# Scattering amplitudes for QCD, gravity and massive particles

arXiv:1802.06730 [hep-ph],
2111.06847, 2207.14597, 2312.14913 [hep-th]

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# Invitation

$n$-gluon MHV formula:

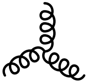
Parke, Taylor '86

$$A(1^-, 2^+, 3^-, 4^+, \ldots, n^+) = \frac{i\langle 1\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle \ldots \langle n\,1\rangle}$$

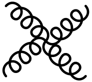
# Reminder*

QCD Feynman rules, color-stripped:

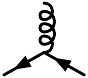
for color ordering see e.g. Dixon's 1995 TASI lectures



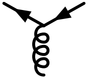

$$= \frac{i}{\sqrt{2}}\left[g^{\lambda\mu}(k-p)^{\nu} + g^{\mu\nu}(p-q)^{\lambda} + g^{\nu\lambda}(q-k)^{\mu}\right],$$




$$= i\,g^{\lambda\nu}g^{\mu\rho} - \frac{i}{2}\left(g^{\lambda\mu}g^{\nu\rho} + g^{\lambda\rho}g^{\mu\nu}\right),$$




$$= \frac{i}{\sqrt{2}}\gamma^{\mu},$$




$$= -\frac{i}{\sqrt{2}}\gamma^{\mu},$$


$$= -\frac{i g_{\mu\nu}}{p^2},$$


$$= \frac{i(\not{p}+m)}{p^2-m^2}.$$

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*Disclaimer: all momenta outgoing

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Parke, Taylor '86

$$A(1^-, 2^+, 3^-, 4^+, \dots, n^+) = \frac{i\langle 1\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle \dots \langle n\,1\rangle}$$

Simplification w.r.t. Feynman rules due to

- gauge invariance
- massless spinor-helicity variables

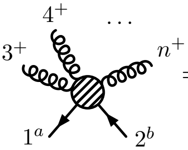
**This talk:**

- possible for **massive** quarks, higher-spin particles, etc!
- concentrate on 2 tree-level methods in 4d
  (loop methods also available and dimreg-compatible)

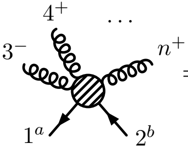
Bern, Dixon, Dunbar, Kosower; Britto, Cachazo, Feng; Forde; Badger;
Frellesvig, Peraro, Zhang; Abreu, Febres Cordero, Ita, Page, etc.

# Results with 2 quarks

AO '18




$$= \frac{i\,m\langle 1^a 2^b\rangle [3|\prod_{j=3}^{n-2}\{\not{P}_{13\dots j}\,\not{p}_{j+1} + (s_{13\dots j} - m^2)\}|n]}{(s_{13}-m^2)(s_{134}-m^2)\dots(s_{13\dots(n-1)}-m^2)\,\langle 34\rangle\langle 45\rangle\dots\langle n-1|n\rangle}$$




$$= -\frac{i\langle 3|1|2|3\rangle\,\big(\langle 1^a 3\rangle[2^b|1+2|3\rangle + \langle 2^b 3\rangle[1^a|1+2|3\rangle\big)}{s_{12}\langle 3\,4\rangle\dots\langle n-1|n\rangle\langle 3|1|1+2|n\rangle}$$

$$+\sum_{k=4}^{n-1}\frac{i\,m\langle 3|\not{p}_1\,\not{P}_{3\dots k}|3\rangle\,\big(\langle 1^a 2^b\rangle\langle 3|\not{p}_1\,\not{P}_{3\dots k}|3\rangle + \langle 1^a 3\rangle\langle 2^b 3\rangle s_{3\dots k}\big)}{s_{3\dots k}\,(s_{13\dots k}-m^2)\dots(s_{13\dots(n-1)}-m^2)\,\langle 3\,4\rangle\dots\langle k-1|k\rangle\langle 3|\not{p}_1\,\not{P}_{3\dots k}|k\rangle}$$

$$\times\frac{\langle 3|\not{P}_{3\dots k}\prod_{j=k}^{n-2}\{\not{P}_{13\dots j}\,\not{p}_{j+1} + (s_{13\dots j} - m^2)\}|n]}{\langle 3|\not{p}_1\,\not{P}_{3\dots k}|k+1\rangle\langle k+1|k+2\rangle\dots\langle n-1|n\rangle}$$

KK relations: $$A(1,\beta,2,\alpha) = (-1)^{|\beta|}\sum_{\sigma\in\alpha\sqcup\beta^T} A(1,2,\sigma)$$

Kleiss, Kuijf '88

# Outline

1. Massive spinor helicity
2. 4-pt Compton amplitude
3. $n$-pt amplitudes via BCFW
4. Massive higher spins
5. Rotating BHs

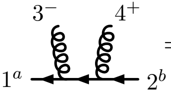
6. Summary & outlook

# Massive spinor helicity

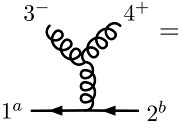
# Feynman-rules calculation of $A(\underline{1}^a, 3^-, 4^+, \overline{2}^b)$

Start at 4 pts, textbook way manageable

$A(\underline{1}^a, 3^-, 4^+, \overline{2}^b) \ni 2$ color-ordered diagrams:




$$= -\frac{i}{2(s_{13}-m^2)} \left(\bar{u}_1^a \not{\varepsilon}_3^- (\not{p}_{13}+m) \not{\varepsilon}_4^+ v_2^b\right)$$




$$= \frac{i}{2s_{34}} \Big\{ (\varepsilon_3^- \cdot \varepsilon_4^+)(\bar{u}_1^a (\not{p}_3 - \not{p}_4) v_2^b) + 2(p_4 \cdot \varepsilon_3^-)(\bar{u}_1^a \not{\varepsilon}_4^+ v_2^b) - 2(p_3 \cdot \varepsilon_4^+)(\bar{u}_1^a \not{\varepsilon}_3^- v_2^b) \Big\}$$

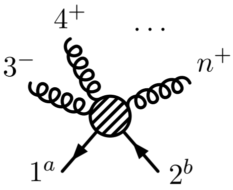
All done?

# Why spinor helicity?

Consider color-ordered QCD amplitude $A(\underline{1}^a, 3^-, 4^+, \dots, n^+, \overline{2}^b)$

Feynman rules give function of

- momenta $p_i^\mu$
- polarization vectors $\varepsilon_\pm^\mu(p_i)$
- external spinors $\bar{u}^a(p_1)$, $v^b(p_2)$



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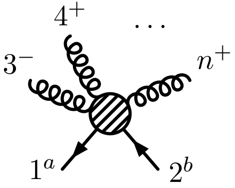


But all vector, spinor indices must be contracted

Remaining indices $\Leftrightarrow$ physical quantum numbers:

- helicities $\pm$ $\Leftrightarrow$ spins $\{\pm 1/2\}_p$, $\{\pm \mathbf{1}\}_{\mathbf{p}}$, etc.
- SU(2) labels $a, b$ $\Leftrightarrow$ spins $\{\pm \mathbf{1/2}\}_{\mathbf{q}}$, $\{\pm 1, 0\}_q$, etc.

Crucial on-shell notion — LITTLE GROUP

# Little groups

- Quantum fields $\Leftarrow$ reps of $\mathrm{SO}(1,3)$
- Quantum states $\Leftarrow$ reps of LITTLE GROUP
  - massless states $\Leftarrow$ $\mathrm{SO}(2)$
  - massive states $\Leftarrow$ $\mathrm{SO}(3)$

# Little groups

- Quantum fields $\Leftarrow$ reps of $\mathrm{SO}(1,3) \quad \subset \quad \mathrm{SL}(2,\mathbb{C})$
- Quantum states $\Leftarrow$ reps of LITTLE GROUP's dbl cover
  - massless states $\Leftarrow$ $\mathrm{SO}(2) \quad \subset \quad \mathbf{U(1)}$
  - massive states $\Leftarrow$ $\mathrm{SO}(3) \quad \subset \quad \mathbf{SU(2)}$

Minor complication: spinorial reps use groups' double covers

U(1) and SU(2) arise naturally in spinor helicity

# Spinor map

Basics of spinor helicity

- Minkowski space isomorphism:*

$$
\begin{aligned}
\mathrm{M}^{2\times 2,\mathbb{C}}_{\text{Hermitian}} &\leftrightarrow \mathbb{R}^{1,3} \\
p_{\alpha\dot\beta} = p_\mu \sigma^\mu_{\alpha\dot\beta} &= \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix} \\
\det\{p_{\alpha\dot\beta}\} &= m^2
\end{aligned}
$$

*$\sigma^0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\sigma^1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma^2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\sigma^3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

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$$\det\{p_{\alpha\dot\beta}\} = m^2$$

- Lorentz group homomorphism:

$$\mathrm{SL}(2,\mathbb{C}) \quad\to\quad \mathrm{SO}(1,3)$$
$$p_{\alpha\dot\delta} \to S_\alpha{}^\beta p_{\beta\dot\gamma}\left(S_\delta{}^\gamma\right)^* \quad\Rightarrow\quad p^\mu \to L^\mu{}_\nu p^\nu, \quad L^\mu{}_\nu = \frac{1}{2}\,\mathrm{tr}\left(\bar\sigma^\mu S \sigma_\nu S^\dagger\right)$$

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# Massless vs massive spinor helicity

Arkani-Hamed, Huang, Huang '17

| MASSLESS | MASSIVE |
|---|---|
| $\det\{p_{\alpha\dot\beta}\} = 0$ | $\det\{p_{\alpha\dot\beta}\} = m^2$ |
| $p_{\alpha\dot\beta} = \lambda_{p\alpha}\tilde\lambda_{p\dot\beta} \equiv \|p\rangle_\alpha [p\|_{\dot\beta}$ | $p_{\alpha\dot\beta} = \lambda_{p\alpha}^{\ a}\epsilon_{ab}\tilde\lambda_{p\dot\beta}^{\ b} \equiv \|p^a\rangle_\alpha [p_a\|_{\dot\beta}$ <br> $\det\{\lambda_{p\alpha}^{\ a}\} = \det\{\tilde\lambda_{p\dot\alpha}^{\ a}\} = m$ |
| $p^\mu = \frac{1}{2}\langle p\|\sigma^\mu\|p]$ | $p^\mu = \frac{1}{2}\langle p^a\|\sigma^\mu\|p_a]$ |
| $p_{\alpha\dot\beta}\tilde\lambda_p^{\dot\beta} = 0$ | $p_{\alpha\dot\beta}\tilde\lambda_p^{a\dot\beta} = m\lambda_{p\alpha}^{\ a}$ |
| $\langle p\, q\rangle = -\langle q\, p\rangle \quad \Rightarrow \quad \langle p\, p\rangle = 0$ <br> $[p\, q] = -[q\, p] \quad \Rightarrow \quad [p\, p] = 0$ <br> $\langle p\, q\rangle [q\, p] = 2p\cdot q$ | $\langle p^a q^b\rangle = -\langle q^b p^a\rangle$ e.g. $\langle p^a p^b\rangle = -m\epsilon^{ab}$ <br> $[p^a q^b] = -[q^b p^a]$ e.g. $[p^a p^b] = m\epsilon^{ab}$ <br> $\langle p^a q^b\rangle [q_b\, p_a] = 2p\cdot q$ |

# Wavefunctions from helicity spinors

$$\varepsilon^\mu_{p+} = \frac{1}{\sqrt{2}} \frac{\langle q|\sigma^\mu|p]}{\langle q\,p\rangle} \qquad\qquad \Rightarrow \qquad \begin{cases} \varepsilon^\pm_p \cdot p = \varepsilon^\pm_p \cdot q = 0 \\ \varepsilon^\mu_{p+}\varepsilon^\nu_{p-} + \varepsilon^\mu_{p-}\varepsilon^\nu_{p+} = -\eta^{\mu\nu} + \dfrac{p^\mu q^\nu + q^\mu p^\nu}{p\cdot q} \\ \varepsilon^{h_1}_p \cdot \varepsilon^{h_2}_p = -\delta^{h_1(-h_2)} \end{cases}$$

$$\varepsilon^\mu_{p-} = \frac{1}{\sqrt{2}} \frac{\langle p|\sigma^\mu|q]}{[p\,q]}$$

# Wavefunctions from helicity spinors

$$\varepsilon^\mu_{p+} = \frac{1}{\sqrt{2}}\frac{\langle q|\sigma^\mu|p]}{\langle q\,p\rangle} \qquad \varepsilon^\mu_{p-} = \frac{1}{\sqrt{2}}\frac{\langle p|\sigma^\mu|q]}{[p\,q]} \quad \Rightarrow \quad \begin{cases} \varepsilon^\pm_p\cdot p = \varepsilon^\pm_p\cdot q = 0 \\ \varepsilon^\mu_{p+}\varepsilon^\nu_{p-} + \varepsilon^\mu_{p-}\varepsilon^\nu_{p+} = -\eta^{\mu\nu} + \dfrac{p^\mu q^\nu + q^\mu p^\nu}{p\cdot q} \\ \varepsilon^{h_1}_p\cdot\varepsilon^{h_2}_p = -\delta^{h_1(-h_2)} \end{cases}$$

$$u^a_p = \begin{pmatrix} |p^a\rangle \\ |p^a] \end{pmatrix} \qquad \bar{u}^a_p = \begin{pmatrix} -\langle p^a| \\ [p^a| \end{pmatrix} \quad \Rightarrow \quad \begin{cases} (\not{p} - m)u^a_p = \bar{u}^a_p(\not{p} - m) = 0 \\ \bar{u}^a_p u^b_p = 2m\epsilon^{ab} \\ \bar{u}^a_p\gamma^\mu u^b_p = 2p^\mu\epsilon^{ab} \\ u^a_p\bar{u}_{pa} = u^a_p\epsilon_{ab}\bar{u}^b_p = \not{p} + m \end{cases}$$

$$v^a_p = \begin{pmatrix} -|p^a\rangle \\ |p^a] \end{pmatrix} \qquad \bar{v}^a_p = \begin{pmatrix} \langle p^a| \\ [p^a| \end{pmatrix} \quad \Rightarrow \quad \begin{cases} (\not{p} + m)v^a_p = \bar{v}^a_p(\not{p} + m) = 0 \\ \bar{v}^a_p v^b_p = 2m\epsilon^{ab} \\ \bar{v}^a_p\gamma^\mu v^b_p = -2p^\mu\epsilon^{ab} \\ v^a_p\bar{v}_{pa} = v^a_p\epsilon_{ab}\bar{v}^b_p = -\not{p} + m \end{cases}$$

# Little-group transformations

Consider Lorentz transformation $p^\mu \to L^\mu{}_\nu p^\nu$

Massless:

$$|p\rangle \to S|p\rangle = e^{i\phi/2}|Lp\rangle \qquad \langle p| \to \langle p|S^{-1} = e^{i\phi/2}\langle Lp|$$

$$|p] \to S^{\dagger\,-1}|p] = e^{-i\phi/2}|Lp] \qquad [p| \to [p|S^{\dagger} = e^{-i\phi/2}[Lp|$$

$\Rightarrow\ \varepsilon_p^\pm \to L\varepsilon_p^\pm \sim e^{\mp i\phi}\varepsilon_{Lp}^\pm$

$e^{ih\phi} \in \mathrm{U}(1)$ encode $2d$ rotations in frame where $p = (E, 0, 0, E)$

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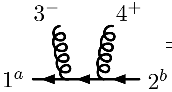
Massive:

$$|p^a\rangle \to S|p^a\rangle = \omega^a{}_b|Lp^b\rangle \qquad |p^a\rangle \to |p^a\rangle S^{-1} = \omega^a{}_b|Lp^a\rangle$$
$$|p^a] \to S^{\dagger\,-1}|p^a] = \omega^a{}_b[Lp^b| \qquad [p^a| \to [p^a|S^\dagger = \omega^a{}_b[Lp^b|$$

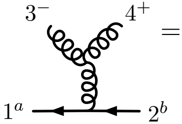
$\omega \in \mathrm{SU}(2)$ encode $3d$ rotations in rest frame where $p = (m, 0, 0, 0)$

# 4-pt Compton amplitude

# Feynman-rules calculation of $A(\underline{1}^a, 3^-, 4^+, \overline{2}^b)$




$$= -\frac{i}{2(s_{13}-m^2)}\,(\bar{u}_1^a\,\not{\varepsilon}_3^-\,(\not{p}_{13}+m)\,\not{\varepsilon}_4^+\,v_2^b)$$




$$= \frac{i}{2s_{34}}\Big\{(\varepsilon_3^-\cdot\varepsilon_4^+)(\bar{u}_1^a(\not{p}_3-\not{p}_4)v_2^b) + 2(p_4\cdot\varepsilon_3^-)(\bar{u}_1^a\,\not{\varepsilon}_4^+\,v_2^b) - 2(p_3\cdot\varepsilon_4^+)(\bar{u}_1^a\,\not{\varepsilon}_3^-\,v_2^b)\Big\}$$

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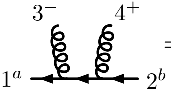
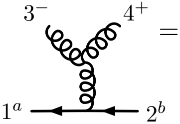
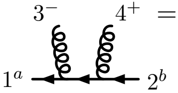



$$= -\frac{i}{2(s_{13}-m^2)} \left(\bar{u}_1^a \not{\varepsilon}_3^- (\not{p}_{13}+m) \not{\varepsilon}_4^+ v_2^b\right)$$

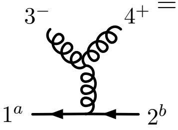



$$= \frac{i}{2s_{34}} \Big\{ (\varepsilon_3^- \cdot \varepsilon_4^+)(\bar{u}_1^a (\not{p}_3 - \not{p}_4) v_2^b) + 2(p_4 \cdot \varepsilon_3^-)(\bar{u}_1^a \not{\varepsilon}_4^+ v_2^b) - 2(p_3 \cdot \varepsilon_4^+)(\bar{u}_1^a \not{\varepsilon}_3^- v_2^b) \Big\}$$

- plug in external wavefunctions:

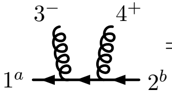



$$= \frac{-i}{(s_{13}-m^2)[3\,q_3]\langle 4\,q_4\rangle} \Big\{ \langle 1^a 3\rangle [q_3|p_{13}|q_4\rangle [4\,2^b] + [1^a q_3]\langle 3|p_{13}|4]\langle q_4 2^b\rangle - m\langle 1^a 3\rangle [q_3 4]\langle q_4 2^b\rangle - m[1^a q_3]\langle 3\,q_4\rangle [4\,2^b] \Big\}$$

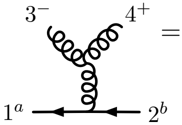



$$= \frac{-i}{s_{34}[3\,q_3]\langle 4\,q_4\rangle} \Big\{ -\frac{1}{2}\langle 3\,q_4\rangle [4\,q_3] \big( \langle 1^a|p_3-p_4|2^b] + [1^a|p_3-p_4|2^b\rangle \big) - \langle 3|4|q_3] \big( \langle 1^a q_4\rangle [4\,2^b] + [1^a 4]\langle q_4 2^b\rangle \big) + \langle q_4|3|4] \big( \langle 1^a 3\rangle [q_3 2^b] + [1^a q_3]\langle 3\,2^b\rangle \big) \Big\}$$

# Feynman-rules calculation of $A(\underline{1}^a, 3^-, 4^+, \overline{2}^b)$

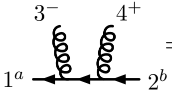



$$= -\frac{i}{2(s_{13}-m^2)}\left(\bar u_1^a\, \not\varepsilon_3^-\,(\not p_{13}+m)\,\not\varepsilon_4^+\, v_2^b\right)$$

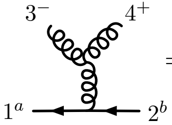



$$= \frac{i}{2s_{34}}\Big\{(\varepsilon_3^-\cdot\varepsilon_4^+)(\bar u_1^a(\not p_3 - \not p_4)v_2^b) + 2(p_4\cdot\varepsilon_3^-)(\bar u_1^a\,\not\varepsilon_4^+\, v_2^b) - 2(p_3\cdot\varepsilon_4^+)(\bar u_1^a\,\not\varepsilon_3^-\, v_2^b)\Big\}$$

- plug in external wavefunctions with $q_3 = p_4$, $q_4 = p_3$:




$$= \frac{i\langle 3|1|4]}{(s_{13}-m^2)s_{34}}\big(\langle 1^a 3\rangle[2^b 4] + [1^a 4]\langle 2^b 3\rangle\big)$$

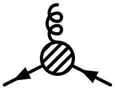



$$= 0$$

- spinor helicity helps **regardless of method**

# 3-pt amplitudes

Modern methods require **on-shell 3-pt input only**

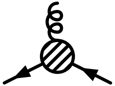



$$\mathcal{A}(1_i^a, 2_{\bar{\jmath}}^b, 3_c^+) = -\frac{iT^c_{i\bar{\jmath}}}{\langle 3\, q\rangle}\big(\langle 1^a q\rangle[2^b 3] + [1^a 3]\langle 2^b q\rangle\big) = -iT^c_{i\bar{\jmath}}\frac{\langle 1^a 2^b\rangle[3|1|q\rangle}{m\langle 3\, q\rangle}$$

$$\mathcal{A}(1_i^a, 2_{\bar{\jmath}}^b, 3_c^-) = \frac{iT^c_{i\bar{\jmath}}}{[3\, q]}\big(\langle 1^a 3\rangle[2^b q] + [1^a q]\langle 2^b 3\rangle\big) = iT^c_{i\bar{\jmath}}\frac{[1^a 2^b]\langle 3|1|q]}{m[3\, q]}$$

# 3-pt amplitudes

Modern methods require **on-shell 3-pt input only**

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$$\mathcal{A}(1_i^a, 2_{\bar{\jmath}}^b, 3_c^-) = \frac{iT^c_{i\bar{\jmath}}}{[3\, q]}\big(\langle 1^a 3\rangle[2^b q] + [1^a q]\langle 2^b 3\rangle\big) = iT^c_{i\bar{\jmath}}\frac{[1^a 2^b]\langle 3|1|q]}{m[3\, q]}$$

NB! Independent of ref. momentum $q$

$$p_2^2 - m^2 = \langle 3|1|3] = 0 \qquad \Rightarrow \qquad \exists\, x_3 \in \mathbb{C} : |1|3] = -m x_3 |3\rangle$$

$$\Rightarrow \qquad x_3 = \frac{[3|1|q\rangle}{m\langle 3\, q\rangle} \quad \text{indep. of } q$$

# BCFW calculation of $A(\underline{1}^a, 3^-, 4^+, \overline{2}^b)$

$$\text{BCFW shift: } \begin{cases} |3] \to |\hat{3}] = |3] - z|4] \\ |4\rangle \to |\hat{4}\rangle = |4\rangle + z|3\rangle \end{cases} \qquad \Rightarrow \qquad \mathcal{A} \to \mathcal{A}(z)$$

Britto, Cachazo, Feng, Witten '05

$$\text{Residue thm: } 0 = \oint \frac{\mathrm{d}z}{2\pi i} \frac{\mathcal{A}(z)}{z} = \mathcal{A}(0) + \sum_{\text{poles of } \mathcal{A}(z)} \frac{1}{z_{\mathrm{p}}} \operatorname*{Res}_{z=z_{\mathrm{p}}} \mathcal{A}(z)$$

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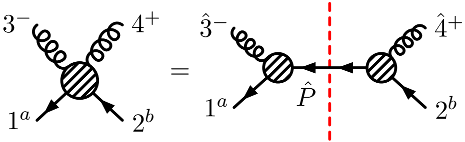


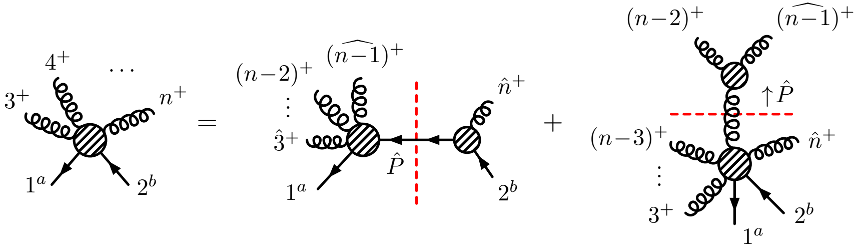

$$= \operatorname*{Res}_{z=z_{13}} A(\underline{1}^a, \hat{3}^-, \hat{4}^+, \overline{2}^b) = A(\underline{1}^a, \hat{3}^-, -\hat{P}^c)\frac{i}{s_{13}-m^2}A(\hat{P}_c, \hat{4}^+, \overline{2}^b)$$

$$= \frac{-i}{(s_{13}-m^2)[3\,4]\langle 4\,3\rangle}\big(\langle 1^a 3\rangle[4\hat{P}^c] - [1^a 4]\langle 3\hat{P}^c\rangle\big)\big(\langle \hat{P}_c\, 3\rangle[2^b 4] + [\hat{P}_c\, 4]\langle 2^b 3\rangle\big)$$

$$= \frac{i\langle 3|1|4]}{(s_{13}-m^2)s_{34}}\big(\langle 1^a 3\rangle[2^b 4] + [1^a 4]\langle 2^b 3\rangle\big)$$

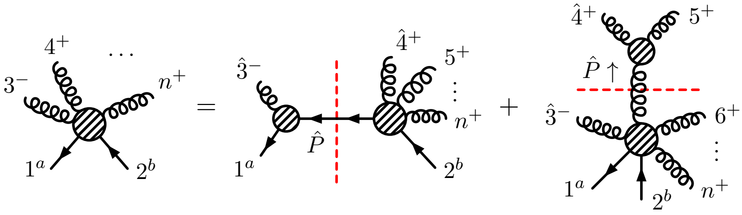
# $n$-pt amplitudes via BCFW

# BCFW recursion for $A(\underline{1}^a, 3^+, \ldots, n^+, \overline{2}^b)$




$$= \ldots = \frac{i\, m \langle 1^a 2^b \rangle [3| \prod_{j=3}^{n-2} \{ \not{P}_{13\ldots j} \not{p}_{j+1} + (s_{13\ldots j} - m^2) \} |n]}{(s_{13} - m^2)(s_{134} - m^2) \ldots (s_{13\ldots(n-1)} - m^2) \, \langle 34 \rangle \langle 45 \rangle \ldots \langle n-1|n \rangle}$$

# BCFW recursion for $A(\underline{1}^a, 3^-, 4^+, \ldots, n^+, \overline{2}^b)$




$$= \ldots = -\frac{i\langle 3|1|2|3\rangle\left(\langle 1^a 3\rangle[2^b|1{+}2|3\rangle + \langle 2^b 3\rangle[1^a|1{+}2|3\rangle\right)}{s_{12}\langle 3\,4\rangle\ldots\langle n{-}1|n\rangle\langle 3|1|1{+}2|n\rangle}$$

$$+\sum_{k=4}^{n-1}\frac{i\,m\langle 3|\not{p}_1\,\not{P}_{3\ldots k}|3\rangle\left(\langle 1^a 2^b\rangle\langle 3|\not{p}_1\,\not{P}_{3\ldots k}|3\rangle + \langle 1^a 3\rangle\langle 2^b 3\rangle s_{3\ldots k}\right)}{s_{3\ldots k}\,(s_{13\ldots k}{-}m^2)\ldots(s_{13\ldots(n-1)}{-}m^2)\,\langle 3\,4\rangle\ldots\langle k{-}1|k\rangle\langle 3|\not{p}_1\,\not{P}_{3\ldots k}|k\rangle}$$

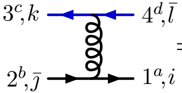
$$\times\frac{\langle 3|\not{P}_{3\ldots k}\prod_{j=k}^{n-2}\{\not{P}_{13\ldots j}\,\not{p}_{j+1} + (s_{13\ldots j} - m^2)\}|n]}{\langle 3|\not{p}_1\,\not{P}_{3\ldots k}|k{+}1\rangle\langle k{+}1|k{+}2\rangle\ldots\langle n{-}1|n\rangle}$$

# Four-quark amplitudes

Lazopoulos, AO, Shi '21

$$\mathcal{A}(\underline{1},\overline{2},\underline{3},\overline{4},5,\ldots,n) = \mathcal{A}(\underline{1},\overline{2},\underline{3},\overline{4},5,\ldots,n) - \mathcal{A}(\underline{1},\overline{2},\underline{3},\overline{4},5,\ldots,n)$$

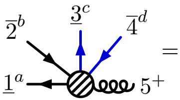
identical flavors from distinct flavors




$$= -\frac{iT^a_{i\bar{\jmath}}T^a_{k\bar{l}}}{s_{12}}\big(\langle 1^a 4^d\rangle[2^b 3^c] + [1^a 4^d]\langle 2^b 3^c\rangle + \langle 1^a 3^c\rangle[2^b 4^d] + [1^a 3^c]\langle 2^b 4^d\rangle\big)$$

Color ordering from adjoint rep.: $T^a_{i\bar{\jmath}} \to \tilde{f}^{ia\bar{\jmath}}$

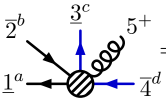
# Adding gluons to four-quark amplitudes




$$= i\Bigg\{ \frac{[1^a5]\langle 2^b3^c\rangle[4^d5] + [1^a5]\langle 2^b4^d\rangle[3^c5]}{(s_{15}-m_1^2)s_{34}} + \frac{[1^a5]\langle 2^b3^c\rangle[4^d5] + \langle 1^a3^c\rangle[2^b5][4^d5]}{s_{12}(s_{45}-m_3^2)}$$

$$+ \frac{\langle 1^a4^d\rangle[2^b5][3^c5] + [1^a5]\langle 2^b3^c\rangle[4^d5] + \langle 1^a3^c\rangle[2^b5][4^d5] + [1^a5]\langle 2^b4^d\rangle[3^c5]}{s_{12}s_{34}}$$

$$+ \frac{s_{12}[5|1|4|5] - (s_{15}-m_1^2)[5|3|4|5]}{s_{12}s_{34}(s_{15}-m_1^2)(s_{45}-m_3^2)} \big(\langle 1^a4^d\rangle[2^b3^c] + [1^a4^d]\langle 2^b3^c\rangle + \langle 1^a3^c\rangle[2^b4^d] + [1^a3^c]\langle 2^b4^d\rangle\big)\Bigg\}$$




$$= i\Bigg\{ \frac{\langle 1^a4^d\rangle[2^b5][3^c5] + [1^a5]\langle 2^b4^d\rangle[3^c5]}{s_{12}(s_{35}-m_3^2)} - \frac{\langle 1^a3^c\rangle[2^b5][4^d5] + [1^a5]\langle 2^b3^c\rangle[4^d5]}{s_{12}(s_{45}-m_3^2)}$$

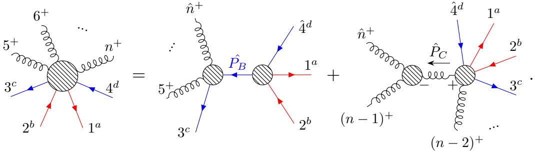
$$- \frac{[5|3|4|5]}{s_{12}(s_{35}-m_3^2)(s_{45}-m_3^2)} \big(\langle 1^a4^d\rangle[2^b3^c] + [1^a4^d]\langle 2^b3^c\rangle + \langle 1^a3^c\rangle[2^b4^d] + [1^a3^c]\langle 2^b4^d\rangle\big)\Bigg\}$$

Recall: fixed quarks 1 and 2 together by KK relations
Moreover, quark 3 maybe be locked in using (gen.) BCJ relations

Bern, Carrasco, Johansson '08; Johansson, AO '15

# Four-quark amplitudes with plus-helicity gluons

Lazopoulos, AO, Shi '21




$$A(1^a,2^b,3^c,5^+,\dots,n^+,4^d) = \frac{-i}{s_{12}\prod_{j=5}^{n-1}\langle j|j+1\rangle}\Bigg\{\frac{1}{\prod_{j=5}^{n} D_{4j\dots n}\langle 5|3|d_5^n]}$$

$$\times\Big[\langle 14\rangle[23][d_5^n|3|P_{45\dots n}|d_5^n] - \langle 14\rangle[2|d_5^n][3|d_5^n]D_{45\dots n} + m\langle 13\rangle[2|d_5^n]\langle 4|3|d_5^n] \qquad (3.10a)$$

$$+\langle 13\rangle[2|P_{45\dots n}|3|d_5^n]\Big([4n]\prod_{j=5}^{n-1} D_{4j\dots n} + m\sum_{i=5}^{n-1}\langle 4|i|d_{i+1}^n]\prod_{j=5}^{i-1} D_{4j\dots n}\Big)\Big]$$

$$+\sum_{i=6}^{n}\frac{m\langle i-1|i\rangle[c_{i-1}^5|d_i^n]}{\prod_{j=5}^{i-1} D_{35\dots j}\prod_{j=i+1}^{n} D_{4j\dots n}\big(D_{35\dots(i-1)}\langle i-1|P_{4i\dots n}|d_i^n] + D_{4i\dots n}\langle i-1|P_{35\dots(i-1)}|d_i^n]\big)}$$

$$\times\Big[\frac{\langle 1|P_{4i\dots n}|d_i^n][2|d_i^n]\langle 34\rangle}{\langle i|P_{4i\dots n}|d_{i+1}^n]} + \frac{[d_i^n|P_{12}|P_{4i\dots n}|d_i^n]}{D_{35\dots i}\langle i|P_{4(i+1)\dots n}|d_{i+1}^n] + D_{4(i+1)\dots n}\langle i|P_{35\dots i}|d_{i+1}^n]} \qquad (3.10b)$$

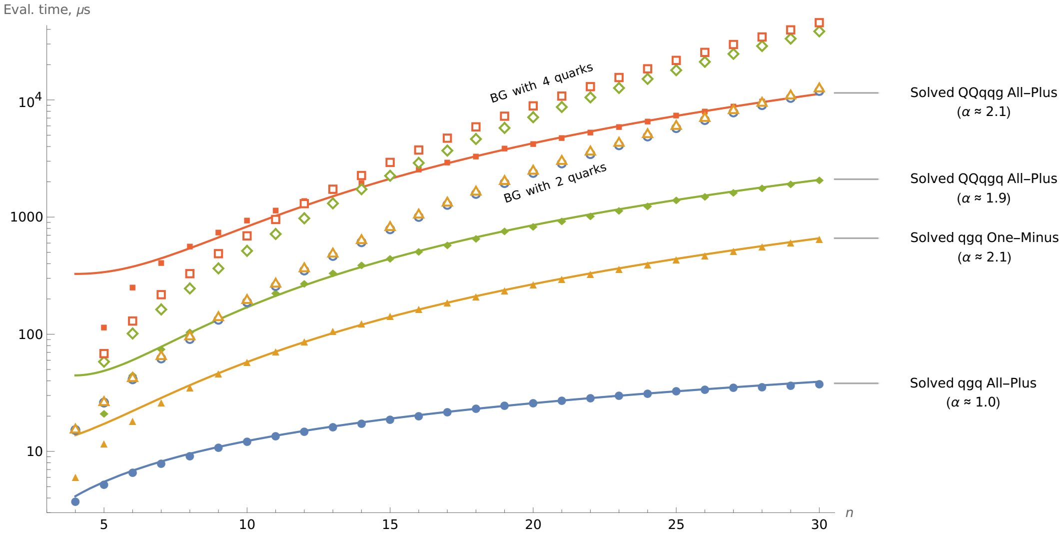
$$\times\Big(\langle 14\rangle[2|P_{12}|3\rangle + \langle 1|P_{4i\dots n}|2]\langle 34\rangle + \frac{\langle 1|i\rangle[2|d_i^n]\langle 34\rangle}{\langle i|P_{4i\dots n}|d_{i+1}^n]}\Big)\Big]\Bigg\}$$

$$+(1\leftrightarrow 2).$$

- gluon insertions between like-flavored quarks $\uparrow$ ✓
- gluon insertions between distinctly flavored quarks ✓

# Formulae against off-shell recursion

Lazopoulos, AO, Shi '21




Num. eval. times against in-house impl. of **off-shell** BG recursion

Berends, Giele '87

# Massive higher spins

# 2-matter all-plus amplitudes

- 2 scalars + $(n-2)$ gluons in gauge theory: Ferrario, Rodrigo, Talavera '06

$$A(1,2,3^+,\dots,n^+) = \frac{im^2[3|\prod_{j=3}^{n-2}\{\not{P}_{13\dots j}\not{p}_{j+1}+(s_{13\dots j}-m^2)\}|n]}{\prod_{j=3}^{n-1}\langle j|j+1\rangle(s_{13\dots j}-m^2)}$$

- 2 quarks + $(n-2)$ gluons in gauge theory: AO '18

$$A(1_a,2^b,3^+,\dots,n^+) = \frac{im\langle 1_a 2^b\rangle[3|\prod_{j=3}^{n-2}\{\not{P}_{13\dots j}\not{p}_{j+1}+(s_{13\dots j}-m^2)\}|n]}{\prod_{j=3}^{n-1}\langle j|j+1\rangle(s_{13\dots,j}-m^2)}$$

- 2 massive higher spins + $(n-2)$ gluons in gauge theory: Lazopoulos, AO, Shi '21

$$A(1_{\{a\}},2^{\{b\}},3^+,\dots,n^+) = \frac{i\langle 1_a 2^b\rangle^{\odot 2s}[3|\prod_{j=3}^{n-2}\{\not{P}_{13\dots j}\not{p}_{j+1}+(s_{13\dots j}-m^2)\}|n]}{m^{2s-2}\prod_{j=3}^{n-1}\langle j|j+1\rangle(s_{13\dots,j}-m^2)}$$

- in other words:

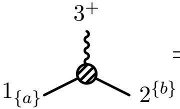
$$A(1_{\{a\}},2^{\{b\}},3^+,\dots,n^+) = \frac{\langle 1_a 2^b\rangle^{\odot 2s}}{m^{2s}}A(1,2,3^+,\dots,n^+)$$

- fails for other helicity configurations already in QCD!

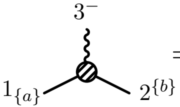
# AHH amplitudes & black holes

3-pt amplitudes singled out (and misnamed as “minimal coupling”):

Arkani-Hamed, Huang, Huang ’17




$$= \frac{\langle 1_a 2^b \rangle^{\odot 2s}}{m^{2s}} \mathcal{M}_3^{(0,+)}$$




$$= \frac{[1_a 2^b]^{\odot 2s}}{m^{2s}} \mathcal{M}_3^{(0,-)}$$

$$\xrightarrow[\text{class. limit}]{} \mathcal{M}_3^{(0,\pm)} \exp\left( \mp \frac{p_3 \cdot S}{m} \right)$$

Guevara, AO, Vines ’18, ’19

Kerr’s spin exp. from Newman-Janis shift, e.g. Arkani-Hamed, Huang, O’Connell ’19

- high interest e.g. in view of GW applications
  - higher-point obstacles (unphysical pole at 4 pts, identifying BH vs NS etc., NLO grav. interactions of Kerr)
- multitude of genuine higher-spin theories still to discover

# Symmetric tensors need transversality

Standard (non-chiral) choice — sym. traceless tensors $\Phi_{\mu_1\dots\mu_s}$

Recall Lorentz group homomorphism:

$$\mathrm{SL}(2,\mathbb{C}) \quad \to \quad \mathrm{SO}(1,3)$$

$$\underbrace{V_\mu \sigma^\mu{}_{\alpha\dot\beta} =: V_{\alpha\dot\beta}}_{\text{SPINOR MAP}} \to S_\alpha{}^\gamma V_{\gamma\dot\delta} \left(S_\beta{}^\delta\right)^* \;\Rightarrow\; V^\mu \to L^\mu{}_\nu V^\nu, \quad L^\mu{}_\nu = \frac{1}{2}\,\mathrm{tr}\left(\bar\sigma^\mu S \sigma_\nu S^\dagger\right)$$

Also: $$\Phi_{\alpha_1\dots\alpha_s\dot\beta_1\dots\dot\beta_s} := \Phi_{\mu_1\dots\mu_s}\sigma^{\mu_1}_{\alpha_1\dot\beta_1}\cdots\sigma^{\mu_s}_{\alpha_s\dot\beta_s}$$

$\Rightarrow$ denote as $(s,s)$ rep. of Lorentz group $\mathrm{SL}(2,\mathbb{C})$

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$\Rightarrow$ denote as $(s,s)$ rep. of Lorentz group $\mathrm{SL}(2,\mathbb{C})$

**Problem:** too many DOFs!

i.e. highly reducible under Wigner's little group $\mathrm{SU}(2)\subset\mathrm{SL}(2,\mathbb{C})$

(decomp. into sym. SU(2) tensors of rank $0,2,\ldots,2s$)

$\Rightarrow$ transversality constraint for irreducibility (also for energy positivity):

$$(\partial^2+m^2)\Phi_{\mu_1\ldots\mu_s}=0, \qquad \partial^\mu\Phi_{\mu\mu_2\ldots\mu_s}=0$$

indeed, # of DOFs: $$\underbrace{\tfrac{1}{2}(s+2)(s+1)}_{\underline{3d}\text{ sym. tensor components}} - \underbrace{\tfrac{1}{2}s(s-1)}_{\text{traces}} = 2s+1$$

# Auxiliary fields & massive gauge invariance

Lagrangian descr. of constr. eqns $(\partial^2 + m^2)\Phi_{\mu_1\dots\mu_s} = 0$, $\partial^\mu \Phi_{\mu\mu_2\dots\mu_s} = 0$ requires aux. fields; originally:

Fierz, Pauli '39; Singh, Hagen '74

$$\text{sym. traceless } \Phi_{\mu_1\dots\mu_s}, \underbrace{\Phi_{\mu_1\dots\mu_{s-2}}, \Phi_{\mu_1\dots\mu_{s-3}}, \dots, \Phi_\mu, \Phi}_{\text{auxiliary}}$$

More recently:

Zinoviev '01

$$\text{sym. double-traceless } \Phi_{\mu_1\dots\mu_s}, \underbrace{\Phi_{\mu_1\dots\mu_{s-1}}, \Phi_{\mu_1\dots\mu_{s-2}}, \dots, \Phi_\mu, \Phi}_{\text{auxiliary}}$$

$$\begin{cases} \delta\Phi_{\mu_1\dots\mu_s} = s\partial_{(\mu_1}\xi_{\mu_2\dots\mu_s)} + \#m\eta_{(\mu_1\mu_2}\xi_{\mu_3\dots\mu_s)} + \dots \\ \delta\Phi_{\mu_1\dots\mu_{s-1}} = \partial_{(\mu_1}\xi_{\mu_2\dots\mu_{s-1})} + m\xi_{\mu_1\dots\mu_{s-1}} + \#m\eta_{(\mu_1\mu_2}\xi_{\mu_3\dots\mu_{s-1})} + \dots \\ \vdots \\ \delta\Phi_{\mu\nu} = \partial_{(\mu}\xi_{\nu)} + m\xi_{\mu\nu} + \#m\eta_{\mu\nu}\xi + \dots \\ \delta\Phi_\mu = \partial_\mu\xi + m\xi_\mu + \dots \\ \delta\Phi = m\xi, \end{cases} \qquad \Phi^{\lambda\mu}{}_{\lambda\mu\mu_5\dots\mu_k} = \xi^\mu{}_{\mu\mu_3\dots\mu_k} = 0$$

- 1st-class constraints instead of 2nd-class (more aux. fields streamline analysis)
- systematic introduction of healthy interactions
- still highly non-trivial, order by order
- no (flat-space) results beyond cubic level (trivalent vertices)

until Cangemi, Chiodaroli, Johansson, AO, Pichini, Skvortsov '22

# Free massive higher-spin theory

AO, Skvortsov '22

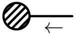
Start chiral, no need to remove DOFs!

$$\underline{\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \Phi^{\alpha_1\dots\alpha_{2s}})(\partial^\mu \Phi_{\alpha_1\dots\alpha_{2s}}) - \frac{m^2}{2}\Phi^{\alpha_1\dots\alpha_{2s}}\Phi_{\alpha_1\dots\alpha_{2s}}}$$

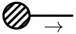
Free field expansion for KFG eqn:

$$\Phi_{\alpha_1\dots\alpha_{2s}}(x) = \int \frac{\hat{d}^3 p}{2p^0}\left[\frac{|p^{(a_1}\rangle_{\alpha_1}\cdots|p^{a_{2s})}\rangle_{\alpha_{2s}}}{m^s}\, a_{a_1\dots a_{2s}}(\vec{p})\, e^{-ip\cdot x} + (-1)^{2s}\frac{|p_{(a_1}\rangle_{\alpha_1}\cdots|p_{a_{2s})}\rangle_{\alpha_{2s}}}{m^s}\, a^{\dagger a_1\dots a_{2s}}(\vec{p})\, e^{ip\cdot x}\right]\Bigg|_{p^0=\sqrt{\vec{p}^2+m^2}}$$

Massive spinor helicity ideal for ext. wavefunctions!



$$p, a_1, \dots, a_{2s} = \frac{1}{m^s}|p^{(a_1}\rangle_{\alpha_1}\cdots|p^{a_{2s})}\rangle_{\alpha_{2s}}$$



$$p, a_1, \dots, a_{2s} = \frac{(-1)^{2s}}{m^s}|p_{(a_1}\rangle_{\alpha_1}\cdots|p_{a_{2s})}\rangle_{\alpha_{2s}}$$

$\Phi_{\{\alpha_1\dots\alpha_{2s}\}}$  $\Phi^{\{\beta_1\dots\beta_{2s}\}}$

$$p \qquad = \frac{i\delta^{(\beta_1}_{\alpha_1}\cdots\delta^{\beta_{2s})}_{\alpha_{2s}}}{p^2 - m^2}$$

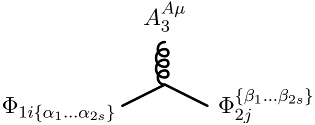
chiral fields for $s \leq 1$: Chalmers, Siegel '97, '01
$s \leq 2$: Delplanque, Skvortsov '24

# Gauge interactions

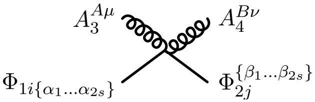
AO, Skvortsov '22

$$\mathcal{L}_g = \frac{1}{2}(D_\mu \Phi^{\{\alpha\}})_i (D^\mu \Phi_{\{\alpha\}})_i - \frac{m^2}{2}\Phi_i^{\{\alpha\}}\Phi_{i\{\alpha\}} - \frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu},$$

$$D_\mu := \partial_\mu + gA_\mu, \quad A_\mu = A_\mu^A t^A, \quad [t^A, t^B] = f^{ABC}t^C, \quad t^A_{ij} = -t^A_{ji}$$




$$= g t^A_{ij}\delta^{(\beta_1}_{\alpha_1}\cdots\delta^{\beta_{2s})}_{\alpha_{2s}}(p_1 - p_2)^\mu$$




$$= -ig^2[t^A_{ik}t^B_{kj} + t^B_{ik}t^A_{kj}]\delta^{(\beta_1}_{\alpha_1}\cdots\delta^{\beta_{2s})}_{\alpha_{2s}}\eta^{\mu\nu}$$

- Amplitudes are simply (not only all-plus!):

$$\mathcal{A}(1_{\{a\}}, 2^{\{b\}}, 3^{h_3}, \ldots, n^{h_n}) = \frac{\langle 1_a 2^b\rangle^{\odot 2s}}{m^{2s}}\mathcal{A}(1, 2, 3^{h_3}, \ldots, n^{h_n})$$

$$\mathcal{A}(1_{\{a\}}, 2^{\{b\}}, 3_{\{c\}}, 4^{\{d\}}, 5^{h_5}, \ldots, n^{h_n}) = \frac{\langle 1_a 2^b\rangle^{\odot 2s}\langle 3_c 4^d\rangle^{\odot 2s}}{m^{4s}}\mathcal{A}(1, 2, 3, 4, 5^{h_5}, \ldots, n^{h_n})$$

$$+(-1)^{2s}\frac{\langle 1_a 4^d\rangle^{\odot 2s}\langle 3_c 2^b\rangle^{\odot 2s}}{m^{4s}}\mathcal{A}(1, 4, 3, 2, 5^{h_5}, \ldots, n^{h_n})$$

# Electromagnetic interactions

AO, Skvortsov '22

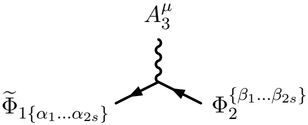
Restrict to SO(2): $\quad f^{ABC} = 0, \quad t_{ij}^{A=1} = \epsilon^{ij}$

$$\Phi^{\{\alpha\}} := \Phi^{\{\alpha\}}_{j=1} + i\Phi^{\{\alpha\}}_{j=2}$$

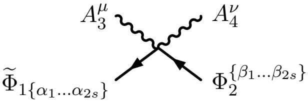
$$\widetilde{\Phi}^{\{\alpha\}} := \Phi^{\{\alpha\}}_{j=1} - i\Phi^{\{\alpha\}}_{j=2}$$

$$D_\mu := \partial_\mu - iQA_\mu$$

$$\mathcal{L}_Q = (\widetilde{D_\mu \Phi^{\{\alpha\}}})(D^\mu \Phi_{\{\alpha\}}) - m^2 \widetilde{\Phi}^{\{\alpha\}} \Phi_{\{\alpha\}} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$




$$= iQ\delta^{(\beta_1}_{\alpha_1} \cdots \delta^{\beta_{2s})}_{\alpha_{2s}} (p_2 - p_1)^\mu$$




$$= 2iQ^2 \delta^{(\beta_1}_{\alpha_1} \cdots \delta^{\beta_{2s})}_{\alpha_{2s}} \eta^{\mu\nu}$$

# Gravitational interactions

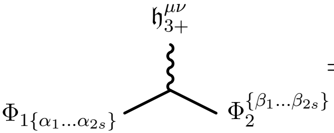
AO, Skvortsov '22

$$\underline{\mathcal{L}_G = \sqrt{-g}\Big\{\frac{1}{2}(\nabla_\mu \Phi^{\{\alpha\}})(\nabla^\mu \Phi_{\{\alpha\}}) - \frac{m^2}{2}\Phi^{\{\alpha\}}\Phi_{\{\alpha\}} + R\Big\}},$$

$$\nabla_\mu \Phi_{\alpha_1 \ldots \alpha_{2s}} = \partial_\mu \Phi_{\alpha_1 \ldots \alpha_{2s}} + 2s\, \omega_{\mu,(\alpha_1}{}^{\beta} \Phi_{\alpha_2 \ldots \alpha_{2s})\beta}$$

Anti-self-dual spin connection $\omega_{\mu,\alpha}{}^{\beta} := \frac{1}{4}\omega_\mu{}^{\hat{\nu}\hat{\rho}} \sigma_{\hat{\nu},\alpha\dot{\gamma}} \bar{\sigma}_{\hat{\rho}}^{\dot{\gamma}\beta} \underset{\text{SDGR}}{\Rightarrow} 0$ e.g. Penrose '76

$\Rightarrow$




$$= i\kappa \delta^{(\beta_1}_{\alpha_1} \cdots \delta^{\beta_{2s})}_{\alpha_{2s}} \Big[p_1^{(\mu} p_2^{\nu)} + \frac{m^2}{2}\eta^{\mu\nu}\Big], \quad \text{etc.}$$

- All-plus amplitudes satisfy

$$\mathcal{M}(1_{\{a\}}, 2^{\{b\}}, 3^+, \ldots, n^+) = \frac{\langle 1_a 2^b \rangle^{\odot 2s}}{m^{2s}} \mathcal{M}(1, 2, 3^+, \ldots, n^+)$$

# Restoring parity

- chiral-field Lagrangian ⇒ chiral interactions (by default)
- for parity, need to add more interactions (natural in EFT)
- extended 3pt-parity-even Lagrangian:

Cangemi, Chiodaroli, Johansson, AO, Pichini, Skvortsov ’23

$$
\begin{aligned}
\mathcal{L}_{\text{AHH}} &= \frac{1}{2}(D_\mu \Phi^{\{\alpha\}})_i (D^\mu \Phi_{\{\alpha\}})_i - \frac{m^2}{2}\Phi_i^{\{\alpha\}}\Phi_{i\{\alpha\}} \\
&\quad - \frac{g}{2}\sum_{k=0}^{2s-1}\frac{1}{m^{2k}}(D_{(\alpha_1(\dot\gamma_1}\cdots D_{\alpha_k)\dot\gamma_k)}\Phi^{\alpha_1\ldots\alpha_k\alpha_{k+1}\gamma_1\ldots\gamma_{2s-1-k}})_i (F_{\alpha_{k+1}}{}^{\beta_{k+1}})_{ij} \\
&\qquad\qquad \times (D^{(\dot\gamma_1(\beta_1}\cdots D^{\dot\gamma_k)\beta_k)}\Phi_{\beta_1\ldots\beta_k\beta_{k+1}\gamma_1\ldots\gamma_{2s-1-k}})_j \\
&= \frac{1}{2}\langle D_\mu\Phi|D^\mu\Phi\rangle - \frac{m^2}{2}\langle\Phi|\Phi\rangle - \frac{g}{2}\sum_{k=0}^{2s-1}\frac{1}{m^{2k}}\langle\Phi|\Big\{\big(|\overleftarrow{D}|\overrightarrow{D}|\big)^{\odot k}\odot|F^-|\Big\}|\Phi\rangle
\end{aligned}
$$

- reproduces both $\mathcal{A}_3^{(s,+)} = \frac{\langle 1_a 2^b\rangle^{\odot 2s}}{m^{2s}}\mathcal{A}_3^{(0,+)}$ **and** $\mathcal{A}_3^{(s,-)} = \frac{[1_a 2^b]^{\odot 2s}}{m^{2s}}\mathcal{A}_3^{(0,-)}$

# Rotating black holes

# QCD (hel. $\pm 1$, spin $1/2$) vs GR (hel. $\pm 2$, spin $s$)

| QCD | GR |
|---|---|
| $A(1^a, 2^b, 3^+) = -ig \frac{\langle 1^a 2^b \rangle}{m} x$ | $\mathcal{M}_3^{(s,+)} = -\frac{\kappa}{2} \frac{\langle 12 \rangle^{\odot 2s}}{m^{2s-2}} x^2$ |
| $A(1^a, 3^-, 4^+, 2^b) = \frac{ig^2 \langle 3\|1\|4] (\langle 1^a 3 \rangle [2^b 4] + [1^a 4] \langle 2^b 3 \rangle)}{(s_{13} - m^2) s_{34}}$ | $\mathcal{M}(1^{\{a\}}, 3^-, 4^+, 2^{\{b\}})$ |

$$\text{3-pt helicity factor } x = -\frac{\sqrt{2}}{m}(p_1 \cdot \varepsilon^+) = \left[\frac{\sqrt{2}}{m}(p_1 \cdot \varepsilon^-)\right]^{-1}$$

$$\mathcal{M}(1^{\{a\}}, 3^-, 4^+, 2^{\{b\}}) = \left(\frac{\kappa}{2}\right)^2 \frac{i \langle 3|1|4]^4}{(s_{13} - m^2)(s_{14} - m^2) s_{34}} \left( \frac{\langle 13 \rangle [24] + [14] \langle 23 \rangle}{\langle 3|1|4]} \right)^{\odot 2s}$$

Together with cl. limit applicable to scattering of black holes!

Guevara, AO, Vines '18

Chung, Huang, Kim, Lee '18

# Removing unphysical pole via higher-spin theory

- extended 3pt-parity-even Lagrangian:

Cangemi, Chiodaroli, Johansson, AO, Pichini, Skvortsov '23

$$\mathcal{L}_{\text{Kerr}} = \sqrt{-g}\Big\{\frac{1}{2}\langle\nabla_\mu\Phi|\nabla^\mu\Phi\rangle-\frac{m^2}{2}\langle\Phi|\Phi\rangle-\frac{1}{4}\sum_{k=0}^{2s-2}\frac{2s-k-1}{m^{2k}}\langle\Phi|\Big\{\big(|\overleftarrow{\nabla}|\overrightarrow{\nabla}|\big)^{\odot k}\odot|R_-|\Big\}|\Phi\rangle\Big\}+\mathcal{O}(R^2)$$

- chiral Riemann/Weyl curvature spinor is

$$R_{-\alpha}{}^{\beta}{}_{\gamma}{}^{\delta} := \frac{1}{4}R_{\hat\lambda\hat\mu\,\hat\nu\hat\rho}\sigma^{\hat\lambda}_{\alpha\dot\varepsilon}\bar\sigma^{\hat\mu,\dot\varepsilon\beta}\sigma^{\hat\nu}_{\gamma\dot\zeta}\bar\sigma^{\hat\rho,\dot\zeta\delta}.$$

- reproduces both $\mathcal{M}_3^{(s,+)} = \frac{\langle 1_a 2^b\rangle^{\odot 2s}}{m^{2s}}\mathcal{M}_3^{(0,+)}$ **and** $\mathcal{M}_3^{(s,-)} = \frac{[1_a 2^b]^{\odot 2s}}{m^{2s}}\mathcal{M}_3^{(0,-)}$

# Removing unphysical pole

Cangemi, Chiodaroli, Johansson, AO, Pichini, Skvortsov '23

$$M(\mathbf{1}^s,\mathbf{2}^s,3^-,4^+) = \frac{\langle 3|1|4]^4 P_1^{(2s)}}{m^{4s}s_{12}t_{13}t_{14}} - \frac{\langle \mathbf{13}\rangle[4\mathbf{2}]\langle 3|1|4]^3}{m^{4s}s_{12}t_{13}}P_2^{(2s)} + \frac{\langle\mathbf{13}\rangle\langle 3\mathbf{2}\rangle[\mathbf{1}4][4\mathbf{2}]}{m^{4s}s_{12}}\left(\langle 3|1|4]^2 P_2^{(2s-1)} + m^4\langle 3|\rho|4]^2 P_4^{(2s-1)}\right)$$
$$+ \frac{\langle\mathbf{13}\rangle\langle 3\mathbf{2}\rangle[\mathbf{1}4][4\mathbf{2}]}{m^{4s-2}s_{12}}\langle 3|1|4]\langle 3|\rho|4]\left(P_2^{(2s-2)} - m^2\langle\mathbf{12}\rangle[\mathbf{12}]P_4^{(2s-2)}\right)$$
$$+ \frac{\langle\mathbf{13}\rangle^2\langle 3\mathbf{2}\rangle^2[\mathbf{1}4]^2[4\mathbf{2}]^2}{2m^{4s-4}}\langle\mathbf{12}\rangle[\mathbf{12}]\left[(1+\eta)P_{5|\varsigma_1}^{(2s-2)} + (1-\eta)P_{5|\varsigma_2}^{(2s-2)}\right] + \alpha\, C_\alpha^{(s)}$$

After classical limit:

Kosower, Maybee, O'Connell '18

Aoude, AO '21

$$\mathcal{M}(\mathbf{1},\mathbf{2},3^-,4^+) = \mathcal{M}_4^{(0)}\Big(e^x\cosh z - w\, e^x \operatorname{sinhc} z + \tfrac{w^2-z^2}{2}E(x,y,z) + (w^2-z^2)(x-w)\tilde{E}(x,y,z)$$
$$- \frac{(w^2-z^2)^2}{2\xi}\left(\mathcal{E}(x,y,z) + \eta\,\tilde{\mathcal{E}}(x,y,z)\right)\Big) + \alpha\, C_\alpha^{(\infty)},$$

$$x := a\cdot q_\perp, \qquad y := a\cdot q,$$
$$z := |a|\frac{p\cdot q_\perp}{m}, \qquad w := \frac{a\cdot\chi\, p\cdot q_\perp}{p\cdot\chi}$$

- meshes well with Teukolsky eqn solutions
- compatible with other amplitude results
- used in cutting-edge 1-loop calculations

Bautista, Guevara, Kavanagh, Vines '21

Bjerrum-Bohr, Chen, Skowronek '23

Bohnenblust, Cangemi, Johansson, Pichini '24
Alessio, Gonzo, Canxin Shi '25

# Summary

- SU(2) covariance $\Leftrightarrow$ arbitrary spin projections
- Elegant form for two-quark amplitudes with
  - all gluons of same helicity (e.g. all plus)
  - one gluon of different helicity (e.g. one minus)

  AO '18
- New analytic results for four-quark amplitudes

  Lazopoulos, AO, Shi '21
- Applicable to any massive particles with spin, black holes

  Guevara, AO, Vines '18, 19
  Aoude, AO '21
- New chiral-field approach to massive higher spins

  AO, Skvortsov '22
- New results for quantum higher spins and rotating black holes

  Cangemi, Chiodaroli, Johansson, AO, Pichini, Skvortsov '22,'23

Thank you!

# Backup slides

# Solution to BCJ relations

Bern, Carrasco, Johansson '08
Johansson, AO '15

BCJ relations:

$$A(\underline{1},\overline{2},\alpha,3,\beta) = \sum_{\sigma\in S(\alpha)\sqcup\!\sqcup\beta} A(\underline{1},\overline{2},3,\sigma)\prod_{i=1}^{|\alpha|}\frac{\mathcal{F}(q,\sigma,1|i)}{s_{2,\alpha_1,\ldots,\alpha_i}-m_2^2}$$

Kleiss-Kuijf basis of $(n-2)!$ primitives $\left\{A(\underline{1},\overline{2},\sigma)\right\}$

$\Rightarrow$ BCJ basis of $(n-3)!$ primitives $\left\{A(\underline{1},\overline{2},3,\sigma)\right\}$

# Solution to BCJ relations for QCD

Johansson, AO '15

General BCJ relations:

$$A(\underline{1},\overline{2},\alpha,\underline{q},\beta) = \sum_{\sigma\in S(\alpha)\sqcup\beta} A(\underline{1},\overline{2},\underline{q},\sigma) \prod_{i=1}^{|\alpha|} \frac{\mathcal{F}(q,\sigma,1|i)}{s_{2,\alpha_1,\ldots,\alpha_i} - m_2^2},$$

where $\alpha$ is purely gluonic

Melia basis of $(n-2)!/k!$ primitives

$$\big\{A(\underline{1},\overline{2},\sigma)\;\big|\;\sigma\in \text{Dyck}_{k-1}\times\{\text{gluon insertions}\}_{n-2k}\big\}$$

$\Rightarrow$ new BCJ basis of $(n-3)!(2k-2)/k!$ primitives

$$\big\{A(\underline{1},\overline{2},\underline{q},\sigma)\;\big|\;\{\underline{q},\sigma\}\in \text{Dyck}_{k-1}\times\{\text{gluon insertions in }\sigma\}_{n-2k}\big\}$$

# Helicity basis

Arkani-Hamed, Huang, Huang '17

Take $p^\mu = (E, P\cos\varphi\sin\theta, P\sin\varphi\sin\theta, P\cos\theta)$

$$|p^a\rangle = \lambda_{p\alpha}^{\ a} = \begin{pmatrix} \sqrt{E-P}\cos\frac{\theta}{2} & -\sqrt{E+P}e^{-i\varphi}\sin\frac{\theta}{2} \\ \sqrt{E-P}e^{i\varphi}\sin\frac{\theta}{2} & \sqrt{E+P}\cos\frac{\theta}{2} \end{pmatrix}$$
$$[p^a| = \tilde{\lambda}_{p\dot{\alpha}}^{\ a} = \begin{pmatrix} -\sqrt{E+P}e^{i\varphi}\sin\frac{\theta}{2} & -\sqrt{E-P}\cos\frac{\theta}{2} \\ \sqrt{E+P}\cos\frac{\theta}{2} & -\sqrt{E-P}e^{-i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$

Then

$$s^\mu(u_p^a) = \frac{1}{2m}\bar{u}_{pa}\gamma^\mu\gamma^5 u_p^a = (-1)^{a-1}s_p^\mu$$
$$s_p^\mu = \frac{1}{m}(P, E\cos\varphi\sin\theta, E\sin\varphi\sin\theta, E\cos\theta)$$

# Comparison with earlier results

Older reference-momentum-dep. spinors:

$$\bar{u}_p^{a=1} = \begin{pmatrix} -\langle p^1| \equiv \frac{m\langle q|}{\langle q\,p^\flat\rangle} \\ [p^1| \equiv [p^\flat| \end{pmatrix} = \bar{u}_p^-(q) \qquad v_p^{a=1} = \begin{pmatrix} -|p^1\rangle \equiv -\frac{m|q\rangle}{\langle p^\flat\,q\rangle} \\ |p^1] \equiv |p^\flat] \end{pmatrix} = v_p^-(q)$$

$$\bar{u}_p^{a=2} = \begin{pmatrix} -\langle p^2| \equiv -\langle p^\flat| \\ [p^2| \equiv -\frac{m[q|}{[q\,p^\flat]} \end{pmatrix} = -\bar{u}_p^+(q) \qquad v_p^{a=2} = \begin{pmatrix} -|p^2\rangle \equiv -|p^\flat\rangle \\ |p^2] \equiv \frac{m|q]}{[p^\flat\,q]} \end{pmatrix} = -v_p^+(q)$$

Kleiss, Stirling '86, Dittmaier '98, Schwinn, Weinzierl '05)

$\Rightarrow$ Analytically retrieve older non-SU(2)-covariant formulae

Schwinn, Weinzierl '07

$$A(\underline{1}^1, 3^-, 4^+, \dots, n^+, \overline{2}^1) = 0$$

$$A(\underline{1}^1, 3^-, 4^+, \dots, n^+, \overline{2}^2) = \frac{-i\langle 2^\flat 3\rangle}{\langle 1^\flat 3\rangle\langle 34\rangle\dots\langle n{-}1|n\rangle} \sum_{k=4}^{n} \frac{\langle 3|\not{p}_1\,\not{p}_{3\dots k}|3\rangle^2}{s_{3\dots k}\langle 3|\not{p}_1\,\not{p}_{3\dots k}|k\rangle}$$
$$\times \left\{ \delta_{k=n} + \delta_{k\neq n} \frac{m^2\langle k|k{+}1\rangle\langle 3|\not{p}_{3\dots k}\prod_{j=k+1}^{n-1}\{(s_{13\dots j}-m^2)-\not{p}_j\,\not{p}_{13\dots j}\}|n]}{(s_{13\dots k}-m^2)\dots(s_{13\dots(n-1)}-m^2)\langle 3|\not{p}_1\,\not{p}_{3\dots k}|k{+}1\rangle} \right\}$$

$$A(\underline{1}^2, 3^-, 4^+, \dots, n^+, \overline{2}^1) = \frac{i\langle 1^\flat 3\rangle}{\langle 2^\flat 3\rangle\langle 34\rangle\dots\langle n{-}1|n\rangle} \sum_{k=4}^{n} \frac{\langle 3|\not{p}_1\,\not{p}_{3\dots k}|3\rangle^2}{s_{3\dots k}\langle 3|\not{p}_1\,\not{p}_{3\dots k}|k\rangle}$$
$$\times \left\{ \delta_{k=n} + \delta_{k\neq n} \frac{m^2\langle k|k{+}1\rangle\langle 3|\not{p}_{3\dots k}\prod_{j=k+1}^{n-1}\{(s_{13\dots j}-m^2)-\not{p}_j\,\not{p}_{13\dots j}\}|n]}{(s_{13\dots k}-m^2)\dots(s_{13\dots(n-1)}-m^2)\langle 3|\not{p}_1\,\not{p}_{3\dots k}|k{+}1\rangle} \right\}$$

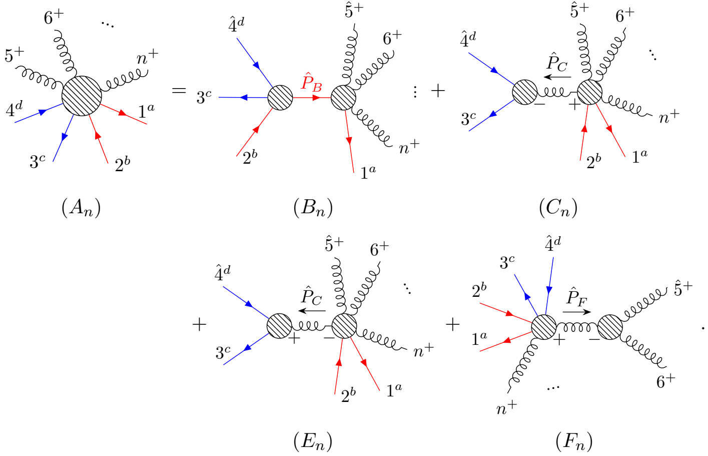
$$A(\underline{1}^2, 3^-, 4^+, \dots, n^+, \overline{2}^2) = \frac{i\langle 1^\flat 2^\flat\rangle}{m\langle 34\rangle\dots\langle n{-}1|n\rangle} \sum_{k=4}^{n} \frac{\langle 3|\not{p}_1\,\not{p}_{3\dots k}|3\rangle^2}{s_{3\dots k}\langle 3|\not{p}_1\,\not{p}_{3\dots k}|k\rangle} \left[1 + \frac{s_{3\dots k}\langle 3\,2^\flat\rangle}{\langle 3|\not{p}_{3\dots k}\,\not{p}_1^\flat|2^\flat\rangle}\right]$$
$$\times \left\{ \delta_{k=n} + \delta_{k\neq n} \frac{m^2\langle k|k{+}1\rangle\langle 3|\not{p}_{3\dots k}\prod_{j=k+1}^{n-1}\{(s_{13\dots j}-m^2)-\not{p}_j\,\not{p}_{13\dots j}\}|n]}{(s_{13\dots k}-m^2)\dots(s_{13\dots(n-1)}-m^2)\langle 3|\not{p}_1\,\not{p}_{3\dots k}|k{+}1\rangle} \right\}$$

# Four-quark amplitudes with plus-helicity gluons

Lazopoulos, AO, Shi '21

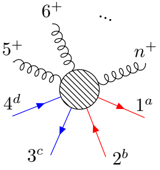
gluon insertions between like-flavored quarks (see above ↑) ✓

gluon insertions between distinctly flavored quarks (see below ↓) ✓



# Four-quark amplitudes with plus-helicity gluons

Lazopoulos, AO, Shi '21




$$A(1^a,2^b,3^c,4^d,5^+,\ldots,n^+) = \frac{-i}{\prod_{j=5}^{n-1}\langle j|j{+}1\rangle}\Bigg\{\frac{1}{\prod_{j=5}^{n-1}D_{4\ldots j}[d_n^5|P_{12}|3|P_{12}|1|d_n^5]}$$

$$\times\Bigg[\frac{[e_n|5][d_n^5|1|P_{123}|d_n^5]}{D_{123}\langle n|P_{123}|d_{n-1}^5]}\Big(\langle 13\rangle[2|d_n^5]+\langle 23\rangle[1|d_n^5]\Big)+\frac{1}{s_{12}\langle n|P_{12}|3|P_{12n}|d_{n-1}^5]}$$

$$\times\Big[M\langle 12\rangle[d_n^5|P_{12}|3|d_n^5]\Big([3|d_n^5]\langle 4|P_{12}|d_n^5]+[e_n|5]\langle 3|P_{12}|d_n^5]\Big)$$

$$+m\langle 34\rangle[d_n^5|2|1|d_n^5]\Big([1|d_n^5]\langle 2|P_{12}|d_n^5]+[2|d_n^5]\langle 1|P_{12}|d_n^5]\Big)\Big] \qquad (3.33a)$$

$$+\frac{[d_n^5|1|P_{123}|d_n^5]}{D_{1n}\langle n|P_{123n}|d_{n-1}^5]-D_{123n}\langle n|P_{1n}|d_{n-1}^5]}\Big[\langle 24\rangle[1|d_n^5][3|d_n^5]+\langle 23\rangle[1|d_n^5][e_n|5]$$

$$-\frac{[d_n^5|1|P_{123}|d_n^5]}{D_{123}}\Big([13]\langle 24\rangle+\langle 14\rangle[23]\Big)+\frac{[d_n^5|1|P_{123}|d_n^5]}{D_{123}^2\langle n|P_{123}|d_{n-1}^5]}$$

$$\times\Big(\langle 13\rangle[2|P_{123}|n\rangle[e_n|5]+\langle 23\rangle[1|P_{123}|n\rangle[e_n|5]-m\langle 13\rangle\langle 4n\rangle[2|d_n^5]-m\langle 23\rangle\langle 4n\rangle[1|d_n^5]\Big)\Big]\Bigg]$$

$$+\sum_{i=5}^{n-1}\frac{M\langle i|i{+}1\rangle[a_{i+1}^n|d_i^5][d_i^5|P_{23}|P_{4\ldots i}|d_i^5]}{[d_i^5|P_{3\ldots i}|2|3|P_{3\ldots i}|d_i^5]\prod_{l=i}^{n-1}D_{2\ldots l}\prod_{k=5}^{i-1}D_{4\ldots k}\big(D_{4\ldots i}\langle i+1|P_{2\ldots i}|d_i^5]-D_{2\ldots i}\langle i+1|P_{4\ldots i}|d_i^5]\big)}$$

$$\times\Bigg[\frac{[d_i^5|P_{23}|P_{4\ldots i}|d_i^5]}{D_{4\ldots(i-1)}\langle i|P_{2\ldots(i-1)}|d_{i-1}^5]-D_{2\ldots(i-1)}\langle i|P_{4\ldots(i-1)}|d_{i-1}^5]}\Big[\langle 1|P_{2\ldots i}|3]\langle 24\rangle+M\langle 14\rangle[23]$$

$$-\frac{1}{D_{4\ldots i}\langle i|P_{4\ldots(i-1)}|d_{i-1}^5]}\Big(\langle 1|P_{23}|P_{4\ldots i}|i\rangle\langle 23\rangle[e_i|5]-D_{4\ldots i}\langle 1i\rangle\langle 24\rangle[3|d_i^5] \qquad (3.33b)$$

$$+m\langle 1|P_{2\ldots i}|d_i^5]\langle 23\rangle\langle 4i\rangle+M\langle 13\rangle\langle i|P_{4\ldots i}|2][e_i|5]+mM\langle 13\rangle[2|d_i^5]\langle 4i\rangle+m^2\langle 1i\rangle\langle 23\rangle[e_i|5]\Big)\Big]$$

$$+\frac{1}{\langle i|P_{4\ldots(i-1)}|d_{i-1}^5]}\Big[\langle 1|P_{23}|d_i^5]\langle 23\rangle[e_i|5]-\langle 1|P_{4\ldots i}|d_i^5][3|d_i^5]\langle 24\rangle+M\langle 13\rangle[2|d_i^5][e_i|5]\Big]\Bigg]$$

$$+\sum_{i=5}^{n-1}\frac{M\langle 12\rangle\langle i|i{+}1\rangle\big([3|d_i^5]\langle 4|P_{3\ldots i}|d_i^5]+\langle 3|P_{3\ldots i}|d_i^5][e_i|5]\big)[d_i^5|P_{3\ldots i}|3|d_i^5]}{s_{3\ldots i}\prod_{l=i}^{n-1}D_{2\ldots l}\prod_{k=5}^{i-1}D_{4\ldots k}\langle i{+}1|P_{3\ldots i}|3|P_{3\ldots i}|d_i^5]\langle i|P_{3\ldots(i-1)}|3|P_{3\ldots(i-1)}|d_{i-1}^5]}$$

$$\times\frac{[a_{i+1}^n|P_{3\ldots i}|2|3|P_{3\ldots i}|d_i^5]}{[d_i^5|P_{3\ldots i}|2|3|P_{3\ldots i}|d_i^5]}$$

$$+\sum_{i=5}^{n-1}\frac{m\langle 34\rangle\langle i|i{+}1\rangle}{s_{3\ldots i}\prod_{k=5}^{i-1}D_{4\ldots k}\langle i|P_{3\ldots(i-1)}|3|P_{3\ldots(i-1)}|d_{i-1}^5]}$$

$$\times\Bigg[\frac{[d_i^5|P_{3\ldots i}|2|1|P_{3\ldots i}|d_i^5]\big(\langle 1|P_{3\ldots i}|d_i^5][2|P_{12}|P_{3\ldots i}|d_i^5]+\langle 2|P_{3\ldots i}|d_i^5][1|P_{12}|P_{3\ldots i}|d_i^5]\big)}{s_{12}\langle n|P_{12}|2|P_{3\ldots i}|d_i^5]\langle i{+}1|P_{3\ldots i}|3|P_{3\ldots i}|d_i^5]}$$

$$+\frac{Ms_{3\ldots i}[d_i^5|2|P_{3\ldots i}|d_i^5][a_{i+1}^n|d_i^5]\big(\langle 12\rangle[d_i^5|P_{3\ldots i}|2|d_i^5]+\langle 1|P_{3\ldots i}|d_i^5]\langle 2|P_{3\ldots i}|d_i^5]\big)}{\prod_{j=i}^{n-1}D_{2\ldots j}\langle i{+}1|P_{3\ldots i}|2|P_{3\ldots i}|d_i^5][d_i^5|P_{3\ldots i}|2|3|P_{3\ldots i}|d_i^5]}$$

$$+\sum_{k=i+1}^{n-1}\frac{M\langle k|k{+}1\rangle[d_i^5|P_{3\ldots i}|2|P_{3\ldots k}|P_{3\ldots i}|d_i^5][a_{k+1}^n|P_{3\ldots k}|P_{3\ldots i}|d_i^5]}{s_{3\ldots k}\prod_{j=k}^{n-1}D_{2\ldots j}\langle k|P_{3\ldots k}|2|P_{3\ldots i}|d_i^5]\langle k{+}1|P_{3\ldots k}|2|P_{3\ldots i}|d_i^5]\langle i{+}1|P_{3\ldots i}|3|P_{3\ldots i}|d_i^5]}$$

$$\times\Big(\langle 12\rangle[d_i^5|P_{3\ldots i}|2|P_{3\ldots k}|P_{3\ldots i}|d_i^5]+s_{3\ldots k}\langle 1|P_{3\ldots i}|d_i^5]\langle 2|P_{3\ldots i}|d_i^5]\Big)\Bigg]\Bigg\}.$$