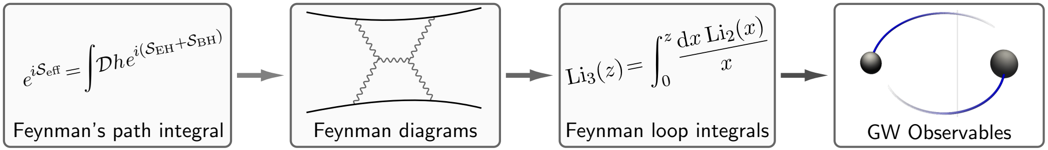


Outline

- Introduction: gravitational two-body problems



- An Effective Field Theory for two-body dynamics



- Gravitational binary dynamics to NNNLO

$$\alpha_1 G + \alpha_2 G^2 + \alpha_3 G^3 + \alpha_4 G^4 + \alpha_5 G^5 + \cdots$$

- Conclusion & Outlook

Gravitational waves from binary coalescences

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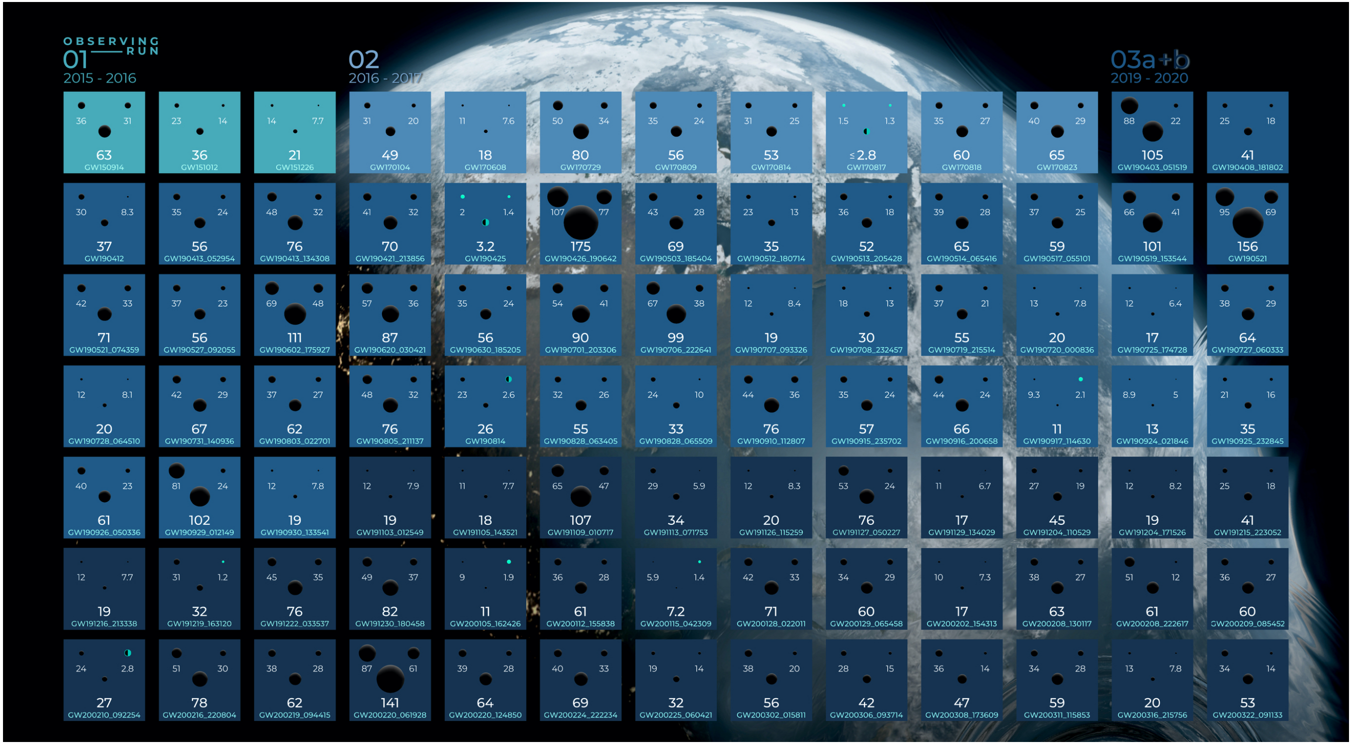

Credit: Carl Knox (OzGrav, Swinburne)

GWTC-3: 90 GW events – the majority are binary black holes (BH), but also several binary neutron stars (NS) and mixed NS-BHs.

gwosc.org

Gravitational waves from binary coalescences

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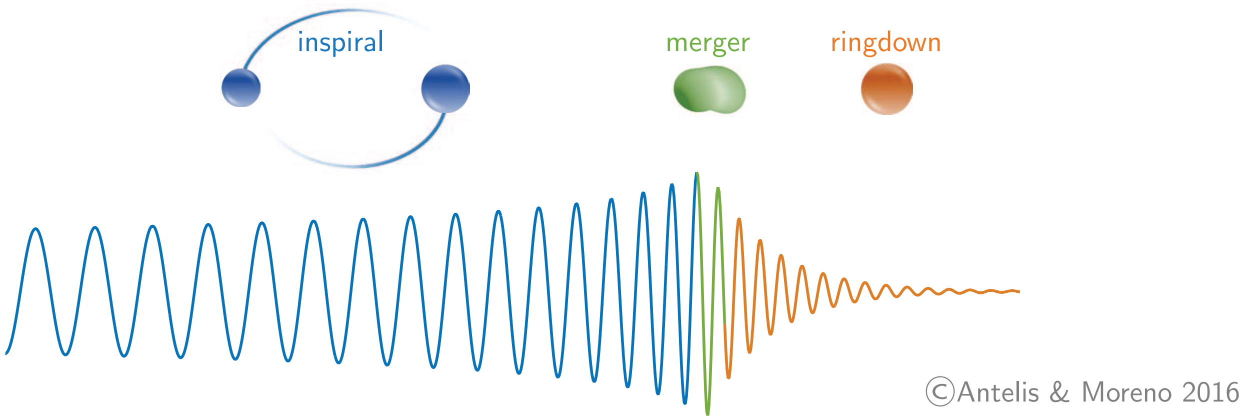


Merger: Numerical Relativity

Ringdown: black hole perturbation theory

Inspiral: the interaction between two bodies is weak

$$v^2 \sim \frac{GM}{r} \ll 1$$

- Numerical Relativity: accurately, but computationally expensive
- Analytic methods: corrections in v or G are perturbatively calculable

Post-Newtonian/post-Minkowskian expansion

- ▶ LHC theory technology, QFT methodology, shown great power!

Effective Field Theory

止於至善

- Gravitational binary system

$$S_{\rm WL} = \sum_{i=1,2} \left[-\frac{m_i}{2} \int dt\, g_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu + \cdots \right] \qquad S_{\rm GR} = \frac{-1}{16\pi G} \int d^4x \sqrt{-g}\, R + \cdots$$

- Effective action for gravitational binary systems

Goldberger-Rothstein hep-th/0409156

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$e^{iS_{\rm eff}[x_a(\tau)]} = \int \mathcal{D}h_{\mu\nu}\, e^{iS_{\rm WL} + iS_{\rm GR}}$$

- Post-Minkowskian expand in powers of G

$$L_{\rm eff} = L_0 + G L_1 + G^2 L_2 + \cdots \qquad L_0 = -\sum_i \frac{m_i}{2} \eta_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu$$

- The equations of motion for trajectories:

Kälin-Porto 2006.01184

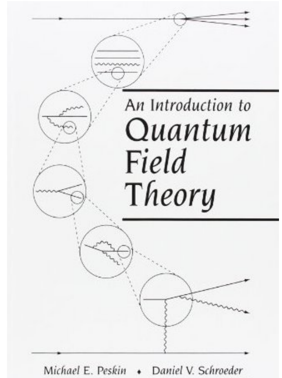
$$m_i \ddot{x}_i^\mu = -\eta^{\mu\nu} \sum_{n=1}^{\infty} G^n \left(\frac{\partial L_n}{\partial x_i^\nu} - \frac{d}{d\tau_i} \frac{\partial L_n}{\partial \dot{x}_i^\nu} \right) \qquad x_i^\mu = b_i^\mu + u_i^\mu \tau_i + \delta x_i^\mu(\tau_i) + \cdots$$

- Physical observables:

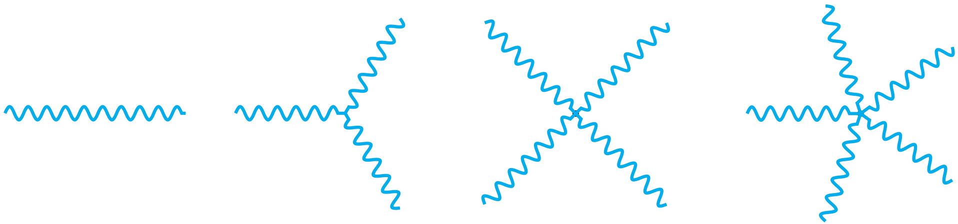
$$\Delta p_i^\mu = p_i^\mu(+\infty) - p_i^\mu(-\infty) = -\eta^{\mu\nu} \sum_{n=1}^{\infty} G^n \int_{-\infty}^{\infty} d\tau_i \left(\frac{\partial L_n}{\partial x_i^\nu} \right)$$

Effective Field Theory

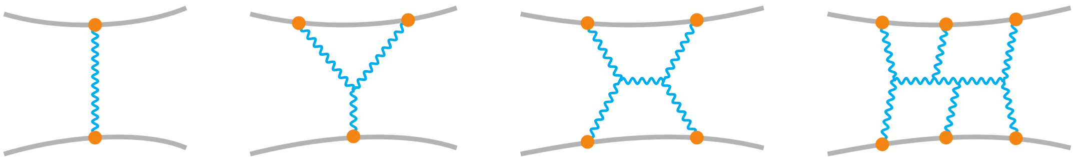
- Worldlines as classical sources in path integral:



- Hilbert-Einstein: $\mathcal{L}_{\text{HE}} = \mathcal{L}_{hh} + \mathcal{L}_{hhh} + \mathcal{L}_{hhhh} + \cdots$



- Classical physics: we use the saddle-point approximation in path integrals.



- Enjoy the advantages of quantum field theory methods and classical physics

  powerful and systematic & purely classical at all steps (simplicity)

Effective Field Theory

止於至善

The in-in effective action is obtained by performing a closed-time-path integral

$$e^{iS_{\text{eff}}[x_{a,1},x_{a,2}]} = \int \mathcal{D}h_1 \mathcal{D}h_2 \, e^{i(S_{\text{GR}}[h_1]-S_{\text{GR}}[h_2]+S_{\text{WL}}[h_1,x_{a,1}]-S_{\text{WL}}[h_2,x_{a,2}])}$$

It is convenient to use the Keldysh basis — Galley PRL 110 (2013) 174301

$$h^-_{\mu\nu} = \frac{1}{2}(h_{1\mu\nu} + h_{2\mu\nu}) \qquad x^\alpha_{a,+} = \frac{1}{2}(x^\alpha_{a,1} + x^\alpha_{a,2})$$

$$h^+_{\mu\nu} = h_{1\mu\nu} - h_{2\mu\nu} \qquad x^\alpha_{a,-} = x^\alpha_{a,1} - x^\alpha_{a,2}$$

for which the matrix of (classical) propagators for gravitions becomes

$$i\begin{pmatrix} 0 & -\Delta_{\text{adv}}(x-y) \\ -\Delta_{\text{ret}}(x-y) & 0 \end{pmatrix}$$

The worldline equations of motion: — Kälin-Neef-Porto JHEP 01 (2023) 140

$$m_i \ddot{x}^\mu_i(\tau) = -\eta^{\mu\nu} \frac{\delta S_{\text{eff, int}}[x_{a,\pm}]}{\delta x^\nu_{i,-}(\tau)}\bigg|_{\text{PL}} \qquad \Delta p^\mu_i = -\eta^{\mu\nu} \int_{-\infty}^{\infty} d\tau \frac{\delta S_{\text{eff, int}}[x_{a,\pm}]}{\delta x^\nu_{i,-}(\tau)}\bigg|_{\text{PL}}$$

Physical Limit (PL): $x_{a,-} \to 0$, $x_{a,+} \to x_a$.

Closed-time path integrals

EQUILIBRIUM AND NONEQUILIBRIUM FORMALISMS MADE UNIFIED

Kuang-chao CHOU, Zhao-bin SU,* Bai-lin HAO and Lu YU

Institute of Theoretical Physics, Academia Sinica, P.O. Box 2735, Beijing, China

Received 5 June 1984

Physics Reports

Volume 118, Issues 1–2, February 1985, Pages 1-131

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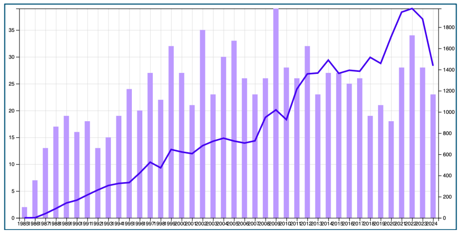

Effective Field Theory

- Effective action:

$$\Longrightarrow \quad -4\pi G m_1 m_2 \int d\tau_1 d\tau_2 \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik\cdot(x_1-x_2)}}{k^2} \Big(2(\dot{x}_1\cdot\dot{x}_2)^2 - \dot{x}_1^2\dot{x}_2^2\Big)$$

- Equation of motion: $\quad x_i^\mu = b_i^\mu + u_i^\mu \tau_i + \delta x_i^\mu(\tau_i) + \cdots$

$$\ddot{x}_1^\mu = 4i\pi G m_2 \int d\tau_2 \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik\cdot(b+u_1\tau_1+u_2\tau_2)}}{k^2} \Big[(2\gamma^2-1)k^\mu - 2(k\cdot u_1)\big(2\gamma u_2^\mu - u_1^\mu\big)\Big] + \mathcal{O}(G^2)$$

$$= 4i\pi G m_2 \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik\cdot b + i(k\cdot u_1)\tau_1}\hat{\delta}(k\cdot u_2)}{k^2} \Big[(2\gamma^2-1)k^\mu - 2(k\cdot u_1)\big(2\gamma u_2^\mu - u_1^\mu\big)\Big] + \mathcal{O}(G^2)$$

- Trajectory to $\mathcal{O}(G)$: $\quad \gamma = u_1\cdot u_2,\ u_i^2 = 1,\ q\cdot u_i = 0$

$$x_1^\mu = b_1^\mu + u_1^\mu \tau_1$$
$$- 4i\pi G m_2 \Big[(2\gamma^2-1)\eta^{\mu\nu} - 2(2\gamma u_2^\mu - u_1^\mu)u_1^\nu\Big] \int \frac{d^D k}{(2\pi)^D} \frac{\hat{\delta}(k\cdot u_2)k_\nu}{k^2} \frac{e^{ik\cdot b + i(k\cdot u_1 - i0)\tau_1}}{(k\cdot u_1 - i0)^2} + \mathcal{O}(G^2)$$

Effective Field Theory

- Effective action:

$$\Longrightarrow \quad -4\pi G m_1 m_2 \int d\tau_1 d\tau_2 \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik\cdot(x_1-x_2)}}{k^2} \left(2(\dot{x}_1\cdot\dot{x}_2)^2 - \dot{x}_1^2\dot{x}_2^2\right)$$

- Impulse:

$$\Delta p_1^\mu = -\eta^{\mu\nu}\frac{\partial S}{\partial x_1^\nu} = 4\pi G m_1 m_2 \int d\tau_1 d\tau_2 \int \frac{d^D k}{(2\pi)^D} \frac{e^{ik\cdot(x_1-x_2)}}{k^2} \left(2(\dot{x}_1\cdot\dot{x}_2)^2 - \dot{x}_1^2\dot{x}_2^2\right)$$

- LO (1PM): $x_i^\mu = b_i^\mu + u_i^\mu \tau_i + \delta x_i^\mu(\tau_i) + \cdots$

$$\Delta p_1^\mu = 4\pi G m_1 m_2 (2\gamma^2-1) \int \frac{d^D q}{(2\pi)^D} \frac{i q^\mu \hat{\delta}(q\cdot u_1)\hat{\delta}(q\cdot u_2) e^{iq\cdot b}}{q^2} = \frac{(2\gamma^2-1)}{\sqrt{\gamma^2-1}} \frac{2G m_1 m_2 b^\mu}{b^2}$$

- NLO (2PM): $\gamma = u_1\cdot u_2,\ u_i^2 = 1,\ q\cdot u_i = 0$

$$\Delta p_1^\mu \sim G^2 \int \frac{d^D q}{(2\pi)^D} \delta(q\cdot u_1)\,\delta(q\cdot u_2) e^{iq\cdot b} \int \frac{d^D k}{(2\pi)^D} \frac{\delta(k\cdot u_2)}{(k\cdot u_1 - i0)^\sharp\,[k^2]\,[(k-q)^2]}$$

Effective Field Theory

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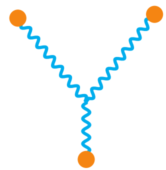


Effective Field Theory

止於至善

- Observables at $\mathcal{O}(G^N)$ Kälin-**ZL**-Porto PRL2020 Dlapa-Kälin-**ZL**-Porto PRL2022 JHEP2024

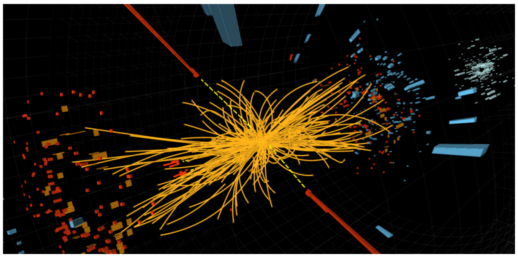
$$\Delta p_i^\mu \sim \int d^D q \, \frac{e^{iq\cdot b}\,\delta(q\cdot u_1)\delta(q\cdot u_2)}{|q^2|^\sharp} \int \left(\prod_{i=1}^{N-1} d^D\ell_i \, \frac{\delta(\ell_j\cdot u_a)}{(\ell_i\cdot u_b - i0)^{\nu_i}} \right) \frac{\mathcal{N}^\mu(q,u_a)}{D_1 D_2 D_3 \cdots}$$

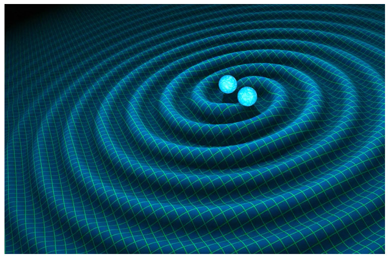
  - Graviton propagators:

$$\frac{1}{D_i} \longrightarrow \frac{1}{\ell^2 + i0} \quad \text{or} \quad \frac{1}{(\ell^0 \pm i0)^2 - \vec{\ell}^{\,2}}$$

  - Cut: always one delta function $\delta(\ell_i\cdot u_a)$ for each loop
  - Kinematics: $q \cdot u_a = 0,\ u_a^2 = 1,\ u_1 \cdot u_2 = \gamma \implies$ single scale γ to all orders

- Multi-loop technology from QFT can be used to solve classical gravitational problems!





QFT toolbox

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- Observables at $\mathcal{O}(G^N)$ Kälin-**ZL**-Porto PRL 2020 Dlapa-Kälin-**ZL**-Porto PRL 2022 Dlapa-Kälin-**ZL**-Neef-Porto PRL 2023

$$\Delta p_i^\mu \sim \int d^D q \, \frac{e^{iq\cdot b}\,\delta(q\cdot u_1)\delta(q\cdot u_2)}{|q^2|^\sharp} \int \left(\prod_{i=1}^{N-1} d^D \ell_i \, \frac{\delta(\ell_j \cdot u_a)}{(\ell_i \cdot u_b - i0)^{\nu_i}} \right) \frac{\mathcal{N}^\mu(q, u_a)}{D_1 D_2 D_3 \cdots}$$

- Perturbative QFT toolbox: Dlapa-Kälin-**ZL**-Porto JHEP 2024

$$\int d^d\ell \cdots \frac{\partial}{\partial \ell^\mu} \frac{v^\mu}{P(\ell, \dots)} = 0$$

$$\boldsymbol{f}(x,\epsilon) = \{I_1, I_2, \dots, I_N\}$$

Integration-By-Parts Relations

$$\frac{d\boldsymbol{f}(x,\epsilon)}{dx} = \boldsymbol{A}(x,\epsilon)\,\boldsymbol{f}(x,\epsilon)$$

Differential Equations (DEs)

$$\boldsymbol{g} = \boldsymbol{T}\,\boldsymbol{f}$$

$$\frac{d\boldsymbol{g}(x,\epsilon)}{dx} = \epsilon\,\mathcal{M}(x)\boldsymbol{g}(x,\epsilon)$$

Canonical DEs

Iterated Integrals

$$\boldsymbol{g}(x,\epsilon) = \sum_n \epsilon^n \boldsymbol{g}^{(n)}(x)$$

$$\boldsymbol{g}^{(n)}(x) = \int_{x_0}^{x} \mathcal{M}(t)\boldsymbol{g}^{(n-1)}(t)\,dt + \boldsymbol{g}_0^{(n)}$$

Initial Conditions

PN limit: $v \to 0$

pot: $\ell^\mu = (\omega, \boldsymbol{\ell}) \sim (v, 1)$

rad: $\ell^\mu = (\omega, \boldsymbol{\ell}) \sim (v, v)$

PM Observables

$$\mathcal{O} = \sum_i c_i I_i$$

- Post-Minkowskian physics can be bootstrapped from Post-Newtonian data using DEs!

NLO: 2PM

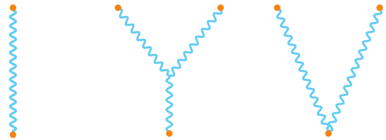
以於至善

Spin interactions $g_{\mu\nu} e^\mu_a e^\nu_b = \eta_{ab}$

$$-\frac{1}{2}\left(\omega^{ab}_\mu S_{ab} v^\mu + \frac{1}{m} R_{\beta\rho\mu\nu} e^\alpha_a e^\beta_b e^\mu_c e^\nu_d S^{ab} S^{cd} v^\rho v_\alpha - \frac{C_{\rm ES}}{m} E_{\mu\nu} e^\mu_a e^\nu_b S^{ac} S_c{}^b + \cdots\right)$$

- EFT provides a systematic way to include spin effects. See Fei's talk
- Needed one-loop integrals are simple

$$\int d^D\ell \, \frac{\delta(\ell\cdot u_1)}{[(\pm\ell\cdot u_2)]^{a_1}} \frac{1}{[\ell^2]^{a_2}\,[(\ell-q)^2]^{a_3}}$$



- Nontrivial to simplify complicated tensor expressions

Observables: **ZL**-Porto-Yang JHEP 2021

$$\Delta p_1^\mu = \frac{\nu G^2 M^3}{|b|^3}\left[3 D_1 \epsilon_{\alpha\rho\beta\sigma} \hat{b}^\mu \hat{b}^\alpha u_1^\beta u_2^\sigma a_1^\rho + \cdots + \frac{D_{20}}{|b|} u_1^\mu (a_1\cdot a_2) + \cdots + \frac{D_{14}}{|b|} u_2^\mu a_1^2 + \cdots\right]$$

$$s_\mu = m a_\mu \equiv \frac{1}{2m}\epsilon_{\mu\nu\alpha\beta} p^\nu S^{\alpha\beta}$$

NNLO: 3PM

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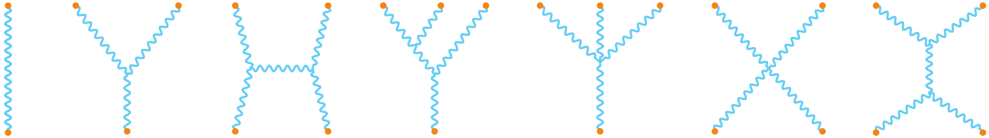


- $\mathcal{O}(G^3)$: two-loop integrals Kälin-**ZL**-Porto PRL 2020 PRD 2020

$$\int \frac{d^D\ell_1 d^D\ell_2\, \delta(\ell_1\cdot u_1)\, \delta(\ell_2\cdot u_2)}{[\ell_1\cdot u_2]^{a_1}\, [\pm\ell_2\cdot u_1]^{a_2}} \frac{1}{[\ell_1^2]^{a_3}\, [\ell_2^2]^{a_4}\, [(\ell_1+\ell_2-q)^2]^{a_5}\, [(\ell_1-q)^2]^{a_6}\, [(\ell_2-q)^2]^{a_7}}$$

- The reduction and evaluation of integrals can be performed in standard techniques.

$$\partial_x \vec{f}(x,\epsilon) = \epsilon \left(\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}\right) \vec{f}(x,\epsilon)$$

- Conservative dynamics at $\mathcal{O}(G^3)$: Kälin-**ZL**-Porto PRL 2020

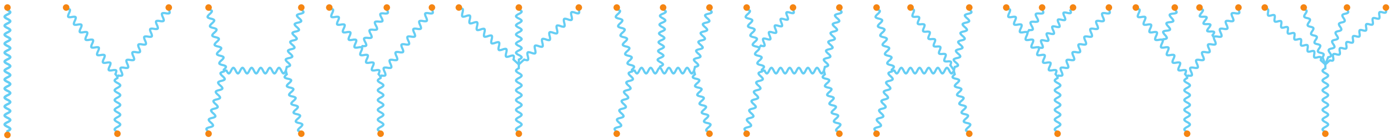
$$\Delta p_1^\mu = \frac{G^3 b^\mu}{|b^2|^2}\left(\frac{8m_1^2m_2^2(4\gamma^4-12\gamma^2-3)}{(\gamma^2-1)}\log\left(\gamma-\sqrt{\gamma^2-1}\right) - \frac{2m_1m_2(m_1^2+m_2^2)(16\gamma^6-32\gamma^4+16\gamma^2-1)}{(\gamma^2-1)^{5/2}} - \frac{4m_1^2m_2^2\gamma(20\gamma^6-90\gamma^4+120\gamma^2-53)}{3(\gamma^2-1)^{5/2}}\right) + \frac{3\pi}{2}\frac{(2\gamma^2-1)\,(5\gamma^2-1)}{(\gamma^2-1)^2}\frac{G^3m_1m_2(m_1+m_2)}{|b^2|^{3/2}}\left((m_1+\gamma m_2)u_2^\mu - (m_2+\gamma m_1)u_1^\mu\right)$$

- We provided the first confirmation for the result from a scattering amplitude clacualtion.

Bern-Cheung-Roiban-Shen-Solon-Zeng 2019

- We computed quadrupolar and octupole tidal corrections at $\mathcal{O}(G^3)$. Kälin-**ZL**-Porto PRD 2020

NNNLO: 4PM



$\mathcal{O}(G^4)$: three-loop integrals Dlapa-Kälin-**ZL**-Porto PRL 2022 PLB 2022 Dlapa-Kälin-**ZL**-Neef-Porto PRL 2023

$$\int d^D\ell_1 d^D\ell_2 d^D\ell_3 \frac{\delta(\ell_1\cdot u_1)\,\delta(\ell_2\cdot u_2)\,\delta(\ell_3\cdot u_2)}{[\ell_1\cdot u_2]^{\alpha_1}\,[\ell_2\cdot u_1]^{\alpha_2}\,[\ell_3\cdot u_1]^{\alpha_3}}\,\frac{D_8^{-\nu_8} D_9^{-\nu_9}}{D_1^{\nu_1} D_2^{\nu_2}\cdots D_7^{\nu_7}}$$

$\{\ell_1^2, \ell_2^2, (\ell_1-q)^2, (\ell_2-q)^2, (\ell_3-q)^2, \ell_3^2, (\ell_1-\ell_2)^2, (\ell_2-\ell_3)^2, (\ell_3-\ell_1)^2\}$

IBP reduction:

conservative: $\mathcal{O}(10^2)$ master integrals full: $\mathcal{O}(10^3)$ master integrals

Differential Equations Dlapa-Kälin-**ZL**-Porto PRL 2022 PLB 2022 Dlapa-Henn-Wagner 2211.16357

$$\frac{d\vec{f}(x,\epsilon)}{dx} = \epsilon\,\mathcal{M}(x)\,\vec{f}(x,\epsilon)$$

- The majority can be solved in terms of multiple polylogarithms.
- Elliptic integrals appear in post-Minkwskian gravity for the first time.

Elliptic differential equations

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- From 4PM order, more complicated functions appear beyond polylogarithms.
- An elliptic example:

$$\frac{d}{dx}\begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = \begin{pmatrix} \frac{1-x^2}{2x(1+x^2)} & \frac{1+x^2}{4x(1-x^2)} & \frac{3x}{(1-x^2)(1+x^2)} \\ -\frac{1-x^2}{x(1+x^2)} & \frac{3(1+x^2)}{2x(1-x^2)} & -\frac{6x}{(1-x^2)(1+x^2)} \\ \frac{1-x^2}{x(1+x^2)} & -\frac{1+x^2}{2x(1-x^2)} & -\frac{1-4x^2+x^4}{x(1-x^2)(1+x^2)} \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} + \mathcal{O}(\epsilon)$$

It can then be written as a third-order differential equation:

$$\left[\frac{d^3}{dx^3} - \frac{6x}{1-x^2}\frac{d^2}{dx^2} - \frac{1-4x^2+7x^4}{x^2(1-x^2)^2}\frac{d}{dx} - \frac{1+x^2}{x^3(1-x^2)}\right] f_1(x) = 0$$

It is easy to find the three solutions:

$$x\,\mathrm{K}^2(1-x^2), \qquad x\,\mathrm{K}(1-x^2)\mathrm{K}(x^2), \qquad x\,\mathrm{K}^2(x^2)$$

Complete elliptic integrals: $\mathrm{K}(x) \equiv \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-xt^2)}}$

Elliptic differential equations

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With the knowledge of leading-ϵ solutions, one may transform the elliptic diagonal block into

$$\frac{d}{dx}\vec{g}(x,\epsilon) = \epsilon\,\tilde{D}_{\text{ell}}(x)\,\vec{g}(x,\epsilon) + \ldots$$

with

$$\tilde{D}_{\text{ell}} = \begin{pmatrix} -\frac{4\left(1+x^2\right)}{3x(1-x^2)} & \frac{\pi^2}{x(1-x^2)\text{K}^2(1-x^2)} & 0 \\ \frac{2\left(1+110x^2+x^4\right)\text{K}^2\left(1-x^2\right)}{3\pi^2 x(1-x^2)} & -\frac{4\left(1+x^2\right)}{3x(1-x^2)} & \frac{\pi^2}{x(1-x^2)\text{K}^2(1-x^2)} \\ \frac{16\left(1+x^2\right)\left(1-18x+x^2\right)\left(1+18x+x^2\right)\text{K}^4\left(1-x^2\right)}{27\pi^2 x(1-x^2)} & \frac{2\left(1+110x^2+x^4\right)\text{K}^2\left(1-x^2\right)}{3\pi^2 x(1-x^2)} & -\frac{4\left(1+x^2\right)}{3x(1-x^2)} \end{pmatrix}$$

- Elliptic integrals appear in the transformation matrix: automated by INITIAL
- Higer $\mathcal{O}(\epsilon)$: Iterated integrals involving elliptic kernels. See Xing's talk

NNNLO: 4PM

The full impulse at $\mathcal{O}(G^4)$: Dlapa-Kälin-**ZL**-Porto PRL 2022 JHEP 2023 Dlapa-Kälin-**ZL**-Neef-Porto PRL 2023

$$\Delta p_1^\mu\Big|_{\rm NNNLO} = \frac{G^4}{|b|^4}\left(c_b\,\frac{b^\mu}{|b|} + c_1\,\frac{\gamma u_2^\mu - u_1^\mu}{\gamma^2-1} + c_2\,\frac{\gamma u_1^\mu - u_2^\mu}{\gamma^2-1}\right)$$

$$\begin{aligned}
\frac{c_b}{\pi} = &-\frac{3h_{34}m_2m_1(m_1^3+m_2^3)}{64v_\infty^5} + \frac{m_1^2m_{12}m_2^2}{4}\Bigg[\frac{3h_6\mathrm{K}^2(w_2)}{4v_\infty^3} - \frac{3h_8\mathrm{K}(w_2)\mathrm{E}(w_2)}{4v_\infty^3} + \frac{21h_5w_3\mathrm{E}^2(w_2)}{8v_\infty^3} - \frac{\pi^2h_{16}v_\infty}{4(\gamma+1)} + \frac{3\gamma h_{10}(\mathrm{Li}_2(w_2)-4\mathrm{Li}_2(\sqrt{w_2}))}{w_3v_\infty^2}\\
&+\log(v_\infty)\left(\frac{h_{32}}{2v_\infty^3} - \frac{3h_{14}\log(\frac{w_3}{2})}{v_\infty} - \frac{3\gamma h_{22}\log(w_1)}{2v_\infty^4}\right)\Bigg] + m_2^2m_1^3\Bigg[\frac{h_{52}}{48v_\infty^6} - \frac{h_{63}}{768\gamma^9w_3v_\infty^5} - \frac{3v_\infty(h_{40}\mathrm{Li}_2(w_2)+2w_3h_{33}\mathrm{Li}_2(-w_2))}{64w_3}\\
&+\frac{3h_{14}\log(\frac{w_3}{2})\log(w_3)}{4v_\infty} + \frac{\gamma h_{39}\log(w_1)}{8w_3^3v_\infty^2} + \frac{3\gamma h_{22}\log(w_3)\log(w_1) - h_{35}\log(\frac{w_3}{2})}{8v_\infty^4} + \frac{h_{56}\log(2) - h_{57}\log(w_3) + 2\gamma h_{55}\log(\gamma)}{32v_\infty^5} - \frac{\gamma h_{51}\log(w_1)}{16v_\infty^7}\Bigg]\\
&+m_1^2m_2^3\Bigg[\frac{h_{58}}{192\gamma^7v_\infty^5} + \frac{h_{53}}{48v_\infty^6} + \frac{\gamma h_{49}\log(w_1)}{16v_\infty^6} - \frac{2\gamma h_{50}\log(w_1) + 3\gamma^2h_{13}\log^2(w_1)}{32v_\infty^7} - \frac{h_{41}\log(\frac{w_3}{2})}{8v_\infty^4} + \frac{3\gamma\log(w_1)\big(5h_{26}\log(2)+8h_{12}\log(w_3)\big)}{8v_\infty^4}\\
&-\frac{h_{36}\log(w_3)}{4v_\infty^3} + \frac{\gamma h_{30}\log(\gamma)}{2v_\infty^3} + \frac{h_{37}\log(2)}{8v_\infty^3} + \frac{3(h_{17}w_3\mathrm{Li}_2(w_2) - 2h_{23}\mathrm{Li}_2(-w_2) + h_{15}\log^2(w_3) - h_9\log^2(2))}{8v_\infty} - \frac{3h_7\log(2)\log(w_3)}{v_\infty}\Bigg]
\end{aligned}$$

$$c_1 = m_1m_2^2\left(\frac{2h_{46}m_{12s}}{v_\infty^6} + \frac{9\pi^2h_1m_{12}^2}{32v_\infty^2}\right) + m_1^2m_2^3\left(\frac{4\gamma h_{47}}{3v_\infty^6} - \frac{8\gamma h_2\log(w_1)}{v_\infty^6} + \frac{16h_{25}\log(w_1)}{v_\infty^3} - \frac{8h_3}{3v_\infty^5}\right)$$

$$\begin{aligned}
c_2 = &-m_1^4m_2\left(\frac{9\pi^2h_1}{32v_\infty^2} + \frac{2h_{46}}{v_\infty^6}\right) + m_2^2m_1^3\Bigg[+\frac{h_{60}}{705600\gamma^8v_\infty^5} - \frac{4\gamma h_{48}}{3v_\infty^6} + \frac{3h_{38}(\mathrm{Li}_2(w_2)-4\mathrm{Li}_2(\sqrt{w_2})) - \gamma h_{21}(\mathrm{Li}_2(-w_1^2) + 2\log(\gamma)\log(w_1))}{16v_\infty^4}\\
&+\frac{3\gamma h_{31}(2\mathrm{Li}_2(-w_1) + \log(w_1)\log(w_3))}{8v_\infty^4} + \frac{h_{62}\log(w_1)}{6720\gamma^9v_\infty^6} + \frac{32\gamma^2h_{44}\log^2(w_1)}{v_\infty^7} + \frac{8\gamma\big(2h_4\log(2) - h_{27}\log(w_1)\big)\log(w_1)}{v_\infty^4} - \frac{32h_{29}\log(w_1)}{3v_\infty^3} + \frac{\pi^2h_{42}}{192v_\infty^4}\Bigg]\\
&+m_2^3m_1^2\Bigg[\frac{h_{59}}{1440\gamma^7v_\infty^5} - \frac{h_{19}(\mathrm{Li}_2(-w_1^2) + 2\log(\gamma)\log(w_1))}{8v_\infty^4} + \frac{h_{43}(\mathrm{Li}_2(w_2) - 4\mathrm{Li}_2(\sqrt{w_2}))}{32v_\infty^4} - \frac{h_{20}(2\mathrm{Li}_2(-w_1) + \log(w_1)\log(w_3))}{4v_\infty^4}\\
&-\frac{h_{61}\log(w_1)}{480\gamma^8v_\infty^6} - \frac{16\gamma^2h_{11}\log^2(w_1)}{v_\infty^4} - \frac{32\gamma h_{45}\log^2(w_1)}{v_\infty^7} + \frac{16\gamma h_{28}\log(w_1)}{5v_\infty^3} - \frac{32h_{24}\log(2)\log(w_1)}{v_\infty^4} - \frac{\pi^2h_{18}}{48v_\infty^4} - \frac{2h_{54}}{45v_\infty^6}\Bigg]
\end{aligned}$$

with $\gamma \equiv u_1\cdot u_2,\ v_\infty = \sqrt{\gamma^2-1},\ w_1 = \gamma - v_\infty,\ w_2 = \frac{\gamma-1}{\gamma+1},\ w_3 = \gamma+1,\ h_i =$ polynomial in γ.

$$\mathrm{Li}_2(z) \equiv \int_0^z dx\,\frac{\log(1-x)}{-x}$$

$$K(z) \equiv \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-zx^2)}}$$

$$E(k) \equiv \int_0^1 dx\,\frac{\sqrt{1-zx^2}}{\sqrt{1-x^2}}$$

NNNLO: 4PM

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The full impulse at $\mathcal{O}(G^4)$: Dlapa-Kälin-**ZL**-Porto JHEP 2023 Dlapa-Kälin-**ZL**-Neef-Porto PRL 2023

$$\Delta p_1^\mu\big|_{\text{NNNLO}} = \frac{G^4}{|b|^4}\left(c_b \frac{b^\mu}{|b|} + c_1 \frac{\gamma u_2^\mu - u_1^\mu}{\gamma^2 - 1} + c_2 \frac{\gamma u_1^\mu - u_2^\mu}{\gamma^2 - 1} \right)$$

- We obtained the full dynamics of binary inspirals at $\mathcal{O}(G^4)$ for the first time.
- Conservative part agrees perfectly with previous derivations.

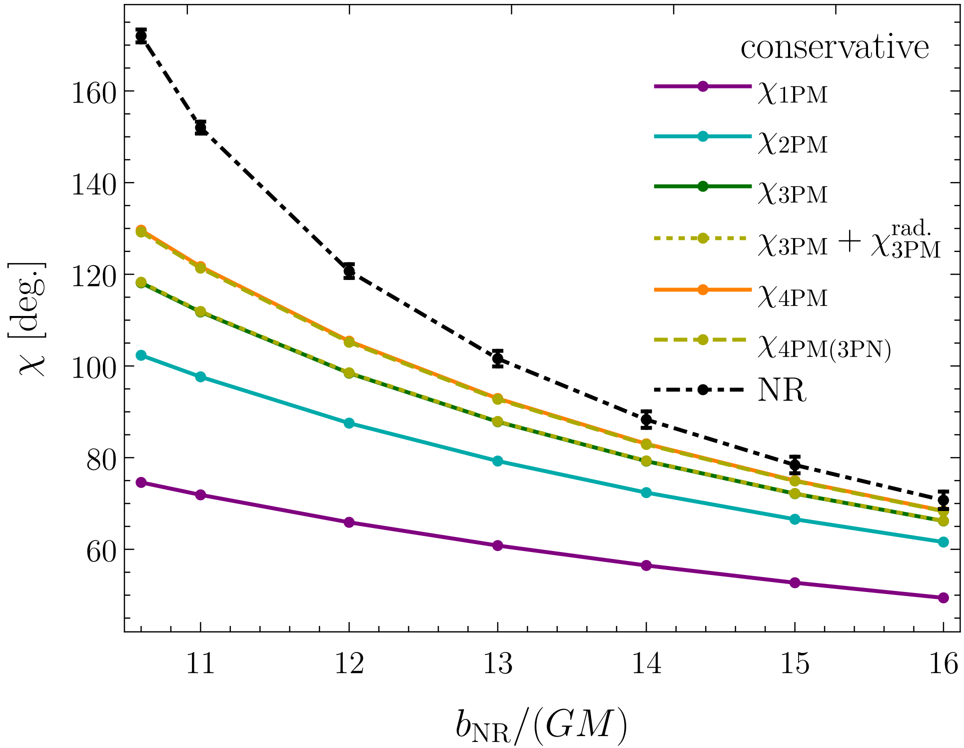
  Bern-Parra-Martinez-Roiban-Ruf-Shen-Solon-Zeng 2022 Dlapa-Kälin-**ZL**-Porto PRL 2022
- Perfect agreement with the state-of-the-art PN computations

  Cho-Dandapat-Gopakumar 2021 Cho 2022 Bini-Geralico 2021 2022 Bini-Damour 2022
- Later, two new calculations confirmed our results.

  Damgaard-Hansen-Planté-Vanhove 2023 (exponentiation of amplitudes)

  Jakobsen-Mogull-Plefka-Sauer-Xu 2023 (worldline formalism)

Analytic vs Numerical Relativity



Khalil-Buonanno-Steinhoff-Vines 2204.05047

NNNLO: local-in-time part

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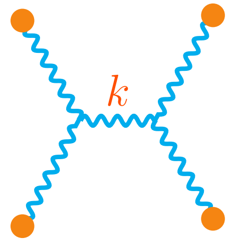
- The full result does not describe generic elliptic-like motion due to nonlocal-in-time effects.
  Damour-Jaranowski-Schäfer 2014 Galley-Leibovich-Porto-Ross 2015 Cho-Kälin-Porto 2021

- Nonlocal-in-time radial action:

$$\mathcal{S}_r^{(\mathrm{nloc})} = -\frac{GE}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{dE}{d\omega}\log\left(\frac{4\omega^2}{\mu^2}e^{2\gamma_E}\right)$$



- The 4PM integrand can be built from 3PM diagrams. $\omega \equiv k\cdot u_{\mathrm{com}}$

$$\int d^D\ell_1 d^D\ell_2 \frac{\delta(\ell_1\cdot u_1)\delta(\ell_2\cdot u_2)}{[\ell_1\cdot u_2][\ell_2\cdot u_1]}\frac{\log(\omega^2)}{[\ell_1^2][\ell_2^2][(\ell_1+\ell_2-q)^2][(\ell_1-q)^2][(\ell_2-q)^2]}$$

- We managed to compute the integrals and obtained nonlocalin-time contribution:

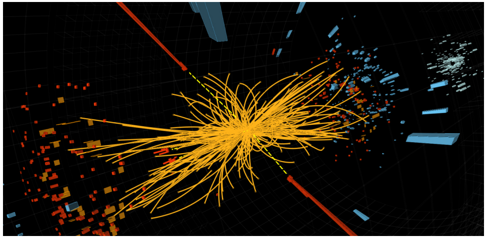
$$\frac{\nu}{(\gamma^2-1)^2}\Bigg[h_1+\frac{\pi^2 h_2}{\sqrt{\gamma^2-1}}+h_3\log\frac{\gamma+1}{2}+\frac{h_4\mathrm{arccosh}(\gamma)}{\sqrt{\gamma^2-1}}+h_5\log\frac{\gamma-1}{8}+h_6\log^2\frac{\gamma+1}{2}+\frac{h_8\log(2)\mathrm{arccosh}(\gamma)}{\sqrt{\gamma^2-1}}$$
$$+h_7\,\mathrm{arccosh}(\gamma)^2+h_9\log\frac{\gamma-1}{8}\log\frac{\gamma+1}{2}+\frac{h_{10}\log\frac{\gamma^2-1}{16}\mathrm{arccosh}(\gamma)}{\sqrt{\gamma^2-1}}+h_{11}\mathrm{Li}_2\frac{\gamma-1}{\gamma+1}+h_{12}\frac{\mathrm{arccosh}^2(\gamma)+4\mathrm{Li}_2(\sqrt{\gamma^2-1}-\gamma)}{\sqrt{\gamma^2-1}}\Bigg]$$

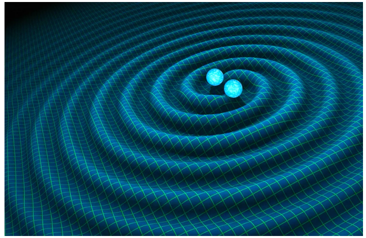
Coefficients h_i: exact-ν (iterated elliptic integrals) and SF-expanded (30SF) forms
Dlapa-Kälin-**ZL**-Porto PRL 2024

- Using 6PN results in the literature, we constructed an improved bound Hamiltonian.

Conclusion & Outlook

Modern techniques from Quantum Field Theory have already proven useful to improve theoretical predictions for gravitational-wave observables.





We have developed an efficient framework and made breakthroughs to NNNLO.

- Conservative spin & tidal effects at NLO JHEP 06 (2021) 012 PRD 102 (2020) 124025
- Conservative dynamics at NNLO PRL 125 (2020) 261103
- Conservative dynamics at NNNLO PLB 822 (2021) 136698 PRL 128 (2022) 161104 PRL 130 (2023) 101401
- Local-in-time & nonlocal-in-time separation to N^4LO PRL 132 (2024) 221401 arXiv:2506.20665
- Novel techniques to evaluate classical loop integrals JHEP 07 (2023) 181 JHEP 08 (2023) 109

Outlook: precision frontier

以於至善

GW science is just starting! discovery potential = precise theoretical predictions!

$$
\begin{array}{lllllllllll}
 & & \text{0PN} & \text{1PN} & \text{2PN} & \text{3PN} & \text{4PN} & \text{5PN} & \text{6PN} & \cdots & \\
\text{1PM} & \left(\frac{GM}{r}\right)\times(& 1 & +\,v^2 & +\,v^4 & +\,v^6 & +\,v^8 & +\,v^{10} & +\,v^{12} & +\cdots) & = \left(\frac{GM}{r}\right)\times(1) \\
\text{2PM} & \left(\frac{GM}{r}\right)^2\times(& & 1 & +\,v^2 & +\,v^4 & +\,v^6 & +\,v^8 & +\,v^{10} & +\cdots) & = \left(\frac{GM}{r}\right)^2\times(1) \\
\text{3PM} & \left(\frac{GM}{r}\right)^3\times(& & & 1 & +\,v^2 & +\,v^4 & +\,v^6 & +\,v^8 & +\cdots) & = \left(\frac{GM}{r}\right)^3\times(1+\nu) \\
\text{4PM} & \left(\frac{GM}{r}\right)^4\times(& & & & 1 & +\,v^2 & +\,v^4 & +\,v^6 & +\cdots) & = \left(\frac{GM}{r}\right)^4\times(1+\nu) \\
\text{5PM} & \left(\frac{GM}{r}\right)^5\times(& & & & & 1 & +\,v^2 & +\,v^4 & +\cdots) & = \left(\frac{GM}{r}\right)^5\times(1+\nu+\nu^2) \\
\text{6PM} & \left(\frac{GM}{r}\right)^6\times(& & & & & & 1 & +\,v^2 & +\cdots) & = \left(\frac{GM}{r}\right)^6\times(1+\nu+\nu^2) \\
\text{7PM} & \left(\frac{GM}{r}\right)^7\times(& & & & & & & 1 & +\cdots) & = \left(\frac{GM}{r}\right)^7\times(1+\nu+\nu^2+\nu^3) \\
 & & & & & & & & & & \quad \text{0SF} \quad \text{1SF} \quad \text{2SF} \quad \text{3SF} \cdots
\end{array}
$$

- 1SF: Driesse-Jakobsen-Mogull-Plefka-Sauer-Usovitsch 2024 D-J-Klemm-M-Nega-P-S-U 2024

Outlook: precision frontier

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GW science is just starting! discovery potential = precise theoretical predictions!

0PN 1PN 2PN 3PN 4PN 5PN 6PN $\cdots$

$$\text{1PM}\quad \left(\frac{GM}{r}\right)\times\left(1+v^2+v^4+v^6+v^8+v^{10}+v^{12}+\cdots\right)=\left(\frac{GM}{r}\right)\times\left(1\right)$$

$$\text{2PM}\quad \left(\frac{GM}{r}\right)^2\times\left(1+v^2+v^4+v^6+v^8+v^{10}+\cdots\right)=\left(\frac{GM}{r}\right)^2\times\left(1\right)$$

$$\text{3PM}\quad \times\left(1+v^2+v^4+v^6+v^8+\cdots\right)=\left(\frac{GM}{r}\right)^3\times\left(1+\nu\right)$$

$$\text{4PM}\quad \left(\frac{GM}{r}\right)^4\times\left(1+v^2+v^4+v^6+\cdots\right)=\left(\frac{GM}{r}\right)^4\times\left(1+\nu\right)$$

$$\text{5PM}\quad \left(\frac{GM}{r}\right)^5\times\left(1+v^2+v^4+\cdots\right)=\left(\frac{GM}{r}\right)^5\times\left(1+\nu+\nu^2\right)$$

$$\text{6PM}\quad \left(\frac{GM}{r}\right)^6\times\left(1+v^2+\cdots\right)=\left(\frac{GM}{r}\right)^6\times\left(1+\nu+\nu^2\right)$$

$$\text{7PM}\quad \left(\frac{GM}{r}\right)^7\times\left(1+\cdots\right)=\left(\frac{GM}{r}\right)^7\times\left(1+\nu+\nu^2+\nu^3\right)$$

0SF 1SF 2SF 3SF $\cdots$

- 1SF: Driesse-Jakobsen-Mogull-Plefka-Sauer-Usovitsch 2024   D-J-Klemm-M-Nega-P-S-U 2024

Outlook: precision frontier

止於至善

GW science is just starting! discovery potential = precise theoretical predictions!

$$\begin{array}{llllllll} & 0\text{PN} & 1\text{PN} & 2\text{PN} & 3\text{PN} & 4\text{PN} & 5\text{PN} & 6\text{PN} \; \cdots \end{array}$$

$$1\text{PM} \quad \left(\frac{GM}{r}\right) \times \left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots \right) = \left(\frac{GM}{r}\right) \times \left(1 \right)$$

$$2\text{PM} \quad \left(\frac{GM}{r}\right)^2 \times \left(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + \cdots \right) = \left(\frac{GM}{r}\right)^2 \times \left(1 \right)$$

$$3\text{PM} \quad \cdots = \left(\frac{GM}{r}\right)^3 \times \left(1 + \nu \right)$$

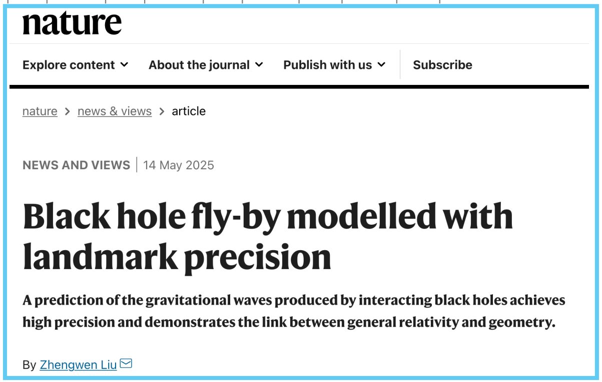
$$4\text{PM} \quad \cdots = \left(\frac{GM}{r}\right)^4 \times \left(1 + \nu \right)$$

$$5\text{PM} \quad \cdots = \left(\frac{GM}{r}\right)^5 \times \left(1 + \nu + \nu^2 \right)$$

$$6\text{PM} \quad \cdots = \left(\frac{GM}{r}\right)^6 \times \left(1 + \nu + \nu^2 \right)$$

$$7\text{PM} \quad \cdots = \left(\frac{GM}{r}\right)^7 \times \left(1 + \nu + \nu^2 + \nu^3 \right)$$

$$0\text{SF} \quad 1\text{SF} \quad 2\text{SF} \quad 3\text{SF} \; \cdots$$



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NEWS AND VIEWS | 14 May 2025

Black hole fly-by modelled with landmark precision

A prediction of the gravitational waves produced by interacting black holes achieves high precision and demonstrates the link between general relativity and geometry.

By Zhengwen Liu

- 1SF: Driesse-Jakobsen-Mogull-Plefka-Sauer-Usovitsch 2024 D-J-Klemm-M-Nega-P-S-U 2024

Outlook: precision frontier

止於至善

GW science is just starting! discovery potential = precise theoretical predictions!

0PN 1PN 2PN 3PN 4PN 5PN 6PN $\cdots$

$$\text{1PM}\quad \left(\frac{GM}{r}\right)\times\left(1+v^2+v^4+v^6+v^8+v^{10}+v^{12}+\cdots\right)=\left(\frac{GM}{r}\right)\times(1)$$

$$\text{2PM}\quad \left(\frac{GM}{r}\right)^2\times\left(1+v^2+v^4+v^6+v^8+v^{10}+\cdots\right)=\left(\frac{GM}{r}\right)^2\times(1)$$

$$\text{3PM}\quad \left(\frac{GM}{r}\right)^3\times(1+\nu)$$

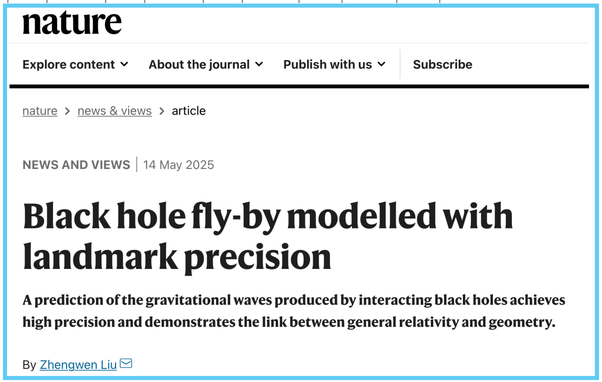
$$\text{4PM}\quad \left(\frac{GM}{r}\right)^4\times(1+\nu)$$

$$\text{5PM}\quad \left(\frac{GM}{r}\right)^5\times(1+\nu+\nu^2)$$

$$\text{6PM}\quad \left(\frac{GM}{r}\right)^6\times(1+\nu+\nu^2)$$

$$\text{7PM}\quad \left(\frac{GM}{r}\right)^7\times(1+\nu+\nu^2+\nu^3)$$

0SF 1SF 2SF 3SF $\cdots$



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NEWS AND VIEWS | 14 May 2025

Black hole fly-by modelled with landmark precision

A prediction of the gravitational waves produced by interacting black holes achieves high precision and demonstrates the link between general relativity and geometry.

By Zhengwen Liu

- 1SF: Driesse-Jakobsen-Mogull-Plefka-Sauer-Usovitsch 2024 D-J-Klemm-M-Nega-P-S-U 2024
- 1SF: local-in-time conservative dynamics: Dlapa-Kälin-**ZL**-Porto 2506.20665

Outlook: precision frontier

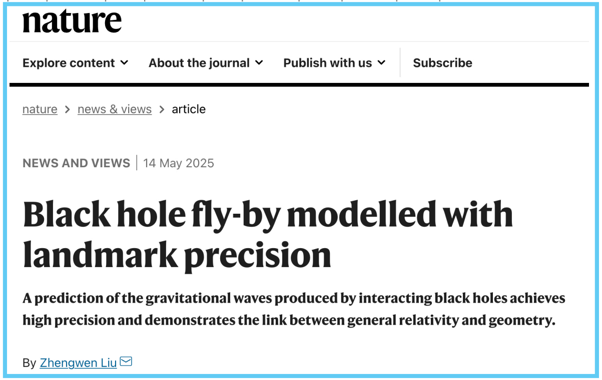
止於至善

GW science is just starting! discovery potential = precise theoretical predictions!

$$
\begin{array}{llllllllll}
& & \text{0PN} & \text{1PN} & \text{2PN} & \text{3PN} & \text{4PN} & \text{5PN} & \text{6PN} & \cdots \\
\text{1PM} & \left(\frac{GM}{r}\right)\times(& 1 & +\,v^2 & +\,v^4 & +\,v^6 & +\,v^8 & +\,v^{10} & +\,v^{12} & +\cdots) = \left(\frac{GM}{r}\right)\times(1) \\
\text{2PM} & \left(\frac{GM}{r}\right)^2\times(& & 1 & +\,v^2 & +\,v^4 & +\,v^6 & +\,v^8 & +\,v^{10} & +\cdots) = \left(\frac{GM}{r}\right)^2\times(1)
\end{array}
$$

$$
\begin{aligned}
\text{3PM} &\quad \left(\frac{GM}{r}\right)^3\times(1+\nu) \\
\text{4PM} &\quad \left(\frac{GM}{r}\right)^4\times(1+\nu) \\
\text{5PM} &\quad \left(\frac{GM}{r}\right)^5\times(1+\nu+\nu^2) \\
\text{6PM} &\quad \left(\frac{GM}{r}\right)^6\times(1+\nu+\nu^2) \\
\text{7PM} &\quad \left(\frac{GM}{r}\right)^7\times(1+\nu+\nu^2+\nu^3)
\end{aligned}
$$

0SF 1SF 2SF 3SF ⋯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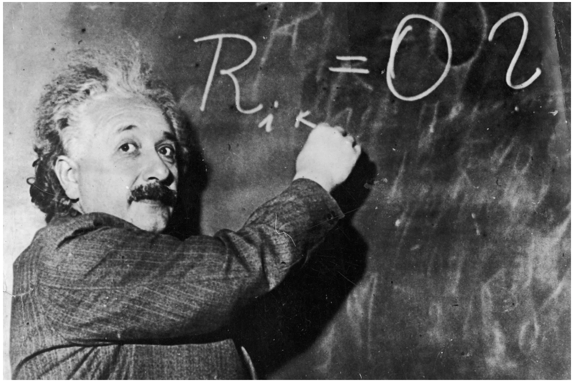
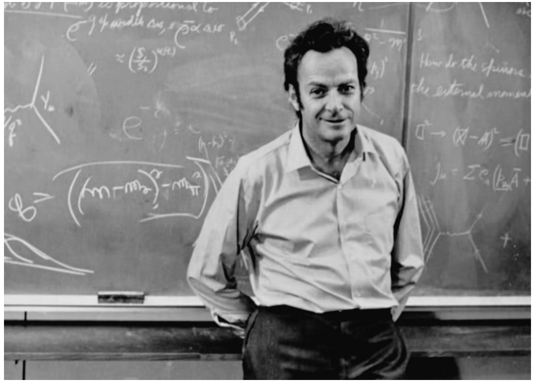
- 1SF: local-in-time conservative dynamics: Dlapa-Kälin-**ZL**-Porto 2506.20665
- 2SF: nonplanar diagram & more complicated functions

Feynman's integrals solve Einstein's equations!





谢谢！